ON THE FIELD THEORY LIMIT OF MULTI-LOOP STRING AMPLITUDES

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Outline

- On string amplitudes
- One- and two-loop examples
- Effective actions in background fields
- Picking diagrams from strings
- Results and outlook

Work in collaboration with R. Russo (QMUL) and S. Sciuto (Torino)

ON STRING AMPLITUDES

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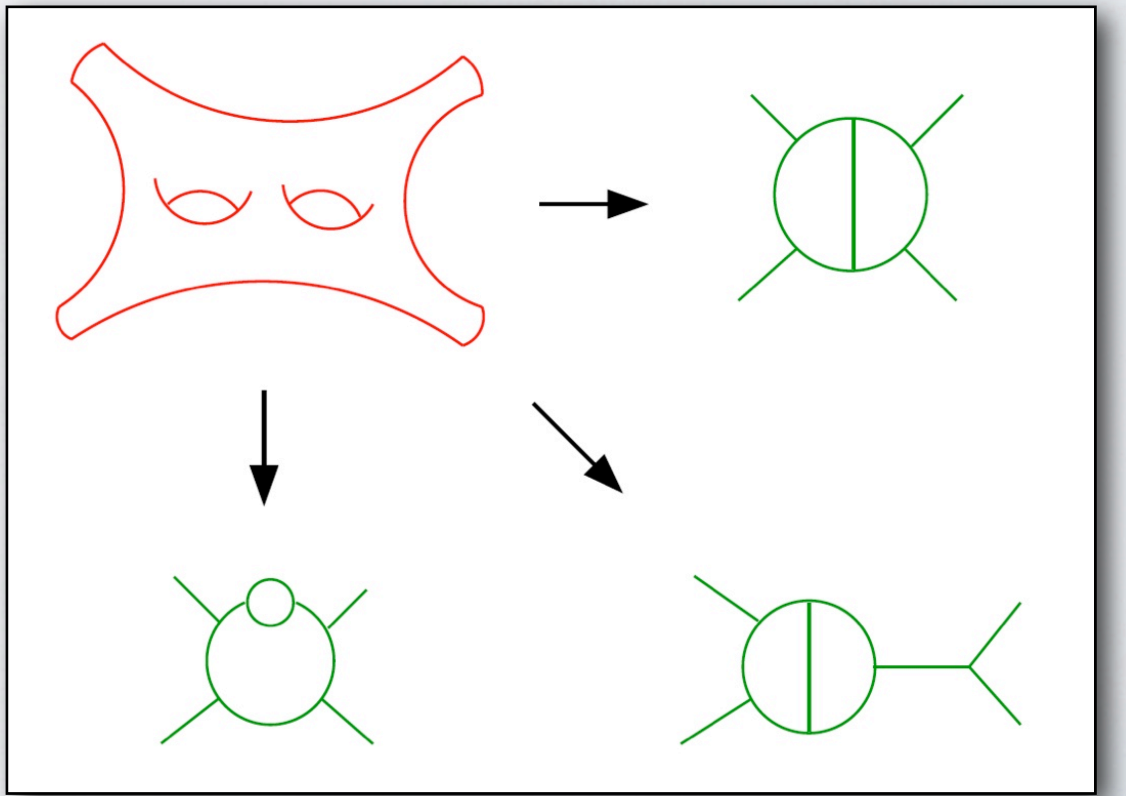
``Strings is a mythological story about the son of a king ... ''

Features of string amplitudes

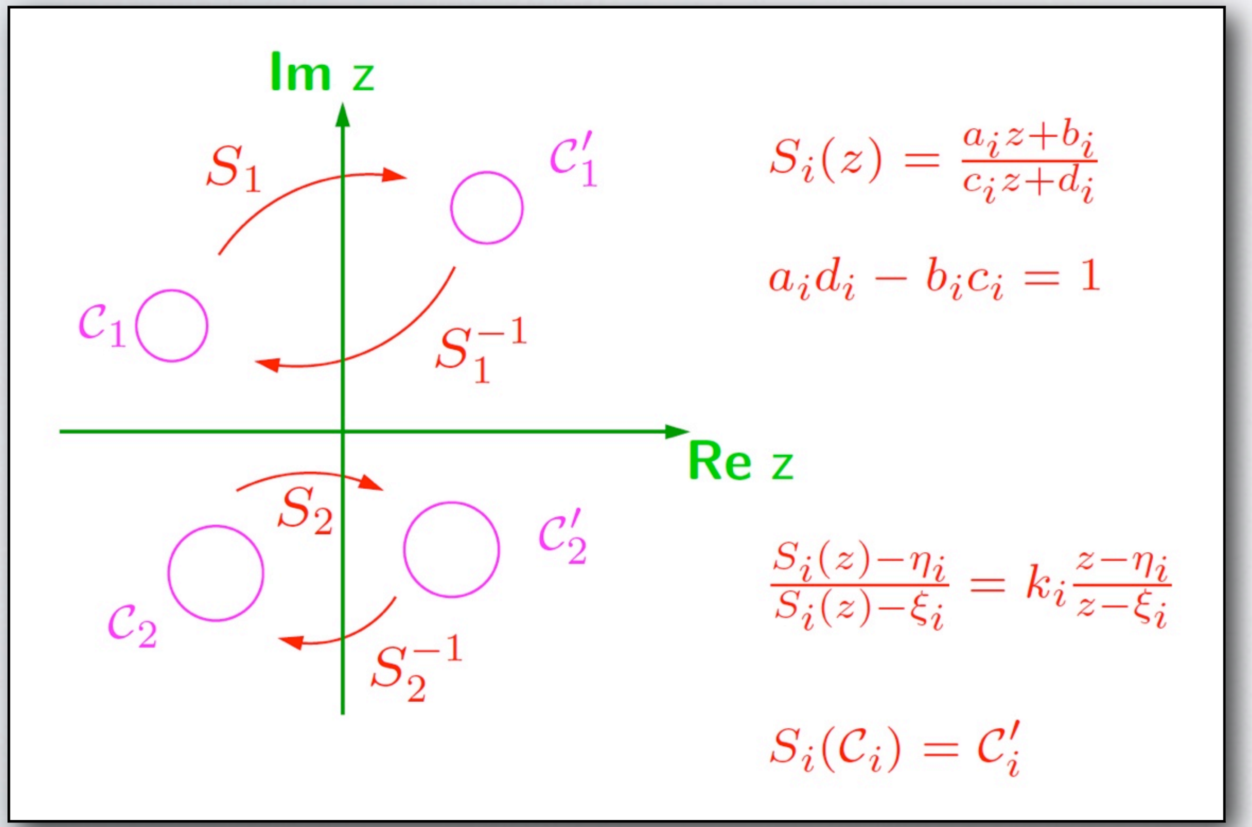
- String theory expresses on-shell scattering amplitudes of a d-dimensional interacting field theory in terms of correlation functions of operators in a two-dimensional free field theory.
  - Master formulas exist for n-point g-loop amplitudes.
- String theory is first-quantized: the number of string loops is fixed at the outset.
  - Computations are performed in a $d = 2$ field theory on a Riemann surface of fixed genus g.
- String theory has an infinite number of massive states.
  - Masses are multiples of the string tension $M_n^2 \propto n/\alpha' \propto n T$.
  - Tuning the limit $\alpha' \to 0$ for different strings one may get different effective field theories.
- In the field theory limit $\alpha' \to 0$ Riemann surfaces degenerate into Feynman-like graphs
  - Only massless (or lowest-lying) excitations circulate in the loops.
  - The g-loop string diagram generates all field theory diagrams from different corners of moduli space.
  - Is it practical?



