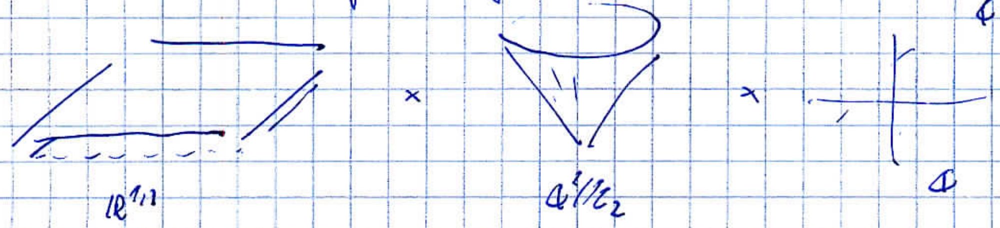
- The embedding of (susy) gauge theories in string theory via D-branes has been fruitful and inspiring in many directions. It's unnecessary to recall that Maldacena's conjecture ~~started from~~ was rooted in the realization of $\mathcal{N}=4$ SYM by means of D3 branes in flat space and in the gravitational profile they induce.
- In less-supersymmetric, non-conformal cases (we will focus on $\mathcal{N}=2$, in $d=4$) the profile of certain bulk fields must depend on some transverse direction, representing the (complexified) energy scale, so as to ~~reproduce~~ account for the running of the gauge theory.
- This was checked quite a long time ago at the perturbative level, i.e. for the logarithmic running. (re
- In $\mathcal{N}=2$, there is a whole series of corrections to the running for the exact coupling due to instantons. Tremendous advances have been made ~~[illegible]~~ within Field Theory regarding instanton effects in the last two decades:
  * the exact solution ~~of~~ for the l.e.e.t. in the Coulomb branch was found by S & W on the basis of symmetry and duality arguments.
  * this was explicitly reproduced by computing multi-instanton contributions by means of localization techniques à la Nekrasov. (Such ideas and techniques were much inspired by string theory issues)
- What is the effect of these non-perturbative corrections on the "gravity" side of $\mathcal{N}=2$ gauge/gravity pairs?
  * One can exploit symmetry and duality arguments to determine some background which incorporates the SW solution, like in Witten's M-theory construction based on D4/NS5 configurations. [Analogue of SW in F.T.]

* One can also try to compute directly the multi-instanton corrections to the bulk field profiles [analogue of Nekrasov for F.T.]
  → This is exactly what we're doing, and what I'll try to sketchily describe here.

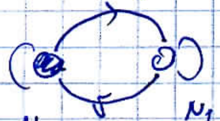
· I concentrate here on a specific prototypical case: fractional D3-branes (fD3) at the non-isolated $\mathbb{C}^2/\mathbb{Z}_2$ singularity:



- fD3 cover $\mathbb{R}^{1,3}$, are stuck at the tip of $\mathbb{C}^2/\mathbb{Z}_2$, are pointlike (but can move) in $\mathbb{C}$

- they come in two types (0 and 1) associated to the two irreps of $\mathbb{Z}_2$

- A collection of N_0, N_1 fD3's of the two types supports an $\mathcal{N}=2$ SYM gauge theory



$S(U(N_0)\times U(N_1))$ w/ hypers $(N_0, \bar{N}_1)$, $(\bar{N}_0, N_1)$ ($\to SU(N_0)\times SU(N_1)$ upon decoupling of the diag. part)

. Though one can proceed in the general case, here i shall; us consider ~~the case $N_0 \times N_1$ ...~~ – the dynamics of the $\underline{SU(N_0)}$ gauge group only. In this case we have a $\underline{SU(N_0)}$ theory with $\underline{2N_1}$ flavours $\Rightarrow$ $\underline{\beta\text{-function} \propto 2(N_0-N_1)}$

- the conformal case $N_1 = N_0 = N$ ($\Rightarrow N_f = 2N$) $\beta \sim 0$. The theory is non-trivial (even perturbatively) in presence of masses and receives instanton corrections.
  N.B. By suitably decoupling (some) flavours one recovers in the end non-conformal cases, including the pure $SU(N)$ theory.

