Conventions

$$k_\mu = (\vec{k}, k_4) = (\vec{k}, i k_0) \qquad k^2 = \vec{k}^2 + k_4^2 = \vec{k}^2 - k_0^2$$

$$k^*_\mu (\vec{k}^*, i k_0^*)$$

Summary FR

$$\mathcal{L}(x) = \psi^*_i V_{ij} \psi_j + \frac{1}{2} \varphi_i W_{ij} \varphi_j + \mathcal{L}_{int}$$

$\psi_i \in \mathbb{C}$, $\varphi_i \in R$ $\qquad$ i = any set of indices

V, W contains derivatives , $\bar{V}, \bar{W}$ (FT) have an inverse

$$\mathcal{L}_{int}(x) = \int d^4x_1 \cdots \alpha_{i_1 \cdots}(x, x_1, \cdots) \psi^*_{i_1}(x_1) \cdots \psi_{i_m}(x_m) \cdots \varphi_{i_n}(x_n) \cdots$$

propagators = - inverse of $\bar{V}, \bar{W}$

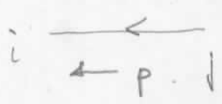
$$S = \int d^4x \, \mathcal{L} \qquad \begin{aligned} \psi_i(x) &= \int d^4k \, e^{ik.x} a_i(k) \\ \psi^*_i(x) &= \int d^4k \, e^{ik.x} b_i(k) \\ \varphi_i(x) &= \int d^4k \, e^{ik.x} c_i(k) \end{aligned}$$

$$iS = \frac{i}{2} \int d^4x \, d^4k \, d^4k'' \, e^{i(k+k').x} c_i(k) W_{ij}(x) c_j(k') + \cdots$$

$$= \frac{i}{2} \int d^4x \, d^4k \ldots d^4k'' \, e^{i(k+k'+k'').x} c_i(k) \bar{W}_{ij}(k'') c_j(k') + \cdots$$

$$= \frac{1}{2} (2\pi)^4 i \int d^4k \, d^4k' \, c_i(k) \bar{W}_{ij}(-k-k') c_j(k') + \cdots$$

$$W \Rightarrow \frac{\partial}{\partial x^\mu} , \qquad \bar{W} \Rightarrow i k_\mu$$



$$\Delta_{ij}(p) = - \frac{1}{(2\pi)^4 i} \left[\bar{V}(p) \right]^{-1}_{ij}$$

$$= - \frac{1}{(2\pi)^4 i} \left[\frac{1}{2} \bar{W}(p) + \frac{1}{2} \bar{W}^{tr}(p) \right]^{-1}_{ij}$$

ex :

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 - \frac{1}{2} M^2 W_\mu W_\mu$$

$$= \frac{1}{2} W_\alpha \left\{ -(\overleftarrow{\partial}_\mu \overrightarrow{\partial}^\mu + M^2)\delta_{\alpha\beta} + \overleftarrow{\partial}_\beta \overrightarrow{\partial}_\alpha \right\} W_\beta$$

$$iS = i \int d^4x\, d^4p\, d^4p'\, e^{ip\cdot x} \frac{1}{2} C_\alpha(p) \{\cdots\}_{\alpha\beta} C_\beta(p')\, e^{ip'\cdot x}$$

$$= \frac{i}{2} \int d^4x\, d^4p\, d^4p'\, e^{i(p+p')\cdot x} C_\alpha(p) \left\{ -(-p\cdot p' + M^2)\delta_{\alpha\beta} - p_\beta p'_\alpha \right\} C_\beta(p')$$

$$= \frac{i}{2} (2\pi)^4 i \int d^4p\, C_\alpha(p) \left\{ -(p^2+M^2)\delta_{\alpha\beta} + p_\alpha p_\beta \right\} C_\beta(-p)$$

$$\overline{W}_{\alpha\beta}(p) = -(p^2+M^2)\delta_{\alpha\beta} + p_\alpha p_\beta \quad , \qquad \overline{W}^{-1}_{\alpha\beta} = A(p^2)\delta_{\alpha\beta} + B(p^2) p_\alpha p_\beta$$

$$\left\{ -(p^2+M^2)\delta_{\alpha\beta} + p_\alpha p_\beta \right\} \left\{ A\delta_{\beta\mu} + B p_\beta p_\mu \right\} = \delta_{\alpha\mu}$$

$$-(p^2+M^2) A \delta_{\alpha\mu} - (p^2+M^2) B p_\alpha p_\mu + A p_\alpha p_\mu + B p^2 p_\alpha p_\mu = \delta_{\alpha\mu}$$

$$A = -\frac{1}{p^2+M^2} \quad , \qquad B = -\frac{1}{M^2(p^2+M^2)}$$

$$\Delta_{\mu\nu} = \frac{1}{(2\pi)^4 i} \frac{1}{p^2+M^2} \left\{ \delta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2} \right\}$$

$$\alpha_{i_1\cdots}(x, x_1, \cdots) = \int d^4k\, d^4k_1 \cdots \overline{\alpha}_{i_1\cdots}(k, k_1, \cdots)$$

$$\times \exp\{ ik\cdot x + ik_1\cdot(x - x_1) + \cdots \}$$

$$i\int d^4x\, \mathcal{L}_{int}(x) = i\int d^4x \int d^4k\, d^4k_1 \ldots \overline{\alpha}_{i_1\cdots}(k, k_1, \cdots)$$

$$\times \exp\{ ik\cdot x + ik_1\cdot(x - x_1) + \cdots \}$$

$$\times \int d^4p_1 \, e^{ip_1\cdot x_1} \, b_{i_1}(p_1) \cdots$$

$$\times \int d^4p_\mu \, e^{ip_m\cdot x_\mu} \, a_{i_\mu}(p_\mu) \cdots$$

$$\times \int d^4p_m \, e^{ip_m\cdot x_m} \, c_{i_\mu}(p_m) \cdots$$

$$= i \int d^4x \int d^4k \, d^4k_1 \cdots \int d^4p_1 \cdots \int d^4p_m \cdots \int d^4p_m \int d^4x_1 \cdots$$

$$\times \bar{\alpha}_{i_1\cdots}(k, k_1, \cdots) \exp\{ik\cdot x + ik_1\cdot(x - x_1) + \cdots$$

$$+ ip_1\cdot x_1 + \cdots + ip_m\cdot x_\mu + \cdots + ip_m\cdot x_m + \cdots\}$$

$$\times b_{i_1}(p_1) \cdots a_{i_\mu}(p_m) \cdots c_{i_m}(p_m) \cdots$$

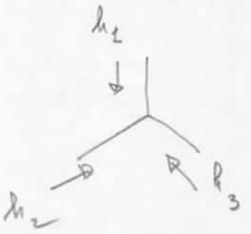
$$= (2\pi)^4 i \int d^4k \cdots \bar{\alpha}_{i_1\cdots}(k, k_1, \cdots) \, \delta^4(k + k_1 + \cdots)$$

$$\times (2\pi)^4 b_{i_1}(k_1) \cdots \times (2\pi)^4 a_{i_\mu}(k_\mu) \cdots \times (2\pi)^4 c_{i_\mu}(k_m) \cdots$$

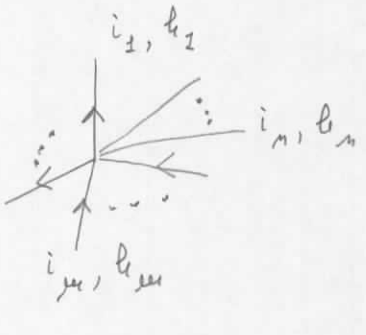
ex: $\mathcal{L}_{int} = \frac{g}{3!} \varphi^3(x)$ $\quad i \int d^4x \, \mathcal{L}_{int}(x) = (2\pi)^4 i \frac{g}{3!} \int \prod_i d^4k_i$

$$\times \delta^4\Big(\sum_i k_i\Big) \, c(k_1) \cdots c(k_3)$$

$$\times (2\pi)^{12}$$



$$(2\pi)^4 i g \, \delta^4(k_1 + k_2 + k_3)$$



$$(2\pi)^4 i \sum_{\{1\cdots\mu-1\}} \sum_{\{\mu\cdots m-1\}} \sum_{\{m\cdots\}} (-1)^P$$

$$\times \bar{\alpha}_{i_1\cdots}(k_1, \cdots) \, \delta^4(k_1 + \cdots)$$

$\{\cdots\}$ permutation of indices and momenta fermion $\xrightarrow{\hspace{2em}}$ $p \rightarrow$ $\qquad \frac{1}{(2\pi)^4 i}\,\frac{-i\not p + m}{p^2+m^2}$

in f $\sqrt{2p_0}\,u(p)$

in A $-\sqrt{2p_0}\,\bar v(p)$

out f $\sqrt{2p_0}\,\bar u(p)$

out A $\sqrt{2p_0}\,v(p)$

$(i\not p + m)\,u(p) = 0 \;,\quad (-i\not p + m)\,v(p) = 0$

$$\sum_{spin} u(p)\,\bar u(-p) = \frac{1}{2p_0}(-i\not p + m)$$

$$\sum_{spin} v(p)\,\bar v(-p) = \frac{1}{2p_0}(-i\not p - m)$$

Canonical transformations

$$\int \prod_i [\mathcal{D}\varphi_i]\, e^{iS[\varphi]} \;,\qquad S[\varphi] = \int d^4x\, \mathcal{L}[\varphi]$$

$\varphi_i = \phi_i + f_i(\varphi, \phi)$ invertible

$$[\mathcal{D}\varphi_i] = \det\frac{\partial\varphi_i}{\partial\phi_j}\,[\mathcal{D}\phi_j]$$

$$\int \prod_{i=1}^{n} [dz_i]\, e^{i(z,Az)} = \frac{(i\pi)^n}{\det A}$$

$$\frac{1}{\det X} = c\int [\mathcal{D}\eta]\, e^{iS[\eta]} \;,\qquad S[\eta] = \int d^4x\, \eta_i^* X_{ij}\,\eta_j$$

$$\uparrow$$

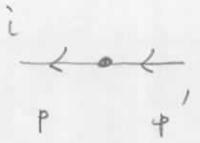
$$\prod_i [\mathcal{D}\eta_i^*][\mathcal{D}\eta_i]$$

$$X_{ij} = \delta_{ij} + \frac{\partial f_i}{\partial\phi_j}$$

$$S[\eta] = \int d^4x\, \eta_i^*(x)\,\eta_i(x) + \int d^4x\, d^4y\, \eta_i^*(x)\, Y_{ij}(x,y)\,\eta_j(y)$$

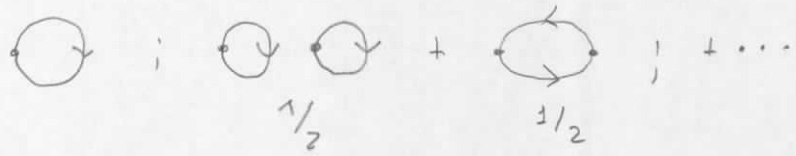
$$Y_{ij}(x,y) = \frac{\partial f_i(x,\phi)}{\partial\phi_j(y)}$$

$= -\frac{1}{(2\pi)^4 i}\delta_{ij}$

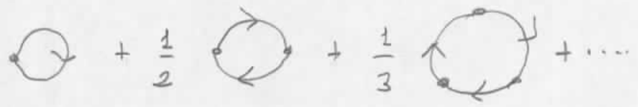


$(2\pi)^4 i\ \overline{Y}_{ij}(p,p')$

$1\ ;$



$$\int[\mathcal{D}\eta]\,e^{iS[\eta]} = e^{iT},\quad T=$$



$$\frac{1}{\det X} \longrightarrow \det X \quad \Rightarrow \quad T \longrightarrow -T$$

$$\int[\mathcal{D}\varphi]\,e^{iS[\varphi]} = \int[\mathcal{D}\phi][\mathcal{D}\eta]\,e^{iS[\phi+f]+iS[\eta]}$$

— · —

Little theorem $\mathcal{L} = \frac{1}{2}\varphi_i V_{ij}\varphi_j + \mathcal{L}_{int}(\varphi) + \frac{1}{2}\sigma_i(\partial^2-m^2)\sigma_i$

$+ J_i\varphi_i + H_i\sigma_i$

$-\frac{1}{(2\pi)^4 i}\overline{V}^{-1}_{ij}$

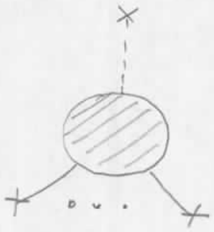
$\frac{1}{(2\pi)^4 i}\frac{\delta_{ij}}{p^2+m^2}$

$\Leftarrow \mathcal{L}_{int}$

α_{ij} may contain derivatives but not fields.

$\varphi_i \longrightarrow \varphi'_i = \varphi_i + \alpha_{ij}\sigma_j$

$\mathcal{L} \to \mathcal{L}'$ $\Longrightarrow$



$= 0$

$$\frac{1}{2}\varphi_i V_{ij}\varphi_j = \frac{1}{2}(\varphi_i + \alpha_{ik}\sigma_k)V_{ij}(\varphi_j + \alpha_{j\ell}\sigma_\ell)$$

$$= \frac{1}{2}\varphi_i V_{ij}\varphi_j$$

$$+\frac{1}{2}\left(\varphi_i V_{ij}\alpha_{j\ell}\sigma_\ell + \alpha_{ik}\sigma_k V_{ij}\varphi_j\right) + \dots$$

$$\downarrow$$

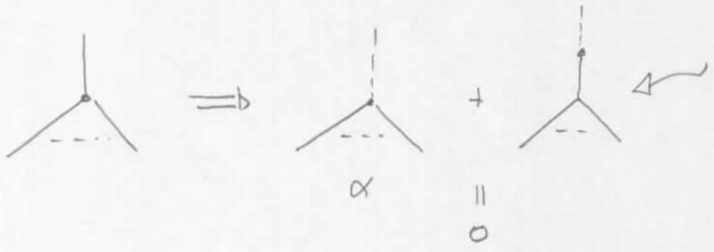
$$\frac{1}{2}\varphi_i V_{i\ell}\alpha_{\ell j}\sigma_j + \frac{1}{2}\varphi_i V_{i\ell}\alpha_{j\ell}\sigma_j \qquad V_{ij} = V_{ji}$$

$$= \varphi_i V_{i\ell}\alpha_{\ell j}\sigma_j$$

$$\mathcal{L}' = \frac{1}{2}\varphi V\varphi + \varphi V\alpha\sigma + \frac{1}{2}\sigma\alpha^{tr}V\alpha\sigma + \frac{1}{2}\sigma(\partial^2 - m^2)\sigma + \mathcal{L}_{int}(\varphi + \alpha\sigma)$$

$$+ J(\varphi + \alpha\sigma) + H\sigma$$

$$(2\pi)^4 i\,(V\alpha)_{ij}$$



$$(2\pi)^4 i\,V\alpha\left(-\frac{V^{-1}}{(2\pi)^4 i}\right) = -\alpha$$

$$= 0$$

$$(2\pi)^4 i\,J\alpha \qquad (2\pi)^4 i\,J\left(-\frac{V^{-1}}{(2\pi)^4 i}\right)(2\pi)^4 i\,V\alpha = -(2\pi)^4 i\,J\alpha$$

QED $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + J_\mu A_\mu$$

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2}(\partial_\mu A_\mu)^2$$

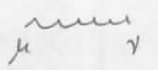
$$\mathcal{L}' = -\frac{1}{2}\left(\partial_\mu A_\nu \partial_\mu A_\nu - \underset{\times}{\partial_\mu A_\nu \partial_\nu A_\mu} + \underset{\times}{\partial_\mu A_\mu \partial_\nu A_\nu}\right) + J_\mu A_\mu$$

$$\partial_\mu(A_\nu \partial_\nu A_\mu) = \partial_\mu A_\nu \partial_\nu A_\mu + A_\nu \partial_\mu\partial_\nu A_\mu = \partial_\mu A_\nu \partial_\nu A_\mu + A_\mu \partial_\mu\partial_\nu A_\nu$$

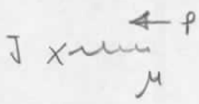
$$\partial_\mu(A_\mu \partial_\nu A_\nu) \qquad = \partial_\mu A_\mu \partial_\nu A_\nu + A_\mu \partial_\mu\partial_\nu A_\nu$$

$$\partial_\mu(\cdots\mu\cdots) = \partial_\mu A_\nu \partial_\nu A_\mu - \partial_\mu A_\mu \partial_\nu A_\nu$$

$$\mathcal{L}' = -\frac{1}{2}(\partial_\mu A_\nu)^2 + J_\mu A_\mu \Rightarrow$$



$$\frac{1}{(2\pi)^4 i}\frac{\delta_{\mu\nu}}{p^2}$$



$$(2\pi)^4 i\, J_\mu(p)$$

$$\mathcal{L}'' = \mathcal{L}' + \frac{1}{2}\varphi\partial^2\varphi + H\varphi \quad , \quad A_\mu \to A_\mu - \varepsilon\partial_\mu\varphi$$

$$\mathcal{L}'' = -\frac{1}{2}(\partial_\mu A_\nu - \varepsilon\partial_\mu\partial_\nu\varphi)^2 + \frac{1}{2}\varphi\partial^2\varphi + J_\mu(A_\mu - \varepsilon\partial_\mu\varphi) + H\varphi$$

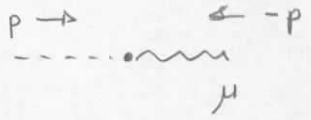
$$-\frac{1}{2}(\partial_\mu A_\nu)^2 + \varepsilon(\partial_\mu A_\nu)\partial_\mu\partial_\nu\varphi + O(\varepsilon^2)$$

$$\partial_\mu(A_\nu\partial_\mu\partial_\nu\varphi) = (\partial_\mu A_\nu)\partial_\mu\partial_\nu\varphi + A_\nu\partial^2\partial_\nu\varphi$$

$$\partial_\nu(A_\nu\partial^2\varphi) = (\partial_\nu A_\nu)\partial^2\varphi + A_\nu\partial^2\partial_\nu\varphi$$

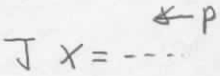
$$\partial_\mu(A_\nu\partial_\mu\partial_\nu\varphi) = (\partial_\mu A_\nu)\partial_\mu\partial_\nu\varphi + \partial_\nu(A_\nu\partial^2\varphi) - (\partial_\nu A_\nu)\partial^2\varphi$$

$$(\partial_\mu A_\nu)\partial_\mu\partial_\nu\varphi \overset{t.d.}{\sim} (\partial_\mu A_\mu)\partial^2\varphi$$

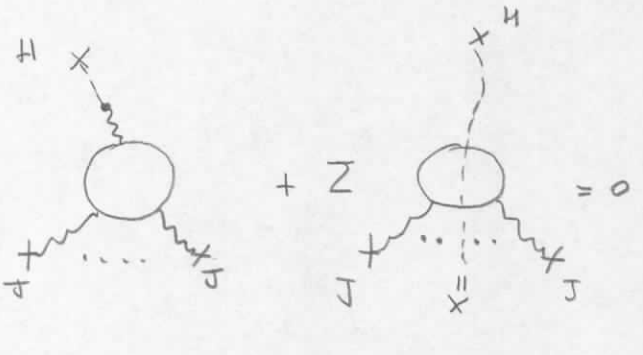


$$(2\pi)^4 i\,\varepsilon(-ip_\mu)(-p^2) = -(2\pi)^4\varepsilon p^2 p_\mu$$

$$-\varepsilon J_\mu\partial_\mu\varphi$$

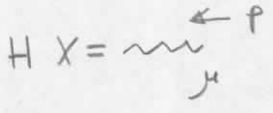


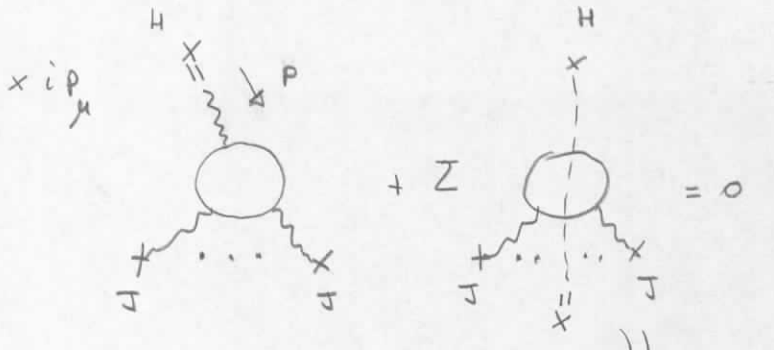
$$-(2\pi)^4 i\,\varepsilon J_\mu(p)(ip_\mu) = (2\pi)^4\varepsilon J_\mu p_\mu$$



$$-(2\pi)^4\varepsilon p^2 p_\mu \frac{1}{(2\pi)^4 i}\frac{1}{p^2} = i\varepsilon p_\mu$$

$\Rightarrow$





S-matrix $\quad p\cdot J = 0 \quad \Rightarrow \quad = 0$

$$\mathcal{L}' = \mathcal{L}_{WV} - \frac{1}{2}(\partial_\mu A_\mu)^2 \,, \qquad A_\mu \to A_\mu - \varepsilon\partial_\mu\varphi$$

$$-\frac{1}{2}(\partial_\mu A_\mu - \varepsilon\partial^2\varphi)^2 = -\frac{1}{2}(\partial_\mu A_\mu)^2 + \varepsilon(\partial_\mu A_\mu)\partial^2\varphi + O(\varepsilon^2)$$

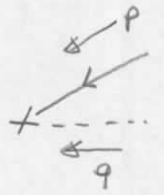
φ free-field $\Rightarrow$ $\partial_\mu A_\mu$ free-field

Generalization : $\mathcal{L}_{WV} \Rightarrow \mathcal{L}_{WV} - \frac{1}{2}C^2 \,,$ $\quad A_\mu \to A_\mu - \partial_\mu\Lambda$, $\quad C \to C + \hat{t}\Lambda$

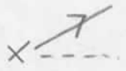
$$\mathcal{L} = \mathcal{L}' - \bar\psi(\not\partial + m)\psi + ie\bar\psi\not{A}\psi + \bar J_e\psi + \bar\psi J_e$$

$$A_\mu \to A_\mu - \varepsilon\partial_\mu\varphi \,, \qquad \psi \to (1 - i\varepsilon e\varphi)\psi + O(\varepsilon^2)$$
$$\bar\psi \to (1 + i\varepsilon e\varphi)\bar\psi + O(\varepsilon^2)$$

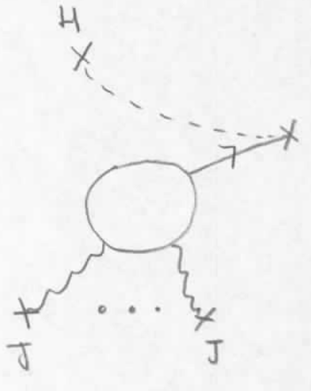
$$i\varepsilon e\varphi(\bar\psi J_e - \bar J_e\psi)$$



$(2\pi)^4 i\{-i\varepsilon e\bar J_e(p+q)\}$



$(2\pi)^4 i\{i\varepsilon e J_e(p+q)\}$

+ 

etc.

$$\mathcal{L} = \mathcal{L}_{inv} - \frac{1}{2} C^2 \quad \begin{cases} C = \partial_\mu A_\mu \\ C = \partial_\mu A_\mu + \lambda A_\mu A_\mu \end{cases}$$

$$A_\mu(x) \rightarrow A_\mu(x) - i\lambda \partial_\mu^x \int d^4y \, \Delta(x-y) A^2(y)$$

$$\Delta(x-y) = \frac{1}{(2\pi)^4 i} \int d^4p \, \frac{e^{ip\cdot(x-y)}}{p^2}$$

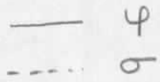
$$\partial_x^2 \Delta(x-y) = -\frac{1}{(2\pi)^4 i} \int d^4p \, e^{ip\cdot(x-y)} = i\delta^4(x-y)$$

$$\partial_\mu A_\mu(x) \rightarrow \partial_\mu A_\mu(x) + i\lambda \int d^4y \, \partial_x^2 \Delta(x-y) A^2(y)$$

$$= \partial_\mu A_\mu(x) + \lambda A^2(x)$$

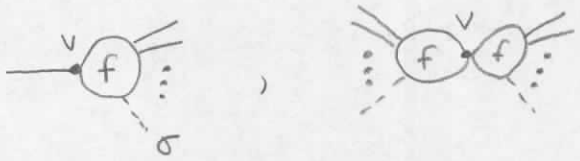
Generalized theorem

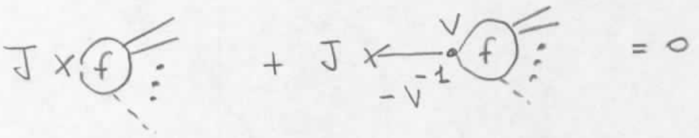
$$\mathcal{L} = \frac{1}{2}\varphi V \varphi + \mathcal{L}_{int}(\varphi) + \frac{1}{2}\sigma\partial^2\sigma + J\varphi + H\sigma$$



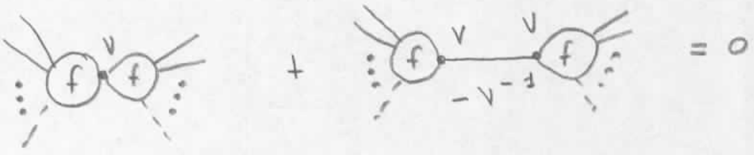
$$\varphi \rightarrow \varphi + f(\varphi, \sigma)$$

$\frac{1}{2}\varphi V \varphi \longrightarrow$

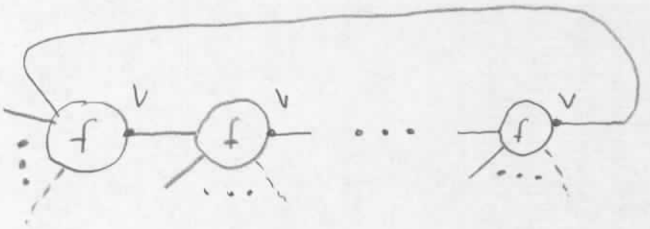




Jf



there are non-vanishing terms in the transformed theory.



$$-\frac{1}{2}(\partial_\mu A_\mu)^2 \longrightarrow -\frac{1}{2}\left\{\partial_\mu A_\mu - i\lambda\partial_x^2\int d^4y\,\Delta(x-y)A^2(y)\right\}^2$$

$O(\lambda)$ $$i\lambda(\partial_\mu A_\mu)\partial_x^2\int d^4y\,\Delta(x-y)A^2(y)$$

$$A_\mu(x) = \int d^4k\, e^{ik\cdot x} a_\mu(k)$$

iS $$-\lambda\int d^4x\, d^4y\, d^4k\, d^4p\, d^4k_1\, d^4k_2\; a_\mu(k)\, a_\nu(k_1)\, a_\nu(k_2)$$

$$\times\left\{\partial_\mu e^{ik\cdot x}\right\}\left\{\frac{1}{(2\pi)^4 i}\partial_x^2\frac{e^{ip\cdot(x-y)}}{p^2}\right\} e^{i(k_1+k_2)\cdot y}$$

$$= -\lambda\int d^4x\, d^4y\int d^4k \ldots d^4k_2\; a_\mu(k)\, a_\nu(k_1)\, a_\nu(k_2)$$

$$\times\{ik_\mu\}\left\{-p^2\frac{1}{(2\pi)^4 i p^2}\right\}\exp\{ik\cdot x + ip\cdot(x-y) + i(k_1+k_2)\cdot y\}$$

$$\downarrow$$

$$i(k+p)\cdot x + i(k_1+k_2-p)\cdot y$$

$$= -\lambda\int d^4k\, d^4p\, d^4k_1\, d^4k_2\; a_\mu(k)\, a_\nu(k_1)\, a_\nu(k_2)$$

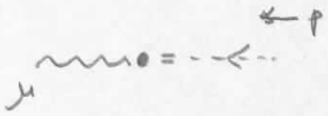
$$\times\{ik_\mu\}\left\{-p^2\frac{1}{(2\pi)^4 i p^2}\right\}(2\pi)^4\delta^4(k+p)\,(2\pi)^4\delta^4(k_1+k_2-p)$$

$$= -\lambda\int d^4p\, d^4k_1\, d^4k_2\; a_\mu(-p)\, a_\nu(k_1)\, a_\nu(k_2)$$

$$\times\{-ip_\mu\}\left\{-p^2\frac{1}{(2\pi)^4 i p^2}\right\}(2\pi)^8\delta^4(k_1+k_2-p)$$

$$= \lambda \int d^4p\, d^4l_1\, d^4l_2\; a_\mu(-p)\, a_\alpha(l_1)\, a_\beta(l_2)$$

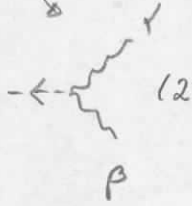
$$\times \left\{ (2\pi)^4 i\, (-ip_\mu)(-p^2) \right\} \frac{1}{(2\pi)^4 i p^2} \left\{ (2\pi)^4 i\, \delta_{\alpha\beta} \right\} \delta^4(l_1 + l_2 - p)$$



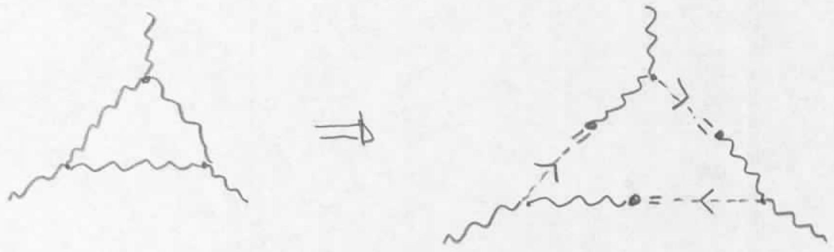
$(2\pi)^4 i\, (ip^2 p_\mu)$

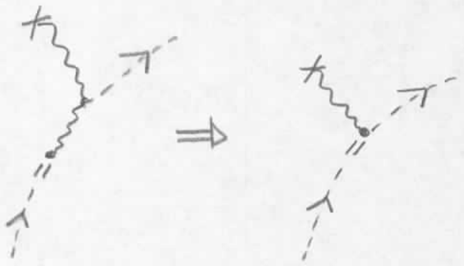
$\frac{1}{(2\pi)^4 i}\ \frac{1}{p^2}$

η

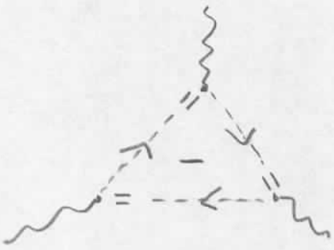


$(2\pi)^4 i\, \lambda\, \delta_{\alpha\beta} \cdot 2$





$(2\pi)^4 i\; 2\lambda i p_\mu$



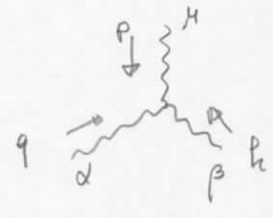
$$\Rightarrow \mathcal{L}_\eta = \eta^* \partial^2 \eta + 2\lambda A_\mu \eta^* \partial^\mu \eta$$

Summary: $C = \partial_\mu A_\mu + \lambda A_\mu A_\mu \to \partial_\mu (A_\mu + \partial_\mu \Lambda) + \lambda (A_\mu + \partial_\mu \Lambda)^2$

$$= C + \partial^2 \Lambda + 2\lambda A_\mu \partial_\mu \Lambda + O(\lambda^2)$$

$$C \to C + O\Lambda \,, \quad O = \partial^2 + 2\lambda A_\mu \partial_\mu \Rightarrow \mathcal{L}_\eta = \eta^* O \eta$$

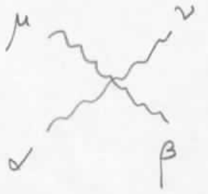
$$-\frac{1}{2}(\partial_\mu A_\mu + \lambda A_\nu A_\nu)^2 \Rightarrow -\lambda (\partial_\mu A_\mu) A^2$$



$$-(2\pi)^4 i \lambda \{ i p_\mu \delta_{\alpha\beta} + \text{perm} \}$$

$$= (2\pi)^4 \lambda\, 2 \{ p_\mu \delta_{\alpha\beta} + q_\alpha \delta_{\mu\beta} + k_\beta \delta_{\mu\nu} \}$$

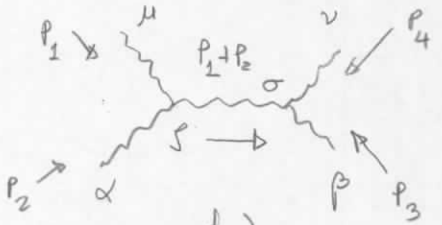
$$\Rightarrow -\frac{1}{2}\lambda^2 (A_\mu A_\mu)^2$$



$$-(2\pi)^4 i \frac{\lambda^2}{2} (\delta_{\mu\sigma} \delta_{\alpha\beta} + \text{perm})$$

$$= -(2\pi)^4 i \frac{\lambda^2}{2} 8 (\delta_{\mu\nu} \delta_{\alpha\beta} + \delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha})$$

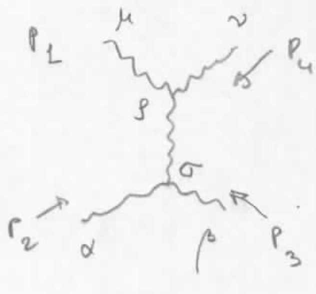
Cancellations: $\gamma\gamma \to \gamma\gamma$ $\qquad p \cdot \varepsilon(p) = 0$



$d_1)$

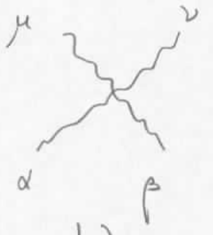
$$d_1 = (2\pi)^8 4\lambda^2 \frac{1}{(2\pi)^4 i} \delta_{\alpha\mu} (-p_1 - p_2)_\rho \frac{\delta_{\rho\sigma}}{(p_1+p_2)^2} \delta_{\nu\beta} (p_1 + p_2)_\sigma$$

$$= (2\pi)^4 i\, 4\lambda^2 \delta_{\alpha\mu} \delta_{\nu\beta}$$



$d_2)$

$d_3)$

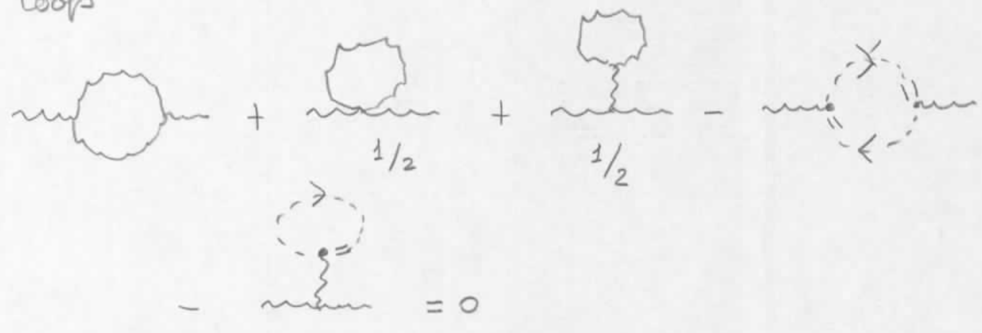


$$-(2\pi)^4 i\, 4\lambda^2(\delta_{\mu\nu}\delta_{\alpha\beta}+\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})$$

$d_4)$

$$\Rightarrow \sum_i d_i = 0$$

loop



ex:

$$\frac{6\times2\times3}{(3!)^2\times2!}=\frac{1}{2}$$

Unitarity $\qquad S \qquad S^\dagger S = 1\!\!1$

$S = 1\!\!1 + iT \qquad (1\!\!1+iT)(1\!\!1-iT^\dagger)=1\!\!1 \;, \quad T-T^\dagger = iT^\dagger T$

$$\langle a|T|b\rangle-\langle a|T^\dagger|b\rangle = i\langle a|T^\dagger T|b\rangle = i\sum_c\langle a|T|c\rangle\langle c|T^\dagger|b\rangle$$

$|b\rangle=|a\rangle \qquad \langle a|T|a\rangle-\langle a|T^\dagger|a\rangle = i\sum_c|\langle a|T|c\rangle|^2$

$$2i\,\mathrm{Im}\langle a|T|a\rangle = i\sum_c|\langle a|T|c\rangle|^2$$

Scalar fields:

$$\Delta_F(z)=\frac{1}{(2\pi)^4 i}\int d^4p\,\frac{e^{ip\cdot z}}{p^2+m^2-i\varepsilon}\;, \quad \Delta_F(z)=\theta(z_0)\Delta^+(z)+\theta(-z_0)\Delta^-(z)$$

$$\Delta^\pm(z)=\frac{1}{(2\pi)^3}\int d^4p\, e^{ip\cdot z}\,\theta(\pm p_0)\,\delta(p^2+m^2)$$

$$\left[\Delta^+(z)\right]^* = \frac{1}{(2\pi)^3}\int d^4p\, e^{-ip\cdot z}\,\theta(p_0)\,\delta(p^2+m^2)$$

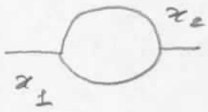
$$\nwarrow \; p_\mu \to -p_\mu$$

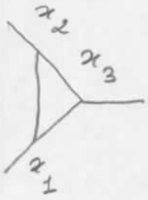
$$= \frac{1}{(2\pi)^3}\int d^4p\, e^{+ip\cdot z}\,\theta(-p_0)\,\delta(p^2+m^2) = \Delta^-(z)$$

$$\Delta^+(-z) = \Delta^-(z)$$

$$\Delta_F^*(z) = \theta(z_0)\,\Delta^-(z) + \theta(-z_0)\,\Delta^+(z)$$

notation $\Delta_{ij} = \Delta_F(x_i - x_j)$, $\Delta^+_{ij} = \Delta^-_{ji}$

ex: 