Different limit of a single string diagram

The Schottky parametrization

- A Riemann surface of genus g can be represented by cutting and identifying g pairs of circles on the Riemann sphere, via projective transformations S_i.
  - The Riemann surface is then $\Sigma_g = (\mathbf{C} \cup \infty) / \mathcal{S}_g$ where $\mathcal{S}_g$ is the genus-g Schottky group generated by the transformations S_i.
- Each projective transformation has three complex parameters, chosen as its fixed points η_i and ξ_i, and the multiplier k_i.
  - The multipliers k_i are proportional to the radii of the circles C_i and they drive the field theory limit as $k_i \to 0$.



Building a two-loop Riemann surface in the Schottky representation

- The shape of the genus-g Riemann surface is determined by 3g - 3 complex moduli (subtracting one overall projective trasformation on the sphere)
  - The number of moduli matches the number of propagators for g-loop vacuum bubbles with cubic vertices.
  - Geometric objects on the Riemann surface can be expressed as series over the Schottky group.

Geometric objects

The string operator formalism provides explicit constructions for geometric objects defined on Riemann surfaces, in terms of series on the Schottky group.

Let $T_\alpha = S_i^a \cdot S_j^b \cdot \ldots$ be an element of the Schottky group. Then one defines

Abelian Differentials:
$$\omega_\mu = \sum_\alpha^{(\mu)} \left(\frac{1}{z - T_\alpha(\eta_\mu)} - \frac{1}{z - T_\alpha(\xi_\mu)} \right) dz$$

Period Matrix:
$$\tau_{\mu\nu} = \frac{1}{2\pi \mathrm{i}} \int_{b_\nu} \omega_\mu(z)$$

Prime Form:
$$E_g(z,w)\sqrt{dz dw} = (z-w) \hat{\prod}_\alpha \frac{z - T_\alpha(w)}{z - T_\alpha(z)} \frac{w - T_\alpha(z)}{w - T_\alpha(w)}$$

Scalar Propagator:
$$\mathcal{G}_g(z,w) = \log\left[E_g(z,w)\right] - \frac{1}{2} \int_z^w \omega_\mu \left[(2\pi\, \mathbf{Im}\tau)^{-1} \right]^{\mu\nu} \int_z^w \omega_\nu$$

- In the field theory limit only a handful of Schottky group elements contribute.
- Relevant terms are easily generated with available software for symbolic manipulations.

Master formulas

String amplitudes are computed by fixing the quantum numbers of the external states and then evaluating correlation functions of the corresponding vertex operators in the two-dimensional theory at the relevant genus.

Since the 2-*d* theory is free, closed form results can be obtained for g-loop, *M*-point amplitudes. For example for open string gluon states one finds

$$A^{(\mathrm{g})}_{M,1}\left(\epsilon_1, p_1; \ldots; \epsilon_N, p_N\right) = C_{\mathrm{g}} \mathcal{N}^M \int [dm]^M_{\mathrm{g}} \prod_{i<j} \exp\left[2\alpha' p_i \cdot p_j G_{\mathrm{g}}(z_i, z_j)\right]$$

$$\times \left\{ \exp\left[\sum_{i \neq j} \sqrt{2\alpha'}\, \epsilon_i \cdot p_j\, \partial_{z_i} G_{\mathrm{g}}(z_i, z_j) + \frac{1}{2} \sum_{i \neq j} \epsilon_i \cdot \epsilon_j\, \partial_{z_i} \partial_{z_j} G_{\mathrm{g}}(z_i, z_j)\right]\right\}_{\mathtt{m.l.}}$$

Integration is over a fundamental region of the g-loop moduli space. Normalization can be computed in terms of the string slope and coupling, and the space-time dimension *d*.

$$C_{\mathrm{g}} = (2\pi)^{-d\,\mathrm{g}} \, (2\alpha')^{-d/2} \, g_s^{2\mathrm{g}-2} \qquad ; \qquad \mathcal{N} = 2\, g_s \, (2\alpha')^{d/4-1/2}$$

The conformal properties of the scalar propagator can be fixed by a choice of local coordinates $V_i(z)$ around the punctures: one writes

$$G_{\mathrm{g}}(z_i, z_j) = \mathcal{G}_g(z_i, z_j) - \frac{1}{2}\log V_i'(0) - \frac{1}{2}\log V_j'(0)$$

where $V'_i(0)$ is required to have conformal weight $w = -1$

Pluses and minuses

- ***Remarkably*** ...
  - Such formulas exist: no such results in field theory.
  - Quantum numbers are well-managed:
    - ✳ color decomposition is already performed via Chan-Paton factors;
    - ✳ loop momentum integration is already performed, so that helicity methods are immediately applicable.
  - Limited off-shell continuation is possible: the gauge chosen by string theory can be identified.
  - While the full perturbative string amplitude is not completely well-defined, the field theory limit is algorithmically implementable.

- ***However*** ...
  - Only a limited set of quantum field theories can be reached: scalars, massless gauge theories, gravity, unbroken SUSY.
    - ✳ Non-supersymmetric fermions are difficult to include.
    - ✳ Theories with several mass scales (SM ...) cannot be handled
  - The problem is reduced to the computation of `scalar integrals with numerators': the method is not competitive with generalized unitarity in terms of speed.
  - It is however still interesting within string theory and for its own sake:
    - ✳ non-perturbative applications via D-branes;
    - ✳ new structures at high loop order;
    - ✳ dualities ...