• The fD3's couple to closed string fields, in particular also to fields from the twisted sector.

They represent sources for the twisted scalars b (NS-NS sector) and c (R-R sector), that can be grouped into a complex scalar

$$t \equiv c + \underset{\substack{\uparrow \\ \text{dilaton-axion}}}{\tau}\, b \longrightarrow c + \underset{\text{constant in our situation}}{\frac{i}{g_s}}\, b$$

This field is actually part of a superfield T collecting the twisted d.o.f., ~~with~~ the relevant components for us are

$$T = t + \dots + \vartheta^4 \frac{\partial^2}{\partial z^2}\bar{t} + \dots$$

The twisted fields are 6-dim, do not propagate along $\mathbb{C}^2/\mathbb{Z}_2$. fD3's are extended along space-time and point-like in $\mathbb{C}$. ~~The fields~~ With all the fD3's in the origin, t gets a non-trivial profile along $\mathbb{C}$ of logarithmic type:

$$i\pi\, t(z) = i\pi\, \tau_0 - 2(\underset{\uparrow}{N_0} - \underset{\uparrow}{N_1}) \log \frac{z}{\mu} \quad \leftarrow \text{cutoff scale (unnecessary for } N_0 = N_1)$$

the two types of branes couple to t with different signs

→ at this level it appears that

$$t \Longleftrightarrow \tau_{gauge} = \frac{\vartheta_{YM}}{\pi} + i\frac{8\pi}{g^2_{YM}} \quad \text{of the } SU(N_0) \overset{N_f = 2N_1}{\text{theory}}.$$

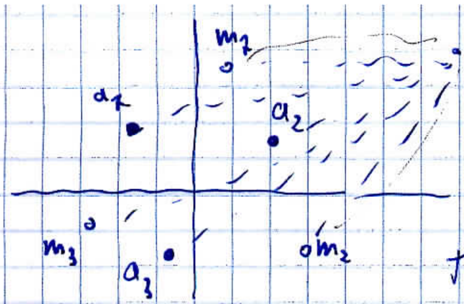
we will see that this simple correspondence has to be modified at the np. level.

The above profile solves the e.o.m

$$\underset{\substack{\uparrow \\ \text{from bulk action}}}{\Box t} = 8i\, \cancel{\delta^2(z)}\, \underset{\substack{\uparrow \\ \text{from fD3's action}}}{(N_0 - N_1)\, \delta^2(z)}$$

- Placing the fD3's at different positions:

$$\underset{\substack{\uparrow \\ \text{adjoint field in the } SU(N_0) \text{ theory}}}{\langle \phi_0 \rangle} = \mathrm{diag}(a_1, a_2, \dots a_N) \quad ; \quad \underset{\substack{\uparrow \\ \text{untwisted fused}}}{\langle \phi_1 \rangle} = \mathrm{diag}(\underset{\substack{\uparrow \\ \to \text{masses of the quarks}}}{m_1, m_2, \dots m_N})$$



the profile becomes

$$i\pi t(z) = i\pi t_0 - 2\,\mathrm{Tr}_{N_0}\log\frac{z-\langle\phi_0\rangle}{\mu} + 2\,\mathrm{Tr}_{N_1}\log\frac{z-\langle\phi_1\rangle}{\mu}$$

this solves the e.o.m. ~~$\nabla t = 8\,J_{cl}\,\delta^2(z)$~~

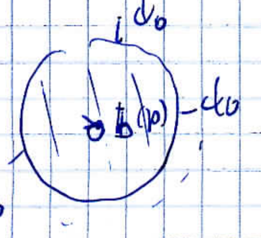
corresponding to δ-sources in each location, which can also be seen as

$$\nabla t = 8\,J_{cl}\,\delta^2(z)$$

with J_{cl} = operator which ~~shifts~~ translates the δ-function to the various singularities.

$$J_{cl} = i\left(\mathrm{Tr}_{N_0}\, e^{i\bar p\langle\phi_0\rangle} - \mathrm{Tr}_{N_1}\, e^{i\bar p\langle\phi_1\rangle}\right) \qquad \left(\bar p = -i\frac{\partial}{\partial \bar z}\right)$$

~~The source~~ This follows from the source term on the fD3's induced by disk-diagrams providing interactions of $\bar t$ with the ϕ_0 (or ϕ_1) open string fields: e.g.