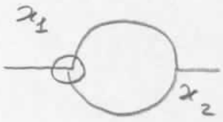
$= (ig)^2\,\Delta_{12}\,\Delta_{21}$, 

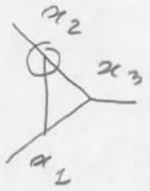
$= (ig)^3\,\Delta_{12}\,\Delta_{23}\,\Delta_{31}$

$F(x_1, x_2)$ $F(x_1, x_2, x_3)$

diagram $\Longleftrightarrow$ $F(x_1, \dots, x_n)$

Definition : $F(x_1, \dots, \underline{x}_i, \dots, x_n)$

$F(\underline{x}_1, x_2)$ 

, $F(x_1, \underline{x}_2, x_3)$ 

rule

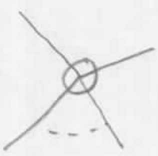
$\Delta_{ij} \longrightarrow$

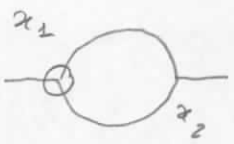
 Δ_{ij}

 Δ^+_{ij}

 Δ^-_{ij}

 Δ^*_{ij}

 $ig \to -ig$

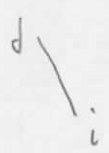


$$-(ig)^2 \Delta^+_{12} \Delta^-_{21}$$

$$x_1, \dots, x_m \qquad x_i^0 > x_j^0 \quad \forall j$$

$$\Rightarrow \quad F(x_1, \dots, \underline{x}_i, \dots, x_m) = -F(x_1, \dots, x_i, \dots, x_m)$$

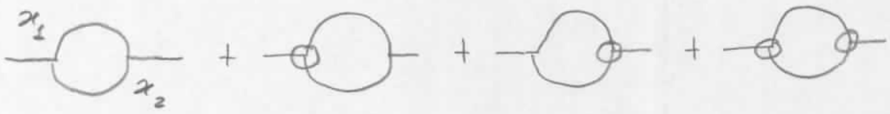
$$\Rightarrow \Delta^-_{ji}$$



$$\begin{aligned} \Delta_{ji} &= \Delta(x_j - x_i) \\ &= \theta(x_j^0 - x_i^0)\Delta^+_{ji} + \theta(x_i^0 - x_j^0)\Delta^-_{ji} \\ &= \Delta^-_{ji} \end{aligned}$$

largest time equation $\quad \sum_S F(x_1, \dots, x_m) = 0$

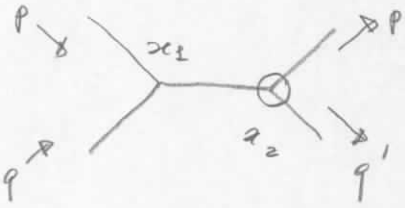
ex:



$$= (ig)^2 \left\{ \Delta_{12}\Delta_{12} - \Delta^+_{12}\Delta^+_{12} - \Delta^-_{12}\Delta^-_{12} + \Delta^*_{12}\Delta^*_{12} \right\} = 0$$

$$x_2^0 > x_1^0 \qquad \Delta_{12} = \Delta^-_{12}, \quad \Delta^*_{12} = \Delta^+_{12}$$

positive energies flow towards circled vertices



$$\Delta^-_{12} = \Delta^-(x_1 - x_2) = \Delta^+(x_2 - x_1)$$

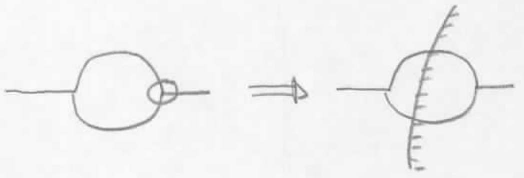
$$\sim \int d^4x_1\, d^4x_2 \int d^4k\, \theta(k_0)\, \delta(k^2 + m^2) \exp\{ i(p+q-k)\cdot x_1 + i(k - p' - q')\cdot x_2 \}$$

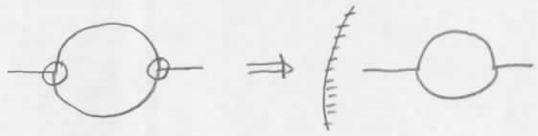
$$\sim \int d^4k\, \theta(k_0)\, \delta(k^2 + m^2)\, \delta^4(p+q-k)\, \delta^4(k - p' - q')$$

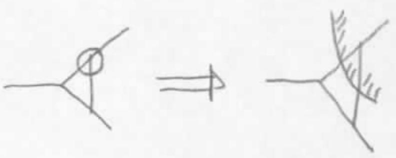
$$= \theta(p_0+q_0)\,\delta((q+p)^2+m^2)\,\delta^4(p+q-p'-q')$$

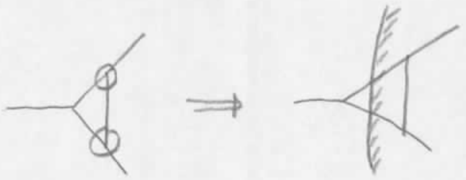
circled (uncircled) points form connected sets

Cuts :





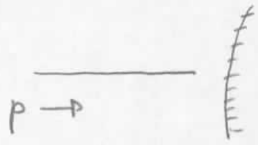




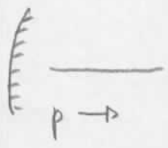
$$F(x_1,\dots,x_n) \longrightarrow \bar F(p_1,\dots,p_n)$$

$$\bar F(p_1,\dots,p_n) + \hat{\bar F}(p_1,\dots,p_n) = -\sum_{cuts} \bar F_{cut}(p_1,\dots,p_n)$$

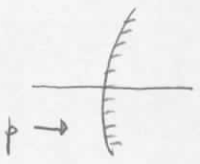
F.R



$\frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+m^2-i\varepsilon}$,

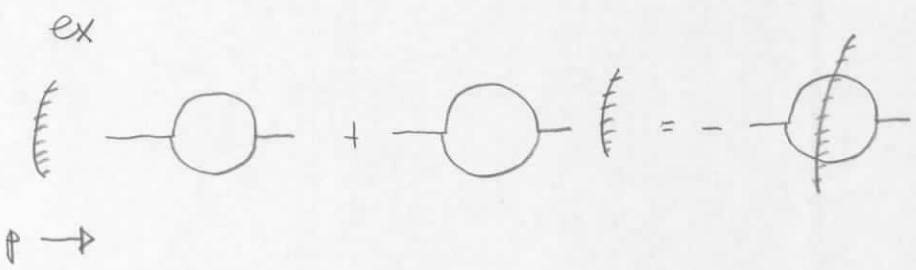


$-\frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+m^2+i\varepsilon}$



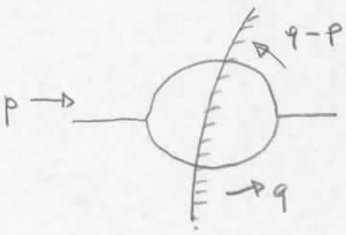
$\frac{1}{(2\pi)^3}\,\theta(p_0)\,\delta(p^2+m^2)$

ex



$$\langle a|T|a\rangle - \langle a|T^\dagger|a\rangle = i\sum_c |\langle c|T|a\rangle|^2$$

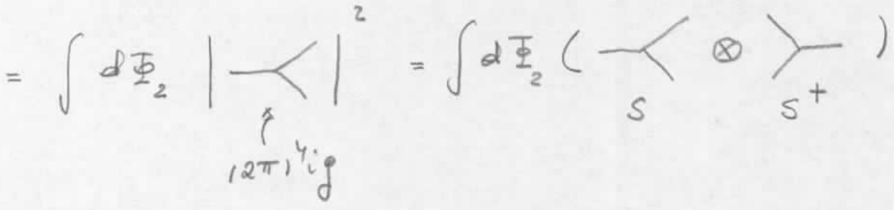
Σ_c in scalar theory



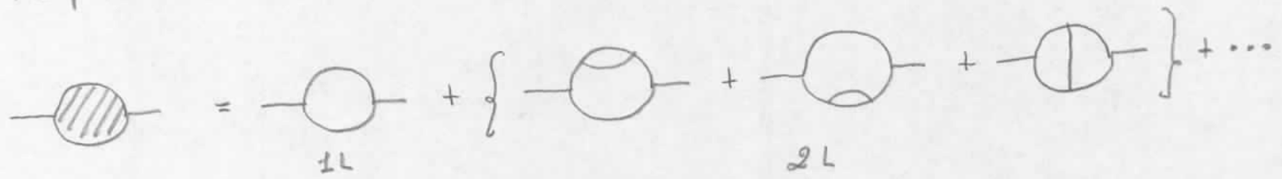
$$= -\left[(2\pi)^4 i g\right]^2 \frac{1}{(2\pi)^6} \int d^4q\ \theta(q_0)\,\delta(q^2+m^2) \times \theta(p_0-q_0)\,\delta((p-q)^2+m^2)$$

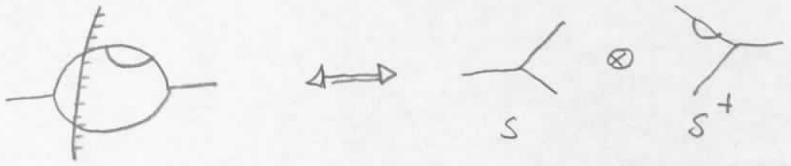
$k = p - q$

$$= (2\pi)^2 g^2 \int d^4q\, d^4k\ \theta(q_0)\,\delta(q^2+m^2)\,\theta(k_0)\,\delta(k^2+m^2)\,\delta^4(p-q-k)$$

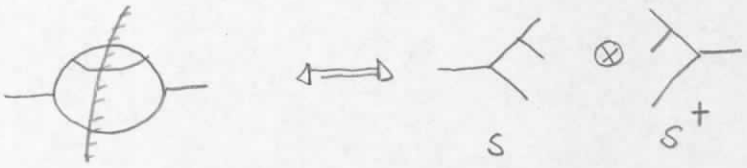


Higher orders



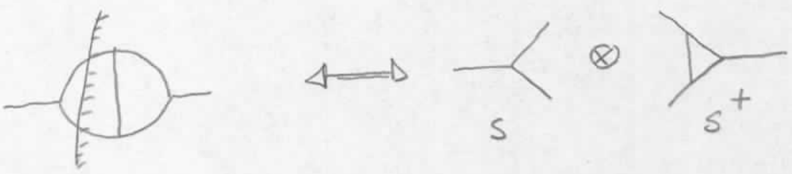


$1 \to 2$	$\otimes$	$1 \to 2$
T		1L

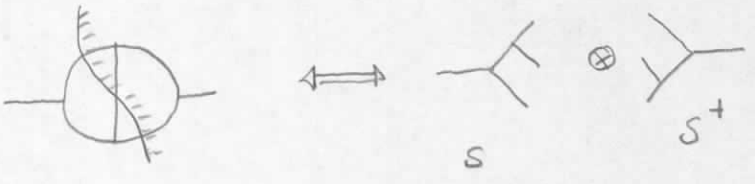


$1 \to 3$	$\otimes$	$1 \to 3$
T		T

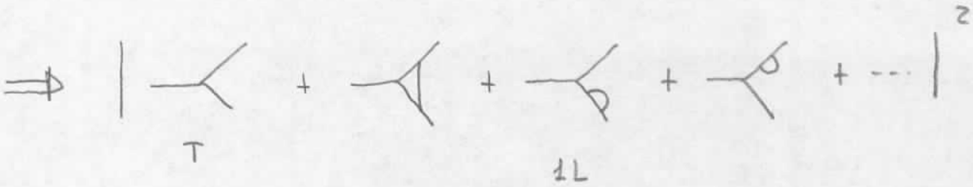
...



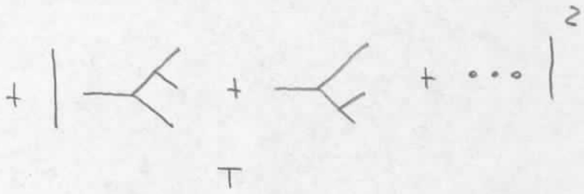
$1 \to 2$	$\otimes$	$1 \to 2$
T		1L



$1 \to 3$	$\otimes$	$1 \to 3$
T		T



1 → 2



1 → 3

+ ...

...

Spin ½

$$S_F(z) = \frac{1}{(2\pi)^4 i}\int d^4p\, e^{ip\cdot z}\,\frac{-i\not p + m}{p^2+m^2-i\varepsilon} \quad , \quad S_F(z) = (-\not\partial + m)\,\Delta_F(z)$$

$$S_F(z) = (-\not\partial + m)\{\theta(z_0)\,\Delta^+(z) + \theta(-z_0)\,\Delta^-(z)\}$$

$$S^\pm(z) = (-\not\partial + m)\,\Delta^\pm(z)$$

$$\not\partial\,\theta(z_0) = \gamma^4\partial_4\theta(z_0) \qquad z_4 = iz_0 \;,\; z_0 = -iz_4$$

$$\partial_4 = \frac{\partial}{\partial z_4} = \frac{dz_0}{dz_4}\frac{\partial}{\partial z_0} = -i\partial_0$$

$$\not\partial\,\theta(z_0) = -i\gamma^4\delta(z_0)$$

$$S_F(z) = \theta(z_0)\,S^+(z) + \theta(-z_0)\,S^-(z) + i\gamma^4\delta(z_0)\{\Delta^+(z) - \Delta^-(z)\}$$

$$\Delta^\pm(z) = \frac{1}{(2\pi)^3}\int d^4p\, e^{ip\cdot z}\,\theta(\pm p_0)\,\delta(p^2+m^2)$$

$$\Delta^\pm(z)\Big|_{z_0=0} = -\frac{i}{(2\pi)^3}\int d^3p\int_{-\infty}^{+\infty} dp_0\, e^{i\vec p\cdot\vec z}\,\theta(\pm p_0)\,\delta(\vec p^{\,2} - p_0^2 + m^2)$$

$$\swarrow$$

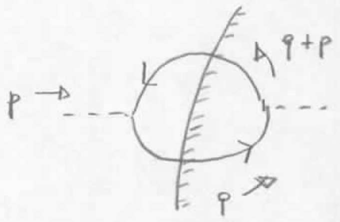
$$p_0 = \pm\sqrt{\vec p^{\,2} + m^2} = \pm\,\omega_p$$

$$\delta(p^2+m^2) = \frac{1}{2\omega_p}\{\delta(p_0+\omega_p) + \delta(p_0-\omega_p)\}$$

$$\Rightarrow \quad \Delta^+(z)\Big|_{z_0=0} = \Delta^-(z)\Big|_{z_0=0}$$

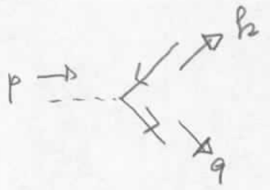
$$\mathcal{L} = -\bar\psi(\not\partial + m)\psi + \frac{1}{2}\varphi(\partial^2 - \mu^2)\varphi + g\bar\psi\psi\varphi$$

$$(2\pi)^4 ig$$



$$= \left[(2\pi)^4 ig\right]^2 \frac{g^2}{(2\pi)^6}\int d^4q\ \theta(q_0)\,\delta(q^2+m^2)\,\theta(p_0-q_0)\,\delta((p-q)^2+m^2)$$

$$\times\, \mathrm{tr}\left\{(-i\not q + m)\left[-i(\not q+\not p)+m\right]\right\}$$



$$M = (2\pi)^4 ig\,2\sqrt{q_0k_0}\ \bar u(q)\,v(k)\,\delta^4(p-q-k)$$

$$M^+ = -(2\pi)^4 ig\,2\sqrt{q_0k_0}\ \bar v(k)\,u(q)\,\delta^4(p'-q-k)$$

$$\sum |M|^2 = (2\pi)^8 g^2\, 4q_0k_0 \int\frac{d^3q}{(2\pi)^3 2q_0}\int\frac{d^3k}{(2\pi)^3 2k_0}\,\delta^4(p-q-k)\,\delta^4(p'-q-k)$$

$$\times \sum_{\text{spin}} \bar v(k)\,u(q)\,\bar u(q)\,v(k)$$

$$\sum_{\text{spin}} \bar v(k)\,u(q)\,\bar u(q)\,v(k) = \mathrm{tr}\sum_{\text{spin}} u(q)\,\bar u(q)\,v(k)\,\bar v(k)$$

$$= \frac{1}{4q_0k_0}\,\mathrm{tr}\{(-i\not q+m)(-i\not k-m)\}$$

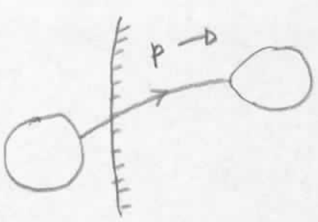
$$\sum |M|^2 = -(2\pi)^2 q^2\int\frac{d^3q\,d^3k}{4q_0k_0}\,\mathrm{tr}\{(-i\not q+m)(i\not k+m)\}\,\delta^4(p-q-k)\,\delta^4(p'-q-k)$$

$$= -(2\pi)^2 g^2\int d^4q\,d^4k\ \theta(q_0)\,\delta(q^2+m^2)\,\theta(k_0)\,\delta(k^2+m^2)\,\underset{\substack{\downarrow \\ k=p-q}}{\delta^4(p-q-k)}\,\delta^4(p'-q-k)$$

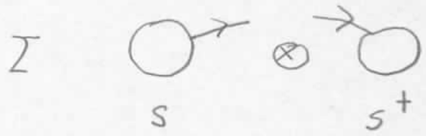
$$\times\,\mathrm{tr}\{\cdots\}$$

$$= -(2\pi)^2 g^2\int d^4q\ \theta(q_0)\,\delta(q^2+m^2)\,\theta(p_0-q_0)\,\delta((p-q)^2+m^2)\,\delta^4(p'-p)$$

$$\times\,\mathrm{tr}\left\{(-i\not q+m)\left[i(\not p-\not q)+m\right]\right\}$$



$$\Rightarrow \frac{1}{(2\pi)^3}(-i\not p + m)\,\theta(p_0)\,\delta(p^2+m^2)$$



$$\Rightarrow 2p_0 \sum u\bar u = 2p_0 \sum_{\text{spin}} \int \frac{d^3p}{(2\pi)^3\, 2p_0}\, u\bar u$$

$$= 2p_0 \int d^4p\, \theta(p_0)\,\delta(p^2+m^2)\, \frac{1}{(2\pi)^3}\, \underbrace{\sum_{\text{spin}} u\bar u}_{\rightarrow \frac{1}{2p_0}(-i\not p + m)}$$

$$S_F(p) = \frac{1}{(2\pi)^4 i}\, \frac{-i\not p + m}{p^2+m^2-i\varepsilon} \qquad \sum_{\text{spin}} u(p)\,\bar u(p) = \frac{1}{2p_0}(-i\not p + m) \Rightarrow \text{unitarity}$$

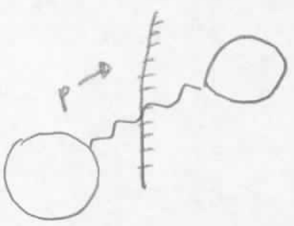
Spin 1

$$\Delta_{F\mu\nu}(z) = \frac{1}{(2\pi)^4 i}\int d^4p\, \frac{e^{ip\cdot z}}{p^2+m^2-i\varepsilon}\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)$$

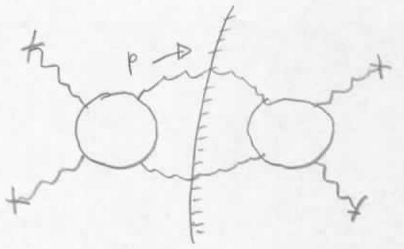
$$= \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2}\right)\Delta_F(z) \qquad \partial_0 \Delta^+(z)\Big|_{z_0=0} \neq \partial_0 \bar\Delta(z)\Big|_{z_0=0} \qquad \text{S-terms}$$

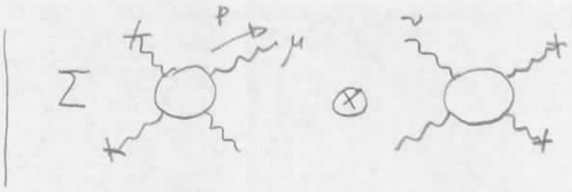
Photon

$$\sum_\lambda e^\lambda_\mu(p)\left[e^\lambda_\mu(p)\right]^* = \delta_{\mu\nu} + \frac{p_\mu \bar p_\nu + \bar p_\mu p_\nu}{\bar p\cdot p}\,, \qquad \bar p_\mu \equiv (-\vec p, ip_0)$$

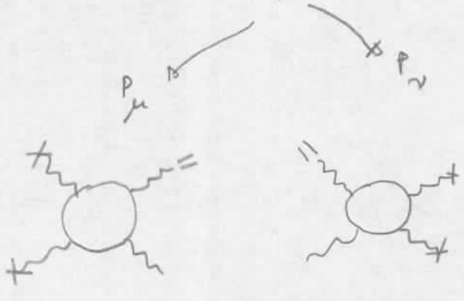


$$\Rightarrow \frac{1}{(2\pi)^3}\,\theta(p_0)\,\delta(p^2)\,\delta_{\mu\nu}$$





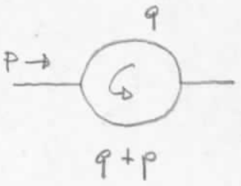
$$\Sigma \Rightarrow \delta_{\mu\nu} + \frac{p_\mu \bar{p}_\nu + \bar{p}_\mu p_\nu}{\bar{p}\cdot p}$$



$= 0$ $\qquad$ $= 0$ $\qquad$ by WI

Loop

ϕ^3



$$I = \int d^4q \, \frac{1}{(q^2+m^2)((q+p)^2+m^2)}$$

FT $\quad \displaystyle \frac{1}{ab} = \int_0^1 dx \left\{ ax + b(1-x) \right\}^{-2}$

$$I = \int_0^1 dx \int d^4q \left\{ (1-x)(q^2+m^2) + x((q+p)^2+m^2) \right\}^{-2}$$

$$= \int_0^1 dx \int d^4q \left\{ q^2 + 2x\, p\cdot q + x p^2 + m^2 \right\}^{-2}$$

$q_\mu \to q_\mu - x p_\mu \qquad I = \displaystyle\int_0^1 dx \int d^4q \, (q^2+M^2)^{-2}$

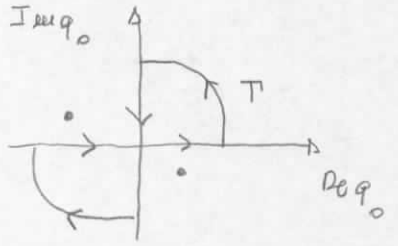
$$M^2 = x(1-x)p^2 + m^2$$

WR $\quad \displaystyle \int d^4q = \int_{-\infty}^{+\infty} dq_0 \int d^3q \qquad q^2 = \vec{q}^{\,2} - q_0^2$

$$q^2 + M^2 - i\varepsilon = \vec{q}^{\,2} + M^2 - i\varepsilon - q_0^2$$

assume $\vec q^{\,2} + M^2 > 0$

$$q_0 = \pm(\vec q^{\,2} + M^2 - i\varepsilon)^{1/2} = \pm(\vec q^{\,2} + M^2) \mp i\varepsilon$$



$$\int_T dq_0 \cdots = 0$$

$$\int_{-\infty}^{+\infty} dq_0 = -\int_{+i\infty}^{-i\infty} dq_0 \qquad q_4 = iq_0 \qquad \Rightarrow i\int_{-\infty}^{+\infty} dq_4 \Rightarrow \int d^4q_M = i\int d^4q_E$$

$$I = i\int_0^1 dx \int d^4q\,(q^2+M^2)^{-2} = i\int_0^1 dx\, 2\pi^2 \int_0^\infty d\omega\, \frac{\omega^3}{(\omega^2+M^2)^2}$$

Regularization

$$I_\Lambda = 2\pi^2 i \int_0^1 dx \int_0^\Lambda d\omega\, \frac{\omega^3}{(\omega^2+M^2)^2} \qquad z = \omega^2 \qquad dz = 2\omega\, d\omega$$

$$I_\Lambda = i\pi^2 \int_0^1 dx \int_0^{\Lambda^2} dz\, \frac{z}{(z+M^2)^2} = i\pi^2 \int_0^1 dx \int_0^{\Lambda^2} dz\, \frac{z+M^2-M^2}{(z+M^2)^2}$$

$$= i\pi^2 \int_0^1 dx \left\{ \ln(z+M^2) - \frac{M^2}{z+M^2} \right\}\Bigg|_0^{\Lambda^2} \qquad I_\Lambda = \int_0^1 dx\, J_\lambda$$

$$J_\Lambda = i\pi^2 \left\{ \ln\frac{\Lambda^2+M^2}{M^2} + M^2\left(\frac{1}{\Lambda^2+M^2} - \frac{1}{M^2}\right)\right\} \underset{\Lambda\to\infty}{\sim} i\pi^2\left\{ \ln\frac{\Lambda^2}{M^2} - 1 + O\left(\frac{1}{\Lambda^2}\right)\right\}$$

$$I_\Lambda \underset{\Lambda\to\infty}{\sim} i\pi^2 \left\{ \ln\lambda^2 - 1 - \int_0^1 dx\, \ln M^2 \right\}$$

DR

$$\int d^4q \to \int d^4q$$

$$\int d^\mu q = \int_0^\infty d\omega\, \omega^{n-1} \int_0^{2\pi} d\theta_1 \prod_{l=3}^{n} \int_0^\pi d\theta_{l-1} (\sin\theta_{l-1})^{l-2}$$

$$\int_0^\pi d\theta \sin^m\theta = \pi^{1/2} \frac{\Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu}{2}+1)}$$

$$I_m = i\int d^\mu q \frac{1}{(q^2+\mu^2)^2} = i\int_0^\infty d\omega \frac{\omega^{\mu-1}}{(\omega^2+\mu^2)^2}\; 2\pi \prod_{l=3}^{n} \int_0^\pi d\theta_{l-1} (\sin\theta_{l-1})^{l-2}$$

$$\pi^{1/2}\frac{\Gamma(1)}{\Gamma(3/2)}\;\pi^{1/2}\frac{\Gamma(3/2)}{\Gamma(2)}\cdots\pi^{1/2}\frac{\Gamma(\frac{\mu-1}{2})}{\Gamma(\frac{\mu}{2})}$$

$$\pi^{\frac{\mu-2}{2}} = \pi^{\mu/2-1}$$

$$I_m = 2i\frac{\pi^{m/2}}{\Gamma(m/2)}\int_0^\infty d\omega \frac{\omega^{\mu-1}}{(\omega^2+\mu^2)^2}$$

$$\int_0^\infty d\omega \frac{\omega^{\mu-1}}{(\omega^2+\mu^2)^\alpha} = \frac{1}{2}\frac{\Gamma(\mu/2)\Gamma(\alpha-m/2)}{\Gamma(\alpha)} \times (\mu^2)^{\mu/2-\alpha}$$

$$I_m = i\pi^{m/2}\,\Gamma(2-m/2)(\mu^2)^{\mu/2-2}$$

$$\mu = 4-\varepsilon\,,\quad \frac{m}{2} = 2-\varepsilon/2 \qquad \Gamma(z) \underset{z\to 0}{\sim} \frac{1}{z} - \gamma + O(z)$$

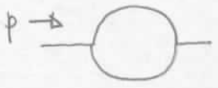
$$a^\varepsilon = e^{\varepsilon \ln a} = 1+\varepsilon\ln a + O(\varepsilon^2)$$

$$I_m = i\pi^{2-\varepsilon/2}\,\Gamma(\tfrac{\varepsilon}{2})(\mu^2)^{-\varepsilon/2}$$

$$= i\pi^2(1-\tfrac{\varepsilon}{2}\ln\pi+\cdots)(\tfrac{2}{\varepsilon}-\gamma+\cdots)(1-\tfrac{\varepsilon}{2}\ln\mu^2+\cdots)$$

$$= i\pi^2(\tfrac{2}{\varepsilon}-\gamma-\ln\pi-\ln\mu^2)+O(\varepsilon)$$

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln \pi \,, \qquad I_m = i\pi^2 \left(\frac{1}{\bar{\varepsilon}} - \ln \mu^2 \right) + O(\varepsilon)$$

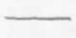
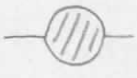
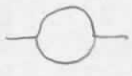
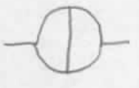
ϕ^3  $$\Sigma = i g^2 \int d^4 q \, \frac{1}{(q^2+m^2)((q+p)^2+m^2)}$$

$$= i g^2 \int_0^1 dx \int d^4 q \, (q^2+\mu^2)^{-2} \,, \qquad \mu^2 = x(1-x)p^2 + m^2$$

$$\Sigma = i\pi^2 g^2 \left\{ \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln \mu^2 \right\} = (2\pi)^4 i \, \frac{g^2}{16\pi^2} \left\{ \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln \mu^2 \right\}$$

Dyson re-summation

 $$\Delta = \frac{1}{(2\pi)^4 i} \, \frac{1}{p^2+m^2}$$

$\bar{\Delta}$ =  +  + ... ,  =  +  + ...

N.B. no Σ insertions

(i.e. no )

$$\bar{\Delta} = \Delta + \Delta \Sigma \Delta + \cdots = \Delta (1 + \Sigma \Delta + \ldots) = \frac{\Delta}{1 - \Sigma \Delta}$$

$$\bar{\Delta} = \frac{1}{(2\pi)^4 i} \, \frac{1}{p^2+m^2} \left\{ 1 - \frac{1}{(2\pi)^4 i} \, \frac{\Sigma}{p^2+m^2} \right\}^{-1} = \frac{1}{(2\pi)^4 i} \left\{ p^2 + m^2 - \frac{\Sigma}{(2\pi)^4 i} \right\}^{-1}$$

$$p^2 = - m^2_{exp} \Rightarrow - m^2_{exp} + m^2 - \frac{1}{(2\pi)^4 i} \, \Sigma(-\underset{\text{pole}}{m^2_{exp}}) = 0$$

$$m^2 = m^2_{exp} + \frac{1}{(2\pi)^4 i} \, \Sigma(-m^2_{exp})$$

renormalized parameter and CT

$$m^2 = m_R^2 + \frac{g^2}{16\pi^2}\delta_m$$

$$m_R^2 + \frac{g^2}{16\pi^2}\delta_m = m_{exp}^2 - \frac{g^2}{16\pi^2}\left\{\frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln\mu^2\Big|_{p^2=-m_{exp}^2}\right\}$$

$$\Rightarrow \quad \delta_m = -\frac{1}{\bar{\varepsilon}} \qquad m_R^2 = m_{exp}^2 + \frac{g^2}{16\pi^2}\int_0^1 dx \ln\mu^2\Big|_{p^2=-m_{exp}^2}$$

$$m \longrightarrow m_R \longrightarrow m_{exp}$$

$$p^2 + m^2 - \frac{\Sigma}{(2\pi)^4 i} = p^2 + m_{exp}^2 + \frac{1}{(2\pi)^4 i}\Sigma(-m_{exp}^2) - \frac{1}{(2\pi)^4 i}\Sigma(p^2)$$

$$\Sigma(p^2) = \Sigma(-m_{exp}^2) + (p^2 + m_{exp}^2)\left\{\Sigma_{WF} + (p^2 + m_{exp}^2)\Sigma_R\right\}$$

$$\bar{\Delta} = \frac{1}{(2\pi)^4 i}\frac{1}{p^2 + m_{exp}^2}\left\{1 + \frac{1}{(2\pi)^4 i}\Sigma_{WF} + O(p^2 + m_{exp}^2)\right\}^{-1}$$

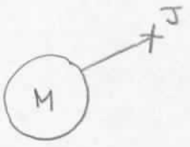
$\phi^3 \longrightarrow$ UV finite

Physical sources $\quad J \to ZJ \quad \sim \quad J\bar{\Delta}J \sim \frac{1}{(2\pi)^4 i}\frac{JJ}{p^2 + m_{exp}^2} \quad p^2 \to -m_{exp}^2$

$$\Rightarrow Z = \left\{1 + \frac{1}{(2\pi)^4 i}\Sigma_{WF}\right\}^{1/2}$$

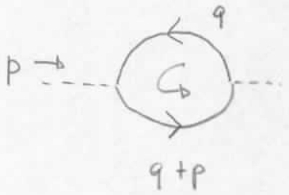
S-matrix

$$JZ\bar{\Delta}M = JZ\frac{1}{(2\pi)^4 i}\frac{1}{p^2 + m_{exp}^2}\left\{Z^2 + O(p^2 + m_{exp}^2)\right\}^{-1} M$$



$$\times (p^2 + m_{exp}^2) \lim_{p^2 \to -m_{exp}^2} = \frac{J}{Z} M \longrightarrow \text{LSZ}$$

$g\bar{\psi}\psi\phi$



$$\Sigma = -ig^2 \int d^4q \, \frac{1}{(q^2+m^2)((q+p)^2+m^2)} \, \mathrm{tr}\left\{(-i\not{q}+m)\left[-i(\not{q}+\not{p})+m\right]\right\}$$

$$\mathrm{tr}\{\cdots\} = -4q\cdot(q+p) + 4m^2$$

$$\Sigma = (2\pi)^4 i \, \frac{g^2}{4\pi^2} \int d^4q \int_0^1 dx \, \frac{q\cdot(q+p) - m^2}{(q^2+2x\,p\cdot q + x p^2 + m^2)^2}$$

requires new integrals

$$I_\alpha = i\int d^nq\,(q^2+2k\cdot q+M^2)^{-\alpha} = i\int d^4q\,(q^2+M^2-k^2)^{-\alpha} = i\int d^4q\,(q^2+\mu^2)^{-\alpha}$$

$$= i\pi^{n/2}\,\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\,(\mu^2)^{n/2-\alpha}$$

$$I^\mu_\alpha = i\int d^nq\,q^\mu(q^2+2k\cdot q+M^2)^{-\alpha} = \frac{1}{2}\,\frac{1}{1-\alpha}\,\frac{\partial}{\partial k_\mu}\,\underbrace{i\int d^nq\,(q^2+2k\cdot q+M^2)^{1-\alpha}}_{I_{\alpha-1}}$$

$$= \frac{1}{2}\,\frac{1}{1-\alpha}\,\frac{\partial}{\partial k_\mu}\, i\pi^{n/2}\,\frac{\Gamma(\alpha-1-n/2)}{\Gamma(\alpha-1)}\,(M^2-k^2)^{n/2+1-\alpha}$$

$$= \frac{i}{2}\,\pi^{n/2}\,\frac{\Gamma(\alpha-1-n/2)}{(1-\alpha)\Gamma(\alpha-1)}\,(M^2-k^2)^{n/2-\alpha}\left(\frac{n}{2}+1-\alpha\right)(-2k^\mu)$$

$$= -i\pi^{n/2}\,\frac{\Gamma(\alpha-1-n/2)}{\Gamma(\alpha-1)}\,\frac{\alpha-1-n/2}{\alpha-1}\,(M^2-k^2)^{n/2-\alpha}\,k^\mu = -i\pi^{n/2}\,\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}\,(M^2-k^2)^{n/2-\alpha}\,k^\mu$$

$$I^\mu_\alpha = -I_\alpha\,k^\mu$$

$$I^{\mu\nu}_\alpha = i\int d^nq\,q^\mu q^\nu(q^2+2k\cdot q+M^2)^{-\alpha} = \frac{1}{2(1-\alpha)}\,\frac{\partial}{\partial k_\nu}\,\underbrace{i\int d^4q\,q^\mu(q^2+2k\cdot q+M^2)^{1-\alpha}}_{I^\mu_{\alpha-1}}$$

$$= -\frac{1}{2(1-\alpha)}\frac{\partial}{\partial k_\nu}\, i\pi^{n/2}\frac{\Gamma(\alpha-1-n/2)}{\Gamma(\alpha-1)}(M^2-k^2)^{\mu/2+1-\alpha}\,k^\mu$$

$$= -\frac{1}{2(1-\alpha)}\, i\pi^{n/2}\frac{\Gamma(\alpha-1-n/2)}{\Gamma(\alpha-1)}\Big\{(\tfrac{\mu}{2}+1-\alpha)(M^2-k^2)^{n/2-\alpha}(-2k^\nu)k^\mu + (M^2-k^2)^{\mu/2+1-\alpha}\delta^{\mu\nu}\Big\}$$

$$= \frac{1}{2}\, i\pi^{n/2}\frac{\Gamma(\alpha-1-n/2)}{\Gamma(\alpha-1)}\,\frac{\alpha-1-n/2}{\alpha-1}(M^2-k^2)^{n/2-\alpha}\Big\{2k^\mu k^\nu+\frac{M^2-k^2}{\alpha-1-n/2}\delta^{\mu\nu}\Big\}$$

$$I_\alpha^{\mu\nu} = I_\alpha\Big\{k^\mu k^\nu + \frac{1}{2}\frac{M^2-k^2}{\alpha-1-n/2}\delta^{\mu\nu}\Big\}$$

$\alpha = 2$

$$I_2 = i\pi^2\Big\{\frac{1}{\bar{\varepsilon}} - \ln\mu^2\Big\} + O(\varepsilon)$$

$$I_2^\mu = -i\pi^2\Big\{\frac{1}{\bar{\varepsilon}} - \ln\mu^2\Big\}k^\mu + O(\varepsilon)$$

$$I_2^{\mu\nu} = i\pi^2\Big\{\frac{1}{\bar{\varepsilon}} - \ln\mu^2\Big\}\Big\{k^\mu k^\nu - \frac{1}{2}(1+\frac{\varepsilon}{2})\mu^2\delta^{\mu\nu}\Big\} + O(\varepsilon)$$