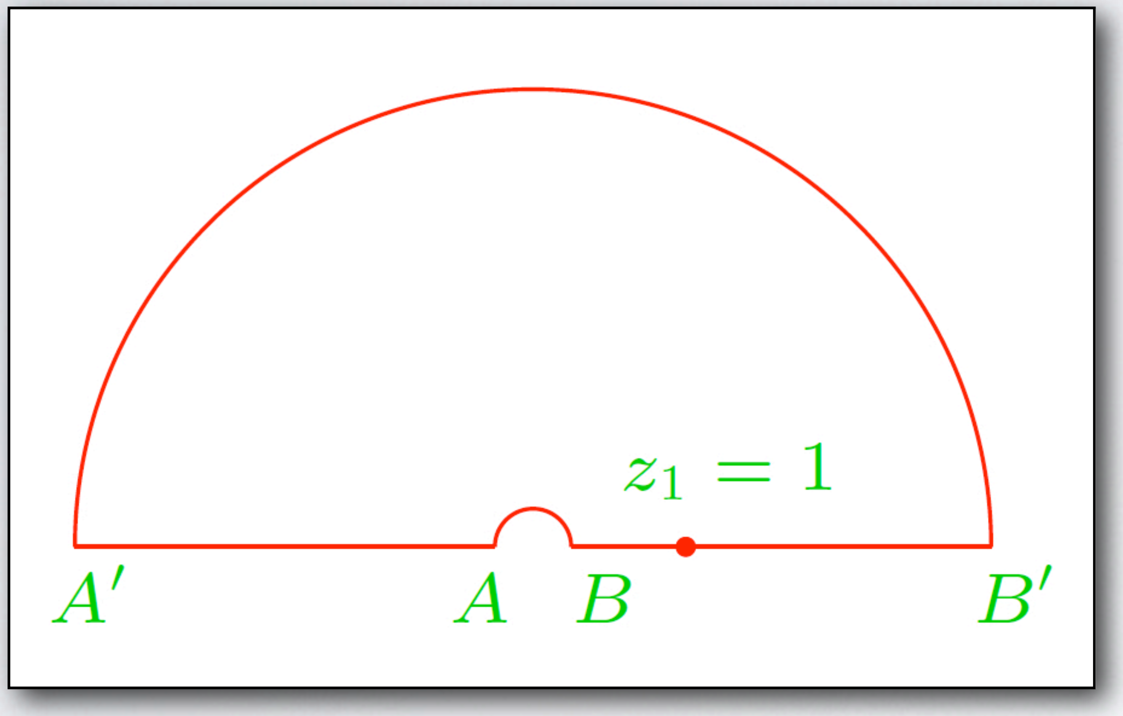
ONE-LOOP GLUON AMPLITUDES

The annulus

At one loop the Schottky group has only one generator: series and products over the group become ordinary Taylor series and products over integers.

One may standardize the representation of the annulus by fixing the projective gauge as

$$\eta = 0 \quad ; \quad \xi = \infty \quad ; \quad z_1 = 1$$



The cut annulus in the Schottky representation

The Schottky generator is then simply

$$S(z) = k\, z$$

which implies

$$B = -A = \sqrt{k}$$

$$B' = -A' = -1/\sqrt{k}$$

External states are cyclically ordered along either one of the boundaries, *AA*' or *BB*'.

The integration region is determined by symmetry and modular invariance:

- The transformation $k \to 1/k$ does not affect the geometry
- One may map $1 < z_i < 1/\sqrt{k} \longrightarrow k < z_i < \sqrt{k}$

Thus one may simply use

$$0 < k < z_i < 1$$

One-loop master formula

The ingredients of the one-loop master formula for gluons are easily determined

Measure of integration

$$[dm]_1^M \;=\; \frac{1}{k^2} \prod_{n=1}^{\infty} (1-k^n)^{2-d} \left(-\frac{\log k}{2}\right)^{-\frac{d}{2}} \prod_{i=2}^{M} dz_i \, \Theta\left(z_i - z_{i+1}\right)$$

Matching the string and the strong coupling, from tree level

$$g_s \;=\; \frac{1}{2}\, g_d \,(2\alpha')^{1-d/4}$$

The scalar propagator

$$G(z_i, z_j) \;=\; \log\left(\left|\sqrt{\frac{z_i}{z_j}} - \sqrt{\frac{z_j}{z_i}}\right|\right) + \frac{1}{2\log k}\left(\log\frac{z_i}{z_j}\right)^2 + \log\left[\prod_{n=1}^{\infty} \frac{\left(1-k^n\frac{z_j}{z_i}\right)\left(1-k^n\frac{z_i}{z_j}\right)}{(1-k^n)^2}\right]$$

where the choice

$$V_i'(0) = (\omega(z_i))^{-1} = z_i$$

insures modular invariance

$$G(z_i/z_j;k) \;=\; G(z_j/z_i;k) \;=\; G(kz_i/z_j;k)$$

The field theory limit

- From the string operator formalism we know that Laurent expansion of the integrand in powers of k counts the mass level of the state propagating in the loop.

$$k^{-2} \longrightarrow \text{tachyon} \; ; \quad k^{-1} \longrightarrow \text{gluon} \; ; \; \ldots$$

- The master formula has an overall power of α'. String moduli defining the shape of the surface must be expressed in units of α', in order to take the limit $\alpha' \to 0$
  - Hint: measure of integration is $d \log k$ ...

- Pedestrian field theory limit (exact for scalars):

$$\log k = -\frac{t}{\alpha'} \qquad ; \qquad \log z_i = -\frac{t_i}{\alpha'}$$

  - Note: t and t_i will be identified with with sums of Schwinger parameters associated with propagators around the loop.

- For gluons, the overall power p of α' after the change of variables is not uniform: instead, $-M/2 < p < 0$. One must locate all further sources of positive powers of α'.
  - Four-point vertices $\quad (t_i - t_{i-1})/\alpha' = \mathcal{O}(1)$
  - Expansion of the exponential $\quad \exp\left[2\alpha' p_i \cdot p_j \, G_{ij}\right] \longrightarrow \exp\left[c_0(t_i) + \alpha' c_1(t_i)\right]$

Some results

- One-loop diagrammatics is fully developed.
  - À la Bern-Kosower (no quartic vertex topology).
  - Direct field theory limit distinguishes cubic and quartic vertices, irreducible and reducible topologies.

- Off-shell continuation, with identification of individual topologies, establishes the gauge choice naturally performed by string theory
  - For irreducible topologies: Background Field Feynman gauge
  - For reducible topologies tree subdiagrams are computed in the Gervais-Neveu non-linear gauge

$$S_{GN} = \int d^d x \left\{ -\frac{1}{4} \mathrm{Tr}\left(F^2\right) - \frac{1}{2} \mathrm{Tr}\left[\left(\partial \cdot A - \mathrm{i} g_d A^2\right)^2\right] \right\}$$

- Bosonic string theory is well-defined only in the critical dimension $d = 26$. This is a bonus in the field theory limit: amplitudes have the correct d dependence (dimensional regularization à la 't Hooft-Veltman). We understand this in a D-brane picture.

