THE LONG-DISTANCE SINGULARITIES OF MASSLESS GAUGE THEORIES

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INTRODUCTION

A subject with a long history ...

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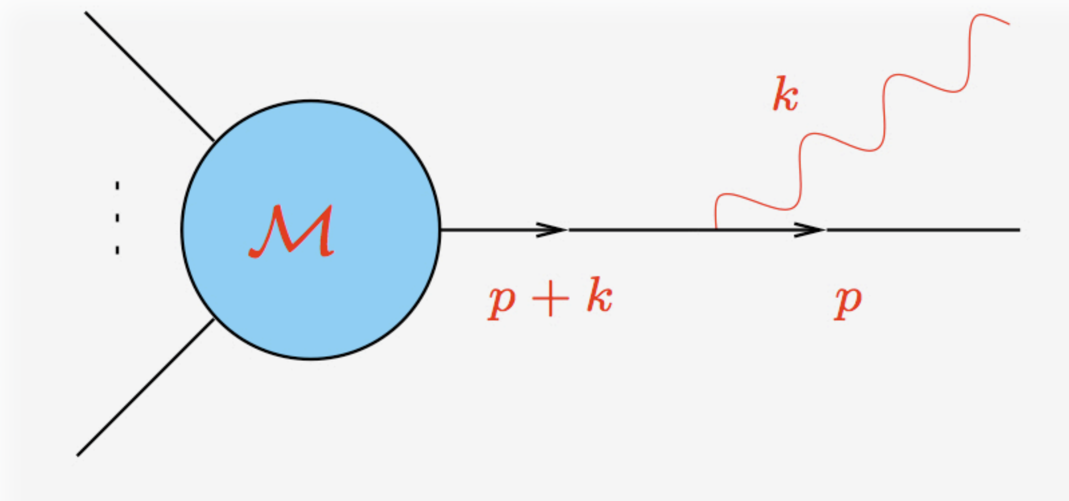
Textbook theory ...



$$\rightarrow \quad -ig\overline{u}(p)\not{\epsilon}(k)t_a \frac{i(\not{p}+\not{k})}{(p+k)^2+i\varepsilon}\mathcal{M}\,,$$

Singularities arise only when propagators go on shell

$$2p\cdot k = 2p_0k_0(1-\cos\theta_{pk}) = 0\,,$$
$$\rightarrow k_0 = 0\ (IR);\quad \cos\theta_{pk} = 1\,.$$

➡ Emission is not suppressed at long distances

➡ Isolated charged particles are not true asymptotic states of unbroken gauge theories

- A serious problem: the S matrix does not exist in the usual Fock space
- Possible solutions: construct finite transition probabilities (KLN theorem)
  construct better asymptotic states (coherent states)
- Long-distance singularities obey a pattern of exponentiation

$$\mathcal{M} = \mathcal{M}_0\left[1-\kappa\frac{\alpha}{\pi}\frac{1}{\epsilon}+\ldots\right] \;\Rightarrow\; \mathcal{M} = \mathcal{M}_0 \exp\left[-\kappa\frac{\alpha}{\pi}\frac{1}{\epsilon}+\ldots\right]$$

... and Practice

Just a formal issue in Quantum Field Theory? Are there practical applications?

- Higher order QCD calculations at colliders hinge upon cancellation of divergences between virtual corrections and real emission contributions
  - Cancellation must be performed analytically before numerical integrations
  - Need local counterterms for matrix elements in all singular regions
  - State of the art: NLO multileg, NNLO for (some) color singlet processes