UV parte
$$I_2\Big|_{UV} = \frac{i\pi^2}{\bar{\varepsilon}}\,, \qquad I_2^\mu\Big|_{UV} = -\frac{i\pi^2}{\bar{\varepsilon}}k^\mu$$

$$I_2^{\mu\nu}\Big|_{UV} = \frac{i\pi^2}{\bar{\varepsilon}}\Big\{k^\mu k^\nu - \frac{1}{2}(M^2-k^2)\delta^{\mu\nu}\Big\}$$

$$I_2^{\mu\mu}\Big|_{UV} = \frac{i\pi^2}{\bar{\varepsilon}}(3k^2-2M^2)$$

$$\Sigma_{UV} = (2\pi)^4 i\,\frac{g^2}{4\pi^2}\,\frac{1}{\varepsilon}\int_0^1 dx\,\Big\{3k^2-2M^2-k\cdot p-m^2\Big\}$$

$$k = xp\,,\quad M^2 = xp^2+m^2$$

$$\Sigma_{UV} = (2\pi)^4 i \, \frac{3g^2}{4\pi^2} \, \frac{1}{\bar{\varepsilon}} \int_0^1 dx \, (x^2 p^2 - x p^2 - \mu^2)$$

$$= -(2\pi)^4 i \, \frac{g^2}{8\pi^2} \, \frac{1}{\bar{\varepsilon}} (p^2 + 6\mu^2)$$

$$\Sigma = \Sigma_{UV} + \Sigma_F \, , \qquad \Sigma_F = (2\pi)^4 i \, \frac{g^2}{8\pi^2} \, \sigma_F (p^2)$$

$$M^2 = M_R^2 + \frac{g^2}{16\pi^2} \delta_M \, , \qquad \mu^2 = \mu_R^2 + \frac{g^2}{16\pi^2} \delta_m$$

$$\bar{\Delta} = \frac{1}{(2\pi)^4 i} \left\{ p^2 + M^2 - \frac{\Sigma}{(2\pi)^4 i} \right\}^{-1} \qquad - M_{\text{exp}}^2 + M^2 - \frac{1}{(2\pi)^4 i} \Sigma(-M_{\text{exp}}^2) = 0$$

$$p^2 + M^2 - \frac{\Sigma}{(2\pi)^4 i} = p^2 + M^2 + \frac{g^2}{8\pi^2} \Big\{ (6\mu_{\text{exp}}^2 - M_{\text{exp}}^2) \frac{1}{\bar{\varepsilon}} - \sigma_F(-M_{\text{exp}}^2)$$

$$+ (p^2 + M_{\text{exp}}^2) \left[\Sigma_{WF} + \Sigma_R (p^2) \right] \Big\}$$

$$\Sigma_{WF} = \frac{\partial \Sigma}{\partial p^2} \bigg|_{p^2 = -M_{\text{exp}}^2} = \frac{1}{\bar{\varepsilon}} + \Sigma_{WF}^F$$

$$p^2 + M_R^2 + \frac{g^2}{16\pi^2} \delta_M + \frac{g^2}{8\pi^2} \Big\{ (6\mu_{\text{exp}}^2 - M_{\text{exp}}^2) \frac{1}{\bar{\varepsilon}} - \sigma_F (p^2)$$

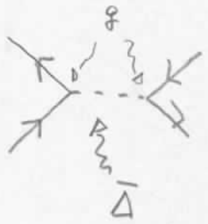
$$+ (p^2 + \mu_{\text{exp}}^2) \left[\frac{1}{\bar{\varepsilon}} + \Sigma_{WF}^F + \Sigma_R (p^2) \right] \Big\}$$

$$\delta_M + 2(6\mu_{\text{exp}}^2 - M_{\text{exp}}^2) \frac{1}{\bar{\varepsilon}} = 0$$

$$p^2 + M_R^2 + \frac{g^2}{8\pi^2} \Big\{ -\sigma_F (p^2) + (p^2 + M_{\text{exp}}^2) \left[\frac{1}{\bar{\varepsilon}} + \Sigma_{WF}^F + \Sigma_R (p^2) \right] \Big\}$$

$$-M_{exp}^2 + M_R^2 - \frac{g^2}{8\pi^2}\sigma_F(-M_{exp}^2) = 0 \,, \qquad M_R^2 = M_{exp}^2 + \frac{g^2}{8\pi^2}\sigma_F(-M_{exp}^2)$$

$$\bar{\Delta} = \frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+M_{exp}^2}\left\{1+\frac{g^2}{8\pi^2}\left(\frac{1}{\bar{\varepsilon}}+\Sigma^F_{WF}+\Sigma_R\right)\right\}^{-1}$$



$$g^2\bar{\Delta} = \frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+M_{exp}^2}\,\frac{g^2}{1+\frac{g^2}{8\pi^2}\left(\frac{1}{\bar{\varepsilon}}+\Sigma^F_{WF}+\Sigma_R\right)}$$

$$g^2 = g_R^2\left(1+\frac{g_R^2}{16\pi^2}\delta_g\right)$$

$$g^2\bar{\Delta} = \frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+M_{exp}^2}\,g_R^2\left(1+\frac{g_R^2}{16\pi^2}\delta_g\right)\left\{1-\frac{g^2}{8\pi^2}\left(\frac{1}{\bar{\varepsilon}}+\Sigma^F_{WF}+\Sigma_R\right)\right\}$$

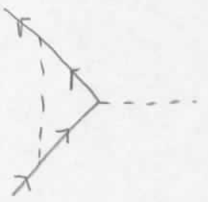
$$1+\frac{g_R^2}{16\pi^2}\left\{\delta_g - 2\left(\frac{1}{\bar{\varepsilon}}+\Sigma^F_{WF}+\Sigma_R\right)\right\}$$

$$\Longrightarrow \quad \delta_g = \frac{2}{\bar{\varepsilon}}$$

$$g^2\bar{\Delta} = \frac{1}{(2\pi)^4 i}\,\frac{g_R^2}{p^2+M_{exp}^2}\left\{1-\frac{g_R^2}{8\pi^2}\left(\Sigma^F_{WF}+\Sigma_R\right)\right\}$$

$p = p_0$ derive $\frac{1}{(2\pi)^4 i}\,\frac{g_{exp}^2}{p^2+M_{exp}^2} \Longrightarrow g_R = g_R(g_{exp})$

BUT



$$\sim \int \frac{d^4q}{q^4} \sim \int \frac{dq}{q} \quad \text{UV divergent}$$

entering renormalization of g

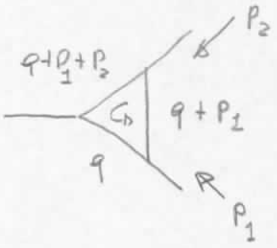
$$g^2\,\bar{\Delta} = \frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+M_{exp}^2}\;\frac{g^2}{\left\{1+\frac{g^2}{8\pi^2}\left(\frac{1}{\bar{\varepsilon}}+\Sigma^F_{WF}\right)\right\}\left\{1+\frac{g^2}{8\pi^2}\underbrace{\Sigma_R}_{O(p^2+M_{exp}^2)}+O(g^4)\right\}}$$

$$= \frac{1}{(2\pi)^4 i}\,\frac{g_R^2}{p^2+M_{exp}^2}\left(1+\frac{g_R^2}{16\pi^2}\delta_g\right)\left\{1+\frac{g^2}{8\pi^2}\left(\frac{1}{\bar{\varepsilon}}+\Sigma^F_{WF}\right)\right\}\frac{1}{1+\frac{g^2}{8\pi^2}\Sigma_R}$$

$$\delta_g = \frac{2}{\bar{\varepsilon}}$$

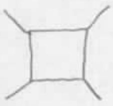
$$= \frac{1}{(2\pi)^4 i}\,\frac{g_R^2}{p^2+M_{exp}^2}\left\{1-\frac{g_R^2}{8\pi^2}\Sigma^F_{WF}\right\}\frac{1}{1+\frac{g^2}{8\pi^2}\Sigma_R}$$

ϕ^3 $\quad\Sigma$ divergent



$$= ig^3\int d^4q\left\{(q^2+m^2)((q+p_1)^2+m^2)((q+p_1+p_2)^2+m^2)\right\}^{-1}$$

$$\sim\int\frac{d^4q}{q^6}\sim\int\frac{dq}{q^2}\quad\text{finite}$$

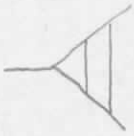


etc.

finite

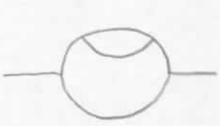
2-loop

$$\sim\int dq\,\frac{q^7}{q^{10}}\sim\int\frac{dq}{q^3}\quad\text{finite}$$

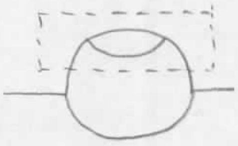


$$\sim\int dq\,\frac{q^7}{q^{12}}\sim\int\frac{dq}{q^5}\quad\text{finite}$$

etc.



overall finite



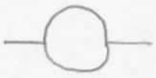
order-by-order renormalization

$$\mathcal{L} = \dots - \frac{1}{2}m^2\phi^2\dots \qquad m^2 = m_R^2+\frac{g^2}{16\pi^2}\delta_m$$

$$-\frac{1}{2}m_R^2\phi^2-\frac{g^2}{32\pi^2}\delta_m\phi^2 \longrightarrow$$

$$-(2\pi)^4 i\,\frac{g^2}{16\pi^2}\delta_m$$

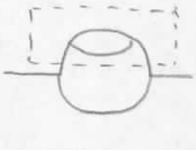
$\Rightarrow$



$+$

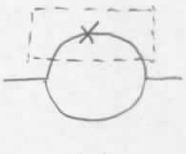
finite by construction

$O(g^4)$



UV+F

$+$



−UV

$$\Rightarrow\ F = -\int_0^1 dx\,\ln\mu^2$$

$$\int_0^1 dx\,\ln\mu^2\Rightarrow\ln q^2 \qquad \Rightarrow\ \int d^4q\,\frac{\ln q^2}{q^6}\quad\text{finite}$$

$$\mu^2 = x(1-x)q^2+m^2$$

$$\downarrow$$

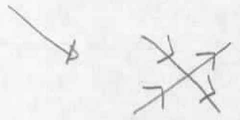
$$>0$$

ϕ^3 super-renormalizable

non - ren

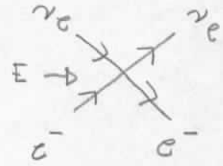
$\mathcal{L} = G_F \, j_\alpha j_\alpha^{+}$, $j_\alpha = \bar{\nu}_e \gamma_\alpha (1+\gamma^5) e + \bar{\nu}_\mu \gamma_\alpha (1+\gamma^5)\mu$

$G_F = \frac{g^2}{m_p^2}$

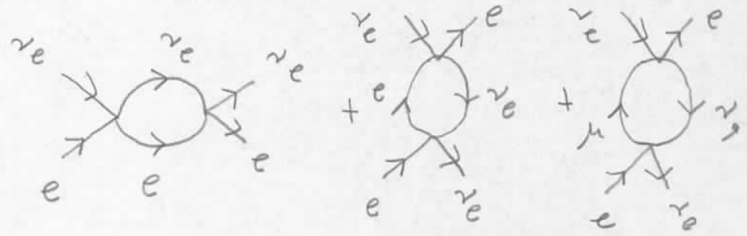


Use Λ-reg   let O be an observable with scale E

$$O = g^\ell \left\{ a_0 + a_2 g^2 + \cdots + a_k g^k \frac{\Lambda^2}{E^2} + a_{k+2} g^{k+2} \frac{\Lambda^4}{E^4} + \cdots \right\}$$

ex: 

$O(g^2)$ $g^2 a_0$



$\sim g^4 \frac{\Lambda^2}{E^2}$

$\sigma(\nu_e e^- \rightarrow \nu_e e^-) = g^2 \left(a_0 + g^2 a_1 \frac{\Lambda^2}{E^2} + \cdots \right)$ divergent BUT g-ren required

use $\Gamma(\mu \rightarrow \nu_\mu \nu_e e) = g^2 \left(b_0 + g^2 b_1 \frac{\Lambda^2}{E^2} + \cdots \right)$ derive $g = g(\Gamma, \Lambda)$

replace in σ $\Longrightarrow$ $\frac{\Lambda^2}{E^2}$ survives $\Rightarrow$ NON-ren theory.

DR as analytic continuation

$$I = \int d^4 q \, \frac{1}{(q^2+m^2)((q+p)^2+M^2)}$$ $p_\mu \equiv (\vec{0}, i\mu)$, $p^2 = -\mu^2$

$a_1 = q^2 + m^2 = \omega^2 - q_0^2 + m^2$, $a_2 = \omega^2 - (q_0+\mu)^2 + M^2$

$$I = \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega\, \omega^{\mu-2} \int d\Omega_{n-1} \frac{1}{a_1 a_2}$$

I defined for

$$= 2 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{\mu-1}{2})} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega \frac{\omega^{\mu-2}}{a_1 a_2} \qquad 1 < \mathrm{Re}\, n < 4$$

$$d\omega\, \omega^{\mu-2} = d\omega\, \omega\, \omega^{\mu-3} = \frac{1}{2} d\omega^2 (\omega^2)^{\frac{\mu-3}{2}} = \frac{1}{2} d\omega^2 \frac{2}{n-1} \frac{d}{d\omega^2} (\omega^2)^{\frac{n-1}{2}}$$

$$I = 2 \frac{\pi^{\frac{\mu-1}{2}}}{\Gamma(\frac{\mu-1}{2})} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega^2 \frac{1}{\mu-1} (\omega^2)^{\frac{\mu-1}{2}} (-\frac{\partial}{\partial \omega^2}) \frac{1}{a_1 a_2} \qquad 1 < \mathrm{Re}\, n < 4$$

$$= \frac{2}{\mu-1} \frac{\pi^{\frac{\mu-1}{2}}}{\Gamma(\frac{\mu-1}{2})} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega\, 2\omega\, \omega^{\mu-1} (-\frac{\partial}{\partial \omega^2}) \frac{1}{a_1 a_2}$$

$$= 2 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{\mu-1}{2}+1)} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega\, \omega^{\mu} (-\frac{\partial}{\partial \omega^2}) \frac{1}{a_1 a_2}$$

$$= 2 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{\mu-1}{2}+\lambda)} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega\, \omega^{\mu-2+2\lambda} (-\frac{\partial}{\partial \omega^2})^{\lambda} \frac{1}{a_1 a_2}$$

$\mathrm{Re}\, \mu - 2 + 2\lambda > -1$, $\mathrm{Re}\, n > 1 - 2\lambda$ $\Rightarrow$ $1 - 2\lambda < \mathrm{Re}\, n < 4$

$$1 = \frac{1}{2} (\frac{dq_0}{dq_0} + \frac{d\omega}{d\omega})$$

$$I = 2 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{\mu-1}{2})} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega \frac{1}{2} (\frac{dq_0}{dq_0} + \frac{d\omega}{d\omega}) \frac{\omega^{\mu-1}}{a_1 a_2} \qquad 1 < \mathrm{Re}\, n < 4$$

$$= - \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{\mu-1}{2})} \int_{-\infty}^{+\infty} dq_0 \int_0^\infty d\omega\, (q_0 \frac{\partial}{\partial q_0} + \omega \frac{\partial}{\partial \omega}) \frac{\omega^{\mu-2}}{a_1 a_2}$$

$$\frac{\partial}{\partial q_0}\frac{1}{a_1 a_2} = -\frac{1}{a_1^2 a_2}(-2q_0) - \frac{1}{a_1 a_2^2}(-2)(q_0+\mu)$$

$$= \frac{2}{a_1 a_2}\left\{\frac{q_0}{a_1} + \frac{q_0+\mu}{a_2}\right\}$$

$$-q_0\frac{\partial}{\partial q_0}\frac{w^{\mu-2}}{a_1 a_2} = -2\frac{w^{\mu-2}}{a_1 a_2}\left\{\frac{q_0^2}{a_1} + \frac{q_0(q_0+\mu)}{a_2}\right\},$$

$$\frac{\partial}{\partial w}\frac{w^{\mu-2}}{a_1 a_2} = (\mu-2)\frac{w^{\mu-3}}{a_1 a_2} - \left\{\frac{2w}{a_1^2 a_2} + \frac{2w}{a_1 a_2^2}\right\} w^{\mu-2}$$

$$-w\frac{\partial}{\partial w}\frac{w^{\mu-2}}{a_1 a_2} = -(\mu-2)\frac{w^{\mu-2}}{a_1 a_2} + 2w^{m}\left(\frac{1}{a_1^2 a_2} + \frac{1}{a_1 a_2^2}\right)$$

$$= \frac{w^{\mu-2}}{a_1 a_2}\left\{2-m+2\frac{w^2}{a_1}+2\frac{w^2}{a_2}\right\} = 2\frac{w^{\mu-2}}{a_1 a_2}\left\{1-\frac{m}{2}+\frac{w^2}{a_1}+\frac{w^2}{a_2}\right\}$$

$$-\left(q_0\frac{\partial}{\partial q_0}+w\frac{\partial}{\partial w}\right)\frac{w^{\mu-2}}{a_1 a_2} = 2\frac{w^{\mu-2}}{a_1 a_2}\left\{-\frac{q_0^2}{a_1} - \frac{q_0(q_0+\mu)}{a_2} + 1 - \frac{m}{2} + \frac{w^2}{a_1} + \frac{w^2}{a_2}\right\}$$

$$= 2\frac{w^{\mu-2}}{a_1 a_2}\left\{1-\frac{m}{2} + \frac{w^2-q_0^2}{a_1} + \frac{w^2-q_0(q_0+\mu)}{a_2}\right\}$$

$$w^2 - q_0^2 = a_1 - m^2, \quad w^2 - q_0(q_0+\mu) = a_2 + (q_0+\mu)^2 - M^2 - q_0(q_0+\mu)$$

$$= a_2 + \mu(q_0+\mu) - M^2$$

$$-\left(q_0\frac{\partial}{\partial q_0}+w\frac{\partial}{\partial w}\right)\frac{w^{\mu-2}}{a_1 a_2} = 2\frac{w^{\mu-2}}{a_1 a_2}\left\{1-\frac{m}{2} + \frac{a_1 - m^2}{a_1} + \frac{a_2 + (q_0+\mu)\mu - M^2}{a_2}\right\}$$

$$\downarrow$$

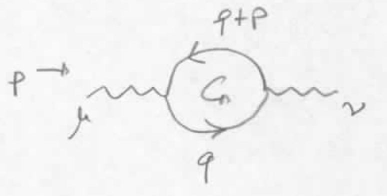
$$3 - \frac{m}{2} - \frac{\mu^2}{a_1} + \frac{\mu(q_0+\mu) - M^2}{a_2}$$

$$I = 2\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{\mu-1}{2}\right)}\int_{-\infty}^{+\infty} dq_0 \int_0^{\infty} dw\, \frac{w^{\mu-2}}{a_1 a_2}\left\{3 - \frac{m}{2} - \frac{\mu^2}{a_1} + \frac{\mu(q_0+\mu) - M^2}{a_2}\right\}$$

$$I = (3 - \frac{n}{2}) I + I' \qquad I' \quad 1 < \mathrm{Re}\, n < 5$$

$$\searrow \quad I = \frac{2}{n-4} I'$$

QED list of UV divergent diagrams



$\Pi_{\mu\nu}$

$$\Pi_{\mu\nu} = ie^2 \int d^4q \, \frac{1}{(q+m^2)((q+p)^2+m^2)} \, \mathrm{tr}\left\{ \gamma_\nu (-i\not{q} + m) \gamma_\mu \left[-i(\not{q}+\not{p}) + m \right] \right\}$$

$$\mathrm{tr}\{\cdots\} = -\mathrm{tr}\, \gamma_\nu \not{q} \gamma_\mu (\not{q}+\not{p}) + m^2 \, \mathrm{tr}\, \gamma_\nu \gamma_\mu$$

$$= 4\left\{ -q_\nu (q+p)_\mu + \delta_{\mu\nu} \, q\cdot(q+p) - (q+p)_\nu q_\mu + m^2 \delta_{\mu\nu} \right\}$$

$$= 4\left\{ -2q_\mu q_\nu - q_\nu p_\mu - q_\mu p_\nu + \delta_{\mu\nu} \left[m^2 + q\cdot(q+p) \right] \right\} = 4 N_{\mu\nu}$$

$$\Pi_{\mu\nu} = 4ie^2 \int_0^1 dx \int d^4q \, N_{\mu\nu} \left\{ \underset{\swarrow}{q^2} + 2xp\cdot q + xp^2 + m^2 \right\}^{-2}$$

$$(q+xp)^2 + x(1-x)p^2 + m^2 , \qquad q' = q + xp$$

$$\mu^2 = x(1-x)p^2 + m^2$$

$$\Pi_{\mu\nu} = 4ie^2 \int_0^1 dx \int d^4q \, N'_{\mu\nu} (q^2 + \mu^2)^{-2}$$

$$N'_{\mu\nu} = -2(q-xp)_\mu (q-xp)_\nu - (q-xp)_\mu p_\nu - (q-xp)_\nu p_\mu$$

$$+ \delta_{\mu\nu} \left\{ m^2 + (q-xp)\cdot(q-xp+p) \right\}$$

$$= -2q_\mu q_\nu - 2x^2 p_\mu p_\nu + 2x p_\mu p_\nu + \delta_{\mu\nu} \left\{ m^2 + q^2 - x(1-x)p^2 \right\} + \text{" q odds"}$$

$$\Pi_{\mu\nu} = 4ie^2\int_0^1 dx \int d^nq\,(q^2+\mu^2)^{-2}\Big\{-2q_\mu q_\nu + 2x(1-x)p_\mu p_\nu$$
$$+\delta_{\mu\nu}\left[m^2+q^2-x(1-x)p^2\right]\Big\}$$

remember $\quad i\int d^nq\,(q^2+\mu^2)^{-2} = i\pi^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)$

$$i\int d^nq\,q_\mu q_\nu(q^2+\mu^2)^{-2} = i\pi^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)\left(-\frac{1}{2}\delta_{\mu\nu}\right)\frac{\mu^2}{n/2-1}$$

$n = 4-\varepsilon$

$\frac{n}{2}-1 = 1-\frac{\varepsilon}{2}$

$\frac{1}{n/2-1} = 1+\frac{\varepsilon}{2}$

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon}-\gamma-\ln\pi\,, \qquad \frac{\varepsilon}{2}\frac{1}{\bar{\varepsilon}} = 1$$

$$= -i\frac{\pi^2}{2}\mu^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2+1\right)\delta_{\mu\nu}$$

$$i\int d^nq\,q^2(q^2+\mu^2)^{-2} = -i\frac{\pi^2}{2}\mu^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)(4-\varepsilon)$$
$$= -i\frac{\pi^2}{2}\mu^2\left(\frac{4}{\bar{\varepsilon}}-4\ln\mu^2-2\right) = -2i\pi^2\mu^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2+\frac{1}{2}\right)$$

$$\Pi_{\mu\nu} = 4i\pi^2e^2\int_0^1 dx\,\Big\{2x(1-x)p_\mu p_\nu\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)$$
$$+\delta_{\mu\nu}\Big[\mu^2\Big(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\underset{\times}{+1}\Big)+m^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)-2\mu^2\Big(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\underset{\times}{+\frac{1}{2}}\Big)\quad\underset{\times}{\Rightarrow 0}$$
$$+x(1-x)p^2\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)\Big]\Big\}$$

$$\Pi_{\mu\nu} = 4i\pi^2e^2\int_0^1 dx\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)\Big\{2x(1-x)p_\mu p_\nu+\delta_{\mu\nu}\big[\mu^2+m^2-2\mu^2$$
$$-x(1-x)p^2\big]\Big\}$$

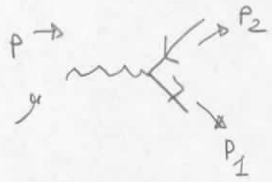
$$[\ldots] = m^2-x(1-x)p^2-\mu^2 = \cancel{m^2}-x(1-x)p^2-x(1-x)p^2-\cancel{m^2}$$
$$= -2x(1-x)p^2$$

$$\Pi_{\mu\nu} = 8i\pi^2e^2\int_0^1 dx\left(\frac{1}{\bar{\varepsilon}}-\ln\mu^2\right)x(1-x)\left(p_\mu p_\nu-p^2\delta_{\mu\nu}\right)$$

$$\Pi_{\mu\nu} = \Pi_0(p^2)\delta_{\mu\nu} + \Pi_1(p^2)p_\mu p_\nu$$

WIs $\quad p^\mu \Pi_{\mu\nu} = \Pi_{\mu\nu} p^\nu = 0 \qquad \Pi_0(p^2)p_\nu + p^2\Pi_1(p^2)p_\nu = 0$

$$\Pi_0 = -p^2\Pi_1$$



$$p_\mu \, \bar{u}(p_1)\gamma^\mu v(p_2) \qquad p_1 + p_2 = p$$
$$= 0$$

$$\bar{\Delta}_{\mu\nu} = \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2 - \frac{1}{(2\pi)^4 i}\Pi_0(p^2)} + \text{" } p_\mu p_\nu \text{ terms"}$$

$\searrow$ requires $\Pi_0(0) = 0$, $\Pi_0 = \Pi(p^2)p^2$

for $m_\gamma \neq 0$

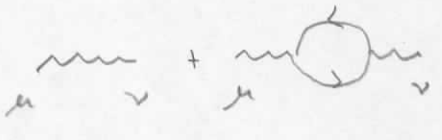
$$|p^2| \ll m^2 \qquad \ln\mu^2 = \ln\{x(1-x)p^2 + m^2\} = \ln m^2 + \ln\{1 + \frac{p^2}{\mu^2}x(1-x)\}$$

$$= \ln m^2 + \frac{p^2}{m^2}x(1-x) + O(\frac{p^4}{\mu^4})$$

$$\Rightarrow \Pi_{\mu\nu} = 8i\pi^2 e^2 \int_0^1 dx \left\{\frac{1}{\varepsilon} - \ln m^2 + \frac{p^2}{m^2}x(1-x) + \cdots\right\} x(1-x)(p_\mu p_\nu - p^2\delta_{\mu\nu})$$

$$\int_0^1 dx\, x(1-x) = \frac{1}{6}, \quad \int_0^1 dx\, x^2(1-x)^2 = \frac{1}{30}$$

$$\Pi_{\mu\nu} = 8i\pi^2 e^2 (p_\mu p_\nu - p^2\delta_{\mu\nu})\left\{\frac{1}{6}(\frac{1}{\varepsilon} - \ln m^2) + \frac{1}{30}\frac{p^2}{m^2} + O(p^4)\right\}$$



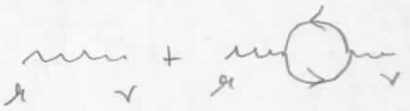
$$= \frac{1}{(2\pi)^4 i}\left\{\frac{\delta_{\mu\nu}}{p^2} + \frac{1}{p^2}\frac{\Pi_{\mu\nu}}{(2\pi)^4 i}\frac{1}{p^2}\right\} + \text{" } p_\mu p_\nu \text{ terms"}$$

$$\Pi_{\mu\nu} = -8i\pi^2 e^2 \delta_{\mu\nu} p^2 (C + \lambda\frac{p^2}{m^2}) + \text{" } p_\mu p_\nu \text{ terms"}$$

$$C = \frac{1}{6}(\frac{1}{\varepsilon} - \ln m^2), \quad \lambda = \frac{1}{30}$$

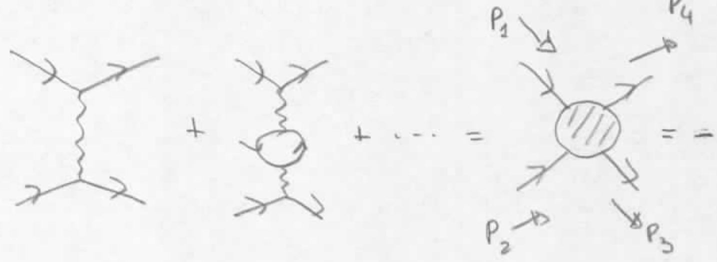
$$\Pi_{\mu\nu} = (2\pi)^4 i \frac{-8\pi^2 e^2}{16\pi^4} p^2 (C + \lambda \frac{p^2}{m^2}) \delta_{\mu\nu} + \ldots$$

$$= -(2\pi)^4 i \frac{e^2}{2\pi^2} p^2 (C + \lambda \frac{p^2}{m^2}) \delta_{\mu\nu}$$



$$= \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2} \left\{ 1 - \frac{e^2}{2\pi^2} (\underset{\substack{\downarrow \\ \text{UV divergent}}}{C} + \lambda \frac{p^2}{m^2}) \right\}$$

Defenition of α



$$= -(2\pi)^4 i \frac{F(p^2)}{p^2} \gamma^\lambda \otimes \gamma^\mu \quad p = p_1 - p_4$$
$$= p_2 - p_3$$

$$F(0) = 4\pi\alpha$$

$$4\pi\alpha = e^2 \left\{ 1 - \frac{e^2}{2\pi^2} C \right\} \qquad e^2 = 4\pi\alpha + x\alpha^2$$

$$(4\pi\alpha + x\alpha^2)(1 - \frac{4\pi\alpha}{2\pi^2} C) = 4\pi\alpha, \quad (4\pi\alpha + x\alpha^2)(1 - 2\frac{\alpha}{\pi} C) = 4\pi\alpha$$

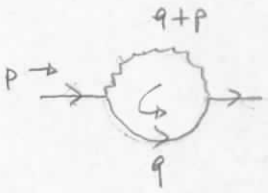
$$\Rightarrow x = 8C$$

$$e^2 = 4\pi\alpha + 8\alpha^2 C = 4\pi\alpha (1 + 2\frac{\alpha}{\pi} C)$$

$$\Rightarrow (2\pi)^4 i (-\frac{1}{p^2}) 4\pi\alpha (1 + 2\frac{\alpha}{\pi} C) \left\{ 1 - \frac{2\alpha}{\pi} (C + \lambda \frac{p^2}{m^2} \right\} \gamma^\mu \otimes \gamma^\mu$$

$$= -(2\pi)^4 i \frac{4\pi\alpha}{p^2} \left\{ 1 - 2\frac{\alpha}{\pi} \lambda \frac{p^2}{m^2} \right\} \gamma^\mu \otimes \gamma^\mu$$

$$\Pi_{\mu\nu}\Big|_{UV} = (2\pi)^4 i \, \frac{e^2}{12\pi^2} \left(p_\mu p_\nu - p^2 \delta_{\mu\nu} \right) \frac{1}{\underline{\underline{\varepsilon}}}$$



$$\Sigma = - e^2 i \int d^4 q \, \frac{\gamma^\mu (- i \not{q} + m) \gamma^\mu}{\underset{1-x}{(q^2 + m^2)} \underset{x}{(q+p)^2}}$$

$$\mu^2 = x(1-x) p^2 + (1-x) m^2$$

$$\Sigma = - e^2 i \int_0^1 dx \int d^4 q \left\{ q^2 + 2x p \cdot q \underset{+ x p^2}{} + (1-x) m^2 \right\}^{-2} \gamma^\mu (- i \not{q} + m) \gamma^\mu$$

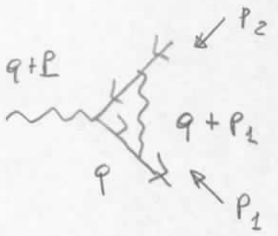
$$\gamma^\mu \gamma^\mu = n \; , \quad \gamma^\mu \not{q} \gamma^\mu = - \gamma^\mu \gamma^\mu \not{q} + 2 \not{q} = (2-n) \not{q}$$

$$\Sigma = - e^2 \int_0^1 dx \, i \int d^4 q \left\{ q^2 + 2x p \cdot q + x p^2 + (1-x) m^2 \right\}^{-2} \left\{ (n-2) i \not{q} + n m \right\}$$

$$= - i e^2 \pi^2 \int_0^1 dx \left(\frac{1}{\underline{\underline{\varepsilon}}} - \ln \mu^2 \right) \left\{ (n-2) i x \not{p} + n m \right\}$$

$$\Sigma\Big|_{UV} = - \frac{i e^2 \pi^2}{\underline{\underline{\varepsilon}}} \int_0^1 dx \left\{ 2 i x \not{p} + 4 m \right\} = - \frac{i e^2 \pi^2}{\underline{\underline{\varepsilon}}} (i \not{p} + 4 m)$$

$$= - (2\pi)^4 i \, \frac{e^2}{16\pi^2} \frac{1}{\underline{\underline{\varepsilon}}} (i \not{p} + 4 m)$$



$$\underline{P} = p_1 + p_2$$

$$\Lambda^\mu = (ie)^3 \, i \int d^4 q \left\{ (q^2 + m^2)(q + p_1)^2 ((q + \underline{P}) + m^2) \right\}^{-1}$$

$$\times \gamma^\alpha (- i \not{q} + m) \gamma^\mu \left[- i (\not{q} + \not{\underline{P}}) + m \right] \gamma^\alpha$$

C. identities

$$\gamma^\alpha\gamma^\mu\gamma^\alpha = -\gamma^\alpha\gamma^\alpha\gamma^\mu + 2\gamma^\mu = (2-n)\gamma^\mu$$

$$\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\alpha = -\gamma^\alpha\gamma^\mu\gamma^\alpha\gamma^\nu + 2\gamma^\nu\gamma^\mu = (n-2)\gamma^\mu\gamma^\nu + 2\gamma^\nu\gamma^\mu$$

$$= (n-2)\gamma^\mu\gamma^\nu + 2(-\gamma^\mu\gamma^\nu + 2\delta^{\mu\nu}) = 4\delta^{\mu\nu} + (n-4)\gamma^\mu\gamma^\nu$$

$$\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha = -\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta + 2\gamma^\beta\gamma^\mu\gamma^\nu = -4\delta^{\mu\nu}\gamma^\alpha\gamma^\beta + (n-4)\gamma^\mu\gamma^\nu\gamma^\beta$$
$$+ 2\gamma^\beta\gamma^\mu\gamma^\nu$$

$$= -4\delta^{\mu\nu}\cancel{\gamma^\alpha}\gamma^\beta + 2\gamma^\beta(-\gamma^\nu\gamma^\mu + 2\cancel{\delta^{\mu\nu}}) + (n-4)\gamma^\mu\gamma^\nu\gamma^\beta$$

$$= -2\gamma^\beta\gamma^\nu\gamma^\mu + (n-4)\gamma^\mu\gamma^\nu\gamma^\beta$$

etc

$$\int d^n q \, \frac{q_\mu q_\nu}{(\ )(\ )(\ )}$$ UV divergat

$$\Rightarrow -\gamma^\alpha \not{q}\,\gamma^\mu \not{q}\,\gamma^\alpha \Big|_{n=4} = 2\not{q}\,\gamma^\mu\not{q}$$

$(1) = q^2 + m^2$

$(2) = (q + p_1)^2$

$(3) = (q + p)^2 + m^2$

$$\Lambda^\mu\Big|_{UV} \Rightarrow e^3 \int d^n q \, \frac{2\not{q}\,\gamma^\mu\not{q}}{(1)(2)(3)}$$

$$= 2e^3\,\gamma^\alpha\gamma^\mu\gamma^\beta \int d^n q \, \frac{q_\alpha q_\beta}{(1)(2)(3)}$$

General F.T.

$$\frac{1}{\prod_{i=1}^{m} a_i} = \Gamma(m)\int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} dx_{n-1} \Big\{ a_1 x_{n-1} + a_2(x_{n-2} - x_{n-1})$$
$$+ \cdots + a_n(1 - x_1)\Big\}^{-m}$$

$$\int d^4q \, \frac{q_\alpha q_\beta}{\underset{\swarrow}{(1)}\underset{\searrow}{(2)}\underset{\searrow}{(3)}} = 2\int_0^1 dx \int_0^x dy \int d^4q \, \frac{q_\alpha q_\beta}{(q^2+2k\cdot q+M^2)^3}$$

$$1-x \qquad x-y \qquad y$$

$$k = y\,P + (x-y)p_1 = y\,(p_1+p_2) + (x-y)p_1 = x\,p_1 + y\,p_2$$

$$M^2 = (1-x)m^2 + (x-y)\,p_1^2 + y\,(P^2+m^2)$$

$$= (1-x+y)\,m^2 + (x-y)\,p_1^2 + y\,P^2$$

$$\int d^\mu q \, \frac{q_\alpha q_\beta}{(q^2+2k\cdot q+M^2)^3} = \pi^{m/2}\,\frac{\Gamma(3-m/2)}{\Gamma(3)}\left\{k_\alpha k_\beta + \frac{1}{2}\,\frac{M^2-k^2}{\underset{\swarrow}{2-m/2}}\,\delta_{\alpha\beta}\right\}(M^2-k^2)^{m/2-3}$$

$$2-\frac{1}{2}(4-\varepsilon) = \frac{\varepsilon}{2}$$

$$'' \quad \Big|_{UV} = \pi^2\,\frac{1}{\Gamma(3)}\left(\frac{1}{\varepsilon}-\frac{1}{2}\gamma-\frac{1}{2}\ln\pi\right)\delta_{\alpha\beta}$$

$$= \frac{\pi^2}{4}\,\frac{\delta_{\alpha\beta}}{\varepsilon}$$

$$\Lambda^\mu\Big|_{UV} \Rightarrow \cancel{2}e^3\,\frac{\pi^2}{\cancel{4}}\,\gamma^\alpha\gamma^\mu\gamma^\beta\,\delta_{\alpha\beta}\,\cancel{2}\int_0^1 dx\int_0^x dy\,\frac{1}{\varepsilon}$$

$$= \frac{e^3\pi^2}{\varepsilon}(-2\gamma^\mu)\,\frac{1}{2} = -\frac{e^3\pi^2}{\varepsilon}\,\gamma^\mu = +(2\pi)^4 i\,\frac{ie^3}{16\pi^2}\,\frac{1}{\varepsilon}\,\gamma^\mu$$