- Bosonic string theory has a tachyon. It can be decoupled by hand. Tachyons in loops have IR divergences not regulated dimensionally. Tachyon effects remain as contact interactions. Tachyon amplitudes can be used to compute scalar amplitudes in field theory by the replacement

$$dx/x^2 \longrightarrow \left[\exp(\alpha'\, m^2\, \log x)\right] dx/x$$

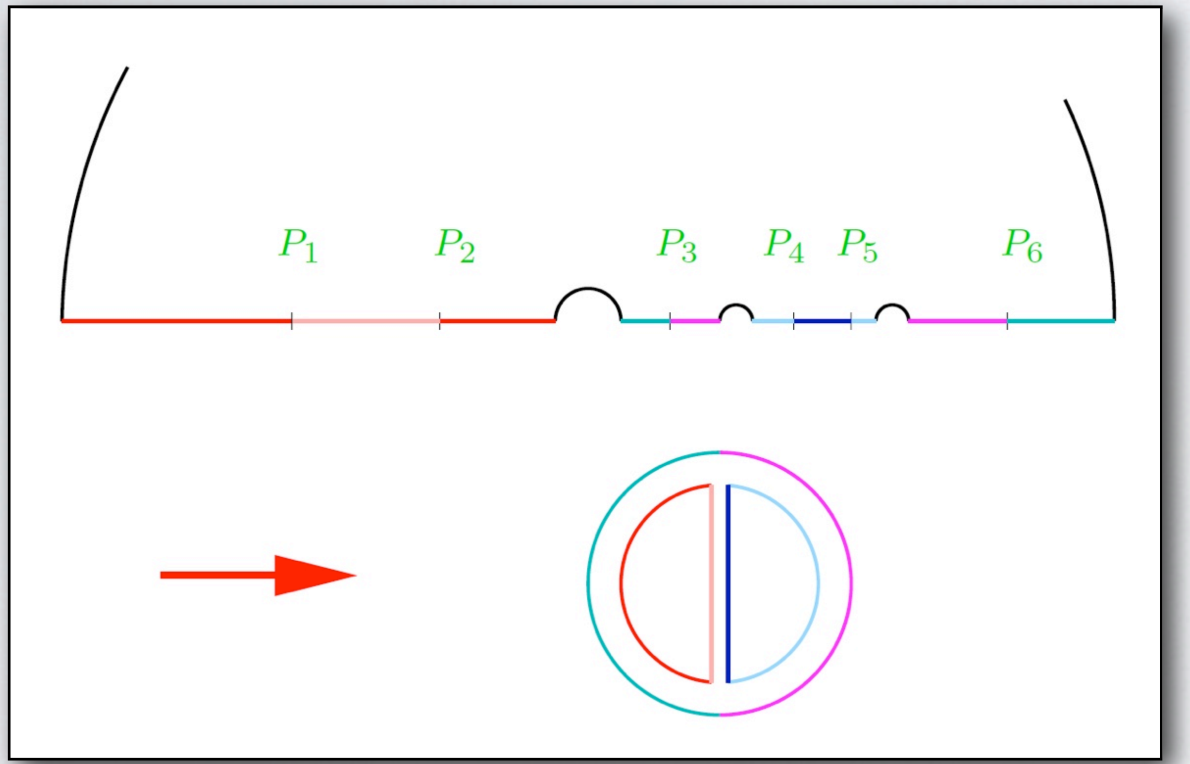
TWO-LOOP AMPLITUDES

The double annulus

At two loops the Schottky group has two generators: however expanding in powers of the multipliers remains simple since $S_i(S_i(z))$ contributes to order k_i^2.

One may standardize the representation of the double annulus by choosing the gauge as

$$\eta_1 = 0 \quad ; \quad \xi_1 = \infty \quad ; \quad \xi_2 = 1$$



The cut double annulus in the Schottky representation

The Schottky generators are then

$$S_1(z) = k_1 z$$

and

$$S_2(z) = \frac{\eta(1-z) + k_2(z-\eta)}{(1-z) + k_2(z-\eta)}$$

It is possible to identify precisely on which propagator and on which boundary the punctures are inserted

Insertion on different boundaries yields different expressions for the integrand of the amplitude, but the results are related by modular transformations, providing highly nontrivial checks on the field theory limit.

Scalar amplitudes

Some ingredients of the two-loop master formula for scalars are given by

- Matching the string and the scalar coupling, from tree level

$$g_s = \frac{1}{4} \lambda \, (2\alpha')^{(6-d)/4}$$

- The scalar propagator to leading order in the multipliers k_1 and k_2

$$\mathcal{G}_2(z_i, z_j) = \log\left(|z_i - z_j|\right) + \frac{1}{2} \frac{\log k_1 \log k_2 - \log^2 S}{\log^2 T \log k_2 + \log^2 U \log k_1 - 2 \log T \log U \log S}$$

where

$$S = \eta_2 \quad ; \quad T = \frac{z_i}{z_j} \quad ; \quad U = \frac{(z_j - \eta_2)(z_i - 1)}{(z_i - \eta_2)(z_j - 1)}$$

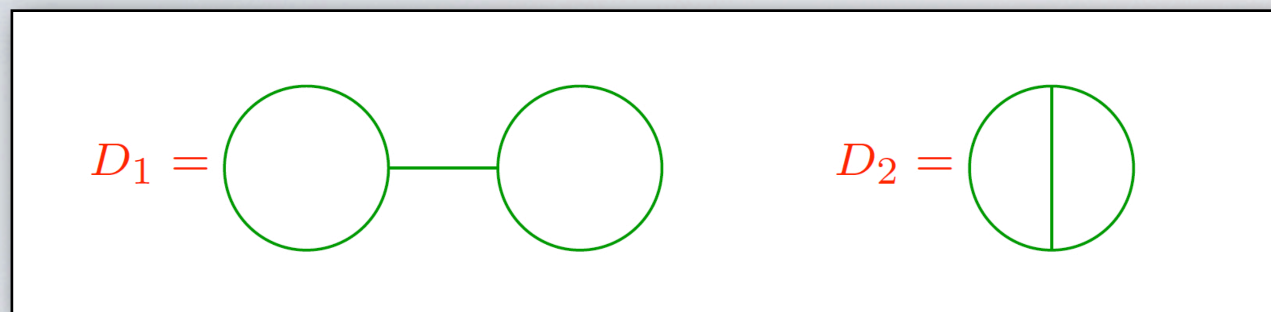
- The choice of local coordinates around the punctures

$$V_i'(0) = \Big[a_1 \omega_1(z_i) + a_2 \omega_2(z_i) \Big]^{-1}$$

with a_i chosen according to the boundary of insertion, insures modular invariance, for example under the transformation exchanging the two inner boundaries

$$z \to \frac{z - \eta_2}{z - 1}$$

Vacuum bubbles



Topologies for scalar two-loop vacuum bubbles

Leading regions in the field theory limit arise as $k_i \rightarrow 0$, and

$$\eta_1 \rightarrow 0 \qquad \text{or} \qquad \eta_1 \rightarrow 1$$

The fixed point η_1 plays the role of `distance between the loops'.