- Cancellations leave behind large logarithms: they must be resummed

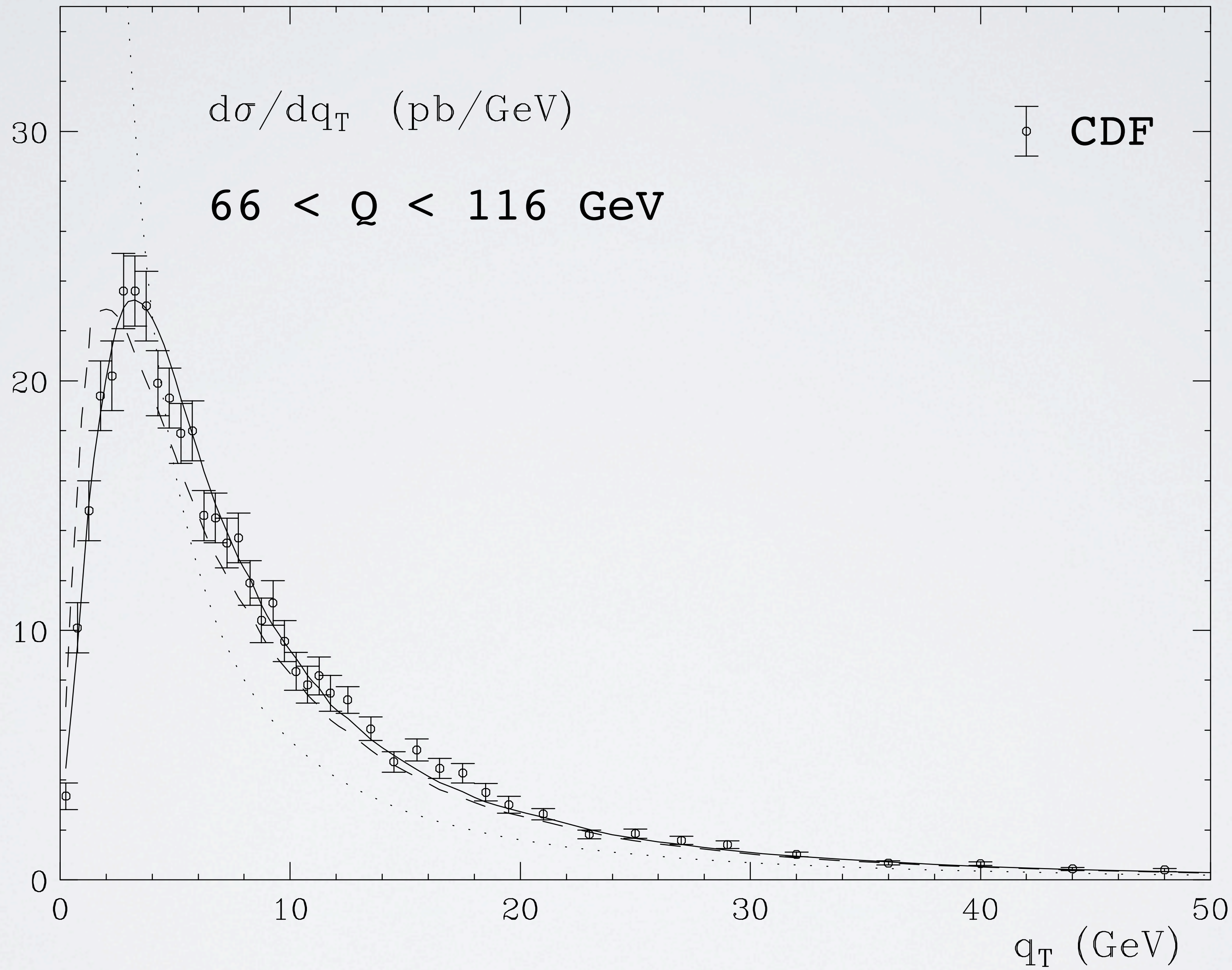
$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)$$

  - For inclusive observables: analytic resummation to high logarithmic accuracy.
  - For exclusive final states: parton shower event generators, (N)LL accuracy.

- Resummation probes the all-order structure of perturbation theory
  - Power-suppressed corrections to QCD cross sections can be studied
  - Links to the strong coupling regime can be established for SUSY gauge theories.

Impact of resummation

Z-boson q_T spectrum at Tevatron (A. Kulesza et al.)