$\sum_\ell \frac{1}{\ell!}$ 

$\xrightarrow{\text{(explicit conf. amplitude)}}$

$$\frac{\pi}{g_s}\sum_\ell \frac{1}{\ell!}\,\mathrm{tr}\langle\phi_0\rangle^\ell\,(i\bar p)^\ell\, b \;\to\; \frac{\pi}{g_s}\,\mathrm{Tr}_{N_0}\, e^{i\bar p\langle\phi_0\rangle}\, b$$

simplify

→ (hypothesis, taking into account fD3's of the other type) → source term

$$-i\pi\left(\mathrm{Tr}_{N_0}\, e^{i\bar p\langle\phi_0\rangle} - \mathrm{Tr}_{N_1}\, e^{i\bar p\langle\phi_1\rangle}\right)\bar t$$

This can be written in susy fashion as a superpotential term depending on the super field T (actually on its fluctuation part δT):

$$\delta\Gamma_{cl} = i\pi\left(\mathrm{Tr}_{N_0}\, e^{i\bar p\langle\phi_0\rangle} - \mathrm{Tr}_{N_1}\, e^{i\bar p\langle\phi_1\rangle}\right)\frac{\delta T}{\bar p^2}$$

Indeed, integrating $\int d^4\theta$ this superpotential term and taking the θ's from δT we retrieve the above source term. ~~.....~~ In this form, we have

$$J_{cl} = \frac{\bar p^2}{\pi}\,\frac{\delta\Gamma_0}{\delta T}\Big|_{\Phi_0 \to \langle\phi_0\rangle}$$

after ∫ on the fD3's d.o.f. ← singular one $(\phi_1 \to \langle\phi_1\rangle)$

this expression is best suited to be extended to include non-perturbative corrections.

- The n.p. corrections we consider correspond to the inclusion of fractional D(-1)-branes, a.k.a. fractional D-instantons. As for the fD3's, fD(-1)'s are of two types, 0 and 1. For simplicity here we consider only those of type 0 (same type as gauged fD3's)

- fD(-1)$_0$ ~~are~~ represent the gauge instantons. Including their contributions to the eff. action for the D3 d.o.f. reproduces Nekrasov's multi-instanton calculations and thus accounts for the "microscopic" derivation of SW results.
  To see this it is crucial to study
  - the spectrum of moduli, i.e. modes of open strings with at least one endpoint on a D(-1)
  - the moduli action describing the interaction of moduli among themselves and with the D3 gauge fs and matter fields. Such interactions arise from (mixed) disk diagrams.
- fD-instanton moduli also interact with closed string (bulk) fields. The moduli action is therefore ~~to be not taken written~~ in the form (for what matters to our purpose)

$$S_{mod}(\mathcal{M}_k, \Phi, T)$$

moduli $\mathcal{M}_k$; gauge fields Φ; twisted closed str. fields T

gauge fields $\Phi = \phi_0 + \ldots\theta\lambda + \theta\sigma^{\mu\nu}\theta\, T_{\mu\nu} + \ldots$

The effective n.p. prepotential is extracted (as usual) by integrating over the moduli:

$$F_{n.p}(\Phi, T) = \sum_k \int d\mathcal{M}_k\, e^{-S_{mod}(\mathcal{M}_k, \Phi, T)}$$

(with care!)