In the two relevant regions dimensionful proper-time variables are defined as

$$\eta_1 \rightarrow 1 : \qquad k_i = \mathrm{e}^{-t_i/\alpha'} \quad ; \quad 1 - \eta_1 = \mathrm{e}^{-t_3/\alpha'}$$

$$\eta_1 \rightarrow 0 : \qquad q_i \equiv k_i/\eta_1 = \mathrm{e}^{-t_i/\alpha'} \quad ; \quad q_3 \equiv \eta_1 = \mathrm{e}^{-t_3/\alpha'}$$

The integration region is found requiring that Schottky circles do not overlap, and simplifies in the field theory limit. Regulating tachyon double poles by treating m^2 as generic one finds

$$D_1 = \frac{N^3}{(4\pi)^d} \frac{g^2}{32} \int_0^\infty dt_3 \int_0^\infty dt_2 \int_0^{t_2} dt_1 \, \mathrm{e}^{-m^2(t_1+t_2+t_3)} \, (t_1 t_2)^{-d/2}$$

$$D_2 = \frac{N^3}{(4\pi)^d} \frac{g^2}{32} \int_0^\infty dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \, \mathrm{e}^{-m^2(t_1+t_2+t_3)} \, (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-d/2}$$

which as expected agrees with field theory, including color and symmetry factors.

EFFECTIVE ACTIONS

Field theory: scalars

Effective actions are useful to study the geometry of moduli space through `vacuum bubbles'.

Consider coupling an adjoint scalar to a constant background field.

$$\mathcal{L} = \mathrm{Tr}\left[D_\mu \Phi D^\mu \Phi - m^2 \Phi^2 + \frac{2}{3}\lambda\, \Phi^3 \right] .$$

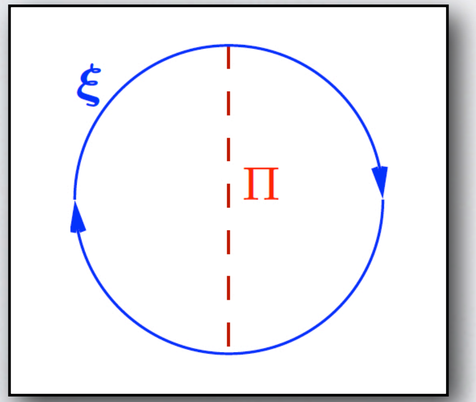
Using a pseudo-abelian constant gauge field (chromo-magnetic) field one may write

$$\mathcal{A}^\mu_{ab} = A^\mu\, \delta_{a,N}\delta_{b,N} \quad ; \quad A_\mu = B\, x_1\, g_{\mu 2} \qquad\qquad \Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\,\Pi(x) & \xi(\mathbf{x}) \\ \xi^\dagger(\mathbf{x}) & \sigma(x) \end{pmatrix}$$

In this case the scalar charged propagator can be computed exactly

$$G_\xi(x,y) = \frac{\mathrm{e}^{-\mathrm{i}B(x_1+y_1)(x_2-y_2)/2}}{(4\pi)^{d/2}} \int_0^\infty dt\, \mathrm{e}^{-m^2 t}\, t^{-d/2+1}\, \frac{B}{\sinh(Bt)}$$
$$\times\, \exp\left[\frac{(x_0-y_0)^2 - (\mathbf{x}_\perp - \mathbf{y}_\perp)^2}{4\ t} - \frac{B\,(x_1-y_1)^2 - (x_2-y_2)^2}{4\tanh(Bt)} \right] .$$

Formal (unrenormalized) expressions for two-loop vacuum bubbles are readily computed



A two-loop charged diagram

$$W^{(2)}_{\xi\Pi}(m,B) = -\mathrm{i}\, V_d\, \frac{\lambda^2}{(4\pi)^d}\, \frac{(N-1)^2}{4} \int_0^\infty dt_1 dt_2 dt_3\ \mathrm{e}^{-m^2(t_1+t_2+t_3)}\ \Delta_0^{-\frac{d}{2}+1}\, \Delta_B^{-1}$$

$$\Delta_0 = t_1 t_2 + t_1 t_3 + t_2 t_3\,, \quad \Delta_B = \frac{1}{B^2}\sinh(Bt_2)\sinh(Bt_3) + \frac{t_1}{B}\sinh\left[B\,(t_2+t_3)\right] .$$

Field theory: gauge bosons

A more challenging and interesting calculation is pure Yang-Mills theory,

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\Big[F^2_{(A+Q)}\Big] - \frac{1}{\xi}\,\mathrm{Tr}\Big[G^2(A,Q)\Big] + \mathcal{L}_{\rm ghost}$$

With string theory in mind, we pick an intricate gauge (Bern, Dunbar), the background field version of the Gervais-Neveu gauge

$$G(A,Q) = D^{(A)}_\mu Q^\mu + \frac{\rm i}{2}\,\alpha\, g\,\{Q_\mu, Q^\mu\}$$

In this form, it is (almost) the most general gauge choice compatible with the BF method.

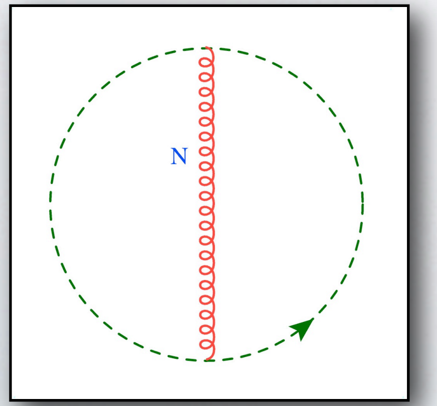
As before, we take a block-diagonal gauge field, non-trivial in a fixed plane.

$$A^{(i)}_\mu = \delta_{a_i b_i}\, B^{(i)}\, x_1\, g_{\mu 2} \quad \longrightarrow \quad F^{(A)}_{12} = \mathrm{diag}\Big\{B^{(i)}\Big\}$$

The gluon propagator has charged polarizations, which can be diagonalized

$$G^+_{ij}(x,y) = -g^{+-}\, G_\xi(x,y)\,, \qquad \text{with} \quad B \to B_i - B_j\,, \quad {\rm e}^{-m^2 t} \to {\rm e}^{-2(B_i - B_j)t}$$

Not withstanding a (well-known) instability, computations can be formally carried out:



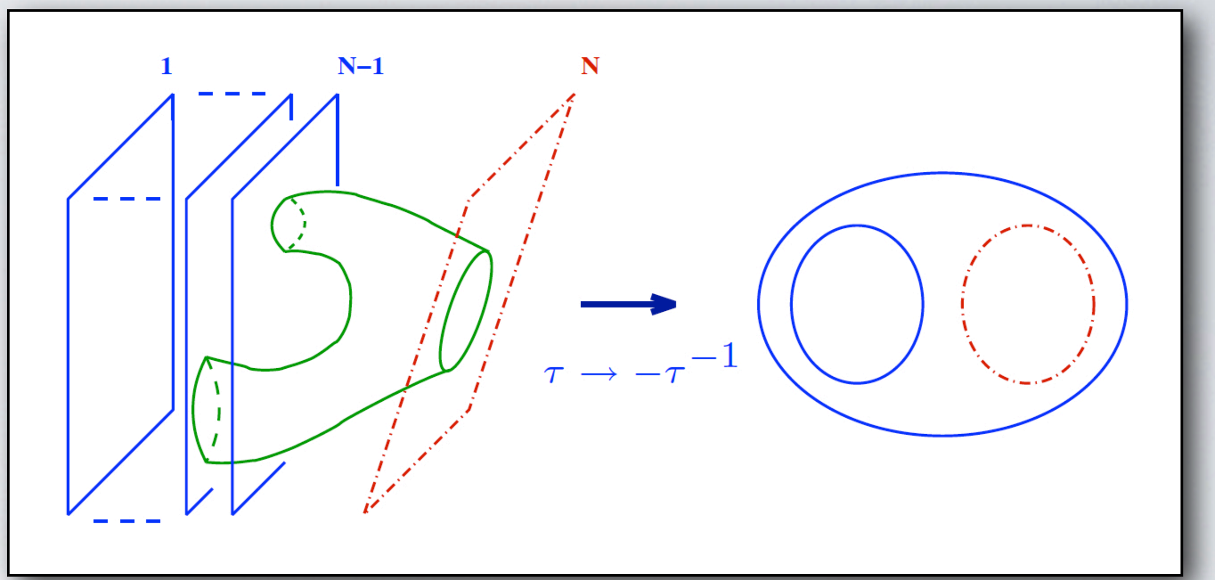
$$= \frac{g^2}{(4\pi)^d}\, N_1 N_2 N_3\, V_d\, \frac{1+\alpha^2}{2}\,(d-2)\int_0^\infty dt_1 dt_2 dt_3\, \frac{t_1+t_2+t_3}{\Delta_0^{d/2}\,\Delta_B\, \prod_{i=1}^3 \cosh\left(B_i t_i\right)}$$

String theory: master formulas

The technology to compute multi-loop string effective actions in these backgrounds was recently developed (Russo, Sciuto).

Basic fact: constant backgrounds simply modify the boundary conditions for the world-sheet fields

$$\left[\partial_\sigma X^i + \mathrm{i}\,\partial_\tau X^j F_j{}^{i\,(A)}\right]_{\sigma=0} = 0$$



Dual pictures for the two-loop string effective action

Twisted boundary conditions lead to new geometric objects on the world sheet: twisted determinants, Prym differentials, a twisted period matrix. They are all computable in the Schotty parametrization. The g-loop partition function reads

$$Z_\epsilon(g) = \frac{\mathrm{e}^{2\pi \mathrm{i}\epsilon_g} - 1}{\prod_{\mu=1}^{g} \cos \pi\epsilon_\mu} C_g \int [dm]_g^0 \; \mathrm{e}^{-\mathrm{i}\pi\epsilon\cdot\tau\cdot\epsilon} \frac{\det(\tau)}{\det(\tau_\epsilon)} \mathcal{R}_g(k_\alpha, \epsilon\cdot\tau)$$

where a vector of dimensionless fields was defined by $\tan(\pi\epsilon_i) = 2\pi\alpha'(B_i - B_{i+1})$

New geometric objects contributing to the partition function, such as

$$\mathcal{R}_g(k_\alpha, \epsilon) = \frac{\prod_\alpha{}' \prod_{n=1}^{\infty}(1-k_\alpha^n)^2}{\prod_\alpha{}' \prod_{n=1}^{\infty}(1-\mathrm{e}^{-2\pi\mathrm{i}\epsilon\cdot\mathbf{N}_\alpha}k_\alpha^n)(1-\mathrm{e}^{2\pi\mathrm{i}\epsilon\cdot\mathbf{N}_\alpha}k_\alpha^n)}$$

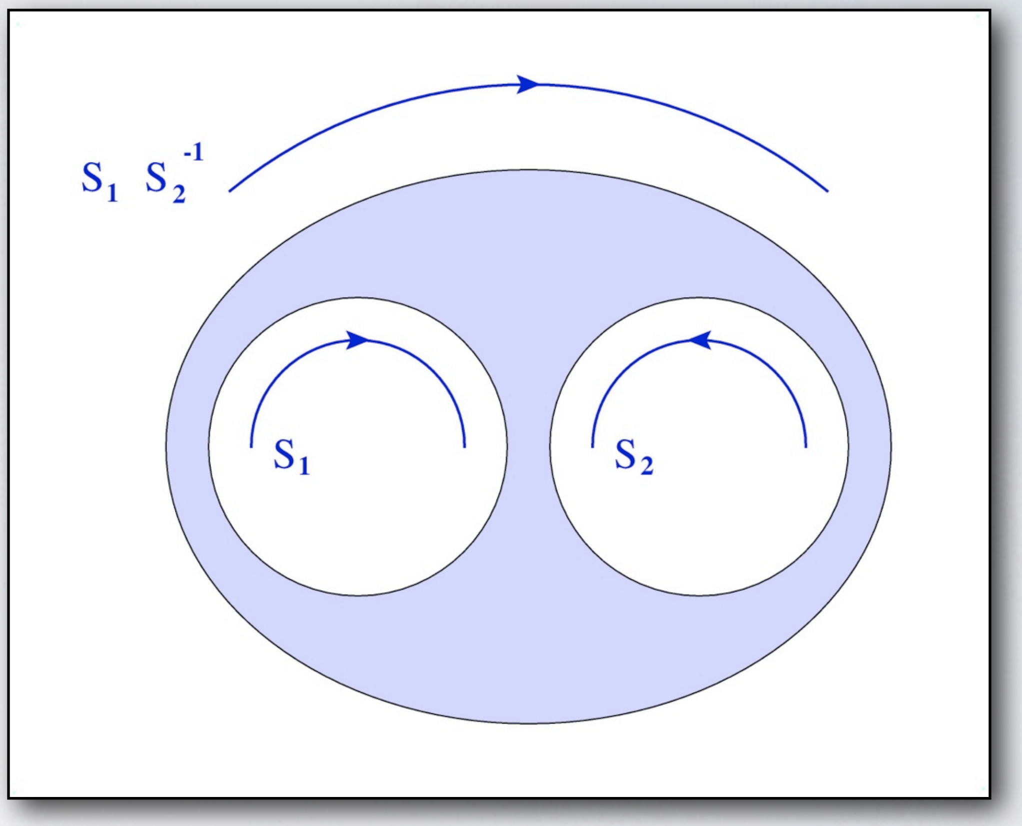
can still be computed perturbatively in the field theory limit.