Summary UV

$$\Pi_{\mu\nu}\Big|_{UV} = (2\pi)^4 i\,\frac{e^2}{12\pi^2}\,(p_\mu p_\nu - p^2\delta_{\mu\nu})\,\frac{1}{\varepsilon}$$

$$\Sigma\Big|_{UV} = -(2\pi)^4 i\,\frac{e^2}{16\pi^2}\,(i\not{p}+4m)\,\frac{1}{\varepsilon}$$

$$\Lambda^\mu\Big|_{UV} = (2\pi)^4 i\,\frac{ie^3}{16\pi^2}\,\gamma^\mu\,\frac{1}{\varepsilon}$$

N.B. most general form of Λ^μ compatible with Lorentz and T-invariance would be

$$\Lambda^\mu = \Lambda_C \gamma^\mu + \Lambda_M \sigma_{\mu\nu}(p_1 - p_2)^\nu$$

$$\Lambda_M \big|_{UV} = 0$$

Renormalization $\qquad \mathcal{L} = \mathcal{L}(\{p\}, \{\phi\})$

parameter ren. $\qquad p = Z_p \, p_R \,, \quad Z_p = 1 + e_R^2 \, \delta Z_p^{(1)} + O(e_R^4)$

P1) p-ren. is sufficient to obtain finite S-matrix elements when WFR for external on-shell particles is included

$$\phi = Z_\phi \, \phi_R \qquad \ldots$$

P2) Green functions can be made finite

P2 $\qquad \mathcal{L}(\{p\}, \{\phi\}) = \mathcal{L}(\{p_R\}, \{\phi\}_R) + e_R^2 \, \delta\mathcal{L}_{CT}^{(1)} + O(e_R^4)$

QED $\qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2}(\partial_\mu A_\mu)^2 - \bar{\psi}(\not{\partial} + m)\psi + ie\bar{\psi}\not{A}\psi$

$$A_\mu = Z_\gamma^{1/2} A_\mu^R \,, \quad \psi = Z_\psi^{1/2} \psi^R \,, \quad e = Z_e \, e_R \,, \quad m = Z_m \, m_R$$

$$Z_i = 1 + e_R^2 \underset{\delta Z_i}{\underset{\parallel}{\delta Z_i^{(1)}}} + O(e_R^4)$$

$$\mathcal{L} = -\frac{1}{4}(1 + e_R^2 \delta Z_\gamma) F_{\mu\nu}^R F_{\mu\nu}^R - \frac{1}{2}(1 + e_R^2 \delta Z_\gamma)(\partial_\mu A_\mu^R)^2$$

$$- (1 + e_R^2 \delta Z_\psi) \bar{\psi}^R \not{\partial} \psi^R - (1 + e_R^2 \delta Z_\psi)(1 + e_R^2 \delta Z_m) m_R \bar{\psi}^R \psi^R$$

$$+ ie_R (1 + e_R^2 \delta Z_\psi)(1 + e_R^2 \delta Z_e)(1 + \frac{1}{2} e_R^2 \delta Z_\gamma) \bar{\psi}^R \not{A}^R \psi^R$$

$$\delta\mathcal{L}_1 = -\frac{1}{4}\delta Z_\gamma F^R_{\mu\nu}F^R_{\mu\nu} - \frac{1}{2}\delta Z_\gamma (\partial_\mu A^R_\mu)^2 - \delta Z_\psi \bar\psi^R \not\partial \psi^R$$

$$- (\delta Z_\psi + \delta Z_m) m_R \bar\psi^R\psi^R + i e_R (\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_\gamma)\bar\psi^R \not A^R \psi^R$$

$$\Pi_{\mu\nu} = \Pi(p^2)p^2\delta_{\mu\nu} + \text{“}p_\mu p_\nu \text{ terms”}$$

$$\bar\Delta_{\mu\nu} = \frac{1}{(2\pi)^4 i}\frac{\delta_{\mu\nu}}{p^2}\left\{ Z_\gamma - \frac{\Pi(p^2)}{(2\pi)^4 i}\right\}^{-1}$$

$$\swarrow \qquad\qquad \searrow$$

$$e_R^2\delta Z_\gamma \qquad -\frac{e_R^2}{12\pi^2}\frac{1}{\varepsilon} \qquad \Rightarrow \delta Z_\gamma = -\frac{1}{12\pi^2}\frac{1}{\varepsilon}$$

Σ

$\bar S =$

$$\bar S = \frac{1}{(2\pi)^4 i}\left\{ Z_\psi\left[i\not p + m_R\delta Z_m\right] - \frac{1}{(2\pi)^4 i}\Sigma\right\}^{-1}$$

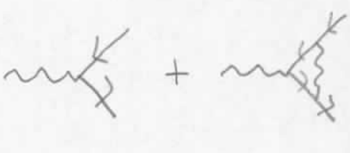
$$\swarrow$$

$$(1+e_R^2\delta Z_\psi) i\not p + \left[1 + e_R^2(\delta Z_\psi + \delta Z_m)\right] m_R$$

UV only $\quad (1+e_R^2\delta Z_\psi)(i\not p + m_R) + e_R^2\delta Z_m m_R + \frac{e_R^2}{16\pi^2}(i\not p + 4m_R)\frac{1}{\varepsilon}$

$$= (i\not p + m_R)\left[1 + e_R^2\delta Z_\psi + \frac{e_R^2}{16\pi^2}\frac{1}{\varepsilon}\right] + e_R^2\delta Z_m m_R + 3\frac{e_R^2}{16\pi^2}m_R\frac{1}{\varepsilon}$$

$$\Rightarrow \delta Z_\psi = -\frac{1}{16\pi^2}\frac{1}{\varepsilon} \quad , \quad \delta Z_m = -\frac{3}{16\pi^2}\frac{1}{\varepsilon}$$



$$(2\pi)^4 i\left\{ ie_R + ie_R^3(\delta Z_\psi + \delta Z_e + \frac{1}{2}\delta Z_\gamma) + i\frac{e_R^3}{16\pi^2}\frac{1}{\varepsilon}\right\}\gamma^\mu$$

$$\delta Z_e - \cancel{\frac{1}{16\pi^2}}\frac{1}{\varepsilon} - \frac{1}{24\pi^2}\frac{1}{\varepsilon} + \cancel{\frac{1}{16\pi^2}}\frac{1}{\varepsilon} = 0 \quad , \quad \delta Z_e = \frac{1}{24\pi^2}\frac{1}{\varepsilon}$$

$$\Rightarrow \delta\mathcal{L}_1 = \frac{1}{\pi^2 \bar{\varepsilon}} \left\{ \frac{1}{48} F^R_{\mu\nu} F^R_{\mu\nu} + \frac{1}{24} (\partial_\mu A^R_\mu)^2 - \frac{1}{16} \bar{\psi}^R \not{\partial} \psi^R + \frac{1}{4} m_R \bar{\psi}^R \psi^R \right.$$
$$\left. - \frac{i e_R}{16} \bar{\psi}^R \not{A}^R \psi^R \right\}$$

$p \to p_R \longrightarrow p_{phys}$ ex :

$$e^2 \bar{\Delta}_{\mu\nu} \longrightarrow e^2 \left\{ 1 - \frac{e^2}{2\pi^2} \left[C + O(p^2) \right] \right\} , \quad C = \frac{1}{6} \left(\frac{1}{\bar{\varepsilon}} - \ln m^2 \right)$$

$e \to e_R$

$$e_R^2 (1 + 2 e_R^2 \delta Z_e) \left\{ 1 - \frac{e_R^2}{2\pi^2} \left[C + F(p^2) \right] \right\}$$

$$= e_R^2 \left\{ 1 + e_R^2 \left[2 \delta Z_e - \frac{1}{2\pi^2} (C + F) \right] \right\}$$

$$\swarrow$$

$$2 \delta Z_e - \frac{1}{12\pi^2} \left\{ \frac{2}{\varepsilon} - \gamma - \ln \pi - \ln m^2 \right\} + \cdots$$

1) MS $\quad \delta Z_e = \dfrac{1}{12\pi^2} \dfrac{1}{\varepsilon}$, 2) $\overline{\text{MS}}$ $\quad \delta Z_e = \dfrac{1}{24\pi^2} \dfrac{1}{\bar{\varepsilon}}$

$$\downarrow$$

$$e_R^2 \left\{ 1 + e_R^2 \left[\frac{1}{12\pi^2} (\gamma + \ln \pi + \ln m^2) - \frac{1}{2\pi^2} F(p^2) \right\} \right.$$

$$e_R^2 \left\{ 1 + e_R^2 \left[\frac{1}{12\pi^2} \ln m^2 - \frac{1}{2\pi^2} F(p^2) \right] \right\}$$

$$4\pi\alpha = e_R^2 \left\{ 1 + e_R^2 \left[\frac{1}{12\pi^2} (\gamma + \ln \pi + \ln m^2) - \frac{1}{2\pi^2} F(0) \right] \right\}$$

$$= e_R^2 \left\{ 1 + e_R^2 \left[\frac{1}{12\pi^2} \ln m^2 - \frac{1}{2\pi^2} F(0) \right] \right\}$$

$\Downarrow$ replace into $e^2 \bar{\Delta}_{\mu\nu}$, p^2-independent terms cancel

P1) $e \to e_R \ , \ m \to m_R$

$$\bar{S} = \frac{1}{(2\pi)^4 i} \left\{ i\not{p} + (1 + e_R^2 \delta Z_m) m_R - \frac{1}{(2\pi)^4 i} \Sigma \right\}^{-1}$$

$$\swarrow$$

$$i\not{p} + m_R + e_R^2 \delta Z_m m_R - \frac{1}{(2\pi)^4 i} \Sigma(i\not{p} = -m_e) - \frac{1}{(2\pi)^4 i} (i\not{p} + m_e) \left\{ \Sigma_{WF} + \Sigma_R \right\}$$

$$\Sigma_R = O(i\not{p} + m_e)$$

1) $$e_R^2 \delta Z_m m_R - \frac{1}{(2\pi)^4 i} \Sigma_{UV}(i\not{p} = -m_e) = 0$$

2) $$m_R - \frac{1}{(2\pi)^4 i} \Sigma_F(i\not{p} = -m_e) = m_e$$

$$\bar{S} = \frac{1}{(2\pi)^4 i} \left\{ \left[1 - \frac{1}{(2\pi)^4 i} (\Sigma_{WF} + \Sigma_R) \right] (i\not{p} + m_e) \right\}^{-1}$$

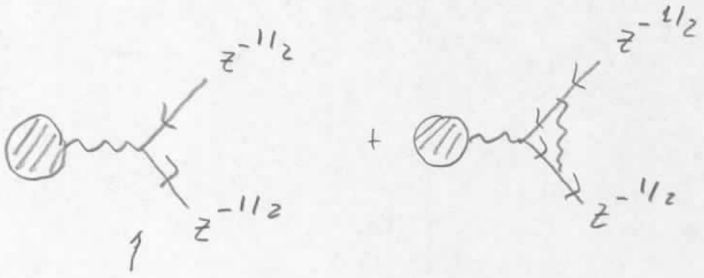
WFR $\quad u \to Z^{1/2} u$

$$\bar{u}\,\bar{S}\,u = \frac{1}{(2\pi)^4 i} Z \bar{u} \left[1 - \frac{1}{(2\pi)^4 i} (\Sigma_{WF} + \Sigma_R) \right]^{-1/2} \frac{1}{i\not{p} + m_e} \left[1 - \frac{1}{(2\pi)^4 i} (\Sigma_{WF} + \Sigma_R) \right]^{-1/2} u$$

$$p^2 \to -m_e^2 \quad \Rightarrow \quad Z = 1 - \frac{1}{(2\pi)^4 i} \Sigma_{WF}$$

$$Z^{1/2} \bar{u} \frac{1}{i\not{p} + m_e} \frac{1}{1 - \frac{1}{(2\pi)^4 i} (\Sigma_{WF} + \Sigma_R)} M$$

S-matrix $\Rightarrow Z^{-1/2} \bar{u} M$



$Z = 1 - \frac{1}{(2\pi)^4 i} \Sigma_{WF}$ $\qquad\qquad$ $Z = 1$

Loops

$$i\pi^2 A_0(m) = i\int d^nq \, \frac{1}{q^2+m^2} = i\pi^{n/2}\,\Gamma(1-n/2)\,(m^2)^{n/2-1}$$

$$(1-\tfrac{n}{2})\,\Gamma(1-\tfrac{n}{2}) = \Gamma(2-n/2) \qquad \mu = 4-\varepsilon \qquad \frac{\mu}{2} = 2-\frac{\varepsilon}{2}$$

$$= i\pi^{2-\varepsilon/2}\,\frac{\Gamma(\varepsilon/2)}{-1+\varepsilon/2}\,(m^2)^{1-\varepsilon/2} = -i\pi^2 m^2\left(1-\frac{\varepsilon}{2}\ln\pi\right)\left(\frac{2}{\varepsilon}-\gamma\right)\left(1-\frac{\varepsilon}{2}\ln m^2\right)$$

$$\times\left(1+\frac{\varepsilon}{2}\right.$$

$$= -i\pi^2\left\{\frac{2}{\varepsilon}-\gamma-\ln\pi-\ln m^2+1\right\} =$$

$$A_0(m) = -m^2\left(\frac{1}{\bar{\varepsilon}}-\ln m^2+1\right)$$

$$\chi = -p^2x^2+(p^2+m_2^2-m_1^2)x+m_1^2$$

$$i\pi^2 B_0(p^2;m_1,m_2) = i\int d^4q\,\frac{1}{(q^2+m^2)((q+p)^2+m^2)} = \frac{1}{\bar{\varepsilon}}-\int_0^1 dx\,\ln\chi$$

$$\chi = -p^2(x-x_-)(x-x_+) \Rightarrow \ln\chi = \ln(-p^2-i0)+\sum_{\lambda=\pm}\ln(x-x_\lambda)$$

$$\chi \to \chi - i0$$

$$B_0(p^2;m_1,m_2) = \frac{1}{\bar{\varepsilon}}-\ln(-p^2-i0)-\sum_\lambda\int_0^1 dx\,\ln(x-x_\lambda)$$

special case $p^2=0$

$$\Rightarrow \int_0^1 dx\,\ln\chi = \int_0^1 \underset{\frac{dx}{dx}}{dx}\,\ln\{(m_1^2-m_2^2)x+m_1^2\} = x\ln\{(m_1^2-m_2^2)x+m_1^2\}\Big|_0^1$$

$$-\int_0^1 dx\,\frac{x}{x-y}$$

$$y = \frac{m_1^2}{m_1^2-m_2^2} \qquad \int_0^1 dx\,\frac{x}{x-y} = \int_0^1 dx\,\frac{x-y+y}{x-y} = \int_0^1 dx\left(1+\frac{y}{x-y}\right)$$

$$= 1+y\ln\frac{1-y}{-y} = 1+y\ln\left(1-\frac{1}{y}\right)$$

$p^2=0$, $m_1=m_2=m$ $\qquad B(0;m,m)=\frac{1}{\varepsilon}-\ln m^2$

$$i\pi^2 B_\mu(p^2;m_1,m_2)=\int d^4q\,\frac{q_\mu}{(q^2+m_1^2)((q+p)^2+m_2^2)}=i\pi^2 B_1(p^2;m_1,m_2)\,p_\mu$$

$(1)=q^2+m_1^2$, $(2)=(q+p)^2+m_2^2$

$$p\cdot q=\frac{1}{2}\left\{(2)-(1)-p^2-m_2^2+m_1^2\right\}$$

$$i\pi^2 p^2 B_1=i\int d^4q\,\frac{p\cdot q}{(1)(2)}=\frac{i}{2}\int d^4q\left\{\frac{1}{(1)}-\frac{1}{(2)}-\frac{p^2+m_2^2-m_1^2}{(1)(2)}\right\}$$

$$p^2 B_1(p^2;m_1,m_2)=\frac{1}{2}\left\{A_0(m_1)-A_0(m_2)-(p^2+m_2^2-m_1^2)B_0\right\}$$

$$B_1(p^2;m,m)=-\frac{1}{2}B_0(p^2;m,m)$$

$$i\pi^2 B_{\mu\nu}(p^2;m_1,m_2)=i\int d^4q\,\frac{q_\mu q_\nu}{(1)(2)}$$

$$B_{\mu\nu}=B_{21}\,p_\mu p_\nu+B_{22}\,\delta_{\mu\nu}$$

$$B_{\mu\nu}=\frac{1}{\pi^2}\int d^4q\,\frac{q_\mu q_\nu}{(1)(2)}=B_{21}\,P_\mu P_\nu+B_{22}\,\delta_{\mu\nu}$$

$$p^2 B_{21}+n\,B_{22}=\frac{1}{\pi^2}\int d^4q\,\frac{q^2}{(1)(2)}=\frac{1}{\pi^2}\int d^4q\,\frac{(1)-m_1^2}{(1)(2)}$$

$$=A_0(m_2)-m_1^2 B_0$$

$$(B_{21}p^2+B_{22})P_\mu=\frac{1}{\pi^2}\int d^4q\,\frac{p\cdot q\;q_\mu}{(1)(2)}=\frac{1}{2\pi^2}\int d^4q\,\frac{q_\mu}{(1)(2)}\left\{(2)-(1)-(p^2+m_2^2-m_1^2)\right\}$$

$$=\frac{1}{2\pi^2}\int d^4q\left\{\frac{q_\mu}{(1)}-\frac{q_\mu}{(2)}-\frac{p^2+m_2^2-m_1^2}{(1)(2)}\,q_\mu\right\}$$

$$\int d^4q \, \frac{q_\mu}{(1)} = 0 \quad , \quad i\int d^4q \, \frac{q_\mu}{(2)} = i\int d^4q \, \frac{q_\mu}{(q+p)^2+m_2^2)} \qquad q' = q+p$$

$$= i\int d^4q \, \frac{q_\mu - p_\mu}{q^2+m_2^2} = -p_\mu A_0(m_2)$$

$$B_{21} p^2 + B_{22} = \frac{1}{2} A_0(m_2) - \frac{1}{2}(p^2+m_2^2-m_1^2) B_1$$

$$B_{21} p^2 + n B_{22} = A_0(m_2) - m_1^2 B_0$$

$$(n-1) B_{22} = \frac{1}{2} A_0(m_2) - m_1^2 B_0 + \frac{1}{2}(p^2+m_2^2-m_1^2) B_1$$

$$i\int d^nq \, \frac{q_\mu q_\nu}{(q^2+2k\cdot q+M^2)^2} = i\pi^{n/2}\Gamma(2-\frac{n}{2})\left\{k_\mu k_\nu + \frac{1}{2}\delta_{\mu\nu}\frac{M^2-k^2}{1-n/2}\right\}(M^2-k^2)^{n/2-2}$$

$$\hookrightarrow B_{22} = \frac{1}{2}\pi^{n/2}\Gamma(1-n/2)\int_0^1 dx \, x^{n/2-1}$$

$$= -\frac{1}{\varepsilon}\int_0^1 dx \, x + O(1)$$

$$= -\frac{1}{\varepsilon}(\frac{1}{6}p^2 + \frac{1}{2}m_1^2 + \frac{1}{2}m_2^2) + O(1)$$

$$(n-1) B_{22} = 3B_{22} + \frac{1}{6}p^2 + \frac{1}{2}(m_1^2+m_2^2)$$

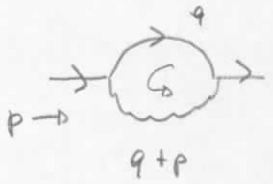
$$3B_{22} = \frac{1}{2} A_0(m_2) - m_1^2 B_0 + \frac{1}{2}(p^2+m_2^2-m_1^2) B_1 - \frac{1}{2}(\frac{1}{3}p^2+m_1^2+m_2^2)$$

$$p^2 B_{21} = \frac{1}{3} A_0(m_2) + \frac{1}{3} m_1^2 B_0 - \frac{2}{3}(p^2+m_2^2-m_1^2) B_1 + \frac{1}{6}(\frac{1}{3}p^2+m_1^2+m_2^2)$$

$$\Pi_{\mu\nu} = \Pi_0 \delta_{\mu\nu} + \text{"} p_\mu p_\nu \text{"} \quad , \quad \Pi_0 = 8i\pi^2 e^2 (B_{21} + B_1) p^2$$

$$\swarrow$$

$$(p^2; m, m)$$



$$\Sigma = -ie^2 \int d^4q \, \frac{1}{(q^2+\mu^2)(q+p)^2} \gamma^\alpha (i\not{q} + \mu) \gamma^\alpha$$

$$= -ie^2 \int d^4q \, \frac{1}{(q^2+\mu^2)(q+p)^2} \left\{ (2-n) i\not{q} + n m \right\}$$

$$= -i\pi^2 e^2 \left\{ (2-n) B_1(p^2; \mu, 0) i\not{p} + n m B_0(p^2; \mu, 0) \right\}$$

$$B_0 = \frac{1}{\bar{\varepsilon}} + O(1), \quad B_1 = -\frac{1}{2\bar{\varepsilon}} + O(1) \qquad n = 4 - \varepsilon \qquad \frac{\varepsilon}{\bar{\varepsilon}} = 2$$

$$(2-n) B_1 = (-2+\varepsilon) B_1 = -2B_1 - 1$$

$$n B_0 = (4-\varepsilon) B_0 = 4B_0 - 2$$

$$\Sigma = -i\pi^2 e^2 \left\{ -2\left(B_1 + \frac{1}{2}\right) i\not{p} + 4\left(B_0 - \frac{1}{2}\right) m \right\}$$

$$= (2\pi)^4 i \, \frac{-e^2}{16\pi^2} \left\{ -2\left(B_1 + \frac{1}{2}\right) i\not{p} + 4\left(B_0 - \frac{1}{2}\right) m \right\}$$

$$\bar{S} = \frac{1}{(2\pi)^4 i} \left\{ i\not{p} + \mu - \frac{1}{(2\pi)^4 i} \Sigma \right\}$$

$$\Sigma = \Sigma(i\not{p} = -m_e) + (i\not{p} + \mu_e) \Sigma_{WF} + O((i\not{p} + \mu_e)^2)$$

$$\Sigma_{WF} = \frac{\partial}{\partial i\not{p}} \Sigma \Big|_{i\not{p} = -m_e}$$

$$\frac{1}{(2\pi)^4 i} \Sigma \Big|_{i\not{p} = -m_e} = -\frac{e^2}{16\pi^2} \left\{ 2\mu_e \left(B_1 + \frac{1}{2}\right) + 4\mu_e \left(B_0 - \frac{1}{2}\right) \right\}$$

$$\mu = Z_\mu \mu_R = (1 + e_R^2 \delta Z_\mu) \mu_R$$

$$e_R^2 \delta Z_m m_R = \frac{1}{(2\pi)^4 i} \Sigma_{UV} (i\not{p} = -\mu_e)$$

$$m_R - \frac{1}{(2\pi)^4 i} \Sigma_F (i\not p = -m_e) = m_e$$

$$\bar{S} = \frac{1}{(2\pi)^4 i} \frac{1}{i\not p + m_e} \left\{ 1 - \frac{1}{(2\pi)^4 i} (\Sigma_{WF} + \Sigma_R) \right\}^{-1}$$

$$p^2 = -(i\not p)(i\not p) , \quad \frac{\partial}{\partial p^2} = \frac{\partial i\not p}{\partial p^2} \frac{\partial}{\partial i\not p} , \quad \frac{\partial p^2}{\partial i\not p} = -2i\not p$$

$$= -\frac{1}{2i\not p} \frac{\partial}{\partial i\not p}$$

$$\frac{1}{(2\pi)^4 i} \Sigma_{WF} = \frac{1}{(2\pi)^4 i} \frac{\partial}{\partial i\not p} \Sigma \Big|_{i\not p = -m_e} = -\frac{e^2}{16\pi^2} \frac{\partial}{\partial i\not p} \Big\{ -2(B_1 + \frac{1}{2}) i\not p$$

$$+ 4 m_e (B_0 - \frac{1}{2}) \Big\} \Big|_{i\not p = -m_e}$$

$$= -\frac{e^2}{16\pi^2} \Big\{ -2(B_1 + \frac{1}{2}) - 2i\not p(-2i\not p) \frac{\partial B_1}{\partial p^2} - 2i\not p \, 4 m_e \frac{\partial B_0}{\partial p^2} \Big\} \Big|_{i\not p = -m_e}$$

$$= -\frac{e^2}{16\pi^2} \Big\{ -2B_1 - 1 - 4p^2 \frac{\partial B_1}{\partial p^2} - 8 m_e i\not p \frac{\partial B_0}{\partial p^2} \Big\} \Big|_{i\not p = -m_e}$$

$$= -\frac{e^2}{16\pi^2} \Big\{ -2B_1 - 1 + 4 m_e^2 B_1' + 8 m_e^2 B_0' \Big\} \qquad B_n' = \frac{\partial B_n}{\partial p^2}$$

$$\searrow (-m_e^2; m_e, 0) \swarrow$$

$$\chi = -p^2 x^2 + (p^2 + m_2^2 - m_1^2) x + m_1^2 \Rightarrow m_e^2 x^2 + 2 m_e^2 x + m_e^2 = m_e^2 (1-x)^2$$

$$B_0(-m_e^2; m_e, 0) = \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln \chi = \frac{1}{\bar{\varepsilon}} - \ln m_e^2 - 2\int_0^1 dx \ln x = \frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 2$$

$$p^2 B_1 = \frac{1}{2} \{ A_0(m_1) - A_0(m_2) - (p^2 + m_2^2 - m_1^2) B_0 \}$$

$$\Downarrow$$

$$-m_e^2 B_1 = \frac{1}{2} \{ A_0(m_e) + 2 m_e^2 B_0 \} = \frac{1}{2} \Big\{ -m_e^2 (\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 1)$$

$$+ 2 m_e^2 (\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 2) \Big\}$$

$$B_1(-m_e^2; m_e, 0) = -\frac{1}{2}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 3\right)$$

$$B_0'(-m_e^2; m_e, \lambda) \qquad \lambda \to 0$$

$$= -\frac{\partial}{\partial p^2}\int_0^1 dx \ln\left\{-p^2x^2 + (p^2+\lambda^2-m_e^2)x + m_e^2\right\}\Big|_{p^2=-m_e^2}$$

$$= -\int_0^1 dx \frac{x(1-x)}{\chi}\Big|_{p^2=-m_e^2} = -\int_0^1 dx \frac{x(1-x)}{m_e^2(1-x)^2+\lambda^2 x} = -\int_0^1 dx \frac{x(1-x)}{m_e^2x^2+\lambda^2(1-x)}$$

$$= -\int_0^1 dx \frac{x}{m_e^2x^2+\lambda^2(1-x)} + \int_0^1 dx \frac{1}{m_e^2}$$

$$= \frac{1}{m_e^2} - \frac{1}{2m_e^2}\int_0^1 dx \frac{d}{dx}\ln\left\{m_e^2x^2+\lambda^2(1-x)\right\} + O(\lambda^2)$$

$$= \frac{1}{m_e^2} - \frac{1}{2m_e^2}\ln\frac{m_e^2}{\lambda^2} \qquad \ln\frac{\lambda}{m_e} = L$$

$$= \frac{1}{m_e^2}(1+L)$$

$$p^2 B_1 = \frac{1}{2}\left\{A_0(m_e) - (p^2-m_e^2)B_0\right\}$$

$$B_1 + p^2 B_1' = -\frac{1}{2}B_0 - \frac{1}{2}(p^2-m_e^2)B_0'$$

$$m_e^2 B_1' = B_1 + \frac{1}{2}B_0 - m_e^2 B_0' = \frac{1}{2}\left\{-\cancel{\frac{1}{\bar{\varepsilon}}} + \cancel{\ln m_e^2} - 3 + \cancel{\frac{1}{\bar{\varepsilon}}} - \cancel{\ln m_e^2} + 2\right\} - 1 - L$$

$$= -\frac{3}{2} - L$$

$$B_1' = -\frac{1}{m_e^2}\left(\frac{3}{2} + L\right)$$

$$\frac{1}{(2\pi)^4 i}\Sigma_{WF} = -\frac{e^2}{16\pi^2}\left\{\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 3 - 1 + 8(1+L) - 4\left(\frac{3}{2}+L\right)\right\}$$

$$= -\frac{e^2}{16\pi^2}\left\{\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 4L + 4\right\}$$

$$\Lambda^\mu = e^3\int d^4q\,\frac{1}{(1)(2)(3)}\,\gamma^\alpha(-i\not{q}+m)\gamma^\mu\left[-i(\not{q}+\not{\ell})+m\right]\gamma^\alpha$$

$$= e^3\int d^4q\,\frac{1}{(1)(2)(3)}\left\{-\gamma^\alpha\not{q}\gamma^\mu\not{q}\gamma^\alpha - \gamma^\alpha\not{q}\gamma^\mu\not{\ell}\gamma^\alpha\right.$$

$$\left. - im\left[\gamma^\alpha\not{q}\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu(\not{q}+\not{\ell})\gamma^\alpha\right] + m^2\gamma^\alpha\gamma^\mu\gamma^\alpha\right\}$$

$$= \pi^2 e^3\left\{-\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\rho\gamma^\alpha\left[C_{21}P_{1\lambda}P_{1\rho} + C_{22}P_{2\lambda}P_{2\rho} + C_{23}(P_{1\lambda}P_{2\rho} + P_{1\rho}P_{2\lambda}) + C_{24}\delta_{\lambda\rho}\right]\right.$$

$$- \gamma^\alpha\gamma^\lambda\gamma^\mu\not{\ell}\gamma^\alpha(C_{11}P_{1\lambda} + C_{12}P_{2\lambda})$$

$$-2im\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\alpha(\quad '' \quad)$$

$$-im\gamma^\alpha\gamma^\mu\not{\ell}\gamma^\alpha C_0$$

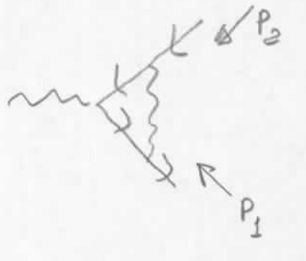
$$\left. + m^2\gamma^\alpha\gamma^\mu\gamma^\alpha C_0\right\}$$

$$\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\lambda\gamma^\alpha = (2-n)^2\gamma^\mu$$

$$\Lambda^\mu = \pi^2 e^3\left\{-(2-n)^2 C_{24}\gamma^\mu + 2\gamma^\rho\gamma^\mu\gamma^\lambda\left[C_{21}P_{1\lambda}P_{1\rho} + C_{22}P_{2\lambda}P_{2\rho} + C_{23}(P_{1\lambda}P_{2\rho} + P_{1\rho}P_{2\lambda})\right]\right.$$

$$+ 2\not{\ell}\gamma^\mu\gamma^\lambda(C_{11}P_{1\lambda} + C_{12}P_{2\lambda}) - 8im_e(C_{11}P_1^\mu + C_{12}P_2^\mu)$$

$$\left. - 4im\,\ell^\mu C_0 - 2m^2\gamma^\mu C_0\right\}$$



$$\bar{u}(-P_1)\Lambda^\mu u(P_2) = \Lambda_c\,\bar{u}(-P_1)\gamma^\mu u(P_2) + \cdots$$

$$\bar{u}(-p_1)(-i\not{p}_1 + m_e) = 0 \quad , \quad (i\not{p}_2 + m_e)u(p_2) = 0$$

$$\bar{u}\not{p}_1 = -i m_e \bar{u} \quad , \quad \not{p}_2 u = +i m_e u$$

1) $\quad i(p_1 + p_2)_\mu \bar{u} \quad u = 0$

2) $\quad i(p_1 - p_2)^\mu \bar{u} u = 2m_e \bar{u}\gamma^\mu u - \bar{u}\sigma^{\mu\nu}(p_1 + p_2)_\nu u$

$$2ip_1^\mu \longrightarrow 2m_e\gamma^\mu \quad , \qquad p_1^\mu \longrightarrow -i m_e \gamma^\mu$$

$$-2ip_2^\mu \longrightarrow 2m_e\gamma^\mu \quad , \qquad p_2^\mu \longrightarrow +i m_e \gamma^\mu$$

$$\bar{u}\Lambda^\mu u \Rightarrow \pi^2 e^3 \bar{u}\Big\{ -(2-n)^2 C_{24}\gamma^\mu + 2\not{p}_1\gamma^\mu\not{p}_1 C_{21} + 2\not{p}_2\gamma^\mu\not{p}_2 C_{22}$$

$$+ 2\not{p}_1\gamma^\mu\not{p}_2 C_{23} + 2\not{p}_2\gamma^\mu\not{p}_1 C_{23} - 8im_e(C_{11}p_1^\mu + C_{12}p_2^\mu)$$

$$+ 2\not{\underline{P}}\gamma^\mu\not{p}_1 C_{11} + 2\not{\underline{P}}\gamma^\mu\not{p}_2 C_{12} - 2m_e^2 C_0 \gamma^\mu\Big\}$$

$$\not{p}_1\gamma^\mu\not{p}_1 = -\not{p}_1\not{p}_1\gamma^\mu + 2p_1^\mu\not{p}_1 = m_e^2\gamma^\mu + 2p_1^\mu\not{p}_1 \longrightarrow m_e^2\gamma^\mu - 2m_e^2\gamma^\mu = -m_e^2\gamma^\mu$$

$$\not{p}_2\gamma^\mu\not{p}_2 = -\gamma^\mu\not{p}_2\not{p}_2 + 2p_2^\mu\not{p}_2 = m_e^2\gamma^\mu + 2p_2^\mu\not{p}_2 \longrightarrow m_e^2\gamma^\mu - 2m_e^2\gamma^\mu = -m_e^2\gamma^\mu$$

$$\not{p}_1\gamma^\mu\not{p}_2 \longrightarrow m_e^2\gamma^\mu$$

$$\not{p}_2\gamma^\mu\not{p}_1 = -\not{p}_2\not{p}_1\gamma^\mu + 2p_1^\mu\not{p}_2 = \not{p}_1\not{p}_2\gamma^\mu - 2p_1\cdot p_2\gamma^\mu + 2p_1^\mu\not{p}_2$$

$$= -\not{p}_1\gamma^\mu\not{p}_2 + 2p_2^\mu\not{p}_1 - 2p_1\cdot p_2\gamma^\mu + 2p_1^\mu\not{p}_2 \longrightarrow (3m_e^2 - 2p_1\cdot p_2)\gamma^\mu$$

$$\underline{P}^2 = (p_1 + p_2)^2 = -2m_e^2 + 2p_1\cdot p_2 \quad , \quad -2p_1\cdot p_2 = -\underline{P}^2 - 2m_e^2$$

$$\downarrow$$

$$m_e^2 - \underline{P}^2$$

$$\Lambda_c = \pi^2 e^3 \Big\{ -(2-m)^2 C_{24} + 2m_e^2 \left[-C_0 - 4C_{11} + 4C_{12} - C_{21} - C_{22} + 2C_{23} \right]$$

$$- 2\underline{P}^2 (C_{11} + C_{23})$$

$$\underline{P}^2 = 0$$

$$\pi^2 C_0 = \int d^4q \; \frac{1}{(1)(2)(3)} = 2\int_0^1 dx \int_0^x dy \int d^4q \Big\{ y(3) + (x-y)P_1 + (1-x)(2) \Big\}^{-3}$$

$$y\left\{ (q+p_1+p_2)^2 + m_e^2 \right\} + (x-y)(q^2+m_e^2) + (1-x)\left\{ (q+p_1)^2 + \lambda^2 \right\}$$

$$= q^2 + 2k\cdot q + M^2$$

$$k = y(p_1+p_2) + (1-x)p_1 = (1-x+y)p_2 + y\,p_2$$

$$M^2 = y(\underline{P}^2 + m_e^2) + (x-y)m_c^2 + (1-x)(p_1^2 + \lambda^2)$$

$$\underline{P}^2 = 0 \;, \quad p_1^2 = -m_e^2 \qquad M^2 = m_c^2(2x-1) + \lambda^2(1-x)$$

$$k^2 - M^2 = (1-x+y)^2 p_1^2 + y^2 p_2^2 + 2y(1-x+y)\, p_1\cdot p_2 - (2x-1)m_e^2 - \lambda^2(1-x)$$

$$p_1\cdot p_2 \Big|_{\underline{P}^2=0} = m_e^2 \qquad k^2 - M^2 = m_c^2\left\{ -(1-x+y)^2 - y^2 + 2y(1-x+y) - (2x-1) \right\}$$

$$- \lambda^2(1-x)$$

$$= m_e^2 \left\{ -(1-x+\cancel{y}-\cancel{y})^2 - (2x-1) \right\} - \lambda^2(1-x)$$

$$= -m_e^2 x^2 - \lambda^2(1-x) = -\mathcal{X}$$

$$C_0 = \int_0^1 dx \int_0^x dy \, \frac{1}{\mathcal{X}} = \int_0^1 dx \, \frac{x}{\mathcal{X}} = \frac{1}{2m_e^2} \int_0^1 dx \, \frac{d}{dx} \ln \mathcal{X} = \frac{1}{2m_c^2} \ln \frac{m_c^2}{\lambda^2} = -\frac{1}{m_c^2} L$$

$$i\int d^4q \, \frac{q_\mu}{(q^2 + 2k\cdot p + M^2)^\alpha} = -I_\alpha k_\mu$$

$$i\int d^n q\, \frac{q_\mu q_\nu}{(q^2+2k\cdot q+M^2)^\alpha} = I_\alpha \left\{ k_\mu k_\nu - \frac{1}{2}\delta_{\mu\nu}\frac{M^2-k^2}{n/2-\alpha+1} \right\},$$

$$k_\mu = (1-x+y)\,p_{1\mu} + y\,p_{2\mu}$$

$$I_\alpha = i\pi^{n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}(M^2-k^2)^{n/2-\alpha}, \qquad \alpha=3 \Rightarrow \frac{i\pi^2}{2}\chi^{-1}$$

$$C_{11} = -\int_0^1 dx\int_0^x dy\, \frac{1-x+y}{\chi} = -\int_0^1 dx\,\frac{1}{\chi}\left\{(1-x)x+\frac{1}{2}x^2\right\}$$

$$\int_0^1 dx\,\frac{x}{\chi} = -\frac{L}{\mu_c^2}, \quad \int_0^1 dx\,\frac{x^2}{\chi} = \frac{1}{\mu_c^2}, \quad \int_0^1 dx\,\frac{x^3}{\chi} = \frac{1}{2\mu_c^2}$$

$$C_{11} = -\int_0^1 \frac{dx}{\chi}\left(x-\frac{1}{2}x^2\right) = -\frac{1}{\mu_c^2}\left(-L-\frac{1}{2}\right) = \frac{1}{\mu_c^2}\left(L+\frac{1}{2}\right) \qquad \text{IR}$$

$$C_{12} = -\int_0^1 dx\int_0^x dy\,\frac{y}{\chi} = -\frac{1}{2}\int_0^1 dx\,\frac{x^2}{\chi} = -\frac{1}{2\mu_c^2}$$

$$C_{21} = \int_0^1 dx\int_0^x dy\,\frac{(1-x+y)^2}{\chi} = \int_0^1 dx\int_0^x dy\,\frac{1}{\chi}\left\{(1-x)^2+2y(1-x)+y^2\right\}$$

$$= \int_0^1 dx\,\frac{1}{\chi}\left\{(1-x^2)x+(1-x)x^2+\frac{1}{3}x^3\right\}$$

$$= \int_0^1 \frac{dx}{\chi}\left\{x-x^2+\frac{1}{3}x^3\right\} = \frac{1}{\mu_c^2}\left\{-L-1+\frac{1}{6}\right\} = -\frac{1}{\mu_c^2}\left(L+5/6\right) \qquad \text{IR}$$

$$C_{22} = \int_0^1 dx\int_0^x dy\,\frac{y^2}{\chi} = \frac{1}{3}\int_0^1 dx\,\frac{x^3}{\chi} = \frac{1}{6\mu_c^2}$$

$$C_{23} = \int_0^1 dx\int_0^x dy\,\frac{(1-x+y)y}{\chi} = \int_0^1 \frac{dx}{\chi}\left\{(1-x)\frac{1}{2}x^2+\frac{1}{3}x^3\right\}$$

$$= \int_0^1 \frac{dx}{x}\left(\frac{1}{2}x^2 - \frac{1}{6}x^3\right) = \frac{1}{m_e^2}\left(\frac{1}{2} - \frac{1}{12}\right) = \frac{5}{12 m_e^2}$$

$$C_{24} = \frac{1}{4}\frac{1}{\bar{\varepsilon}} - \frac{1}{2}\int_0^1 dx \int_0^x dy \, \ln x = \frac{1}{4\bar{\varepsilon}} - \frac{1}{2}\int_0^1 dx \, x \ln(m_e^2 x^2)$$

$$= \frac{1}{4}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 1\right)$$

$$\int_0 \frac{dQ}{Q} = IR \qquad \int d^4q \, \frac{1}{(1)(2)(3)}$$

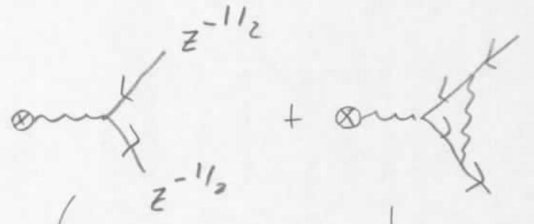
$$(1) = q^2 + m_e^2 = -2Q\cdot p_1 + O(Q^2)$$

$$(2) = Q^2$$

$$(3) = (q + p_1 + p_2)^2 + m_e^2 = 2Q\cdot p_2 + O(Q^2)$$

$$Q = q + p_1 \quad , \quad q = Q - p_1$$

$$\int d^4q \, \frac{q_\mu}{(1)(2)(3)} = \int d^4Q \, \frac{(Q - p_1)_\mu}{(1)(2)(3)} \Longrightarrow C_{11} \; IR \quad \text{etc.}$$



$$\to (2\pi)^4 \frac{ie^3}{16\pi^2} \Lambda_c \gamma^\mu + \text{“}\Lambda_M\text{-terms”}$$