Taking the linear term, $-\delta T$ and expanding the exponential $e^{i\pi\chi}$ we get

$$\delta F_{n.p} = \lim_{\varepsilon_{1,2}\to 0} \frac{\varepsilon_1\varepsilon_2}{Z} \sum_k q_0^k \sum_\ell \frac{(i\pi)^\ell}{\ell!} \int d\mathcal{M}_k\, e^{-S''}\, \mathrm{Tr}\,\chi^\ell \,(-i\pi\,\delta T)$$

(calculated at $T=0$ → i.e. "gauge" instanton partition function

"χ-correlators in the instanton moduli space"

Ref

. It is possible to show (refs. ai romani) that such integrals are exactly those that arise when computing, in the gauge theory, the chiral correlators. (elements of the chiral ring)

. In fact, one has

$$\frac{\langle \mathrm{Tr}\,\phi_0^{\ell+2}\rangle_{inst}}{(\ell+2)!} = -\frac{1}{\ell!} \lim_{\varepsilon_{1,2}\to 0} \frac{\varepsilon_1\varepsilon_2}{Z} \sum_k q_0^k \int d\mathcal{M}_k\, e^{-S''}\, \mathrm{tr}\,\chi^\ell$$

⇒ substituting back, with some reshuffling we express in the end ~~the linear part of~~ $\frac{\delta F^{n.p.}}{\delta T}$, hence the classical source for the twisted fields, in a very simple way:

adding the classical and n.p. part, i.e. $\delta F = \delta F_{cl} + \delta F_{n.p.}$, we can in the end write $\delta F = \delta F_{cl} + \delta F_{n.p.}$

$$\frac{\delta F}{\delta T} = \frac{i\pi}{\hbar^2}\left(\langle \mathrm{Tr}_{N_0}\, e^{i\pi\phi_0}\rangle - \mathrm{Tr}_{N_1}\, e^{i\pi\langle\phi_1\rangle}\right)$$

exact ↑ the effect of the instanton corrections is to promote $\mathrm{Tr}_{N_0} e^{i\pi\langle\phi_0\rangle}$ to $\langle \mathrm{Tr}_{N_0} e^{i\pi\phi_0}\rangle$

big v.e.v. exact correlator !

→ ~~Lorentz-(all five) simple~~

⇒ the effect is thus extremely easy to describe in this way

~~(Proof of some the~~

. The modification of the profile of the field t which solves the e.o.m. with the ~~source~~ source is then just of the same type

$$\boxed{\lim t(z) = i\pi t_0 - 2\left\langle \mathrm{Tr}_{N_0} \log\frac{z-\phi_0}{\mu}\right\rangle + 2\,\mathrm{Tr}_{N_1}\log\frac{z-\langle\phi_1\rangle}{\mu}}$$

exact correlator

. The chiral correlators $\langle \mathrm{Tr}\, \phi_0^{\ell} \rangle$ and hence the correlator appearing in this exact profile can be computed order by order in q_0 via multi-instanton calculus and localization techniques as sketched above.

. However, once the link to this gauge theory exact correlator has been demonstrated by "microscopic" techniques, we can resort to the exact description of the gauge theory à la SW. Ref

As shown in (...) the SW curve (in appearance ~~with multi-instanton calcs~~) can be written as

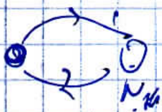
$$y^2 = P^2(z) - g^2 Q(z) \qquad \text{with} \qquad P(z) = \prod_{u=1}^{N} (z - e_u) \sim \sum_{\ell=0}^{N} u_\ell\, z^{N-\ell}$$

$$Q(z) = \prod_{u=1}^{N} (z - m_u)^2$$

$$\left(g^2 = \frac{4 q_0}{(1+q_0)^2} \right)$$

e_u ~ quantum analogs of the classical positions a_u $\qquad \sum e_u = 0$

m = masses of the $2N$ hypers



$(m_1, \dots m_N, m_1, \dots m_N)$ $\qquad \sum m_u = 0$

The l.e. exact coupling matrix is then $\tau^{ij} = \dfrac{\partial a_{Di}}{\partial a_j}$ where

megai

$$a_i(e) = \frac{1}{2\pi i} \oint_{\gamma_i} \lambda \,, \qquad a_{Di}(e) = \frac{1}{2\pi i} \oint_{\tilde{\gamma}_i} \lambda \qquad \{\gamma_i, \tilde{\gamma}_i\} \text{ symplectic basis of cycles}$$