Picking the right variables

The difficulty in extending two-loop calculations beyond the string ground state can be traced to a failure of the pedestrian choice of variables for the field theory limit

A better choice is dictated by geometry, and modular invariance: each boundary must be decomposed as the product of two propagators in a modularly covariant way

One can postulate an exact factorization of the multipliers associated with each boundary as



Schottky actions on the two-loop annulus

$$k\left(S_1\right) \equiv p_1\, p_3\,, \quad k\left(S_2\right) \equiv p_2\, p_3\,, \quad k\left(S_1 \cdot S_2^{-1}\right) \equiv p_1\, p_2\,,$$

The third definition appears to lead to complicated square-root singularities. Remarkably, it can be simply inverted as

$$k_1 = p_1\, p_3\,, \quad k_2 = p_2\, p_3\,, \quad \eta = \frac{(1+p_1)\,(1+p_2)\,p_3}{(1+p_3)\,(1+p_1 p_2 p_3)}$$

The field theory limit for this topology is then driven by expanding in p_i, using

$$\log p_i \equiv -t_i/\alpha'$$

Picking diagrams

Further progress stems form reconstructing the origin of individual contributions to the the string partition function.

- Contributions to the measure arising from the the determinant of the *(b-c)* system (world-sheet ghosts) correspond to the propagation of space-time ghosts (Polchinski).
- Contributions of world-sheet scalars correspond to un-magnetized gluon propagation
- One may distinguish the space-time dimension from the string critical dimension, allowing for propagation of adjoint scalars left over from extra-dimensional gluons.

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Consider the untwisted contribution to the two-loop integrand for the effective action

$$\mathcal{I}_2\,(\epsilon \to 0) \;=\; \frac{dk_1\,dk_2\,d\eta}{k_1^2\,k_2^2}\,\frac{(1-k_1)^2\,(1-k_2)^2}{(1-\eta)^2}\,\Big[\det(\tau)\Big]^{-d/2+1} \prod_\alpha{}' \left[\prod_{n=1}^{\infty}(1-k_\alpha^n)^{-d}\,\prod_{n=2}^{\infty}(1-k_\alpha^n)^2\right]$$

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Ghosts

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Ghosts

Un-magnetized gluons

Picking diagrams

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- Contributions to the measure arising from the the determinant of the *(b-c)* system (world-sheet ghosts) correspond to the propagation of space-time ghosts (Polchinski).
- Contributions of world-sheet scalars correspond to un-magnetized gluon propagation
- One may distinguish the space-time dimension from the string critical dimension, allowing for propagation of adjoint scalars left over from extra-dimensional gluons.

Consider the untwisted contribution to the two-loop integrand for the effective action

$$\mathcal{I}_2\left(\epsilon \to 0\right) = \frac{dk_1\, dk_2\, d\eta}{k_1^2\, k_2^2} \, \frac{(1-k_1)^2\,(1-k_2)^2}{(1-\eta)^2} \left[\det\left(\tau\right)\right]^{-d/2+1} {\prod_\alpha}' \left[\prod_{n=1}^{\infty}\left(1-k_\alpha^n\right)^{-d} \prod_{n=2}^{\infty}\left(1-k_\alpha^n\right)^2\right]$$

Ghosts

Un-magnetized gluons

d = 4 - 2 ε

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Ghosts

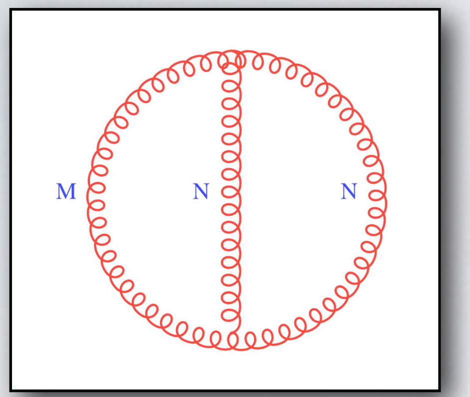
Un-magnetized gluons

d = 4 - 2 ε

d → D = d + s

Sample results

With this technology all relevant two-loop diagrams can be extracted. For example



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d \, (d-2) \int_0^\infty dt_1 dt_2 dt_3 \frac{1}{\Delta_0^{d/2-1} \, \Delta_B^2 \, \prod_{i=1}^3 \cosh^2 [B_i t_i]}$$

$$\times \left[2 \, \frac{\sinh(B_1 t_1)}{B_1} \cosh\left(2B_1 t_1 - B_2 t_2 - B_3 t_3\right) + \mathcal{O}(1-\alpha^2) \right]$$

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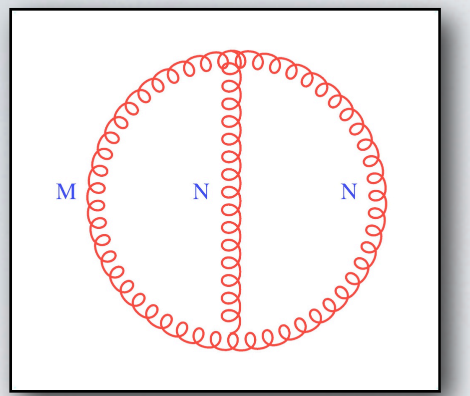

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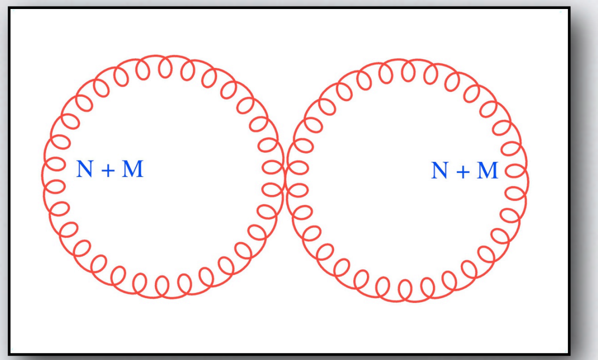

$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d\, \frac{1+\alpha^2}{2} \int_0^\infty dt_1 dt_2\, \frac{B_1 t_1\, B_2 t_2}{\sinh(B_1 t_1)\ \sinh(B_2 t_2)}$$
$$\times \left[(d-2) + 2\cosh\left(2B_1 t_1 + 2B_2 t_2\right)\ +\ \mathcal{O}(1-\alpha^2)\right]$$

- Isolating the four-point vertex requires integrating exactly over η after expanding in k_i, discarding tachyon-related singularities.