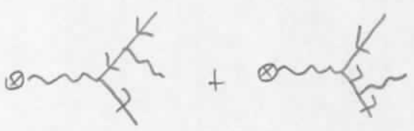
$$\to \left(1 + \frac{e^2}{8\pi^2}\Sigma_{WF}\right)(2\pi)^4 i \, (ie\gamma^\mu)$$

$$p^2 = 0 \qquad \Lambda_c(p^2 = 0) + 2\Sigma_{WF} = 0 \quad , \quad \Lambda_c(p^2 = 0) = -2\Sigma_{WF}$$

$$\text{total} \quad (2\pi)^4 i \, \frac{ie^3}{16\pi^2}\left\{\Lambda_c(p^2) + 2\Sigma_{WF}\right\}\gamma^\mu = (2\pi)^4 i \, \frac{ie^3}{16\pi^2}\underbrace{\left\{\Lambda_c(p^2) - \Lambda_c(0)\right\}}_{\text{UV finite}}\gamma^\mu$$

$$\text{Virtual} \qquad V = V_B + V_{1L} \quad , \quad |V|^2 = \underset{O(e^2)}{|V_B|^2} + \underset{O(e^4)}{2\,\mathrm{Re}\, V_B^\dagger V_{1L}}$$

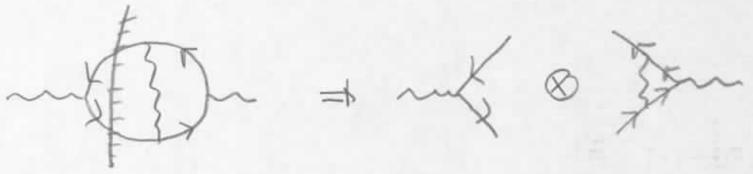
Real



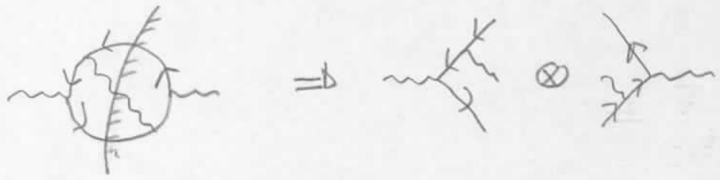
$$O(e^2)$$

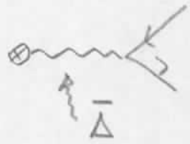
KLN-theorem $|V|^2 + |R|^2 =$ IR finite

ex:  IR finite $\Rightarrow \sum_{\text{cuts}}$  = IR finite



etc.



Renormalization 

$$\Pi_{\mu\nu} = \Pi_0(p^2)\, p^2 \delta_{\mu\nu} + \text{"}p_\mu p_\nu\text{-terms"}, \quad \Pi_0(p^2) = (2\pi)^4 i \frac{e^2}{2\pi^2} \underset{\overset{\swarrow}{(p^2;\, m_e, m_e)}}{(B_{21} + B_1)}$$

$$B_1(p^2; m_e, m_e) = -\frac{1}{2} B_0(p^2; m_e, m_e)$$

$$B_0(p^2; m_e, m_e) = \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln \chi, \quad \chi = -p^2 x^2 + p^2 x + m_e^2$$

$$B_0(0; m_e, m_e) = \frac{1}{\bar{\varepsilon}} - \ln m_e^2$$

$$p^2 B_{21}(p^2; m_e, m_e) = \frac{1}{3}\left\{ A_0(m_e) + (p^2 + m_e^2) B_0(p^2; m_e, m_e) + \frac{1}{6} p^2 + m_e^2 \right\}$$

$$= \frac{1}{3}\left\{ -m_e^2\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + 1\right) + (p^2 + m_e^2)\left(\frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln \chi\right) + \frac{1}{6} p^2 + m_e^2 \right\}$$

$$= \frac{1}{3} p^2 \left(\frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln \chi + \frac{1}{6}\right) - \frac{1}{3} m_e^2 \int_0^1 dx \ln \frac{\chi}{m_e^2}$$

$$\ln \frac{\chi}{m_e^2} = \ln\left\{1 + \frac{p^2}{m_e^2} x(1-x)\right\} \underset{p^2 \to 0}{\sim} \frac{p^2}{m_e^2} x(1-x)$$

$$B_{21} \sim \frac{1}{3}\left(\frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln x + \frac{1}{6}\right) - \frac{1}{3}\frac{\cancel{m_e^2}}{\cancel{p^2}}\frac{p^2}{m_e^2}\int_0^1 dx\, x(1-x)$$

$$B_{21}(0; m_e, m_e) = \frac{1}{3}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2 + \frac{1}{6}\right) - \frac{1}{3}\frac{1}{6} = \frac{1}{3}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)$$

$$B_{21}(0; m_e, m_e) + B_1(0; m_e, m_e) = -\frac{1}{6}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)$$

$$\bar{\Delta}_{\mu\nu} = \frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2}\left\{1 - \frac{e^2}{2\pi^2}\Pi(p^2)\right\}^{-1} \qquad \Pi(p^2) = B_{21}(p^2; m_e, m_e) + B_1(p^2; m_e, m_e)$$

$$e^2\bar{\Delta}_{\mu\nu} \Rightarrow \frac{e^2}{1 + \frac{e^2}{12\pi^2}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)} = 4\pi\alpha$$

$$e^2 = 4\pi\alpha\left\{1 + \frac{e^2}{12\pi^2}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)\right\}, \qquad \left\{1 - \frac{\alpha}{3\pi}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)\right\} e^2 = 4\pi\alpha$$

$$e^2 = \frac{4\pi\alpha}{1 - \frac{\alpha}{3\pi}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)}$$

$$\Rightarrow \frac{e^2}{1 - \frac{e^2}{2\pi^2}\Pi(p^2)} = \frac{4\pi\alpha}{1 - \frac{\alpha}{3\pi}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right)}\;\frac{1}{1 - \frac{2\alpha}{\pi}\Pi(p^2)}$$

$$= 4\pi\alpha\left\{1 - \frac{\alpha}{\pi}\left[\frac{1}{3}\left(\frac{1}{\bar{\varepsilon}} - \ln m_e^2\right) + 2\left(B_{21} + B_1\right)\right]\right\}^{-1} = \frac{4\pi\alpha}{1 - \frac{\alpha}{\pi}\Pi_R(p^2)}$$

$$\Pi_R(p^2) = \frac{1}{3}\left(\cancel{\frac{1}{\bar{\varepsilon}}} - \ln m_e^2\right) + 2\left\{\frac{1}{3}\left(\cancel{\frac{1}{\bar{\varepsilon}}} - \int_0^1 dx \ln x + \frac{1}{6}\right) - \frac{1}{3}\frac{m_e^2}{p^2}\int_0^1 dx \ln\frac{x}{m_e^2}\right.$$
$$\left. - \frac{1}{2}\left(\cancel{\frac{1}{\bar{\varepsilon}}} - \int_0^1 dx \ln x\right)\right\}$$

$$\Pi_R(p^2) = \frac{1}{9} - \frac{2}{3}\frac{m_e^2}{p^2}\int_0^1 dx \ln\frac{x}{m_e^2} + \frac{1}{3}\int_0^1 dx \ln\frac{x}{m_e^2}$$

$$\Pi_R(p^2) = \frac{1}{9} + \frac{1}{3}\left(1 - 2\frac{\mu_e^2}{p^2}\right)\int_0^1 dx \ln \frac{x}{\mu_e^2}$$

$$\Rightarrow \quad \frac{4\pi\alpha}{1 - \frac{\alpha}{\pi}\Pi_R(p^2)} = 4\pi\alpha(p^2)$$

N.B. $\quad \Delta_{\mu\nu} = \frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2}\,, \qquad \Pi_{\mu\nu} = \left(\Pi_0\,\delta_{\mu\nu} + \Pi_1\, p_\mu p_\nu\right)(2\pi)^4 i$

$$\Rightarrow \bar{\Delta}_{\mu\nu} = \frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2 - \Pi_0} + \frac{\Pi_1\, p_\mu p_\nu}{(p^2 - \Pi_0)(p^2 - \Pi_0 - p^2\Pi_1)}$$

QED $\qquad \Pi_0 + p^2\Pi_1 = 0$

$$\bar{\Delta}_{\mu\nu} = \frac{1}{(2\pi)^4 i}\,\frac{1}{p^2 - \Pi_0}\left\{\delta_{\mu\nu} + \Pi_1\frac{p_\mu p_\nu}{p^2}\right\}$$

$$\int_0^1 dx \ln\frac{x}{\mu_e^2} = \int_0^1 dx \ln\left\{1 + \frac{p^2}{\mu_e^2}x(1-x)\right\} \qquad p^2 = -s\,,\ s > 0$$

$$= \int_0^1 dx \ln\left\{1 - \frac{s}{\mu_e^2}x(1-x)\right\} \qquad \frac{s}{\mu_e^2} = z > 0$$

$$\swarrow$$

$$z x^2 - z x + 1 - i0 = 0\,, \qquad x_\pm = \frac{1}{2z}\left\{z \pm \sqrt{z^2 - 4(z - i0)}\right\}$$

$$= \frac{1}{2}\left\{1 \pm \sqrt{1 - 4/z + i0}\right\}$$

$$x_\pm = \frac{1}{2}\left\{1 \pm \sqrt{1 - 4\frac{\mu_e^2}{s} + i0}\right\} \qquad \beta^2 = 1 - 4\frac{\mu_e^2}{s}\,, \quad s >> 4\mu_e^2 \Rightarrow \beta^2 > 0$$

$$x_\pm = \frac{1}{2}\left\{1 \pm \sqrt{\beta^2 + i0}\right\} = \frac{1}{2}\left\{1 \pm \beta \pm i0\right\}$$

$$s >> 4\mu_e^2 \Rightarrow s \to \infty\,, \qquad \beta \sim 1 - 2\frac{\mu_e^2}{s}$$

$$x_+ \sim \frac{1}{2}\left\{2 - 2\frac{\mu_e^2}{s} + i0\right\} \sim 1 - \frac{\mu_e^2}{s} + i0$$

$$x_- \sim \frac{1}{2}\left\{2\frac{\mu_e^2}{s} - i0\right\} \sim \frac{\mu_e^2}{s} - i0$$

$1-x_+ \sim \frac{\mu_e^2}{s} - i0 \quad , \quad 1-x_- \sim 1 - \frac{\mu_e^2}{s} + i0$

$$I = \int_0^1 dx \, \ln\left\{ \frac{s}{\mu_e^2}(x-x_-)(x-x_+) \right\} = \ln\frac{s}{\mu_e^2} + \sum_{\lambda=\pm} \int_0^1 dx \, \ln(x-x_\lambda)$$

$$= \ln\frac{s}{\mu_e^2} + \sum_{\lambda=\pm} \left\{ (x-x_\lambda)\ln(x-x_\lambda) - x \right\}\Big|_0^1$$

$$= \ln\frac{s}{\mu_e^2} - 2 + \sum_{\lambda=\pm} \left\{ (1-x_\lambda)\ln(1-x_\lambda) + x_\lambda \ln(-x_\lambda) \right\}$$

$(1-x_+)\ln(1-x_+) \to 0 \quad , \quad (1-x_-)\ln(1-x_-) \to 0$

$x_-\ln(-x_-) \to 0 , \quad x_+\ln(-x_+) \sim \ln(-1-i0) = -i\pi$

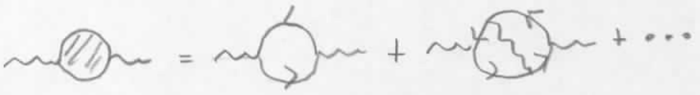
$$I \underset{s\to\infty}{\sim} \ln\frac{s}{\mu_e^2} - i\pi \quad , \quad \Pi_R(s) \underset{s\to\infty}{\sim} \ln\frac{s}{\mu_e^2} - i\pi \quad , \quad \ell = \ln\frac{s}{\mu_e^2}$$

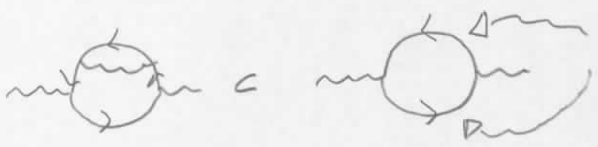
$\Downarrow$

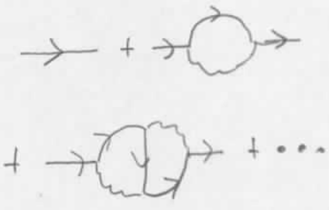
 $\sim \frac{\alpha}{\pi}\ell$,  $\sim \left(\frac{\alpha}{\pi}\ell\right)^2$

$\bar\Delta_{\mu\nu} \Leftarrow$ re-summation of $\left(\frac{\alpha}{\pi}\ell\right)^n$?

$\bar\Delta_{\mu\nu} =$ 



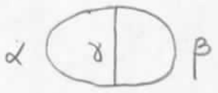
N.B. 

$\bar S =$ 

explicit calculation $\Rightarrow$

 $\sim \left(\frac{\alpha}{\pi}\right)^2 \ell \quad \Rightarrow \quad \frac{4\pi\alpha}{1-\frac{\alpha}{\pi}\Pi_R(p^2)} \Leftrightarrow 4\pi\alpha\left\{1+\frac{\alpha}{\pi}\Pi_R(p^2)\right\}$

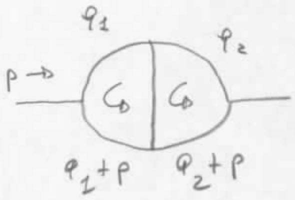
2 - loops



$$I_{\alpha\beta\gamma} = \int d^4q_1\, d^4q_2\, (q_1^2 + m_1^2)^{-\alpha} (q_2^2 + m_2^2)^{-\beta} ((q_1 - q_2 + p)^2 + m_3^2)^{-\gamma}$$

↗ basic integral

Ex :



$$\int d^4q_1\, d^4q_2 \Big\{ \overbrace{(q_1^2 + m^2)((q_1+p)^2 + m_2^2)}^{x} \\ \times ((q_1 - q_2)^2 + m^2) \\ \times \underbrace{(q_2^2 + m^2)((q_2 + p)^2 + m^2)}_{y} \Big\}^{-1}$$

$$= \int_0^1 dx\, dy \int d^4q_1\, d^4q_2 \Big\{ (q_1^2 + 2x\, p\cdot q_1 + x p^2 + m^2)^{-2} ((q_1 - q_2)^2 + m^2)^{-1} \\ \times (q_2^2 + 2y\, p\cdot q_2 + y p^2 + m^2)^{-2} \Big\}$$

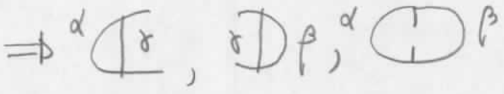
$q_1 + x p_1 = q_1'$, $q_2 + y p_2 = q_2'$, $q_1 - q_2 = q_1' - q_2' - (x-y)p$

$M_1^2 = x(1-x)p^2 + m^2$, $M_2^2 = y(1-y)p^2 + m^2$, $M_3 = m$, $P = (y-x)p$

$$= \int_0^1 dx\, dy\, I_{221}$$

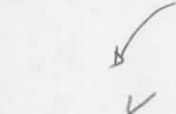
Subdiagrams

$\Rightarrow$



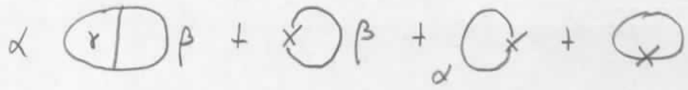
QED $\mathcal{L} = \mathcal{L}_R + e_R^2\, \delta\mathcal{L}^{(1)} + O(e_R^4)$

$$\delta\mathcal{L}^{(1)} = -\frac{1}{4\pi^2\bar{\varepsilon}} \Big\{ -\frac{1}{12} F^R_{\mu\nu} F^R_{\mu\nu} - \frac{1}{6} (\partial_\mu A^R_\mu)^2 - \frac{1}{4} \bar{\psi}^R \not{\partial} \psi^R - m_R \bar{\psi}^R \psi^R + \frac{i}{4} e_R \bar{\psi}_R \not{A}_R \psi_R \Big\}$$

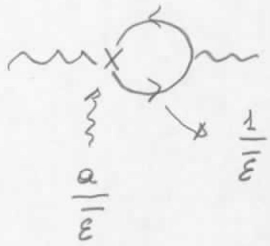


$\Rightarrow O(e_R^4)$

etc.



Unitarity



$\searrow \frac{1}{\varepsilon} - \int_0^1 dx \ln x$

$\Rightarrow$ 1) double pole $\frac{1}{\varepsilon^2} \Rightarrow Z_i + e_R^2 \delta Z_i^{(1)} + e_R^4 \delta Z_i^{(2)} + \cdots$

2) single pole $- \frac{a}{\varepsilon} \int_0^1 dx \ln x$

$\searrow$ imaginary part

= overlapping divergencies

1L - analytical continuation

$$I = \int d^\mu q \prod_{i=1}^{\ell} \left((q+p_i)^2 + m_i^2 \right)^{-\alpha_i} \qquad \mathrm{Re}\, m < 2 \sum_i \alpha_i$$

$$\uparrow$$

$$\frac{1}{m} \frac{\partial q_\mu}{\partial q_\mu} + \mathrm{IBP} \Longrightarrow I = \frac{I' \leftsquigarrow}{2 \sum_i \alpha_i - m} \quad \mathrm{Re}\, m < 2 \sum_i \alpha_i + 1$$

single pole at $\mu = 2 \sum_i \alpha_i$

pole

$$I_{\alpha\beta\gamma} \qquad \frac{1}{m} \frac{\partial q_{1\mu}}{\partial q_{1\mu}} \longrightarrow \mu = 2(\alpha+\gamma)$$

$$\frac{1}{m} \frac{\partial q_{2\mu}}{\partial q_{2\mu}} \longrightarrow \mu = 2(\beta+\gamma)$$

$$q'_2 = q_1 - q_2 \oplus \frac{1}{m} \frac{\partial q_{1\mu}}{\partial q_{1\mu}} \longrightarrow m = 2(\alpha+\beta)$$

$$\frac{1}{2m} \left(\frac{\partial q_{1\mu}}{\partial q_{1\mu}} + \frac{\partial q_{2\mu}}{\partial q_{2\mu}} \right) \rightarrow m = \alpha + \beta + \gamma$$

$$I_{\alpha\beta\gamma} \begin{cases} \text{conv.} & \alpha+\beta+\gamma > 4 \\ \text{div} & \alpha+\beta+\gamma \leq 4 \end{cases} , \qquad I_{\alpha\beta} \begin{cases} \text{conv.} & \alpha+\beta > 2 \\ \text{div} & \alpha+\beta \leq 2 \end{cases}$$

ex: $I_{\alpha\beta\gamma}$, $\alpha+\gamma=2$, $\beta=2$ $\left\{ \begin{array}{l} I_{\alpha\beta\gamma} \text{ div} \\ I_{\alpha\gamma} \text{ div} \end{array} \right.$

$$I_{\alpha\beta\gamma} = \int d^n q_1\, d^n q_2\, (q_1^2+m_1^2)^{-\alpha} (q_2^2+m_2^2)^{-\beta} ((q_1-q_2+p)^2+m_3^2)^{-\gamma}$$

$$\underbrace{\qquad\qquad\qquad}_{x}$$

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \{ x a + (1-x) b \}^{-\alpha-\beta} \qquad \begin{array}{l} \mathrm{Re}\,\alpha>0 \\ \mathrm{Re}\,\beta>0 \end{array}$$

$$I_{\alpha\beta\gamma} = \int_0^1 dx \int d^n q_1\, d^n q_2\, x^{\gamma-1}(1-x)^{\alpha-1} \frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha)\Gamma(\gamma)} (q_1^2+2k\cdot q_1+M^2)^{-\alpha-\gamma} (q_2^2+m_2^2)^{-\beta}$$

$$k = x(p-q_2), \quad M^2 = (1-x)m_1^2 + x\{(p-q_2)^2+m_3^2\}$$

$$= \frac{\cancel{\Gamma(\alpha+\gamma)}}{\Gamma(\alpha)\Gamma(\gamma)}\, i\pi^{n/2} \frac{\Gamma(\alpha+\gamma-n/2)}{\cancel{\Gamma(\alpha+\gamma)}} \int_0^1 dx \int d^n q\, x^{\gamma-1}(1-x)^{\alpha-1}(q^2+m_2^2)^{-\beta}$$
$$\times (M^2-k^2)^{n/2-\alpha-\gamma}$$

$$M^2-k^2 = (1-x)m_1^2 + x m_3^2 + x(1-x)(q-p)^2$$
$$= x(1-x)\{(q-p)^2+M_1^2\} \qquad M_1^2 = \frac{(1-x)m_1^2+x m_3^2}{x(1-x)}$$
$$M_2^2 = p^2+m_2^2$$

$$q' = q-p$$

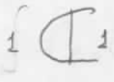
$$I_{\alpha\beta\gamma} = i\pi^{n/2} \frac{\Gamma(\alpha+\gamma-n/2)}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^1 dx\, x^{n/2-\alpha-1}(1-x)^{n/2-\gamma-1} \int d^n q\, (q^2+M_1^2)^{n/2-\alpha-\gamma}$$
$$\times (q^2+2p\cdot q+M_2^2)^{-\beta}$$

now $\alpha=\gamma=1$, $\beta=2$

$$I_{121} = i\pi^{n/2} \frac{\Gamma(2-n/2)}{\Gamma^2(1)} \int_0^1 dx\, x^{n/2-2}(1-x)^{n/2-2}$$

$$\times \int d^n q\, (q^2+M_1^2)^{n/2-2} (q^2+2p\cdot q+M_2^2)^{-2}$$

 $2 \qquad \frac{2}{\mu-4}\int d^4q\,(q^2+2p\cdot q+M_2^2)^{-2}$

following  $\Big|_{div.}$

Strategy : single poles with potentially log residue (= Im part)

$$x^{\mu/2-2} = 1+(\tfrac{\mu}{2}-2)\ln x \quad \text{but } 0<x<1 \text{ etc.}$$

| double poles ×
| ε × ln

$$\Rightarrow\; I = \int d^4q\left\{\frac{\Gamma(2-m/2)}{(q^2+M_1^2)^{2-m/2}} + \frac{2}{\mu-4}\right\}(q^2+2p\cdot q+M_2^2)^{-2} = I_1+I_2$$

$$I_1 = \Gamma(2-\tfrac{m}{2})\int d^4q\,\frac{q^2+M_1^2}{\underbrace{(q^2+M_1^2)^{3-m/2}(q^2+2p\cdot q+M_2^2)^2}_{y}}$$

$$= \Gamma(2-\tfrac{m}{2})\,\frac{\Gamma(5-m/2)}{\Gamma(3-m/2)}\int_0^1 dy\,(1-y)^{2-m/2}\,y\int d^4q\left\{q^2+2y\,p\cdot q+yM_2^2+(1-y)M_1^2\right\}^{\mu/2-5}(q^2+M_1^2)$$

$$= i\pi^{m/2}\,\Gamma(2-\tfrac{m}{2})\,\frac{\Gamma(5-m/2)}{\Gamma(3-m/2)}\,\frac{1}{\Gamma(5-m/2)}\int_0^1 dy\,(1-y)^{2-m/2}\,y$$

$$\times\left\{\Gamma(5-m)(M_1^2+y^2p^2)\,\mathcal{X}^{\mu-5}+\frac{m}{2}\,\Gamma(4-m)\,\mathcal{X}^{\mu-4}\right\}$$

$$\mathcal{X} = yM_2^2+(1-y)M_1^2-y^2p^2, \qquad \mathcal{X}\Big|_{y=1} = M_2^2-p^2$$

$\Longrightarrow$ $m=4$ if not for $\frac{\text{Im}}{\varepsilon}$

$$= i\pi^2\,\Gamma(2-\tfrac{m}{2})\int_0^1 dy\,y\left\{\frac{M_1^2+y^2p^2}{\mathcal{X}} + 2\,\frac{\Gamma(4-m)}{\mathcal{X}^{4-m}}\right\}$$

$$\int_0^1 dy\, y\, \chi^{n-4} = \frac{1}{2}\int_0^1 dy\, \chi^{n-4}\,\frac{dy^2}{dy} = \frac{1}{2}\, y^2 \chi^{n-4}\Big|_0^1 - \frac{1}{2}\int_0^1 dy\, y^2 (n-4)\, \chi^{n-5}\,\frac{\partial\chi}{\partial y}$$

$$= \frac{1}{2}\,(M_2^2-p^2)^{n-4} - \frac{m-4}{2}\int_0^1 dy\, y^2\, \chi^{n-5}\,(M_2^2-M_1^2-2yp^2)$$

$$\Rightarrow i\pi^2\,\Gamma(2-\tfrac{m}{2})\left\{\int_0^1 dy\, y\,\frac{M_1^2+y^2p^2}{\chi} + 2\,\Gamma(4-m)\left[\frac{1}{2}\,(M_2^2-p^2)^{n-4}\right.\right.$$

$$\left.\left. - \frac{1}{2}\,(n-4)\int_0^1 dy\,\frac{M_2^2-M_1^2-2yp^2}{\chi}\, y^2\right\}$$

$$= i\pi^2\,\Gamma(2-\tfrac{m}{2})\left\{\Gamma(4-m)\,(M_2^2-p^2)^{n-4} + \int_0^1 dy\, y\,\frac{M_1^2+y^2p^2}{\chi}\right.$$

$$\left. + \int_0^1 dy\, y^2\,\frac{M_2^2-M_1^2-2yp^2}{\chi}\right\}$$

$$y\left\{M_1^2+y^2p^2+y\,(M_2^2-M_1^2)-2y^2p^2\right\}$$

$$= y\left\{-y^2p^2+y\,M_2^2+(1-y)\,M_1^2\right\} = y\,\chi$$

$$\Downarrow \qquad\qquad \Rightarrow \int_0^1 dy\, y = \frac{1}{2}$$

$$\Gamma(2-\tfrac{m}{2})\,\Gamma(4-m)\,(M_2^2-p^2)^{n-4}$$

$$= \left\{\frac{2}{4-m}+\cdots\right\}\left\{\frac{1}{4-m}+\cdots\right\}\left\{1+(n-4)\ln(M_2^2-p^2)+\cdots\right\}$$

$\Rightarrow \quad \dfrac{2}{(4-m)^2}$ double pole, $\quad \dfrac{3}{4-m}$ single pole

$$-\frac{2}{4-m}\ln(M_2^2-p^2)$$

$$I_2 = \frac{2}{n-4}\int d^nq\,(q^2+2p\cdot q+M_2^2)^{-2} = 2i\pi^{m/2}\,\frac{\Gamma(2-m/2)}{m-4}\,(M_2^2-p^2)^{n/2-2}$$

$$\Rightarrow 2i\pi^2\,\frac{\Gamma(2-m/2)}{n-4}\,(M_2^2-p^2)^{n/2-4}$$

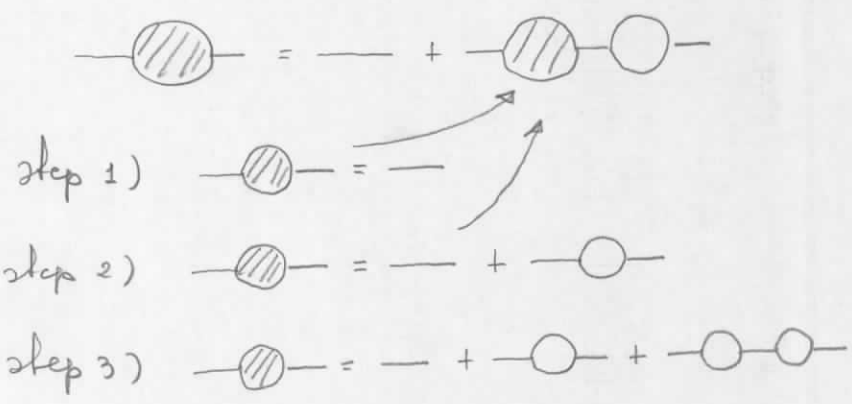
$$\Rightarrow 2\left\{\frac{2}{4-m}+\cdots\right\}\frac{1}{n-4}\left\{1+\frac{n-4}{2}\ln(M_2^2-p^2)+\cdots\right\} \Rightarrow \frac{2}{4-m}\ln(M_2^2-p^2)$$

$$\bar{S} = \frac{1}{(2\pi)^4 i} \left\{ i\not{p} + m - \Sigma \right\}^{-1}$$

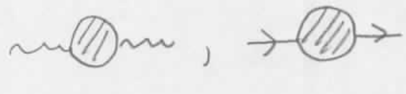
$$T_\mu = \gamma_\mu + \Lambda_\mu$$

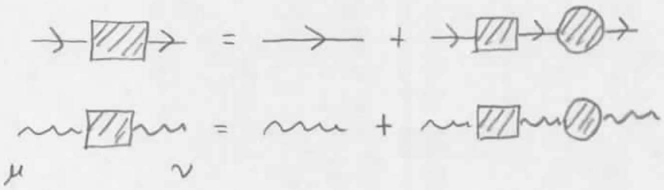
$$\bar{\Delta}_{\mu\nu} = \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2} \left\{ 1 - \Pi_0(p^2) \right\}^{-1}$$

Integral equations : scalar case



etc.