λ (SW differential) is given by

$$\lambda = z\, \psi'(z)\, dz$$

$$\psi(z) = \log \frac{P(z) + \sqrt{P^2(z) - g^2 Q(z)}}{\mu^N}$$

μ^N ← scale

. $\psi''(z)$ represent the generating function for chiral correlators

$$\psi'(z) = \left\langle \mathrm{Tr} \frac{1}{z - \phi} \right\rangle$$

Integrating w.r.t z (and fixing suitably $O(z^0)$ terms) one gets [illegible]

$$\langle \mathrm{Tr} \log \frac{z-\phi}{\mu} \rangle = \frac{1}{2} \log \frac{P+\sqrt{P^2-g^2 Q}}{P-\sqrt{P^2-g^2 Q}} + \text{suitable ~~[illegible]~~}$$

so that at the end, the twisted profile reads [illegible] we will also use

$$\boxed{i\pi t(z) = \log \frac{P(z)-\sqrt{P^2(z)-g^2 Q(z)}}{P(z)-\sqrt{P^2(z)-g^2 Q(z)}}} \to q = e^{i\pi t} = \frac{P-\sqrt{P^2-g^2 Q}}{P+\sqrt{P^2-g^2 Q}} = [illegible]$$

- This ~~solution~~ agrees with the solution put forward (in the pure SU(N) case) by Stefano Cremonesi, in ... (after a series of works in the literature); on ~~the basis~~ of [illegible] arguments linking the IIB set-up to the IIA [illegible]-NS5 config realized by Witten by embedding into M-theory. (references) ... [I do not know if [illegible] going into some detail or not..]

- Relation between the twisted scalar t and the effective gauge coupling

The relation between $i\pi t(z)$ and the effective gauge coupling matrix τ_{ij} is not as simple as it appears at the perturbative level.
To investigate it, we limit ourselves to a ~~core~~ slice of the moduli space (~~which we~~ (called "special vacuum,,) which is ~~[illegible]~~ characterized by having

$$\begin{cases} \langle \mathrm{Tr}\, \phi^\ell \rangle = 0 & \text{for } \ell = 1..N-1 \quad \frac{1}{N}\langle \mathrm{Tr}\, \phi^N \rangle = [illegible] \\ \mathrm{Tr}\, m^\ell = 0 & \quad // \qquad\quad , \quad \mathrm{Tr}\, m^N = m^N \end{cases}$$

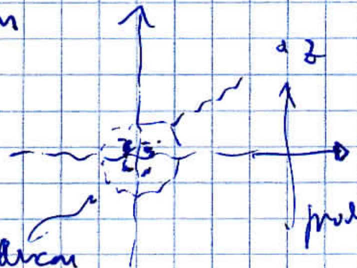
~ symmetric disposition like roots of 1 both of a_i's and m_u's

→ the special vacuum is parametrized only by V (and m); the adimensional coupling matrix can only be a function $\tau_{ij}\left(\frac{m^N}{V^{1/N}}, q_0\right)$

- We propose (and check) that this coupling matrix at the special vacuum is associated in a simple but non-trivial way to the twisted profile $t(z)$

produced by fD3's placed in a configuration which is a "very special" vacuum,
where $\langle \mathrm{Tr}\, \phi^{\ell} \rangle = 0 \quad \ell = 1..N$ (so $V = \mathrm{Tr}\,\phi^N \to 0$ in this case)

This corresponds to $e_u = 0$, → (classically) to a_u making ("enhancon ring")
of radius $\sim m$ (the fD3 remain on such a config..)



electron branes

produce a profile $i\pi\tau(z)$ in each point z
$\Rightarrow q^{(z)} = e^{i\pi\tau(z)}$ is actually $\to q(\frac{m}{z}, q_0)$

⇒ If we identify $\boxed{V = z^N}$, then the coupling of the special vacuum $\cancel{\tau^{ij}(\frac{m}{V^{1/N}}, q_0)}$

~~is written~~ is given by

$$\boxed{\tau^{ij}\left(\frac{m}{z}, q_0\right) = \tau^{ij}\left(0, q\left(\frac{m}{z}, q_0\right)\right)} \quad ☆$$