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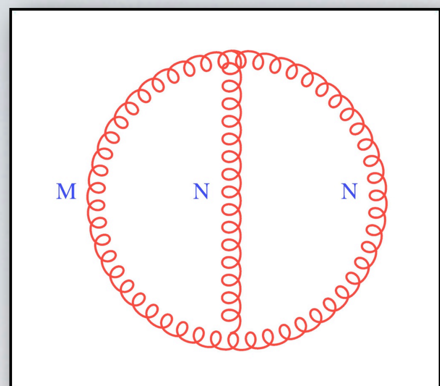

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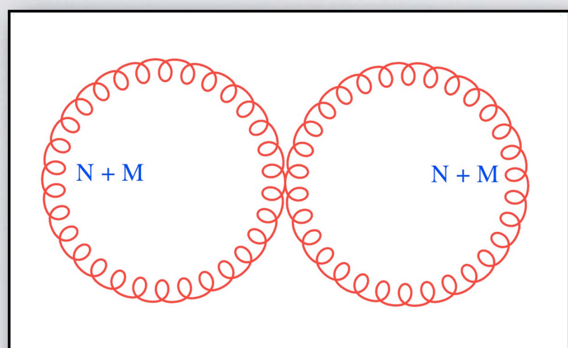

$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 \text{[illegible]} \, \frac{1+\alpha^2}{\text{[illegible]}} \int^\infty \text{[illegible]} \frac{B_1 t_1 \, B_2 t_2}{\text{[illegible]} \, \sinh(B_2 t_2)}$$

$$\times \left[(d \text{[illegible]} \, \mathcal{O}(1-\alpha^2) \right]$$

Preliminary ...

- Isolating the four-point vertex requires integrating exactly over η after expanding in k_i, discarding tachyon-related singularities. A universal regularization applies.

Sample results

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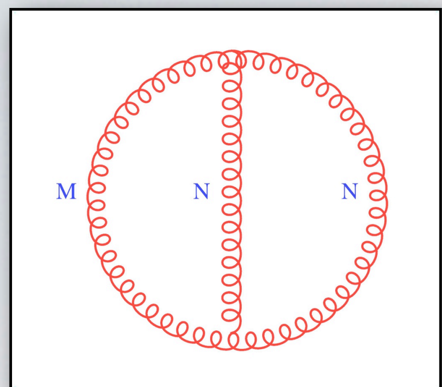

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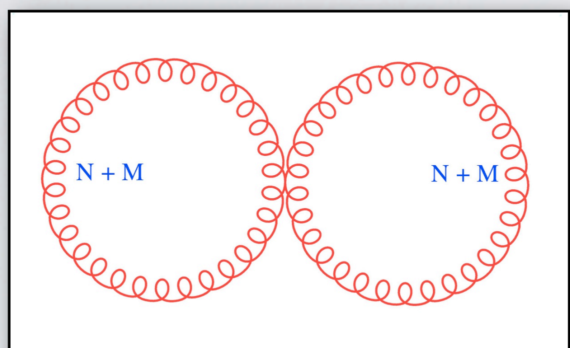

$$= -\frac{g^2}{(4\pi)^d}\, N_1 N_2 N_3 \ldots \frac{1+\alpha^2}{\ldots} \int^\infty \ldots \frac{B_1 t_1\, B_2 t_2}{\ldots \sinh(B_2 t_2)}$$

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Preliminary ...

- Isolating the four-point vertex requires integrating exactly over η after expanding in k_i, discarding tachyon-related singularities. A universal regularization applies.
- All Schwinger parameter integrands are reproduced by string theory with $\alpha = 1$

Sample results

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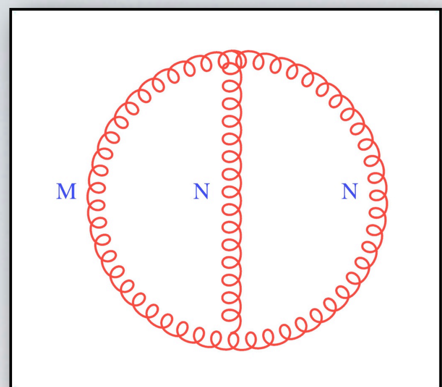

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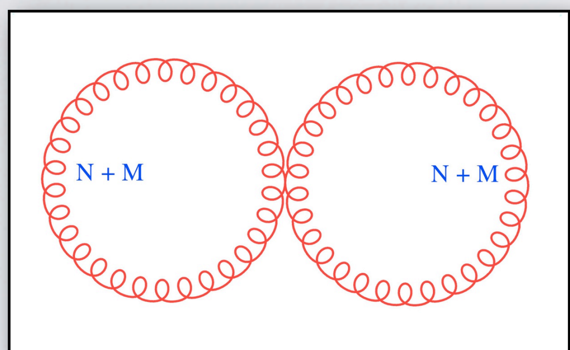

$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 \; \ldots \; 1 + \alpha^2 \; \int^\infty \ldots \frac{B_1 t_1 \, B_2 t_2}{\ldots \sinh(B_2 t_2)}$$

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Preliminary ...

- Isolating the four-point vertex requires integrating exactly over η after expanding in k_i, discarding tachyon-related singularities. A universal regularization applies.
- All Schwinger parameter integrands are reproduced by string theory with $\alpha = 1$
- Integrals are highly divergent and IR singular. To test for a physical result we have reproduced the finite renormalized two-loop effective action of scalar electrodynamics computed by Ritus in the 70's.

OUTLOOK

To summarize

- Our understanding of the field theory limit of perturbative string amplitudes grows at widely separated time intervals.
- Pushing beyond one loop and beyond the string ground state proved difficult so far.
- Studying effective actions in constant background fields at multi-loops is now possible and useful to understand the fragmentation of moduli space in the field theory limit.
- We have made significant progress.
  - The proper variables to identify each vacuum bubble topology have been identified, in a manner generalizable to higher genera.
  - Reconstructing the origin of each factor of the string partition function it is possible to identify not only individual topologies but individual diagrams.
  - String theory naturally computes diagrams in a specific gauge: the Gervais-Neveu Background Field (GNBF) gauge with parameter $\alpha = 1$. This applies to all genera.
- Massless and massive scalar fields, massless gauge bosons and ghosts can be handled using the open bosonic string. Gravitons are expected to follow the same pattern in the closed string channel. Fermions must await further technical developments for superstring amplitudes.
- Applications can be envisaged to perturbative dualities, non-perturbative field theory effects (D-branes and instantons), string phenomenology and further theory developments.

THANK YOU!