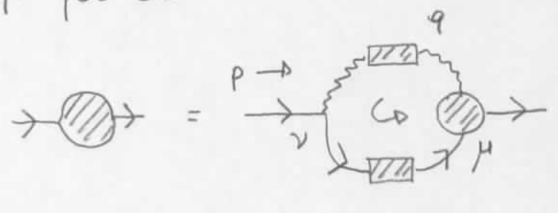
QED Self-energy 

Dyson propagators : 

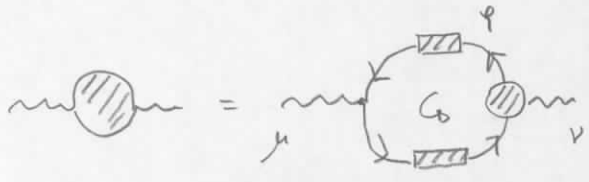
$\bar{S}$

$\bar{\Delta}_{\mu\nu}$

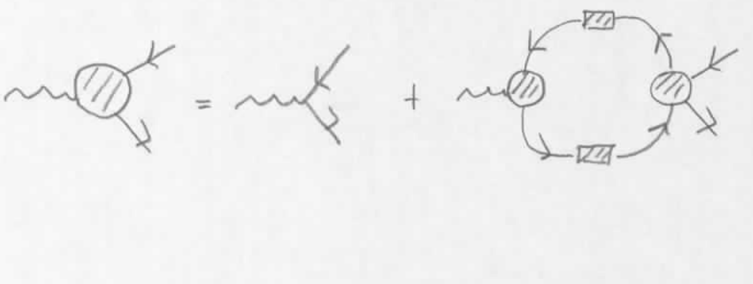
Eqs. for SE



$$\Sigma(p) = -e^2 \int d^4q \, T^\mu(q+p, q) \, \bar{S}(q+p) \gamma^\nu \bar{\Delta}_{\mu\nu}(q)$$



$$\Pi_{\mu\nu} = -e^2 \int d^4q \, \mathrm{tr} \left\{ \gamma^\mu \bar{S}(q) T^\nu(q+p, q) \bar{S}(q+p) \right\}$$



etc

$G \Rightarrow \mathcal{D}(G) = 4\ell - 2b - f$

ex:  $\mathcal{D} = 4 - 2 = 2$,  $\mathcal{D} = 4\times 2 - 2 - 4 = 2$

 $\mathcal{D} = 4\times 3 - 4 - 6 = 2$

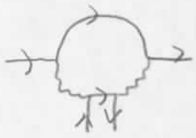
1L → 2L → 3L $\quad \Delta\ell = 1 , \ \Delta b = 1 , \ \Delta f = 2 \quad \Rightarrow \Delta\mathcal{D} = 4 - 2 - 2 = 0$

$\mathcal{D} = 2$

$\mathcal{D} = 1$

$$\frac{\partial\mathcal{D}}{\partial B} = -1$$

$\mathcal{D} = 1$



$\mathcal{D} = -2$

$$\frac{\partial\mathcal{D}}{\partial F} = -\frac{3}{2}$$

$\Rightarrow \mathcal{D}(G) = -B - \frac{3}{2}F + k$

$\mathcal{D} = 0$

$-1 - 3 + k = 0 , \quad k = 4$

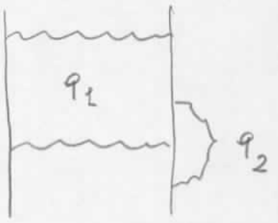
$\mathcal{D}(G) = 4 - B - \frac{3}{2}F$ # of external legs only

$\mathcal{D} = 2 \Rightarrow G \sim \Lambda^2 \ \ldots , \ \mathcal{D} = 0 \Rightarrow G \sim \ln\Lambda^2$

QED	$\mathcal{D}$	
Σ	1	
$\Pi_{\mu\nu}$	2	
Γ_μ	0	
	1	= 0 C-invariance
	0	

Weinberg theorem: a F integral converges if $\mathcal{D}(G) < 0$

and $\mathcal{D}(S) < 0$, $\forall\, S \subset G$



$\mathcal{D} = -2 \Rightarrow$ radial integration

$$G \sim \int \frac{d^8 q}{q^{10}}$$

but q_1 small $\quad G \sim \underset{\checkmark}{\int} d^4 q_1 \times \int \frac{d^4 q_2}{q_2^4} \Rightarrow \mathcal{D} = 0 \quad$ etc

$\mathcal{D}(\Pi_{\mu\nu}) = 2 \quad , \quad \Pi_{\mu\nu} = \Pi_0 (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \Rightarrow \mathcal{D}(\Pi_0) = 0$

$\mathcal{D}(\Sigma) = 1 \qquad i(p'^\mu - p^\mu)\, \Gamma_\mu(p', p) = \bar{S}^{-1}(p') - \bar{S}^{-1}(p)$

ex: $\quad \Gamma_\mu \to \gamma_\mu \qquad \bar{S}^{-1}(p') \to i\not{p}' + m \qquad \bar{S}^{-1}(p) \to i\not{p} + m$

$$i(p'-p)^\mu \Lambda_\mu = \Sigma(p) - \Sigma(p')$$

$\mathcal{D} = 0$

gauge invariance requires $\sim F^4 \longrightarrow 4p\,;\ (2p)\delta\,;\ \delta\delta$

decomposition

$p_1 \cdots p_4 \ | \ \mu_1 \cdots \mu_4$

$\mathcal{D}_{eff} = -4$

Effect of CT $\qquad \mathcal{D} \to \mathcal{D} - 1$

ex

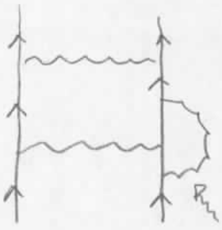
$$\frac{1}{(q^2+m^2)((q+p)^2+m^2)}$$

subtraction at $p = 0$

$-$

$$\frac{1}{(q^2+m^2)^2}$$

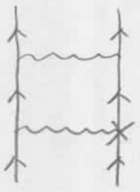
$$= \frac{1}{(\ )^2(\ )} \left\{ q^2 + m^2 - (q+p)^2 - m^2 \right\} = \frac{O(q)}{(\ )^2(\ )}$$



$$G \sim \int \frac{d^3 q}{q^{10}} \qquad \mathcal{D} < 0$$

$$\sim \int \frac{d^4 q}{q^4} \quad \mathcal{D}(S) = 0$$

$+$



$\mathcal{D}(S) = -1$

$$I = 8 i e^2 \pi^{n/2} \Gamma(2 - \frac{n}{2}) \int_0^1 dx\, x(1-x)\, \chi^{n/2 - 2}$$

$$\times \frac{J_\mu}{p^2} (p_\mu p_\nu - p^2 \delta_{\mu\nu}) \frac{J_\nu}{p^2}, \qquad \chi = x(1-x) p^2 + m_e^2, \quad J \cdot p = 0$$

$$I_{CT} = \frac{8}{3} i e^2 \pi^2 \frac{1}{n-4} \frac{J_\mu}{p^2} (p_\mu p_\nu - p^2 \delta_{\mu\nu}) \frac{J_\nu}{p^2}$$

$$I_R = I + I_{CT} = \frac{J \cdot J}{p^2} \left\{ C + 8 i e^2 \pi^2 \int_0^1 dx\, x(1-x) \ln \chi \right\}$$

$$p_\mu \to \lambda p_\mu, \quad m_e \to \lambda m_e \qquad I \to \lambda^{-2} I, \quad I_R \to \lambda^{-2} \ln \lambda^2 I_R$$

$$[\mathcal{L}] = [m]^n \implies \mathcal{L} \to \mu^{-\varepsilon} \mathcal{L}, \quad n - 4 = \varepsilon$$

$\Rightarrow$ prop $\to \mu^{+\varepsilon}$, vert $\to \mu^{-\varepsilon}$

MS-scheme $\quad \eta = \dfrac{1}{8\pi^2 \varepsilon}, \quad m_e = \mu M$

$$\mathcal{L} = -\frac{1}{4} \mu^{-\varepsilon} (1 - \frac{4}{3} e_R^2 \eta) F^R_{\mu\nu} F^R_{\mu\nu} - \frac{1}{2} \mu^{-\varepsilon} (1 - \frac{4}{3} e_R^2 \eta) (\partial_\mu A^R_\mu)^2$$

$$- \mu^{-\varepsilon} (1 - e_R^2 \eta) \bar{\psi}^R \partial\!\!\!/ \psi^R - \mu^{1-\varepsilon} M (1 - 4 e_R^2 \eta) \bar{\psi}^R \psi^R$$

$$+ i \mu^{-\varepsilon} e_R (1 - e_R^2 \eta) \bar{\psi}^R A\!\!\!/^R \psi^R$$

all - orders

$$\mathcal{L} = -\frac{1}{4} B F^R_{\mu\nu} F^R_{\mu\nu} - \frac{1}{2} B (\partial_\mu A^R_\mu)^2 - C \bar\psi^R \not\partial \psi^R - D \bar\psi^R \psi^R + i E \bar\psi^R \not A^R \psi^R$$

$$B = \mu^{-\varepsilon} \sum_{\mu=0}^{\infty} \frac{b_m}{\varepsilon^m} \quad , \quad b_m = \sum_{\ell=m}^{\infty} b_{m,2\ell}\, e_R^{2\ell}$$

$$C = \mu^{-\varepsilon} \sum_{\mu=0}^{\infty} \frac{c_m}{\varepsilon^m} \quad , \quad D = \mu^{1-\varepsilon} M \sum_{\mu=0}^{\infty} \frac{d_m}{\varepsilon^m} \quad , \quad E = \mu^{-\varepsilon} \sum_{\mu=0}^{\infty} \frac{e_m}{\varepsilon^m}$$

$$b_0 = 1 \ , \quad b_1 = -\frac{4}{3} e_R^2 \frac{1}{8\pi^2} = -\frac{e_R^2}{6\pi^2} + O(e_R^4)$$

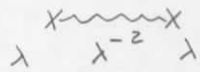
$$c_0 = 1 \ , \quad c_1 = -e_R^2 \frac{1}{8\pi^2} = -\frac{e_R^2}{8\pi^2} + O(e_R^4)$$

$$d_0 = 1 \ , \quad d_1 = -4 e_R^2 \frac{1}{8\pi^2} = -\frac{e_R^2}{2\pi^2} + O(e_R^4)$$

$$e_0 = e_R \ , \quad e_1 = -\frac{e_R^3}{8\pi^2} + O(e_R^5)$$

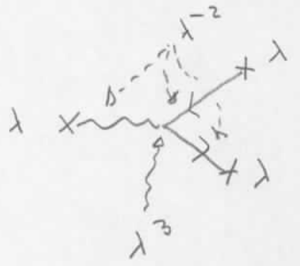
$$b_m \equiv b_m(e_R) \text{ etc} \qquad e_R \to e'_R \implies b'_m = b_m(e'_R)$$

$$A_\mu \to \lambda A_\mu$$



inv.

$$\psi \to \lambda \psi$$



inv

$$\mathcal{L} + J_\mu A_\mu + \bar J \psi + \bar\psi J \ , \qquad A_\mu \to B^{-1/2} A_\mu \ , \quad \psi \to C^{-1/2} \psi \ , \quad \bar\psi \to C^{-1/2} \bar\psi$$

$$\downarrow$$

$$-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu A_\mu)^2 - \bar\psi \not\partial \psi - \underbrace{\frac{D}{C}}_{V} \bar\psi \psi + i \underbrace{\frac{E}{B^{1/2} C}}_{G} + B^{-1/2} J_\mu A_\mu + C^{-1/2} (\bar J \psi + \bar\psi J)$$

$$V = \frac{D}{C} = \mu M \left(\sum_m d_m \varepsilon^{-m}\right)\left(\sum_\mu c_\mu \varepsilon^{-m}\right)^{-1} = \mu M \sum_m v_\mu \varepsilon^{-m}$$

$$= \mu M (1 + d_1 \varepsilon^{-1} + \cdots)(1 - c_1 \varepsilon^{-1} + \cdots) = \mu M \left\{ 1 + (d_1 - c_1)\varepsilon^{-1} + \cdots \right\}$$

$$v_0 = 1 \, , \quad v_1 = d_1 - c_1 = -\frac{3}{8}\frac{e^2}{\pi^2} + O(e^4)$$

$$G = \frac{E}{B^{1/2}C} = \mu^{\varepsilon/2} \sum_\mu g_m \varepsilon^{-m} = (e + e_1 \varepsilon^{-1} + \cdots)(1 - \tfrac{1}{2} b_1 \varepsilon^{-1} + \cdots)(1 - c_1 \varepsilon^{-1} + \cdots)$$

$$g_0 = e \, , \quad g_1 = e_1 - \tfrac{1}{2} e b_1 - e c_1 = \frac{e^3}{12\pi^2} + O(e^5)$$

Transformation $\quad e, M, \mu \longrightarrow e', M', \mu'$ $\Leftarrow$ functional form invariant

Let $\quad \mu = \mu'(1+\delta) , \quad e = e' + \varepsilon de_1 + de_2 \, , \quad M = M'(1 + dM)$

$$g_i(e) = g_i(e') + g_i'(e')(\varepsilon de_1 + de_2) \, , \quad g_i' = \frac{dg_i}{de}$$

$$G = \mu^{\varepsilon/2} \sum_{\mu=0}^{\infty} g_m \varepsilon^{-m} = (\mu')^{\varepsilon/2}(1 + \tfrac{\varepsilon}{2}\delta)\Big\{ e' + \varepsilon de_1 + de_2$$

$$+ \sum_{\mu=1}^{\infty} (g_m + g_m' \varepsilon de_1 + g_m' de_2)\varepsilon^{-m} \Big\}$$

$$= (\mu')^{\varepsilon/2}(1 + \tfrac{\varepsilon}{2}\delta)\Big\{ e' + \varepsilon de_1 + de_2 + \sum_{\mu=1}^{\infty}(g_m + g_m' de_2)\varepsilon^{-m} + \sum_{\mu=1}^{\infty} g_m' de_1 \varepsilon^{1-m} \Big\}$$

& $1 - m = -m'$

$\mu = \mu' + 1$

$$G = (\mu')^{\varepsilon/2}(1 + \tfrac{\varepsilon}{2}\delta)\Big\{ e' + \varepsilon de_1 + de_2 + \sum_{\mu=1}^{\infty}(g_m + g_\mu' de_2)\varepsilon^{-m} + \sum_{\mu=0}^{\infty} g_{m+1}' de_1 \varepsilon^{-m} \Big\}$$

$$= (\mu')^{\varepsilon/2}(1 + \tfrac{\varepsilon}{2}\delta)\Big\{ e' + \varepsilon de_1 + de_2 + g_1' de_1 + \sum_{\mu=1}^{\infty}(g_m + g_\mu' de_2 + g_{m+1}' de_1)\varepsilon^{-m} \Big\}$$

$$= (\mu')^{\varepsilon/2}\Big\{ e' + \varepsilon de_1 + de_2 + \tfrac{1}{2}\delta e' \varepsilon + g_1' de_1 + \tfrac{1}{2} g_1 \delta$$

$$+ \sum_{\mu=1}^{\infty}(g_m + g_m' de_2 + g_{\mu+1}' de_1)\varepsilon^{-m} \Big\}$$

$$de_1 = -\frac{1}{2} e' \delta \sim -\frac{1}{2} e \delta$$

$$de_2 + g_1' de_1 + \frac{1}{2} g_1 \delta = 0 \qquad de_2 = \frac{1}{2} e g_1' \delta - \frac{1}{2} g_1 \delta = \frac{1}{2} (e g_1' - g_1) \delta$$

$$g_\mu + g_m' de_2 + g_{m+1}' de_1 + \frac{1}{2} g_{\mu+1} \delta = g_m$$

$$-e g_{m+1}' + (e g_1' - g_1) g_m' + g_{m+1} = 0$$

$$V = \mu H + \sum_\mu v_\mu \varepsilon^{-m} = \mu' H' (1 + \delta + dH) \left\{ 1 + \sum_{\mu=1}^{\infty} (v_\mu + v_m' \varepsilon de_1 + v_\mu' de_2) \varepsilon^{-m} \right\}$$

$$= \mu' H' (1 + \delta + dH) \left\{ 1 + \sum_{\mu=1}^{\infty} (v_\mu + v_\mu' de_2) \varepsilon^{-m} + \sum_{\mu=0}^{\infty} v_{\mu+1}' de_1 \varepsilon^{-m} \right\}$$

$$= \mu' H' (1 + \delta + dH) \left\{ 1 + v_1' de_1 + \sum_{\mu=1}^{\infty} (v_\mu + v_m' de_2 + v_{\mu+1}' de_1) \varepsilon^{-m} \right\}$$

$$\delta + dH + v_1' de_1 = 0 , \quad dH = -\delta + \frac{1}{2} e v_1' \delta = (\frac{1}{2} e v_1' - 1) \delta$$

$$(\cancel{1} + \delta + dH) v_m + v_m' de_2 + v_{\mu+1}' de_1 = \cancel{v}_m$$

$$B = \mu^{-\varepsilon} \sum_\mu b_m \varepsilon^{-m} = (\mu')^{-\varepsilon} (1 - \delta \varepsilon) \left\{ 1 + \sum_{\mu=1}^{\infty} (b_m + b_m' de_1 \varepsilon + b_m' de_2) \varepsilon^{-m} \right\}$$

$$= (\mu')^{-\varepsilon} (1 - \delta \varepsilon) \left\{ 1 + b_1' de_1 + \sum_{\mu=1}^{\infty} (b_m + b_m' de_2 + b_{\mu+1}' de_1) \varepsilon^{-m} \right\}$$

$$B^{-1/2} J_\mu A_\mu , \qquad J_\mu \to J_\mu' (1 - \varepsilon \delta)^{1/2} (1 + b_1' de_1)^{1/2}$$

$$B^{-1/2} J_\mu A_\mu = J_\mu' A_\mu (1 + b_1' de_1)^{1/2} (\mu')^{\varepsilon/2} \left\{ 1 + b_1' de_1 + \sum_{\mu=1}^{\infty} (b_m + b_m' de_2 + b_{\mu+1}' de_1) \varepsilon^{-m} \right\}^{-1/2}$$

$$= J_\mu' A_\mu (\mu')^{\varepsilon/2} \left\{ 1 + \frac{1}{1 + b_1' de_1} \sum_{\mu=1}^{\infty} (b_m + b_m' de_2 + b_{\mu+1}' de_1) \varepsilon^{-m} \right\}^{-1/2}$$

$$\Rightarrow \left\{ 1 + \sum_{\mu=1}^{\infty} \left[- b'_1 b_m de_1 + b_m + b'_\mu de_2 + b'_{\mu+1} de_1 \right] \varepsilon^{-m} \right\}^{-1/2}$$

$$- b'_1 b_m de_1 + b'_\mu de_2 + b'_{\mu+1} de_1 = 0$$

$$- \frac{1}{2} e \delta (b'_{\mu+1} - b_m b'_1) + \frac{1}{2} (e g'_1 - g_1) b'_m \delta = 0$$

$$(e g'_1 - g_1) b'_\mu - e (b'_{\mu+1} - b_m b'_1) = 0$$

$\mu = 1$ $\quad b_1 = - \frac{e^2}{6\pi^2}, \; g_1 = \frac{e^3}{12\pi^2} \qquad b'_1 = - \frac{e}{3\pi^2}, \; g'_1 = \frac{e^2}{4\pi^2}$

$$(e g'_1 - g_1) b'_1 - e (b'_2 - b_1 b'_1) = 0$$

$$- e b'_2 + e \left(- \frac{e^2}{6\pi^2} \right) \left(- \frac{e}{3\pi^2} \right) + e \frac{e^2}{4\pi^2} \left(- \frac{e}{3\pi^2} \right) - \frac{e^3}{12\pi^2} \left(- \frac{e}{3\pi^2} \right) = 0$$

$$- e b'_2 + \frac{e^4}{\pi^2} \left(\frac{1}{18} - \frac{1}{12} + \frac{1}{36} \right) = 0 \quad \Rightarrow \quad e b'_2 = O(e^6), \quad b'_2 = O(e^5)$$

$$b_2 = O(e^6)$$

$$\mathcal{L} = - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu A_\mu)^2 - \bar{\psi} (\not{\partial} + m_0) \psi + i e_0 \bar{\psi} \not{A} \psi + J_\mu A_\mu + \bar{J} \psi + \bar{\psi} J$$

p_0 = bare parameter

$$\mathcal{L} = - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu A_\mu)^2 - \bar{\psi} \not{\partial} \psi - V \bar{\psi} \psi + i G \bar{\psi} \not{A} \psi + B^{-1/2} J_\mu A_\mu + C^{-1/2} (\bar{J} \psi + \bar{\psi} J)$$

$T_0^{ij} (\{p\}, m_0, e_0)$ $\qquad$ i - external γ legs

$\qquad\qquad\qquad\qquad\quad$ j - " $\quad$ f legs

$$T^{ij} (\{p\}, \mu, M, e) = B^{-i/2} C^{-j/2} T_0^{ij} (\{p\}, m_0, e_0) \Big|_{\substack{m_0 = V \\ e_0 = G}}$$

$$\Rightarrow \mu \frac{d}{d\mu} T^{ij} \Big|_{\substack{\mu_0 \\ e_0 \text{ fixed}}} = 0$$

$$\frac{\partial}{\partial\mu} B^{i/2} = \frac{i}{2} B^{i/2 - 1} \frac{\partial B}{\partial\mu} = \frac{i}{2} B^{i/2} \frac{\partial}{\partial\mu} \ln B \quad \text{etc.}$$

$$\Rightarrow \left\{ \mu \frac{\partial}{\partial\mu} + \mu \frac{\partial M}{\partial\mu}\frac{\partial}{\partial\mu} + \mu \frac{\partial e}{\partial\mu}\frac{\partial}{\partial e} + \frac{i}{2}\mu \frac{\partial \ln B}{\partial\mu} + \frac{j}{2}\mu \frac{\partial \ln C}{\partial\mu} \right\} T^{ij}(\{p\}, \mu, M, e) = 0$$

define $\beta = \lim_{\varepsilon\to 0} \mu \frac{\partial e}{\partial\mu}$, $\delta = \lim_{\varepsilon\to 0} \frac{\mu}{M}\frac{\partial M}{\partial\mu}$

$\gamma_p = \lim_{\varepsilon\to 0} \mu \frac{\partial \ln B}{\partial\mu}$, $\gamma_e = \lim_{\varepsilon\to 0} \mu \frac{\partial \ln C}{\partial\mu}$

$\Rightarrow$ CS - equation $\left\{ \mu \frac{\partial}{\partial\mu} + \delta M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial e} + \frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e \right\} T^{ij} = 0$

existence of CS-eq. T UV finite

$\mu \frac{\partial T}{\partial\mu}$ "

$$\mu \frac{\partial M}{\partial\mu} = \sum_{n=0}^{\infty} a_n \varepsilon^{-n}, \quad \mu \frac{\partial e}{\partial\mu} = \sum_{n=0}^{\infty} b_n \varepsilon^{-n}, \quad \mu \frac{\partial \ln B}{\partial\mu} = \sum_{n=0}^{\infty} c_n \varepsilon^{-n}, \quad \mu \frac{\partial \ln C}{\partial\mu} = \sum_{n=0}^{\infty} d_n \varepsilon^{-n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(a_n \frac{\partial}{\partial M} + b_n \frac{\partial}{\partial e} + \frac{i}{2} c_n + \frac{j}{2} d_n \right) T^{ij} \varepsilon^{-n} = 0$$

$n = 0$ is the CS - eq.

$$a_n = \sum_{k=0}^{\infty} a_{nk} e^k, \quad \text{etc.}, \quad T = \sum_{k=0}^{\infty} T_k e^k$$

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} e^k \left(a_{nk} \frac{\partial}{\partial M} + b_{nk} \frac{\partial}{\partial e} + \frac{i}{2} c_{nk} + \frac{j}{2} d_{nk} \right) \sum_{k'=0}^{\infty} T^{ij}_{k'} e^{k'} \varepsilon^{-n} = 0$$

$$\sum_{k,k'=0}^{\infty} e^{k+k'} \left(a_{nk} \frac{\partial}{\partial M} + \frac{i}{2} c_{nk} + \frac{j}{2} d_{nk} \right) T^{ij}_{k'} + \sum_{k,k'=0}^{\infty} k' b_{nk} T^{ij}_{k'} e^{k+k'-1} = 0$$

$k + h' = K \,, \quad h' = K - k \,, \qquad h' \geq 0 \Rightarrow k \leq K$

$$\sum_{K=0}^{\infty} \sum_{k=0}^{K} \Big\{ e^{K} \Big(a_{mk} \frac{\partial}{\partial H} + \frac{i}{2} c_{mk} + \frac{j}{2} d_{mk} \Big) T^{ij}_{K-k}$$

$$+ (K-k)\, b_{mk}\, T^{ij}_{K-k}\, e^{K-k} \Big\} = 0$$

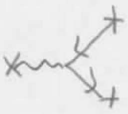
$O(e^{0})$ $\quad \Big(a_{\mu 0} \frac{\partial}{\partial H} + \frac{i}{2} c_{m0} + \frac{j}{2} d_{m0} \Big) T^{ij}_{0} + b_{m0}\, T^{ij}_{1} = 0$

$i = 0 \,, \; j = 2 \qquad T_0 =$  $, \; T_1 = 0 \qquad T^{02} = \bar{J}(p) \dfrac{-i\not{p} + \mu H}{p^2 + \mu^2 H^2} J(p)$

$\Big(a_{\mu 0} \frac{\partial}{\partial H} + d_{m0} \Big) T^{02}_{0} \quad \Rightarrow \quad a_{\mu 0} = b_{m0} = 0$ $\quad$ for arbitrary p

$\Rightarrow \qquad \frac{i}{2} c_{m0}\, T^{ij}_{0} + b_{m0}\, T^{ij}_{1}$

$i = 2 \,, \; j = 0 \qquad T_0 =$  $, \quad T_1 = 0 \qquad \Rightarrow c_{m0} = 0$

$\Rightarrow \; b_{m0}\, T^{ij}_{1} = 0 \qquad i = 1 \,, \; j = 2 \qquad T_0 = 0 \,, \quad T_1 \neq 0$  $\Rightarrow b_{m0} = 0$

$O(e)$ $\quad \displaystyle\sum_{k=0}^{1} \Big\{ \Big[a_{\mu k} \frac{\partial}{\partial H} + \frac{i}{2} c_{\mu k} + \frac{j}{2} d_{\mu k} \Big] T^{ij}_{1-k} + \sum_{k=0}^{2} (2-k)\, b_{mk}\, T^{ij}_{2-k} = 0$

$\Rightarrow \quad \Big(a_{\mu 1} \frac{\partial}{\partial H} + \frac{i}{2} c_{m1} + \frac{j}{2} d_{\mu 1} \Big) T^{ij}_{0} + b_{m1}\, T^{ij}_{i} = 0$

etc $\qquad \Rightarrow \quad a_{\mu k} = \cdots = 0$

$$\mu \frac{\partial e}{\partial \mu} = \beta + R(\varepsilon) \,, \qquad R(0) = 0$$

$$e_0 = G = \mu^{\varepsilon/2} \Big(e + \sum_{\mu=1}^{\infty} g_m \varepsilon^{-m} \Big)$$

$$0 = \mu \frac{\partial e_0}{\partial \mu} = \frac{\varepsilon}{2}\mu^{\varepsilon/2}\left(e + \sum_{\mu=1}^{\infty} g_m \varepsilon^{-m}\right) + \mu^{\varepsilon/2}\left(\mu\frac{\partial e}{\partial \mu}\right)\left(1 + \sum_{\mu=1}^{\infty} g'_m \varepsilon^{-m}\right)$$

$$\frac{\varepsilon}{2}\left(e + \sum_{\mu=1}^{\infty} g_m \varepsilon^{-m}\right) + (\beta + R)\left(1 + \sum_{\mu=1}^{\infty} g'_m \varepsilon^{-m}\right) = 0$$

$$\frac{e}{2}\varepsilon + \frac{1}{2}\sum_{\mu=0}^{\infty} g_{m+1}\varepsilon^{-m} + (\beta + R)\left(1 + \sum_{\mu=1}^{\infty} g'_m \varepsilon^{-m}\right) = 0$$

$$\Rightarrow \quad R = -\frac{e}{2}\varepsilon$$

$$\frac{1}{2}\sum_{\mu=0}^{\infty} g_{\mu+1}\varepsilon^{-m} + \beta + \beta\sum_{\mu=1}^{\infty} g'_m \varepsilon^{-m} - \frac{e}{2}\sum_{\mu=0}^{\infty} g'_{\mu+1}\varepsilon^{-m} = 0$$

$$\frac{1}{2} g_1 + \beta - \frac{e}{2} g'_1 = 0 , \quad \beta = \frac{1}{2}(e g'_1 - g_1)$$

$$\frac{1}{2} g_{m+1} + \beta g'_m - \frac{e}{2} g'_{\mu+1} = 0$$

$$m_0 = V = \mu H\left(1 + \sum_{\mu=1}^{\infty} v_\mu \varepsilon^{-m}\right)$$

$$0 = \mu\frac{\partial m_0}{\partial \mu} = \mu H\left(1 + \sum_{\mu=1}^{\infty} v_\mu \varepsilon^{-m}\right) + \mu\frac{\partial H}{\partial \mu}\,\mu\left(1 + \sum_{\mu=1}^{\infty} v_\mu \varepsilon^{-m}\right)$$

$$+ \mu\frac{\partial e}{\partial \mu}\,\mu H \sum_{\mu=1}^{\infty} v'_\mu \varepsilon^{-m}$$

$$\frac{\mu}{H}\frac{\partial H}{\partial \mu} = \delta + S(\varepsilon) , \quad S(0) = 0$$

$$1 + \sum_{\mu=1}^{\infty} v_\mu \varepsilon^{-m} + (\delta + S)\left(1 + \sum_{\mu=1}^{\infty} v_\mu \varepsilon^{-m}\right) + \left(\beta - \frac{e}{2}\varepsilon\right)\sum_{\mu=1}^{\infty} v'_\mu \varepsilon^{-m} = 0$$

$$\left(1 + \sum_{\mu=1}^{\infty} v_\mu \varepsilon^{-m}\right)(1 + \delta + S) + \beta\sum_{\mu=1}^{\infty} v'_\mu \varepsilon^{-m} - \frac{e}{2}\sum_{\mu=0}^{\infty} v'_{\mu+1}\varepsilon^{-m} = 0$$

$$\Rightarrow \quad S = 0$$

$$(1+\delta)(1+\sum_{\mu=1}^{\infty} v_{\mu}\varepsilon^{-n}) + \beta\sum_{\mu=1}^{\infty} v'_{\mu}\varepsilon^{-n} - \frac{e}{2}\sum_{\mu=0}^{\infty} v'_{\mu+1}\varepsilon^{-n} = 0$$

$$\Rightarrow \quad 1+\delta-\frac{e}{2}v'_1 = 0 \,, \qquad \delta = \frac{e}{2}v'_1 - 1$$ etc.

$$\mu\frac{\partial \ln B}{\partial\mu} = \frac{\mu}{B}\frac{\partial B}{\partial\mu} \,, \qquad B = \mu^{-\varepsilon}(1+\sum_{\mu=1}^{\infty} b_m\varepsilon^{-m})$$

$$= \gamma_p + \rho(\varepsilon) \,, \qquad \rho(0) = 0$$

$$(\gamma_p+\rho)\mu^{-\varepsilon}(1+\sum_{\mu=1}^{\infty} b_m\varepsilon^{-m}) = -\varepsilon\mu^{-\varepsilon}(1+\sum_{\mu=1}^{\infty} b_m\varepsilon^{-m}) + \mu^{-\varepsilon}\mu\frac{\partial e}{\partial\mu}\sum_{\mu=1}^{\infty} b'_{\mu}\varepsilon^{-n}$$

$$(\gamma_p+\rho)(1+\sum_{\mu=1}^{\infty} b_m\varepsilon^{-m}) = -\varepsilon - \sum_{\mu=0}^{\infty} b_{\mu+1}\varepsilon^{-n} + \beta\sum_{\mu=1}^{\infty} b'_{\mu}\varepsilon^{-m} - \frac{e}{2}\sum_{\mu=0}^{\infty} b'_{\mu+1}\varepsilon^{-n}$$

$$\Rightarrow \quad \rho = -\varepsilon$$

$$\gamma_p(1+\sum_{\mu=1}^{\infty} b_m\varepsilon^{-m}) - \sum_{\mu=0}^{\infty} \cancel{b}_{\mu+1}\varepsilon^{-n} = -\sum_{\mu=0}^{\infty} \cancel{b}_{m+1}\varepsilon^{-m} + \beta\sum_{\mu=1}^{\infty} b'_{\mu}\varepsilon^{-m} - \frac{e}{2}\sum_{\mu=0}^{\infty} b'_{\mu+1}\varepsilon^{-m}$$

$$\Rightarrow \quad \gamma_p = -\frac{e}{2}b'_1$$ etc

similarly $\gamma_e = -\frac{e}{2}c'_1$ "

$$\left\{\mu\frac{\partial}{\partial\mu} + \delta M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial e} + \frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e\right\} T^{ij} = 0$$

$$\mu = e^{-t}, \quad t = -\ln\mu \,, \quad \frac{\partial}{\partial\mu} = -\frac{1}{\mu}\frac{\partial}{\partial t} \,, \quad \mu\frac{\partial}{\partial\mu} = -\frac{\partial}{\partial t}$$

$$\left\{-\frac{\partial}{\partial t} + \delta M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial e} + \frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e\right\} T^{ij}(\{p\}, e^{-t}, M, e) = 0$$

ansatz $T^{ij}(\{p\}, e^{-t}, M, e) = f(T)\, T^{ij}(\{p\}, e^{-(t-T)}, M(T), e(T))$

$$\frac{d}{dT} T^{ij} = 0 = \frac{df}{dt} T^{ij} + f \left\{ -\frac{\partial}{\partial t} + \frac{dM}{dT}\frac{\partial}{\partial H} + \frac{de}{dT}\frac{\partial}{\partial e} \right\} T^{ij}$$

$$\left\{ -\frac{\partial}{\partial t} + \frac{dH}{dT}\frac{\partial}{\partial H} + \frac{de}{dT}\frac{\partial}{\partial e} + \frac{d}{dT} \ln f(T) \right\} T^{ij} = 0$$

$$\Rightarrow \quad \frac{dH}{dT} = \delta M , \quad \frac{de}{dT} = \beta , \quad \frac{d}{dT} \ln f = \frac{i}{2}\gamma_p + \frac{i}{2}\gamma_e$$

$$H(0) = M , \quad e(0) = e$$

$$T^{ij}(\{p\}, \mu) \qquad T^{ij}(\lambda\{p\}, \lambda\mu) = \lambda^{cd} T^{ij}(\{p\},\mu)$$

$$T^{ij}\left(\frac{\{p\}}{\lambda}, \frac{\mu}{\lambda}\right) = \lambda^{-cd} T^{ij}(\{p\},\mu)$$

$$T^{ij}(\{p\},\mu) = \lambda^{cd} T^{ij}\left(\frac{\{p\}}{\lambda}, \frac{\mu}{\lambda}\right) \qquad \lambda = e^{-T}$$

$$= \lambda^{cd} f(T)\, T^{ij}\left(\frac{\{p\}}{\lambda}, \mu, H(T), e(T)\right)$$

$$\frac{de}{dT} = \beta , \quad \beta = \frac{1}{2}(e g_1' - g_1) \qquad \text{QED} \quad g_1 = \frac{e^3}{12\pi^2} + O(e^5)$$

$$e g_1' = \frac{1}{4}\frac{e^3}{\pi^2} + O(e^5)$$

$$\beta_{QED} = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{12}\right)\frac{e^3}{\pi^2} + O(e^5) = \frac{e^3}{12\pi^2} + O(e^5)$$

$$\beta = c e^3 + O(e^5) \qquad \frac{de}{dT} = c e^3 \qquad \frac{de}{e^3} = cT \qquad e(0) = e$$

$$-\frac{1}{2}\left\{ e^{-2}(T) - e^{-2} \right\} = cT , \quad e^{-2}(T) - e^{-2} = -2cT , \quad e^{-2}(T) = e^{-2} - 2cT$$

$$e^2(T) = \frac{e^2}{1 - 2c e^2 T} \qquad \text{QED}, \quad c > 0$$

L pole $\quad 1 - 2\frac{e^2}{12\pi^2}T = 1 - \frac{e^2 T}{6\pi^2} = 0$

$$T = \frac{6\pi^2}{e^2} = \frac{6\pi^2}{4\pi\alpha} = \frac{3}{2}\frac{\pi}{\alpha}$$

$$\frac{dM}{dT} = \delta M \,, \quad \frac{dM}{M} = \delta dT = \delta\frac{dT}{de}\,de = \frac{\delta}{\beta}\,de$$

$$\delta = e v_1' - 1 = \frac{3}{8}\frac{e^2}{\pi^2} - 1 \,, \quad \beta = \frac{e^3}{12\pi^2}$$

$$M(T) = M \exp\left\{\int_e^{e(T)} \frac{\delta(x)}{\beta(x)}\,dx\right\}$$

$$\int_e^{e(T)} dx\,\frac{\frac{3}{8}\frac{x^2}{\pi^2} - 1}{\frac{x^3}{12\pi^2}} = 12\pi^2\int_e^{e(T)} dx\left\{-\frac{1}{x^3} + \frac{3}{8\pi^2}\frac{1}{x}\right\}$$

$$= 12\pi^2\left\{\frac{1}{2}x^{-2} + \frac{3}{8\pi^2}\ln x\right\}\Big|_e^{e(T)} \qquad \frac{e^2(T)}{e^2} = \left(1 - \frac{e^2}{6\pi^2}T\right)^{-1}$$

$$2\ln\frac{e(T)}{e} = -\ln\left(1 - \frac{e^2}{6\pi^2}T\right)$$

$$= 12\pi^2\left\{-\frac{T}{12\pi^2} - \frac{3}{16\pi^2}\ln\left(1 - \frac{e^2}{6\pi^2}T\right)\right\} = -T - \frac{9}{4}\ln\left(1 - \frac{e^2}{6\pi^2}T\right)$$

$$M(T) = M\,e^{-T}\left(1 - \frac{e^2}{6\pi^2}T\right)^{-9/4}$$

$$\gamma_p = -\frac{1}{2}e b_1' \,, \quad \gamma_e = -\frac{1}{2}e c_1' \qquad \gamma_p = \frac{e^2}{6\pi^2} \,, \quad \gamma_e = \frac{e^2}{8\pi^2}$$

$$\frac{d}{dT}\ln f = \frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e \,, \quad \frac{df}{f} = \left(\frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e\right)dT$$

$$= \left(\frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e\right)\frac{dT}{de}\,de = \left(\frac{i}{2}\gamma_p + \frac{j}{2}\gamma_e\right)\frac{de}{\beta}$$

$$f = \exp\left\{ \int_e^{e(T)} dx \, \frac{i \frac{x^2}{6\pi^2} + j \frac{x^2}{8\pi^2}}{\frac{x^3}{12\pi^2}} \, \frac{1}{2} = \exp\left\{ 6 \int_e^{e(T)} \frac{dx}{x} \left(\frac{i}{6} + \frac{j}{8} \right) \right\}\right.$$

$$= \exp\left\{ \int_e^{e(T)} dx \left(i + \frac{3}{4} j \right) \frac{1}{x} \right\} = \exp\left\{ -\frac{1}{2} \left(i + \frac{3}{4} j \right) \ln\left(1 - \frac{e^2 T}{6\pi^2} \right) \right\}$$

$$= \left(1 - \frac{e^2 T}{6\pi^2} \right)^{-\left(\frac{i}{2} + \frac{3}{8} j \right)}$$

$$T = \ln\lambda \quad \Rightarrow \quad e^2(\lambda) = \frac{e^2}{1 - \frac{e^2}{6\pi^2} \ln\lambda} \, , \qquad \lambda = \frac{p}{p_0}$$

$$e^2 \bar{\Delta}_{\mu\nu} \rightarrow \frac{e^2}{1 - \frac{e^2}{2\pi^2} \left(B_{21} + B_1 \right)}$$

$\overline{MS}$ $\quad e = Z_e e_R$

$$B_{21} + B_1 = \frac{1}{6} \left(-\frac{1}{\bar{\varepsilon}} + \ln \frac{p^2}{\mu^2} + c \right)$$

$$\Rightarrow \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \left\{ \ln \frac{p^2}{\mu^2} + C \right\}}$$

$$B_{21} + B_1 \Big|_{p^2 = 0} \Rightarrow \frac{1}{6} \left(-\frac{1}{\bar{\varepsilon}} + \ln \frac{m_e^2}{\mu^2} \right)$$

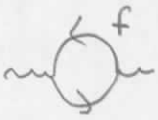
$$\Rightarrow p^2 = 0 \qquad \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln \frac{m_c^2}{\mu^2}}$$

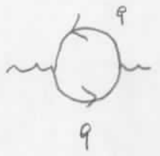
after subtraction $\quad \dfrac{4\pi\alpha}{1 - \frac{\alpha}{3\pi} \left\{ \ln \frac{p^2}{m_e^2} + c \right\}}$

$\ln \dfrac{p^2}{m_e^2}$ can be re-summed, but not c

QED $\quad \alpha \rightarrow \alpha(s) = \dfrac{\alpha}{1 - \frac{\alpha}{3\pi} \Pi_R(s)}$