$\boxed{\text{Justified directly by manipulation of SW curves}}$

coupling in a massive theory at special vacuum with $V = z^N$

UV cplg

coupling in a massless, conformal theory

with UV coupling $q(\frac{m}{z}, q_0)$

Indeed $q = e^{i\pi\tau(z)} = \dfrac{P(z) - \sqrt{P^2(z) - q^2 b}}{P + \sqrt{P^2 - q^2 b}} = q_0\left(1 - \dfrac{m^N}{z^N}\right)^2\left(1 + q_0\, \dfrac{2m^{2N}}{z^{2N}} + \ldots\right)$

is actually a function $q(\frac{m}{z}, q_0)$.

. So in the r.h.s. of ☆ it appears the non-trivial relation in a massless $N_f = 2N$ theory between the so-called "UV coupling" $q_0^{\,i\pi\tau_0}$ (the one appearing in the ~~Nekrasov~~ "Nek." multi-instanton calculus, weighing the instanton contributions) and the gauge coupling.

• For instance in the SU(2) case with $N_f = 4$, ~~we have~~ one has the UV/IR relation

$$i\pi \overbrace{\tau(0, q)}^{\equiv \tau} = \log q_0 + i\pi - \log 16 + \frac{1}{2} q_0 + \frac{13}{64} q_0^2 + \frac{23}{192} q_0^3 + \ldots$$

whose inverse reads $q_0 = -16\left(e^{i\pi\tau} + 8\, e^{2\pi i\tau} + 44\, e^{3\pi i\tau} \dots\right) = -16\, \frac{\eta^8(4\tau)}{\eta^8(\tau)} \quad \left(= \frac{\vartheta_2^4}{\vartheta_4^4}\right)$ (can be proven from the SW curve)

And indeed if we compute (using our solution for $q(\frac{m}{z}, q)$) we get

$$i\pi\,\tau\left(0, q\left(\frac{m}{z}, q_0\right)\right) = \log q_0 + i\pi - \log 16 + 2\log\left(1-\frac{m^2}{z^2}\right) + q_0\left(\frac{1}{2} - \frac{m^2}{z^2} + \frac{5}{2}\frac{m^4}{z^4} + \dots\right)$$

which is exactly the running coupl. of SU(2), $N_f=4$ with masses $(m,-m,m,-m)$ at special vacuum with $v = \mathrm{Tr}\,\phi^2 = z^2$ that arises from multi-inst. instanton calculus

- SU(3) In this the coupling matrix has ~~[illegible]~~ exactly classical structure

$$2\pi i\, \tau^{ij} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \pi i \tau$$

which is preserved by perturb. and n.p. corrections

At zero masses one gets the UV/IR relation

$$i\pi\tau(0, q_0) = i\pi\tau = \log q_0 + i\pi - \log 27 + \frac{4}{9} q_0 + \frac{14}{81} q_0^2 + \frac{1948}{19683} q_0^3 + \dots$$

inverting it, $q_0 = -27\left(e^{i\pi\tau} + 12\, e^{2\pi i\tau} + 90\, e^{3\pi i\tau} + \dots\right) = -27\, \frac{\eta^{12}(3\tau)}{\eta^{12}(\tau)}$ (can be proven by comparing the genus 2 curve)

We find that

$$i\pi\tau\left(0, q\left(\frac{m^3}{z^3}, q_0\right)\right) = \log q_0 + i\pi - \log 27 + 2\log\left(1 - \frac{m^3}{z^3}\right) + q_0\left(\frac{4}{9} - \frac{8m^3}{9z^3} + \frac{22}{9}\frac{m^6}{z^6}\right) + \dots$$

is exactly the running coupling for $N=2$, SU(3) with 6 masses at special vacuum with $v = \frac{1}{3}\langle \mathrm{Tr}\,\phi^3\rangle = z^3$.

- SU(4) also works/ ~~but the~~ counting. Here we have two independent functions in the param. of τ^{ij}, both can be recovered via two different modular functions