$\Pi_R(s) = \Pi(s) - \Pi(0)$

QED $\subset$ SM $\quad \sum_f$  + bosonic $\qquad f = \ell, q$

Im {  +  + ... } $\propto \sigma(\gamma^* \to \text{hadrons})$

$$f(s) = \frac{1}{\pi}\int_0^\infty ds' \, \frac{\operatorname{Im} f(s')}{s' - s - i0}$$

— · —

SU(2) $a = 1, \dots, 3$

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon^{abc} W^b_\mu W^c_\nu$$

1) $\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$ $W^a_\mu \to W^a_\mu - \partial_\mu \lambda^a + g\varepsilon^{abc}\lambda^b W^c_\mu$

or $\vec{F}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\vec{W}_\mu \times \vec{W}_\nu \Rightarrow \vec{F}_{\mu\nu} \to \vec{F}_{\mu\nu} + g\vec{\lambda}\times\vec{F}_{\mu\nu}$

2) $\mathcal{L}_{YM} + \mathcal{L}_S$, $\mathcal{L}_S = -(D_\mu K)^\dagger D_\mu K - \mu^2 K^\dagger K - \frac{1}{2}\lambda(K^\dagger K)^2$

$K = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi + i\varphi^0 \\ \sqrt{2} i\varphi^- \end{pmatrix}$ $\varphi^0 = \varphi^3$, $\varphi^\pm = \frac{1}{\sqrt{2}}(\varphi^1 \mp i\varphi^2)$ $D_\mu = \partial_\mu - \frac{i}{2} g W^a_\mu \tau_a$

$\mathcal{L}$ inv $W^a_\mu \to W^a_\mu - \partial_\mu\lambda^a + g\varepsilon^{abc}\lambda^b W^c_\mu$, $K \to (1 - \frac{i}{2} g\lambda^a \tau_a) K$

$\mathcal{L}_S$: $-\partial_\mu K^\dagger \partial_\mu K - \mu^2 K^\dagger K - \frac{1}{2}\lambda(K^\dagger K)^2$

$\psi = H + \sqrt{2}F \Rightarrow K = \frac{1}{\sqrt{2}}\begin{pmatrix} H + \sqrt{2}F + i\varphi^0 \\ \sqrt{2} i\varphi^- \end{pmatrix}$

$K^\dagger = \frac{1}{\sqrt{2}}(H + \sqrt{2}F - i\varphi^0, -\sqrt{2} i\varphi^+)$

$K^\dagger K = \frac{1}{2}\{(H + \sqrt{2}F)^2 + \varphi^0\varphi^0 + 2\varphi^+\varphi^-\}$, $\Phi^2 = \varphi^0\varphi^0 + 2\varphi^+\varphi^-$

$$\mathcal{L}_s : -\frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}\partial_\mu \varphi^0 \partial_\mu \varphi^0 - \partial_\mu \varphi^+ \partial_\mu \varphi^-$$

$$-\frac{1}{2}\mu^2 \left\{ (H+\sqrt{2}F)^2 + \Phi^2 \right\} - \frac{1}{8}\lambda \left\{ (H+\sqrt{2}F)^2 + \Phi^2 \right\}^2$$

$$\Rightarrow \quad -\frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}\mu^2 (H+\sqrt{2}F)^2 - \frac{1}{8}\lambda (H+\sqrt{2}F)^4$$

$$O(H) \qquad -\frac{1}{2}\mu^2 \, 2\sqrt{2} F H - \frac{1}{8}\lambda \, 4\cdot 2\cdot\sqrt{2} \, F^3 H \qquad \Leftarrow \; = 0$$

$$o\text{-----}H \qquad -\mu^2 F - \lambda F^3 = 0 \qquad \begin{cases} F = 0 \\ \mu^2 + \lambda F^2 = 0 \end{cases}$$

Consider $-(D_\mu k)^\dagger D_\mu k$, $\quad D_\mu k = \frac{1}{\sqrt{2}}\left(\partial_\mu - \frac{i}{2} g W^a_\mu \tau_a\right)\begin{pmatrix} H+\sqrt{2}F+i\varphi^0 \\ \sqrt{2}i\varphi^- \end{pmatrix}$

$$W^a_\mu \tau_a = \begin{pmatrix} W^3_\mu & \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu & -W^3_\mu \end{pmatrix}$$

$$D_\mu k \Rightarrow -\frac{i}{2}g\begin{pmatrix} W^3_\mu & \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu & -W^3_\mu \end{pmatrix}\begin{pmatrix} F \\ 0 \end{pmatrix} = -\frac{i}{2}gF\begin{pmatrix} W^3_\mu \\ \sqrt{2}W^-_\mu \end{pmatrix}$$

$$(D_\mu k)^\dagger \Rightarrow +\frac{i}{2}gF(W^3_\mu, \sqrt{2}W^+_\mu)$$

$$-(D_\mu k)^\dagger D_\mu k \Rightarrow -\frac{1}{4}g^2F^2(W^3_\mu W^3_\mu + 2W^+_\mu W^-_\mu) \qquad M^2 = \frac{1}{2}g^2F^2$$

$\mu^2 > 0 \qquad F = 0, \; M = 0 \qquad$ but $\; m_\phi^2 < 0$

$\mu^2 < 0 \qquad F^2 = -\frac{\mu^2}{\lambda^2} > 0 \Rightarrow M^2 > 0$

$$M = \frac{gF}{\sqrt{2}}, \quad F = \sqrt{2}\frac{M}{g}$$

$$k = \frac{1}{\sqrt{2}}\begin{pmatrix} H + 2\frac{M}{g} + i\varphi^0 \\ \sqrt{2}i\varphi^- \end{pmatrix} \Rightarrow k = \frac{1}{\sqrt{2}}\left(H + 2\frac{M}{g} + i\varphi^a\tau_a\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D_\mu K = \frac{1}{\sqrt{2}}\left(\partial_\mu - \frac{i}{2} g W^a_\mu \tau_a\right)\left(H + 2\frac{M}{g} + i\varphi^a\tau_a\right)\binom{1}{0}$$

$$= \frac{1}{\sqrt{2}}\left\{\partial_\mu H + i\partial_\mu\varphi^a\tau_a - \frac{i}{2} g W^a_\mu \tau_a\left(H + 2\frac{M}{g}\right) + \frac{1}{2} g W^a_\mu \varphi^b \tau_a\tau_b\right\}\binom{1}{0}$$

$$\tau_a\tau_b = \delta_{ab} + i\varepsilon_{abc}\tau_c$$

$$D_\mu K = \frac{1}{\sqrt{2}}\left\{\partial_\mu H + i\partial_\mu\varphi^a\tau_a - \frac{i}{2}g\left(H + 2\frac{M}{g}\right)W^a_\mu\tau_a - \frac{i}{2} g\,\varepsilon_{abc}\varphi^b W^c_\mu\tau_a\right\}\binom{1}{0}$$

$$+\tfrac{1}{2} g W^a_\mu \varphi^a$$

$$= \frac{1}{\sqrt{2}}\left\{\partial_\mu H + \frac{1}{2} g W^a_\mu \varphi_a + i\left[\partial_\mu\varphi^a - \frac{1}{2}g\left(H + 2\frac{M}{g}\right)W^a_\mu - \frac{1}{2} g\,\varepsilon_{abc}\varphi^b W^c_\mu\right]\tau_a\right\}\binom{1}{0}$$

$$= \frac{1}{\sqrt{2}}\left\{\partial_\mu H + \frac{1}{2} g \vec{W}_\mu\cdot\vec{\varphi} + i\left[\partial_\mu\vec{\varphi} - \frac{1}{2}g\left(H + 2\frac{M}{g}\right)\vec{W}_\mu - \frac{1}{2} g\,\vec{\varphi}\times\vec{W}_\mu\right]\cdot\vec{\tau}\right\}\binom{1}{0}$$

$$(a + i\vec{b}\cdot\vec{\tau})^\dagger(a + i\vec{b}\cdot\vec{\tau}) = a^2 + \vec{b}\cdot\vec{b}$$

$$-(D_\mu K)^\dagger D_\mu K = -\frac{1}{2}\left\{\left[\partial_\mu H + \frac{1}{2} g\vec{W}_\mu\cdot\vec{\varphi}\right]^2 + \left[\partial_\mu\vec{\varphi} - \frac{1}{2} g\left(H + 2\frac{M}{g}\right)\vec{W}_\mu - \frac{1}{2} g\,\vec{\varphi}\times\vec{W}_\mu\right]^2\right\}$$

$$= -\frac{1}{2}\partial_\mu H\partial_\mu H - \frac{1}{2}\partial_\mu\vec{\varphi}\,\partial_\mu\vec{\varphi} - \frac{1}{2}\left\{g\vec{W}_\mu\cdot\vec{\varphi}\,\partial_\mu H + \frac{1}{4}g^2(\vec{W}_\mu\cdot\vec{\varphi})^2\right.$$

$$+ \frac{1}{4}g^2\left(H + 2\frac{M}{g}\right)^2\vec{W}_\mu\cdot\vec{W}_\mu + \frac{1}{4}g^2(\vec{\varphi}\times\vec{W}_\mu)^2$$

$$\left.- g\left(H + 2\frac{M}{g}\right)\vec{W}_\mu\cdot\partial_\mu\vec{\varphi} - g(\partial_\mu\vec{\varphi})\cdot(\vec{\varphi}\times\vec{W}_\mu) + \frac{1}{2}g^2\left(H + 2\frac{M}{g}\right)\vec{W}_\mu\cdot(\vec{\varphi}\times\vec{W}_\mu)\right\}$$

$$\Rightarrow -\frac{1}{2}M^2\vec{W}_\mu\cdot\vec{W}_\mu \qquad\qquad -g\,2\frac{M}{g}\vec{W}_\mu\cdot\partial_\mu\vec{\varphi} = -2M\vec{W}_\mu\cdot\partial_\mu\vec{\varphi}$$

$$H \to \left(1 - \frac{i}{2} g \lambda^a \tau_a\right) k \quad , \quad k = S\begin{pmatrix}1\\0\end{pmatrix}$$

$$S = \frac{1}{\sqrt{2}}\left(H + 2\frac{M}{g} + i\varphi^a \tau_a\right)$$

$$\left(H + 2\frac{M}{g} + i\varphi^a\tau_a\right) \to \left(1 - \frac{i}{2} g \lambda^b \tau_b\right)\left(H + 2\frac{M}{g} + i\varphi^a\tau_a\right)$$

$$= H + 2\frac{M}{g} + i\varphi^a\tau_a - \frac{i}{2} g \left(H + 2\frac{M}{g}\right)\lambda^a\tau_a + \frac{1}{2} g \lambda^b \varphi^a \tau_b\tau_a$$

$$\swarrow$$

$$\frac{1}{2} g \lambda^a \varphi_a - \frac{i}{2} g\, \varepsilon_{abc} \lambda^b \varphi^c \tau_a$$

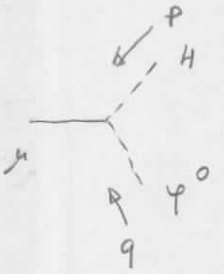
$$H \to H + \frac{1}{2} g \lambda^a \varphi_a$$

$$\varphi^a \to \varphi^a - \frac{i}{2} g \left(H + 2\frac{M}{g}\right)\lambda^a - \frac{i}{2} g\, \varepsilon_{abc} \lambda^b \varphi^c$$

ex : $-\frac{1}{2} g \vec{W}_\mu \cdot \vec{\varphi}\, \partial_\mu H + \frac{1}{2} g H \vec{W}_\mu \cdot \partial_\mu \vec{\varphi}$

$$\vec{W} \cdot \vec{\varphi} = W^3_\mu \varphi^0 + W^+_\mu \varphi^- + W^-_\mu \varphi^+$$

$$\Rightarrow \frac{1}{2} g \left(W^3_\mu H \partial_\mu \varphi^0 - W^3_\mu \varphi^0 \partial_\mu H \right) \Rightarrow$$



$$\Rightarrow (2\pi)^4 i \frac{ig}{2} (q-p)_\mu$$

etc

$$K^+ K = \frac{1}{2} \left\{ \left(H + 2\frac{M}{g} \right)^2 + \Phi^2 \right\}$$

$$-\mu^2 K^+K - \frac{\lambda}{2} (K^+K)^2 = -\frac{1}{2}\mu^2 \left\{ \left(H + 2\frac{M}{g}\right)^2 + \Phi^2 \right\} - \frac{\lambda}{8} \left\{ \left(H + 2\frac{M}{g}\right)^2 + \Phi^2 \right\}^2$$

$$\mu^2 + \lambda F^2 = \beta , \quad \mu^2 = \beta - \lambda F^2 , \quad F = \sqrt{2}\frac{M}{g}$$

$$\mu^2 = \beta - 2\lambda \frac{M^2}{g^2} = \beta - \frac{1}{2} m_H^2$$

$$4\lambda \frac{M^2}{g^2} = m_H^2 , \qquad \lambda = \frac{1}{4} g^2 \frac{m_H^2}{M^2}$$

$$O(H^0) \quad -\frac{1}{2}\mu^2 4 \frac{M^2}{g^2} - \frac{\lambda}{8} 16 \frac{M^4}{g^4} = -2\mu^2 \frac{M^2}{g^2} - 2\lambda \frac{M^4}{g^4}$$

$$= -2\frac{M^2}{g^2} \left(\mu^2 + \lambda \frac{M^2}{g^2} \right) = -2 \frac{M^2}{g^2} \left(\beta - \frac{1}{2} m_H^2 + \frac{1}{4} g^{\cancel{2}} \frac{m_H^2}{\cancel{M^2}} \frac{\cancel{M^2}}{\cancel{g^2}} \right)$$

$$= -2 \frac{M^2}{g^2} \left(\beta - \frac{1}{2} m_H^2 \right)$$

$$O(H^1) \quad -\frac{1}{2}\mu^2 4 \frac{M}{g} - \frac{\lambda}{8} 4 \cdot 8 \frac{M^3}{g^3} = -2\mu^2 \frac{M}{g} - 4\lambda \frac{M^3}{g^2}$$

$$= -2\frac{M}{g} \left(\mu^2 + 2\lambda \frac{M^2}{g^2} \right) = -2\frac{M}{g} \left(\beta - \frac{1}{2} m_H^2 + \frac{1}{2} g^{\cancel{2}} \frac{m_H^2}{\cancel{M^2}} \frac{\cancel{M^2}}{\cancel{g^2}} \right) = -2 \frac{M}{g} \beta$$

$$O(H^2) = -\frac{1}{2}\mu^2 - \frac{\lambda}{8}\,6\cdot 4\,\frac{M^2}{g^2} = -\frac{1}{2}\mu^2 - 3\lambda\frac{M^2}{g^2} = -\frac{1}{2}\left(\beta - \frac{1}{2}\mu_H^2\right)$$

$$-\frac{3}{4}\cancel{g^2}\frac{\mu_H^2}{\cancel{M^2}}\frac{\cancel{M^2}}{\cancel{g^2}} = -\frac{1}{2}\beta + \frac{1}{4}\mu_H^2 - \frac{3}{4}\mu_H^2 = -\frac{1}{2}\mu_H^2 - \frac{1}{2}\beta$$

$$O(H^3) \quad -\frac{\lambda}{8}\,4\cdot 2\,\frac{M}{g} = -\lambda\frac{M}{g} = -\frac{1}{4}g^{\cancel{2}}\frac{\mu_H^2}{M^{\cancel{2}}}\frac{\cancel{M}}{\cancel{g}} = -\frac{1}{4}g\frac{\mu_H^2}{M}$$

$$O(\Phi^2) \quad -\frac{1}{2}\mu^2 - \frac{\lambda}{4}\left(2\frac{M}{g}\right)^2 = -\frac{1}{2}\mu^2 - \lambda\frac{M^2}{g^2} = -\frac{1}{2}\mu^2 - \frac{1}{4}\cancel{g^2}\frac{\mu_H^2}{\cancel{M^2}}\frac{\cancel{M^2}}{\cancel{g^2}}$$

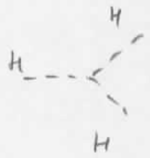
$$= -\frac{1}{2}\beta + \frac{1}{4}\mu_H^2 - \frac{1}{4}\mu_H^2 = -\frac{1}{2}\beta$$

etc.

$\mathcal{L}$ $\quad -2\frac{M}{g}\beta$

$(2\pi)^4 i\left(-2\frac{M}{g}\beta\right)$

$-\frac{1}{4}g\frac{\mu_H^2}{M}$



$(2\pi)^4 i\left(-\frac{1}{4}g\frac{\mu_H^2}{M}\right)6 = (2\pi)^4 i\left(-\frac{3}{2}g\frac{\mu_H^2}{M}\right)$

etc.

$$-2\frac{M}{g}\beta - \frac{3}{4}\frac{g}{16\pi^2}\frac{\mu_H^2}{M}A_0(\mu_H) + \cdots = 0$$

$$\beta = -\frac{3}{128\pi^2}g^2\frac{\mu_H^2}{M}A_0(\mu_H) + \cdots$$

β requires ren.

mass spectrum

W	M
H	μ_H
φ	0

?

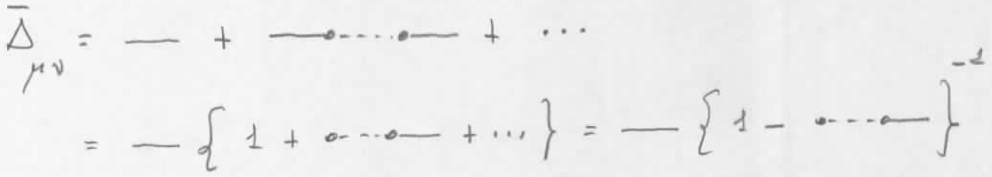
$W^3_\mu \equiv Z_\mu$

$$-M\varphi^a\partial_\mu W^a_\mu = M\left(Z_\mu\partial_\mu\varphi^0 + W^+_\mu\partial_\mu\varphi^- + W^-_\mu\partial_\mu\varphi^+\right)$$

$$\frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+M^2}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right)$$

$$\frac{1}{(2\pi)^4 i}\,\frac{1}{p^2}$$

$$(2\pi)^4 i\,(-iMp_\mu)$$



$$\bar{\Delta}_{\mu\nu}=\frac{1}{p^2+M^2}\left(\delta_{\mu\alpha}+\frac{p_\mu p_\alpha}{M^2}\right)X^{-1}_{\alpha\nu}$$

$$X_{\alpha\nu}=\delta_{\alpha\nu}-(iMp_\alpha)\frac{1}{p^2}(-iMp_\beta)\frac{1}{p^2+M^2}\left(\delta_{\beta\nu}+\frac{p_\beta p_\nu}{M^2}\right)$$

$$=\delta_{\alpha\nu}-\frac{M^2}{p^2}\,\frac{1}{p^2+M^2}\left(p_\alpha p_\nu+\frac{p^2}{M^2}p_\alpha p_\nu\right)$$

$$=\delta_{\alpha\nu}-\frac{\cancel{M^2}}{p^2}\,\frac{1}{\cancel{p^2+M^2}}\,\frac{\cancel{p^2+M^2}}{\cancel{M^2}}\,p_\alpha p_\nu=\delta_{\alpha\nu}-\frac{p_\alpha p_\nu}{p^2}$$

$X_{\alpha\nu}p_\nu=0 \Rightarrow X_{\mu\nu}$ singular

gauge-fixing $\quad C^a=-\frac{1}{\xi}\partial_\mu W^a_\mu+\xi M\varphi^a, \quad R_\xi$

$$\mathcal{L}=\mathcal{L}_{inv}-\frac{1}{2}C^aC^a$$

$$W^a_\mu\to W^a_\mu-\partial_\mu\lambda^a+g\,\varepsilon^{abc}\lambda^b W^c_\mu$$

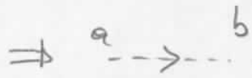
$$\varphi^a\to\varphi^a-\frac{1}{2}g\left(H+2\frac{M}{g}\right)\lambda^a+\frac{1}{2}g\,\varepsilon^{abc}\lambda^b\varphi^c$$

$$C^a\to-\frac{1}{\xi}\partial_\mu\left(W^a_\mu-\partial_\mu\lambda^a+g\,\varepsilon^{abc}\lambda^b W^c_\mu\right)+\xi M\Big\{\varphi^a-\frac{1}{2}g\left(H+2\frac{M}{g}\right)\lambda^a$$
$$+\frac{1}{2}g\,\varepsilon^{abc}\lambda^b\varphi^c$$

$$= C^a + \frac{1}{\xi}\partial^2\Lambda^a - \frac{1}{\xi} g\, \varepsilon^{abc}\partial_\mu(\Lambda^b\varphi^c) - \frac{1}{2}\xi M g\left(M + 2\frac{M}{g}\right)\Lambda^a$$

$$+ \frac{1}{2}\xi M g\, \varepsilon^{abc}\Lambda^b\varphi^c = C^a + M^{ab}X^b + g\, L^{ab}\Lambda^b$$

$$\frac{1}{\xi}\partial^2\Lambda^a - \xi M^2\Lambda^a \implies M^{ab} = \delta^{ab}\left(\frac{1}{\xi}\partial^2 - \xi M^2\right)$$

$\implies$ 

$$\frac{1}{(2\pi)^4 i}\,\delta^{ab}\,\frac{\xi}{p^2+\xi^2M^2}$$

$$\mathcal{L}_{FP} = \eta^*_a\,(M_{ab} + g\,L_{ab})\,\eta_b$$

$$\mathcal{L}_{inv} - \frac{1}{2}C^aC_a \implies -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a - \frac{1}{2}M^2W^a_\mu W^a_\mu - M\varphi^a\partial_\mu W^a_\mu$$

$$- \frac{1}{2\xi^2}(\partial_\mu W^a_\mu)^2 + M\varphi^a\partial_\mu W^a_\mu - \frac{1}{2}\xi^2M^2\varphi^a\varphi^a + \cdots$$

1) $$-\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) - \frac{1}{2\xi^2}(\partial_\mu W^a_\mu)^2 - \frac{1}{2}M^2W^a_\mu W^a_\mu$$

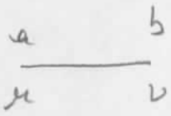
$$= \frac{1}{2}W^a_\alpha\left\{-\left(\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\mu + M^2\right)\delta_{\alpha\beta} + \overleftarrow{\partial}_\beta\overrightarrow{\partial}_\alpha - \frac{1}{\xi^2}\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta\right\}W^a_\beta$$

$$\implies V_{\mu\nu} = -(p^2+M^2)\delta_{\mu\nu} + \left(1 - \frac{1}{\xi^2}\right)p_\mu p_\nu$$

$$V^{-1}_{\mu\nu} = A(p^2)\delta_{\mu\nu} + B(p^2)p_\mu p_\nu$$

$$\implies A = -\frac{1}{p^2+M^2}, \quad B = (1-\xi^2)\frac{1}{(p^2+M^2)(p^2+\xi^2M^2)}$$

$$\Delta^{ab}_{\mu\nu} = \frac{1}{(2\pi)^4 i}\,\frac{\delta^{ab}}{p^2+M^2}\left\{\delta_{\mu\nu} + (\xi^2-1)\frac{p_\mu p_\nu}{p^2+\xi^2M^2}\right\}$$



$$\frac{1}{(2\pi)^4 i}\,\frac{1}{p^2+\xi^2M^2}$$

't H-F gauge $\xi = 1$ $$\Delta^{ab}_{\mu\nu} = \frac{1}{(2\pi)^4 i} \frac{\delta^{ab}\delta_{\mu\nu}}{p^2+M^2} \quad \text{etc}$$

U-gauge $\xi \to \infty$ $$\Delta^{ab}_{\mu\nu} = \frac{1}{(2\pi^4 i} \frac{\delta^{ab}}{p^2+M^2} \left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{M2}\right)$$

$$\Delta^{ab} \to 0, \quad \Delta^{ab}_{FP} \to 0$$

$$\left.\begin{array}{l} \xi = 1 \Leftarrow \text{ren.} \\ \xi \to +\infty \Leftarrow \text{unitarity} \end{array}\right\} \quad S\text{-matrix } \xi\text{-independent}$$

$SU(2) \otimes U(1)$

$$B^a_\mu, B^0_\mu \qquad \mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} F^0_{\mu\nu} F^0_{\mu\nu}, \quad F^0_{\mu\nu} = \partial_\mu B^0_\nu - \partial_\nu B^0_\mu$$

$$\mathcal{L}_S ;\ k \qquad D_\mu k = (\partial_\mu - \frac{i}{2} g B^a_\mu \tau_a - \frac{i}{2} g g_1 B^0_\mu \tau^0) k, \quad \tau_0 \equiv 1\!\!1_{2\times 2}$$

$$k = \frac{1}{\sqrt{2}} \begin{pmatrix} H + \sqrt{2}F + i\varphi^0 \\ \sqrt{2} i \varphi^- \end{pmatrix}$$

$$D_\mu k \Rightarrow -\frac{i}{2} g F \begin{pmatrix} B^3_\mu + g_1 B^0_\mu & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & -B^3_\mu + g_1 B^0_\mu \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{i}{2} g F \begin{pmatrix} B^3_\mu + g_1 B^0_\mu \\ \sqrt{2} W^-_\mu \end{pmatrix}$$

$$(D_\mu k)^+ \Rightarrow +\frac{i}{2} g F (B^3_\mu + g_1 B^0_\mu, \sqrt{2} W^+_\mu)$$

$$-(D_\mu k)^+ D_\mu k \Rightarrow -\frac{1}{4} g^2 F^2 \left\{ (B^3_\mu + g_1 B^0_\mu)^2 + 2 W^+_\mu W^-_\mu \right\}$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} B^3_\mu \\ B^0_\mu \end{pmatrix} \qquad s = \sin\theta_W, \ c = \cos\theta_W$$

$$Z_\mu = c B^3_\mu - s B^0_\mu, \quad A_\mu = s B^3_\mu + c B^0_\mu$$

$$B^3_\mu = c Z_\mu + s A_\mu, \quad B^0_\mu = -s Z_\mu + c A_\mu$$

$$B^3_\mu + g_1 B^0_\mu = c Z_\mu + s A_\mu + g_1(-s Z_\mu + c A_\mu) = (c - g_1 s) Z_\mu + (s + g_1 c) A_\mu$$

$$g_1 = -\frac{s}{c} = -\tan\vartheta_W \quad \longrightarrow \quad c + \frac{s^2}{c} = \frac{1}{c} \qquad \Downarrow \ \frac{1}{c} Z_\mu$$

$$\Rightarrow -M^2 W^+_\mu W^-_\mu - \frac{1}{2}\frac{M^2}{c^2} Z_\mu Z_\mu$$

$$\begin{array}{lll} \text{LO} & & \\ W^\pm & M_W = M & \\ Z & M_Z = \frac{M}{c} & \end{array} \Bigg\} \Rightarrow c\frac{M_Z}{M_W} = 1 \quad \text{LO}$$

GF-term $(\xi = 1)$

$$-\frac{1}{2}C^2 = -\frac{1}{2}C^a C^a = -\frac{1}{2}(-\partial_\mu B^a_\mu + M\varphi^a)^2 - \frac{1}{2}(\partial_\mu B^0_\mu + \frac{s}{c}M\varphi^0)^2 \qquad (\varphi^0 = \varphi^3)$$

$$X^a X^a = X^3 X^3 + 2X^+ X^-$$

$$C^\pm = -\partial_\mu W^\pm_\mu + M\varphi^\pm$$

$$-\frac{1}{2}C^2 = -C^+C^- - \frac{1}{2}(-\partial_\mu B^3_\mu + M\varphi^0)^2 - \frac{1}{2}(\partial_\mu B^0_\mu + \frac{s}{c}M\varphi^0)^2$$

$$\swarrow$$

$$-\frac{1}{2}\Big\{(\partial_\mu B^3_\mu)^2 - 2M\varphi^0\partial_\mu B^3_\mu + M^2\varphi^0\varphi^0 + (\partial_\mu B^0_\mu)^2 + 2\frac{s}{c}M\varphi^0\partial_\mu B^0_\mu + \frac{s^2}{c^2}M^2\varphi^0\varphi^0\Big\}$$

$$= -\frac{1}{2}\Big\{(\partial_\mu A_\mu)^2 + (\partial_\mu Z_\mu)^2 + \frac{M^2}{c^2}\varphi^0\varphi^0 - 2M\varphi^0\Big[c\,\partial_\mu Z_\mu + s\,\cancel{\partial_\mu A_\mu} - \frac{s}{c}(-s\,\partial_\mu Z_\mu + c\,\cancel{\partial_\mu A_\mu})\Big]\Big\}$$

$$= -\frac{1}{2}\Big\{(\partial_\mu A_\mu)^2 + (\partial_\mu Z_\mu)^2 + \frac{M^2}{c^2}\varphi^0\varphi^0 - 2\frac{M}{c}\varphi^0\partial_\mu Z_\mu\Big\}$$

$$\Rightarrow -\frac{1}{2}C^2 = -C^+C^- - \frac{1}{2}C_A^2 - \frac{1}{2}C_Z^2, \quad C_A = \partial_\mu A_\mu, \quad C_Z = -\partial_\mu Z_\mu + \frac{M}{c}\varphi^0$$

C^a is a free field (WI → WST identities)

QED $\quad C = \partial_\mu A_\mu \Rightarrow$

$= 0$

$C_Z = -\partial_\mu Z_\mu + \frac{M}{c}\varphi^0 \Rightarrow$

$+ \frac{M}{c}$

$= 0$

Fermions $-\ \mathcal{L}_F$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \quad \psi_{L,R} = \frac{1}{2}(1 \pm \gamma^5)\psi, \quad \bar\psi_{L,R} = \frac{1}{2}\bar\psi(1 \mp \gamma^5)$$

L) $T^a = -\frac{i}{2}\tau^a$, $T^0 = -\frac{i}{2} g_2 \tau_0$

R) $t^a = 0$, $t^0 = -\frac{i}{2}\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}$

$$D_\mu \psi_L = (\partial_\mu + g B^i_\mu T_i)\psi_L, \quad D_\mu \psi_R = (\partial_\mu + g B^i_\mu t_i)\psi_R \qquad i = 0, 1, 2, 3$$

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_S + \mathcal{L}_{gf} + \mathcal{L}_{FP} + \mathcal{L}_F$$

$$\mathcal{L}_F = -\bar\psi_L \not{D} \psi_L - \bar\psi_R \not{D} \psi_R$$

$$-\bar\psi_L \not\partial \psi_L - \bar\psi_R \not\partial \psi_R = -\bar\psi \not\partial \psi$$

$$\Rightarrow -g\bar\psi_L \not{B}^i T_i \psi_L - g\bar\psi_R \not{B}^i t_i \psi_R$$

$$B^i_\mu T_i = -\frac{i}{2}\begin{pmatrix} B^3_\mu & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & -B^3_\mu \end{pmatrix} - \frac{i}{2} g_2 \begin{pmatrix} B^0_\mu & 0 \\ 0 & B^0_\mu \end{pmatrix}$$

$$= -\frac{i}{2}\begin{pmatrix} B^3_\mu + g_2 B^0_\mu & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & -B^3_\mu + g_2 B^0_\mu \end{pmatrix}$$

$$B^i_\mu t_i = -\frac{i}{2}\begin{pmatrix} g_3 B^0_\mu & 0 \\ 0 & g_4 B^0_\mu \end{pmatrix}$$

$$\bar\psi_L \gamma^\mu \psi_L = \frac{1}{2}\bar\psi\gamma^\mu(1+\gamma^5)\psi, \quad \bar\psi_R \gamma^\mu \psi_R = \frac{1}{2}\bar\psi\gamma^\mu(1-\gamma^5)\psi$$

$$\Rightarrow -\left(-\frac{i}{2}\right)\frac{g}{2}\bar\psi\gamma^\mu\left\{\begin{pmatrix} B^3_\mu + (g_2+g_3)B^0_\mu & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & -B^3_\mu + (g_2+g_4)B^0_\mu \end{pmatrix}\right.$$

$$\left. + \begin{pmatrix} B^3_\mu + (g_2-g_3)B^0_\mu & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & -B^3_\mu + (g_2-g_4)B^0_\mu \end{pmatrix}\gamma^5\right\}\psi$$

$$\mathcal{L}_{CC} = \frac{i}{4} g \bar\psi \gamma^\mu (1+\gamma^5) \begin{pmatrix} 0 & \sqrt{2} W^+_\mu \\ \sqrt{2} W^-_\mu & 0 \end{pmatrix} \psi$$

$$= \frac{i}{2\sqrt{2}} g (\bar u, \bar d) \gamma^\mu (1+\gamma^5) \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

$$= \frac{ig}{2\sqrt{2}} \left\{ \bar u \gamma^\mu (1+\gamma^5) d\, W^+_\mu + \bar d \gamma^\mu (1+\gamma^5) u\, W^-_\mu \right\}$$

$\mathcal{L}_{NC}$ $\qquad g_i = - \frac{s}{c} \lambda_i$

$$B^3_\mu + (g_2 \pm g_3) B^0_\mu = c Z_\mu + s A_\mu - \frac{s}{c}(\lambda_2 \pm \lambda_3)(-s Z_\mu + c A_\mu)$$

$$= \left\{ c + \frac{s^2}{c}(\lambda_2 \pm \lambda_3) \right\} Z_\mu + s \left\{ 1 - (\lambda_2 \pm \lambda_3) \right\} A_\mu$$

$$- B^3_\mu + (g_2 \pm g_4) B^0_\mu = - c Z_\mu - s A_\mu - \frac{s}{c}(\lambda_2 \pm \lambda_4)(-s Z_\mu + c A_\mu)$$

$$= \left\{ - c + \frac{s^2}{c}(\lambda_2 \pm \lambda_4) \right\} Z_\mu + c \left\{ - 1 - (\lambda_2 \pm \lambda_4) \right\} A_\mu$$

$$\Rightarrow \frac{i}{4} g s \bar\psi \not{A} \left\{ \begin{pmatrix} 1-\lambda_2-\lambda_3 & 0 \\ 0 & -1-\lambda_2-\lambda_4 \end{pmatrix} + \begin{pmatrix} 1-\lambda_2+\lambda_3 & 0 \\ 0 & -1-\lambda_2+\lambda_4 \end{pmatrix} \gamma^5 \right\} \psi$$

$gs = e$

$$1 - \lambda_2 + \lambda_3 = 0, \quad 1 + \lambda_2 - \lambda_4 = 0$$

$$1 - \lambda_2 - \lambda_3 = 4 Q_u, \quad 1 + \lambda_2 + \lambda_4 = -4 Q_d$$

$$\left.\begin{array}{ll} 1 - \lambda_2 = 2 Q_u, & \lambda_2 = 1 - 2 Q_u \\ 1 + \lambda_2 = -2 Q_d, & \lambda_2 = -1 - 2 Q_d \end{array}\right\} \quad 1 - Q_u + Q_d = 0$$

$$\Rightarrow i Q_f e \bar f \gamma^\mu f A_\mu$$

$$\Rightarrow \frac{ig}{2c} \bar f \gamma^\mu (I^3_f - 2 Q_f s^2 + I_{3f} \gamma^5) f Z_\mu$$

$$(2\pi)^4 i \frac{ig}{2c}\gamma^\mu(v_f + a_f\gamma^5)$$

$$v_f = I_{3f} - 2Q_f s^2$$

$$a_f = I_{3f}$$

$$\psi_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad u_R, \quad d_R \qquad k = \frac{1}{\sqrt{2}}\begin{pmatrix} H + 2\frac{M}{g} + i\varphi^0 \\ \sqrt{2}\, i\varphi^- \end{pmatrix}$$

$$k^c_i = \varepsilon_{ij} k^*_j \qquad k^c_1 = k^*_2 = -i\varphi^+, \quad k^c_2 = -k^*_1 = -\frac{1}{\sqrt{2}}(H + 2\frac{M}{g} - i\varphi^0)$$

$$k^c = -\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\, i\varphi^+ \\ H + 2\frac{M}{g} - i\varphi^0 \end{pmatrix}, \qquad k^c = i\tau_2 k^*, \quad \tau_2\tau_a^* = -\tau_a\tau_2$$

Candidates for mass term : $\bar\psi_L k u_R$, $\bar\psi_L k^c d_R$

$$k \to (1 - \frac{i}{2} g\lambda^a\tau_a - \frac{i}{2} g g_1 \lambda^0\tau_0) k, \qquad u_R \to (1 - \frac{i}{2} g g_3 \lambda^0) u_R$$

$$\psi_L \to (1 - \frac{i}{2} g\lambda^a\tau_a - \frac{i}{2} g g_2 \lambda^0\tau_0)\psi_L, \qquad d_R \to (1 - \frac{i}{2} g g_4 \lambda^0) d_R$$

$$\bar\psi_L k \to \bar\psi_L (1 + \frac{i}{2} g\lambda^a\tau_a + \frac{i}{2} g g_2\lambda^0\tau_0)(1 - \frac{i}{2} g\lambda^a\tau_a - \frac{i}{2} g g_1 \lambda^0\tau_0) k$$

$$= \bar\psi_L\{1 + \frac{i}{2} g(g_2 - g_1)\lambda^0\} k$$

$$\bar\psi_L k u_R \to \{1 + \frac{i}{2} g\underset{\overset{\parallel}{0}}{(g_2 - g_1 - g_3)}\lambda^0\}\bar\psi_L k u_R \qquad \text{inv for } SU(2)\otimes U(1)$$

$$k_c = i\tau_2 k^* \to i\tau_2(1 + \frac{i}{2} g\lambda^a\tau_a^* + \frac{i}{2} g g_1\lambda^0\tau_0) k^*$$

$$= (1 - \frac{i}{2} g\lambda^a\tau_a + \frac{i}{2} g g_1\lambda^0\tau_0) k^c$$

$$\bar\psi_L k^c \to \{1 + \frac{i}{2} g(g_2 + g_1)\lambda^0\}\bar\psi_L k^c$$

$$\bar\psi_L k^c d_R \to \{1 + \frac{i}{2} g\underset{\overset{\parallel}{0}}{(g_2 + g_1 - g_4)}\lambda^0\}\bar\psi_L k^c d_R \qquad \text{inv for } SU(2)\otimes U(1)$$

$$\mathcal{L}_{SF} \Rightarrow -\sqrt{2}\alpha \frac{M}{g} \bar{u}_L u_R + \sqrt{2}\beta \frac{M}{g} \bar{d}_L d_R + h.c.$$

$$k \Rightarrow \sqrt{2}\frac{M}{g}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\mathcal{L}_{SF} = -\alpha \bar{\Psi}_L k u_R - \beta \bar{\Psi}_L k^c d_R + h.c.$$

$$k^c \Rightarrow -\sqrt{2}\frac{M}{g}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow -\sqrt{2}\alpha \frac{M}{g} \bar{u} u + \sqrt{2}\beta \frac{M}{g} \bar{d} d$$

$$\bar{\Psi}_L \chi_R = \frac{1}{4} \bar{\Psi} (1-\gamma^5)^2 \chi = \frac{1}{2} \bar{\Psi}(1-\gamma^5)\chi$$

$$(\bar{\Psi}_L \chi_R)^\dagger = \frac{1}{2} \bar{\chi}(1+\gamma^5)\Psi$$

$$\Rightarrow \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L = \bar{\Psi}\Psi \quad , \quad \bar{\Psi}_L \Psi_R - \bar{\Psi}_R \Psi_L = -\bar{\Psi}\gamma^5\Psi$$

$$\sqrt{2}\alpha \frac{M}{g} \mu_u \quad , \quad \alpha = \frac{1}{\sqrt{2}} g \frac{\mu_u}{M} \quad ; \quad -\sqrt{2}\beta \frac{M}{g} = \mu_d \quad , \quad \beta = -\frac{1}{\sqrt{2}} g \frac{\mu_d}{M}$$

$$\mathcal{L}_{SF} = -\frac{1}{\sqrt{2}} g \frac{\mu_u}{M} \bar{\Psi}_L k u_R + \frac{1}{\sqrt{2}} g \frac{\mu_d}{M} \bar{\Psi}_L k^c d_R + h.c$$

$$= -\frac{1}{2} g \frac{\mu_u}{M} (\bar{u}, \bar{d})_L \begin{pmatrix} H + 2\frac{M}{g} + i\varphi^0 \\ \sqrt{2} i \varphi^- \end{pmatrix} u_R - \frac{1}{2} g \frac{\mu_d}{M} (\bar{u}, \bar{d})_L \begin{pmatrix} \sqrt{2} i \varphi^+ \\ H + 2\frac{M}{g} - i\varphi^0 \end{pmatrix} d_R + h.c.$$

$$= -\frac{1}{2} g \frac{\mu_u}{M} \left\{ (H + 2\frac{M}{g} + i\varphi^0) \bar{u}_L u_R + \sqrt{2} i \varphi^- \bar{d}_L u_R \right\}$$

$$- \frac{1}{2} g \frac{\mu_d}{M} \left\{ \sqrt{2} i \varphi^+ \bar{u}_L d_R + (H + 2\frac{M}{g} - i\varphi^0) \bar{d}_L d_R \right\} + h.c$$

$$= -\frac{1}{2} g \frac{\mu_u}{M} \left\{ (H + 2\frac{M}{g})(\bar{u}_L u_R + \bar{u}_R u_L) + i\varphi^0 (\bar{u}_L u_R - \bar{u}_R u_L) \right.$$
$$\left. + \sqrt{2} i \varphi^- \bar{d}_L u_R - \sqrt{2} i \varphi^+ \bar{u}_R d_L \right\}$$

$$- \frac{1}{2} g \frac{\mu_d}{M} \left\{ (H + 2\frac{M}{g})(\bar{d}_L d_R + \bar{d}_R d_L) - i\varphi^0 (\bar{d}_L d_R - \bar{d}_R d_L) \right.$$
$$\left. + \sqrt{2} i \varphi^+ \bar{u}_L d_R - \sqrt{2} i \varphi^- \bar{d}_R u_L \right\}$$

$$-\frac{1}{2} g \frac{m_u}{M} H \bar u u - \frac{1}{2} g \frac{m_d}{M} H \bar d d$$

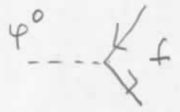
$$+\frac{i}{2} g \frac{m_u}{M} \varphi^0 \bar u \gamma^5 u - \frac{i}{2} g \frac{m_d}{M} \varphi^0 \bar d \gamma^5 d$$

$$-\frac{i}{\sqrt{2}} g \varphi^- \left\{ \frac{m_u}{M} \bar d_L u_R - \frac{m_d}{M} \bar d_R u_L \right\} + \cdots$$

$$\tfrac{1}{2}(1-\gamma^5) \qquad \tfrac{1}{2}(1+\gamma^5$$

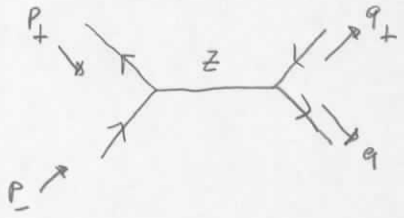
$$\hookrightarrow -i \frac{g}{2\sqrt{2}} \varphi^- \bar d \left\{ \frac{m_u - m_d}{M} - \frac{m_u + m_d}{M} \gamma^5 \right\} u$$

$\Rightarrow$



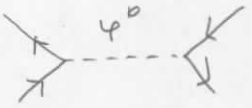
$$(2\pi)^4 i \, ig \frac{m_f}{M} I_{3f} \gamma^5$$

$$M_0 = \frac{M}{c} \qquad \bar f f \to \bar h h \qquad P = p_+ + p_- = q_+ + q_-$$



$$(2\pi)^4 i \left(-\frac{g^2}{4c^2}\right) \gamma^\mu (v_f + a_f \gamma^5) \otimes \gamma^\nu (v_h + a_h \gamma^5)$$

$$\times \frac{1}{P^2 + M_0^2} \left\{ \delta_{\mu\nu} + (\xi^2 - 1) \frac{P_\mu P_\nu}{P^2 + \xi^2 M_0^2} \right\}$$



$$(2\pi)^4 i (-g^2) \frac{m_f m_h}{M^2} I_{3f} I_{3h} \frac{1}{P^2 + \xi^2 M_0^2} \gamma^5 \otimes \gamma^5$$

$$\bar v(p_+) \not{P} (v_f + a_f \gamma^5) u(p_-) = \bar v(p_+)(\not{p}_+ + \not{p}_-)(v_f + a_f \gamma^5) u(p_-)$$

$$= \bar v(p_+) \left\{ \not{p}_+ (v_f + a_f \gamma^5) + (v_f - a_f \gamma^5) \not{p}_- \right\} u(p_-) = -2i m_f a_f \bar v \gamma^5 u$$

$$\bar u(q_-) \not{P} (v_h + a_h \gamma^5) v(q_+) = +2i m_h a_h \bar u \gamma^5 v$$

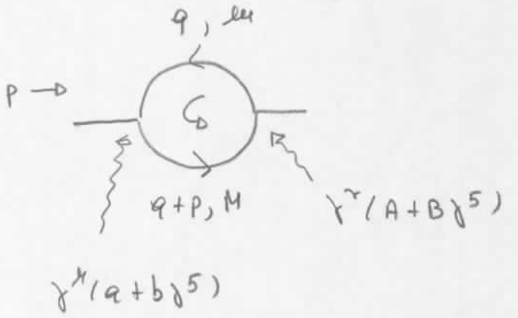
$$\Rightarrow \left(-\frac{g^2}{4c^2}\right) 4 m_f m_h I_{3f} I_{3h} (\xi^2 - 1) \frac{1}{(P^2 + M_0^2)(P^2 + \xi^2 M_0^2)} \gamma^5 \otimes \gamma^5$$

$$-g^2 \frac{m_f m_b}{M^2} I_{3f} I_{3b} \frac{1}{\ell^2+\xi^2 M_0^2} \gamma^5 \otimes \gamma^5$$

$$\Rightarrow \frac{1}{c^2} \frac{\xi^2-1}{\ell^2+M_0^2} + \frac{1}{M^2} = \frac{1}{c^2 M^2 (\ell^2+M_0^2)} \left\{ (\xi^2-1) M^2 + c^2 (\ell^2+M_0^2) \right\}$$

$$\downarrow$$

$$\xi^2 M^2 + c^2 \ell^2 = c^2 (\ell^2 + \xi^2 M_0^2)$$



$$S_{\mu\nu}(a, b ; A, B ; m, M)$$

$$S_{\mu\nu}(\cdots) = i \int d^4 q \frac{1}{(q^2+m^2)((q+p)^2+M^2)} \mathrm{tr} \left\{ \gamma^\mu (a+b\gamma^5)(-i\not{q}+m) \gamma^\nu (A+B\gamma^5) \times \left[-i(\not{q}+\not{p}) + M \right] \right\}$$

$S_{\mu\nu}$ for $\binom{t}{b}$, $S_{\mu\nu} = S \delta_{\mu\nu} + " p_\mu p_\nu "$

W - propagator $$\frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2+M^2} \longrightarrow \frac{1}{(2\pi)^4 i} \delta_{\mu\nu} \left\{ p^2 + M^2 - \frac{g^2}{8} \frac{S_+(p^2)}{(2\pi)^4 i} \right\}^{-1}$$

$$S_+ = S(1, 1 ; 1, 1 ; m_b , m_t)$$

Z - propagator $$\frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu}}{p^2+M_0^2} \rightarrow \frac{1}{(2\pi)^4 i} \delta_{\mu\nu} \left\{ p^2 + M_0^2 - \frac{g^2}{4c^2} \frac{S_0(p^2)}{(2\pi)^4 i} \right\}^{-1}$$

$$S_0 = \sum_{f=b,t} S(v_f, a_f ; v_f, a_f ; m_f , m_f)$$

$v_f = I_{3f} - 2 Q_f s^2$

$a_f = I_{3f}$

LO $\quad M_W = M , \quad M_Z = M_0 = \frac{M}{c} \Rightarrow \rho = \frac{M_W^2}{c^2 M_Z^2} = 1$

1-L $\quad M_W^2 = M^2 + \delta_+ , \quad \delta_+ = -\frac{g^2}{8} \frac{S_+(0)}{(2\pi)^4 i}$

$$M_Z^2 = \frac{M^2}{c^2} + \delta_0 \ , \quad \delta_0 = -\frac{g^2}{4c^2}\,\frac{S_0(0)}{(2\pi)^4 i}$$

$$\rho = \frac{M_W^2}{c^2 M_Z^2} = \frac{M^2+\delta_+}{M^2+c^2\delta_0} = \left(1+\frac{\delta_+}{M^2}\right)\left(1-c^2\frac{\delta_0}{M^2}\right) = 1 + \frac{\delta_+ - c^2\delta_0}{M^2} = 1+\Delta\rho$$

$$\Delta\rho = \delta_+ - c^2\delta_0 = -\frac{g^2}{8}\,\frac{S_+(0)}{(2\pi)^4 i M^2} + \frac{g^2}{4}\,\frac{S_0(0)}{(2\pi)^4 i M^2}$$

$$\Delta\rho = \frac{g^2}{(2\pi)^4 i M^2}\left\{\frac{1}{4}S_0(0) - \frac{1}{8}S_+(0)\right\}$$

$p=0$

$$S_{\mu\nu} = i\int d^4q\,\frac{1}{(q^2+m^2)(q^2+M^2)}\,\mathrm{tr}\{\dots\}$$

$$\mathrm{tr}\{\dots\} = \mathrm{tr}\left\{\gamma^\mu(a+b\gamma5)(-i\not{q}+m)\gamma^\nu(A+B\gamma5)(-i\not{q}+M)\right\}$$

$$= \mathrm{tr}\left\{(a-b\gamma^5)\gamma^\mu(-i\not{q}+m)(A-B\gamma^5)\gamma^\nu(-i\not{q}+M)\right\}$$

$$= \mathrm{tr}\left\{(a-b\gamma^5)(A-B\gamma^5)\gamma^\mu(-i\not{q})\gamma^\nu(-i\not{q}+M)\right\}$$

$$+\mathrm{tr}\left\{(a-b\gamma^5)(A+B\gamma^5)\,m\,\gamma^\mu\gamma^\nu(-i\not{q}+M)\right\}$$

$$= \mathrm{tr}\Big\{\left[aA+bB-(aB+bA)\gamma^5\right]\gamma^\mu(-i\not{q})\gamma^\nu(-i\not{q}+M)$$

$$+m\left[aA-bB+(aB-bA)\gamma^5\right]\gamma^\mu\gamma^\nu(-i\not{q}+M)\Big\}$$

$$= -4(aA+bB)(2q^\mu q^\nu - \delta^{\mu\nu}q^2) + 4mM(aA-bB)\delta^{\mu\nu}$$

1) $$i\int d^4q\,\frac{q^2\delta^{\mu\nu}-2q^\mu q^\nu}{(q^2+m^2)(q^2+M^2)} = i\pi^2\delta^{\mu\nu}\left\{-2B_{22}(0;m,M)\right\} + \delta^{\mu\nu}\,i\int d^4q\,\frac{q^2}{(\ )(\ )}$$

$$= \delta^{\mu\nu}\left\{-2i\pi^2 B_{22}(0;m,M) + \frac{1}{2}\int d^4q\,\frac{(1)+(2)-m^2-M^2}{(1)(2)}\right\}$$

$$= i\pi^2\delta^{\mu\nu}\left\{-2B_{22}(0;m,M) + \frac{1}{2}\left[A_0(m)+A_0(M)-(m^2+M^2)B_0(0;m,M)\right]\right\}$$

2) $$i\int d^4q\,\frac{1}{(1)(2)} = i\pi^2 B_0(0;m,M)$$

$$S = 4i\pi^2 (aA+bB)\Big\{-2B_{22}(0;m,M) + \frac{1}{2}A_0(m) + \frac{1}{2}A_0(M)$$

$$-\frac{1}{2}(m^2+M^2)B_0(0;m,M)\Big\}$$

$$+4i\pi^2(aA-bB)\,mM\,B_0(0;m,M)$$

$$\frac{S}{(2\pi)^4 i} = \frac{1}{4\pi^2}(aA+bB)\{\cdots\} + \frac{1}{4\pi^2}\,mM(aA-bB)\,B_0(0;m,M)$$

1) $$\frac{S_+}{(2\pi)^4 i} = \frac{1}{2\pi^2}\Big\{-2B_{22}(0;m_b,m_t) + \frac{1}{2}A_0(m_b) + \frac{1}{2}A_0(m_t)$$

$$-\frac{1}{2}(m_b^2+m_t^2)B_0(0;m_b,m_t)\Big\}$$

2) $$\frac{S_0}{(2\pi)^4 i} = \frac{1}{4\pi^2}\sum_{f=b,t}\Big\{(v_f^2+a_f^2)\Big[-2B_{22}(0;m_f,m_f) + A_0(m_f) - m_f^2 B_0(0;m_f,m_f)\Big]$$

$$+m_f^2(v_f^2-a_f^2)B_0(0;m_f,m_f)\Big\}$$

$$M_W^2\Delta\rho = -\frac{g^2}{16\pi^2}\Big\{-2B_{22}(0;m_b,m_t) + \frac{1}{2}A_0(m_b) + \frac{1}{2}A_0(m_t)$$

$$-\frac{1}{2}(m_b^2+m_t^2)B_0(0;m_b,m_t)\Big\}$$

$$+\frac{g^2}{16\pi^2}\sum_{f=b,t}\Big\{(v_f^2+a_f^2)\Big[-2B_{22}(0;m_f,m_f) + A_0(m_f) - m_f^2 B_0(0;m_f,m_f)\Big]$$

$$+m_f^2(v_f^2-a_f^2)B_0(0;m_f,m_f)\Big\}$$

UV-part $$A_0(m) \longrightarrow -\frac{m^2}{\bar{\varepsilon}}, \quad B_0 \longrightarrow \frac{1}{\bar{\varepsilon}}, \quad B_{22} \longrightarrow -\frac{1}{4}\frac{m^2+M^2}{\bar{\varepsilon}}$$

$$M_W^2\Delta\rho\Big|_{UV} = -\frac{g^2}{16\pi^2\bar{\varepsilon}}\Big\{\frac{1}{2}(\cancel{m_b^2+m_t^2}) - \frac{1}{2}m_b^2 - \frac{1}{2}m_t^2 - \frac{1}{2}(\cancel{m_b^2+m_t^2})\Big\}$$

$$+\frac{g^2}{16\pi^2\bar{\varepsilon}}\sum_{f=b,t}\Big[(v_f^2+a_f^2)(\cancel{m_f^2} - \cancel{m_f^2} - m_f^2) + (v_f^2-a_f^2)\,m_f^2\Big\}$$

$$M_W^2\,\Delta\rho\Big|_{UV} = -\frac{g^2}{16\pi^2\bar{\varepsilon}}\left\{-\frac{1}{2}(m_b^2+m_t^2) + 2\sum_{f=b,t} m_t^2\,\underset{\overset{\downarrow}{1/4}}{a_t^2}\right\} = 0$$

$$A_0(m) = -m^2\left(\frac{1}{\bar{\varepsilon}} - \ln m^2 + 1\right)$$

$$B_0(0;m,m) = \frac{1}{\bar{\varepsilon}} - \ln m^2$$

$$B_0(0;m,M) = \frac{1}{\bar{\varepsilon}} - \int_0^1 dx\,\ln\left\{(M^2-m^2)x + m^2\right\}$$

$$= \frac{1}{\bar{\varepsilon}} - \left\{\frac{(M^2-m^2)x+m^2}{M^2-m^2}\ln\left[(M^2-m^2)x+m^2\right] - x\right\}\Big|_0^1$$

$$= \frac{1}{\bar{\varepsilon}} - \frac{M^2}{M^2-m^2}\ln M^2 + \frac{m^2}{M^2-m^2}\ln m^2 + 1$$

$$B_{22}(0;m,m) = -\frac{1}{2}m^2\left(\frac{1}{\bar{\varepsilon}} - \ln m^2 + 1\right)$$

$$B_{22}(0;m,M) = -\left(\frac{1}{4\bar{\varepsilon}} + \frac{3}{8}\right)(m^2+M^2) + \frac{1}{4}\frac{1}{M^2-m^2}\left(M^4\ln M^2 - m^4\ln m^2\right)$$

$m_t \to \infty$ $(m_t >> m_b)$, $L_t = \ln m_t^2$

$A_0(m_t) \sim m_t^2(L_t-1)$, $B_0(0;m_t,m_t) \sim -L_t$

$B_0(0;m_b,m_t) \sim 1 - L_t$

$B_{22}(0;m_t,m_t) \sim \frac{1}{2}m_t^2(L_t-1)$, $B_{22}(0;m_b,m_t) \sim -\frac{3}{8}m_t^2 + \frac{1}{4}m_t^2 L_t$

$$\Delta\rho \sim -\frac{g^2}{16\pi^2}\frac{m_t^2}{M_W^2}\left\{\frac{3}{4} - \frac{1}{2}\cancel{L_t} + \frac{1}{2}(\cancel{L_t}-1) - \frac{1}{2}(1-L_t)\right.$$

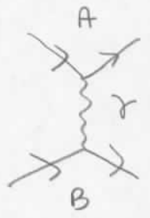
$$\left. - \left(v_t^2+\frac{1}{4}\right)\left[-(\cancel{L_t-1}) + (\cancel{L_t-1}) + L_t\right] + \left(v_t^2-\frac{1}{4}\right)L_t\right\}$$

$$\Delta\rho \sim -\frac{g^2}{16\pi^2}\frac{m_t^2}{M_W^2}\left\{\frac{3}{4} - 1 + \frac{1}{2}L_t - \left(\cancel{v_t^2}+\frac{1}{4}\right)L_t + \left(\cancel{v_t^2}-\frac{1}{4}\right)L_t\right.$$

$$\Delta\rho \sim \frac{g^2}{64\pi^2}\frac{m_t^2}{M_W^2} \qquad (\text{remember } N_f = 3)$$

$$\Longrightarrow \Delta\rho \sim \frac{3g^2}{64\pi^2}\frac{m_t^2}{M_W^2}$$

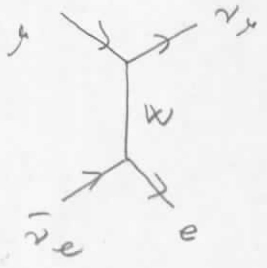
bare parameters g, M, s (+ Yukawa) LO



$$(2\pi)^4 i\,(igs)^2\,\frac{Q_A Q_B}{p^2}\,\gamma^\mu \otimes \gamma^\mu \Longrightarrow g^2 s^2 = 4\pi\alpha$$

$$g^2 s^2 = e^2 \; (= 4\pi\alpha)$$

$$\frac{M^2}{c^2} = M_Z^2$$



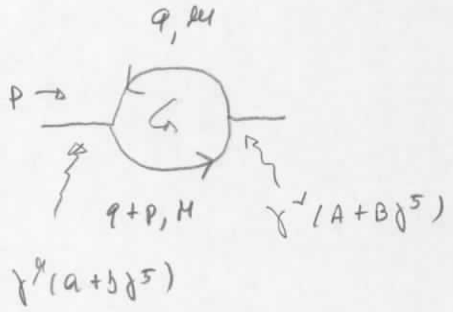
$$\tau_\mu^{-1} = G_F\frac{m_\mu^5}{96\pi^3} \Longrightarrow \frac{g^2}{M^2} = 4\sqrt{2}\,G_F$$

$$M_0 = \frac{M}{C} = M_Z$$

$$\left.\begin{aligned} g^2 s^2 &= 4\pi\alpha \\ \frac{g^2}{M^2} &= 4\sqrt{2}\,G_F \\ \frac{M}{c} &= M_Z \end{aligned}\right\} \quad \frac{g^2}{c^2} = 4\sqrt{2}\,G_F M_Z^2, \quad \frac{c^2}{g^2} = \frac{1}{4\sqrt{2}\,G_F M_Z^2}$$

$$c^2 s^2 = \frac{\pi\alpha}{\sqrt{2}\,G_F M_Z^2} = k \qquad s^2(1-s^2) = k$$

$$s^2 = \frac{1}{2}\left\{1 - \sqrt{1-4k}\right\}$$



$$S^{\mu\nu} = i\int d^4q\,\frac{1}{(q^2+m^2)((q+p)^2+M^2)}\,T^{\mu\nu}$$

$$T^{\mu\nu} = \mathrm{tr}\left\{\gamma^\mu(a+b\gamma^5)(-i\not{q}+m)\gamma^\nu(A+B\gamma^5) \times\left[-i(\not{q}+\not{p})+M\right]\right\}$$

$$S^{\mu\nu} = (2\pi)^4 i\,\frac{i\pi^2}{16i\pi^4}\left(S\delta^{\mu\nu} + R\,p^\mu p^\nu\right) = (2\pi)^4 i\,\frac{1}{16\pi^2}\left(S\delta^{\mu\nu} + R\,p^\mu p^\nu\right)$$

$$T^{\mu\nu} = -\,\mathrm{tr}\left\{\left[aA+bB-(aB+bA)\gamma^5\right]\gamma^\mu \not{q}\,\gamma^\nu(\not{q}+\not{p})\right.$$

$$\left.+\,mH\,\mathrm{tr}\left\{\left[aA-bB+(aB-bA)\gamma^5\right]\gamma^\mu\gamma^\nu\right\}\right.$$

$$= -4\,(aA+bB)\left\{q^\mu(q+p)^\nu - \delta^{\mu\nu}q\cdot(q+p) + q^\nu(q+p)^\mu\right\}$$

$$+\,4\,mH\,(aA-bB)\,\delta^{\mu\nu}$$

$$2q^\mu q^\nu \longrightarrow 2\left(B_{21}\,p^\mu p^\nu + B_{22}\,\delta^{\mu\nu}\right)$$

$$q^\mu p^\nu + q^\nu p^\mu \longrightarrow 2B_1\,p^\mu p^\nu$$

$$-\,q\cdot(q+p)\,\delta^{\mu\nu} \longrightarrow -\,\delta^{\mu\nu}\left\{n B_{22} + p^2 B_{21}\right\}$$

$$S = -4\,(aA+bB)\left\{(2-n)B_{22} - p^2B_{21} - p^2B_1\right\}$$

$$+\,4\,mH\,(aA-bB)\,B_0 \qquad \searrow\; B_i(p^2;m,H)$$

if $m = S = 4(aA+bB)\left\{(n-2)B_{22} + p^2(B_{21}+B_1)\right\} + 4mH(aA-bB)\,B_0$

if $m=H$ $\qquad S = 4\,(aA+bB)\,(2B_{21}-B_0)\,p^2 - 8m^2\,bB\,B_0$

Introduco $\quad \mathcal{B}(p^2;m,m) = 2B_{21}(p^2;m,m) - B_0(p^2;m,m)$

$$S = 4(aA+bB)\,\mathcal{B}(p^2;m,m)\,p^2 - 8m^2\,bB\,B_0(p^2;m,m)$$

1) $\quad S^f_{AA} = (2\pi)^4 i\,[-1]\,(igs)^2\,Q_f^2\,N_f^c \times 4\,\mathcal{B}_f\,p^2 \times \dfrac{1}{16\pi^2}$

$$= (2\pi)^4 i\,\frac{g^2s^2}{16\pi^2}\,4\,Q_f^2\,N_f^c\,\mathcal{B}_f\,p^2 \;=\; (2\pi)^4 i\,\frac{g^2s^2}{16\pi^2}\,\Pi^f_{AA}\,p^2$$

$$\Pi^f_{AA} = 4\,Q_f^2\,N_f^c\,\mathcal{B}_f$$

$$S^f_{ZZ} = +(2\pi)^4 i\,[-1]\left(\frac{ig}{2c}\right)^2 \frac{1}{16\pi^2}\, 4N^c_f \left\{ (v_f^2 + \frac{1}{4}) B_f p^2 - \frac{1}{2} \mu_f^2 B_0 \right\}$$

$$= (2\pi)^4 i\, \frac{g^2}{16\pi^2 c^2} N^c_f \left\{ (\frac{1}{2} - 4Q_f I_{3f} s^2 + 4Q_f^2 s^4) B_f p^2 - \frac{1}{2}\mu_f^2 B_0 \right\}$$

$$v_f = I_{3f} - 2Q_f s^2$$

$$S^f_{ZZ} = (2\pi)^4 i\, \frac{g^2}{16\pi^2 c^2} \Sigma^f_{ZZ}\,, \quad \Sigma^f_{ZZ} = N^c_f \left\{ \frac{1}{2} B_f p^2 - \frac{1}{2} B_0 \mu_f^2 - 4 Q_f I_{3f} B_f p^2 s^2 \right.$$
$$\left. + 4Q_f^2 B_f p^2 s^4 \right\}$$

$$\Sigma^f_{ZZ} = \Sigma^f_{33} - 2s^2 \Sigma^f_{3A} + s^4 \Pi^f_{AA} p^2$$

$$\Sigma^f_{33} = \frac{1}{2} N^c_f (B_f p^2 - B_0 \mu_f^2)\,, \quad \Sigma^f_{3A} = 2N^c_f Q_f I_{3f} B_f p^2 = \Pi^f_{3A} p^2$$

$$S^f_{ZA} = (2\pi)^4 i\,[-]\left(\frac{ig}{2c}\right)(igsQ_f)\frac{1}{16\pi^2}\, 4v_f B_f p^2 N^c_f$$

$$= (2\pi)^4 i\, \frac{g^2 s}{16\pi^2 c} N^c_f Q_f 2B_f p^2 (I_{3f} - 2Q_f s^2)$$

$$S^f_{ZA} = (2\pi)^4 i\, \frac{g^2 s}{16\pi^2 c} \Sigma^f_{ZA} \qquad \Sigma^f_{ZA} = N^c_f 2B_f p^2 (Q_f I_{3f} - 2Q_f^2 s^2)$$

$$= \Sigma^f_{3A} - \Pi^f_{AA} p^2 = (\Pi^f_{3A} - \Pi^f_{AA}) p^2$$

$$(2\pi)^4 i\, \frac{Q_A Q_B}{p^2} (igs)^2 \left\{ 1 - \frac{g^2 s^2}{16\pi^2} \Pi_{\gamma\gamma}(p^2) \right\}^{-1} \gamma^\mu \otimes \gamma^\mu$$

$$\Pi_{\gamma\gamma} = \sum_f \Pi^f_{\gamma\gamma}$$

$$4\pi\alpha = g^2 s^2 \left\{ 1 - \frac{g^2 s^2}{16\pi^2} \Pi_{AA}(0) \right\}^{-1}$$

$$= \left\{ \frac{1}{g^2 s^2} - \frac{1}{16\pi^2} \Pi_{AA}(0) \right\}^{-1} , \qquad \frac{1}{g^2 s^2} - \frac{1}{16\pi^2} \Pi_{AA}(0) = \frac{1}{4\pi\alpha}$$

1) $$\frac{1}{g^2 s^2} = \frac{1}{4\pi\alpha} + \frac{1}{16\pi^2} \Pi_{AA}(0)$$

W) $$S^d_+ = \left\{ (B_{21} + B_1) p^2 + 2 (m_u^2 - m_d^2) B_1 - 2 m_d^2 B_0 \right\}$$

$$S^d_{WW} = (2\pi)^4 i \left(\frac{ig}{2\sqrt{2}} \right)^2 [-1] \frac{1}{16\pi^2} 4 S^d_+ = (2\pi)^4 i \frac{g^2}{16\pi^2} \Sigma^d_{WW}$$

$$\Sigma^d_{WW} = \frac{1}{2} S^d_+ , \qquad \Sigma_{WW} = \sum_d \Sigma^d_{WW}$$

$$\bar{\Delta}_{WW} = \frac{\delta_{\mu\nu}}{(2\pi)^4 i} \left\{ p^2 + M^2 - \frac{1}{(2\pi)^4 i} S_{WW} \right\}^{-1}$$

$$4\sqrt{2} G_F = \frac{g^2}{M^2} \left\{ 1 - \frac{g^2}{16\pi^2 M^2} \Sigma_{WW}(0) \right\}^{-1}$$

$$\frac{1}{\sqrt{2}} G_F = \frac{g^2}{8M^2} \left\{ 1 - \frac{g^2}{8M^2} \frac{\Sigma_{WW}(0)}{2\pi^2} \right\}^{-1} = \left\{ \frac{8M^2}{g^2} - \frac{1}{2\pi^2} \Sigma_{WW}(0) \right\}^{-1}$$

2) $$8 \frac{M^2}{g^2} = \frac{\sqrt{2}}{G} + \frac{1}{2\pi^2} \Sigma_{WW}(0)$$

3) $$M_Z^2 = \frac{M^2}{c^2} - \frac{g^2}{16\pi^2 c^2} \operatorname{Re} \Sigma_{ZZ}(-M_Z^2)$$

Remark on Dyson re-summation

$Z = 1 \quad , \quad A = 2$

$$\Delta_{11} = \frac{1}{(2\pi)^4 i} \frac{1}{p^2 + M_0^2} \quad , \quad \Delta_{22} = \frac{1}{(2\pi)^4 i} \frac{1}{p^2} \quad , \quad M_0 = \frac{M}{c}$$

$$\bar{\Delta}_{ij} = \Delta_{ij} + \Delta_{il} S_{lk} \Delta_{kj} + \cdots = \Delta_{il} \{ \delta_{lj} + S_{lk} \Delta_{kj} + \cdots \}$$

$$X_{lj} = \delta_{lj} + S_{lk} \Delta_{kj}$$

$$\bar{\Delta}_{ij} = \Delta_{il} (1 - S\Delta)^{-1}_{lj} \quad , \quad 1 - S\Delta = \begin{pmatrix} 1 - S_{11}\Delta_{11} & -S_{12}\Delta_{22} \\ -S_{21}\Delta_{11} & 1 - S_{22}\Delta_{22} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad S_{21} = S_{12}$$

$$(1 - S\Delta)^{-1} = \{ (1 - S_{11}\Delta_{11})(1 - S_{22}\Delta_{22}) - S_{12}^2 \Delta_{11}\Delta_{22} \}^{-1}$$

$$\times \begin{pmatrix} 1 - S_{22}\Delta_{22} & S_{12}\Delta_{22} \\ S_{12}\Delta_{11} & 1 - S_{11}\Delta_{11} \end{pmatrix}$$

$$\bar{\Delta}_{11} = \Delta_{1l} (1 - S\Delta)^{-1}_{l1} = \Delta_{11} (1 - S\Delta)^{-1}_{11}$$

$$= \Delta_{11} \{ (1 - S_{11}\Delta_{11})(1 - S_{22}\Delta_{22}) - S_{12}^2 \Delta_{11}\Delta_{22} \}^{-1} (1 - S_{22}\Delta_{22})$$

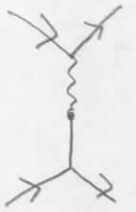
$$= \Delta_{11} \{ 1 - S_{11}\Delta_{11} - S_{12}^2 \frac{\Delta_{11}\Delta_{22}}{1 - S_{22}\Delta_{22}} \}^{-1} = \{ \Delta_{11}^{-1} - S_{11} - \frac{S_{12}^2}{\Delta_{22}^{-1} - S_{22}} \}^{-1}$$

$$\Rightarrow \bar{\Delta}_{ZZ} = \frac{1}{(2\pi)^4 i} \{ p^2 + M_0^2 - S_{ZZ} - \frac{S_{Z\gamma}^2}{p^2 - S_{\gamma\gamma}} \}^{-1} \overset{app.}{\Longrightarrow} \frac{1}{(2\pi)^4 i} \{ p^2 + M_0^2 - S_{ZZ} \}^{-1}$$

$$\bar{\Delta}_{AA} \qquad \cdots \qquad " \qquad \frac{1}{(2\pi)^4 i} \frac{1}{p^2 - S_{AA}}$$

$$\bar{\Delta}_{ZA} = \bar{\Delta}_{AZ} \Longrightarrow \frac{1}{(2\pi)^4 i} \frac{S_{ZA}}{p^2 (p^2 + M_0^2 - S_{ZZ})}$$

N.B. $\sigma^f_{ZA}(0) = 0$



no contribution to renormalization at $p^2 = 0$

1) $$g^2 s^2 = 4\pi\alpha \left\{ 1 + \frac{\alpha}{4\pi} \Pi_{AA}(0) \right\}^{-1}$$

2) $$8 \frac{M^2}{g^2} = \frac{\sqrt{2}}{G_F} + \frac{1}{2\pi^2} \Sigma_{WW}(0)$$

3) $$\frac{M^2}{c^2} = M_Z^2 + \frac{g^2}{16\pi^2 c^2} \mathrm{Re} \left\{ \Sigma_{33} + 2s^2 \Pi_{3A} M_Z^2 - s^4 \Pi_{AA} M_Z^2 \right\}$$

$$\frac{3}{2} \Rightarrow \quad \frac{g^2}{8c^2} \left\{ \frac{\sqrt{2}}{G_F} + \frac{1}{2\pi^2} \Sigma_{WW}(0) \right\} = M_Z^2 + \frac{g^2}{16\pi^2 c^2} \mathrm{Re}\, \Sigma_{ZZ}$$

$$\frac{1}{8} \left(\frac{\sqrt{2}}{G} + \frac{1}{2\pi^2} \Sigma_{WW} \right) \frac{g^2}{c^2} = M_Z^2 + \frac{1}{16\pi^2} \mathrm{Re}\, \Sigma_{ZZ} \frac{g^2}{c^2}$$

$$M_Z^2 \frac{c^2}{g^2} = \frac{1}{4\sqrt{2} G_F} + \frac{1}{16\pi^2} (\Sigma_{WW} - \mathrm{Re}\, \Sigma_{ZZ})$$

$$\times 1 \Rightarrow \quad s^2 c^2 = \frac{4\pi\alpha}{1 + \frac{\alpha}{4\pi} \Pi_{AA}(0)} \frac{1}{M_Z^2} \left\{ \frac{1}{4\sqrt{2} G_F} + \frac{1}{16\pi^2} (\Sigma_{WW} - \mathrm{Re}\, \Sigma_{ZZ}) \right\}$$

neglecting corrections

$$s^2 c^2 = \frac{\pi\alpha}{\sqrt{2} G_F M_Z^2} = \hat{s}^2 \hat{c}^2 \quad \Rightarrow \quad \frac{1}{G_F M_Z^2} = \sqrt{2} \frac{\hat{s}^2 \hat{c}^2}{\pi\alpha}$$

$$\Rightarrow \quad s^2 c^2 = \frac{4\pi\alpha}{1 + \frac{\alpha}{4\pi} \Pi_{AA}} \left\{ \frac{1}{4\cancel{\sqrt{2}}} \cancel{\sqrt{2}} \frac{\hat{s}^2 \hat{c}^2}{\pi\alpha} + \frac{1}{16\pi^2} (\Sigma_{WW} - \mathrm{Re}\, \Sigma_{ZZ}) \right\}$$

expand $s^2 = \hat{s}^2 + \frac{\alpha}{4\pi} s_1 + O(\alpha^2)$

$$s^2 c^2 = s^2 - s^4 = \hat{s}^2 \hat{c}^2 + \frac{\alpha}{4\pi} (\hat{c}^2 - \hat{s}^2) s_1$$

derive s_1, etc.