CDF data on Z production compared with QCD predictions at fixed order (dotted), with resummation (dashed), and with the inclusion of power corrections (solid).

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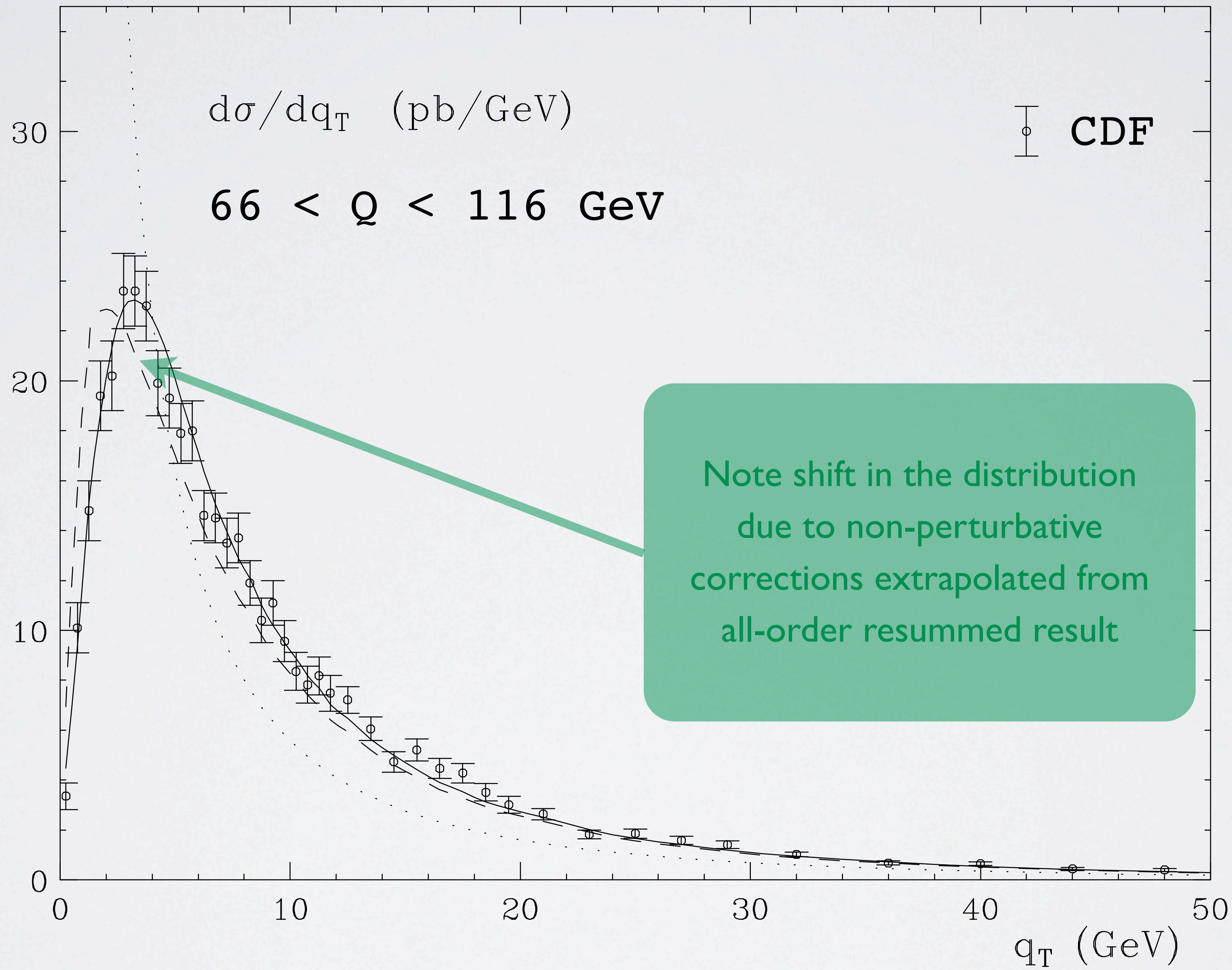
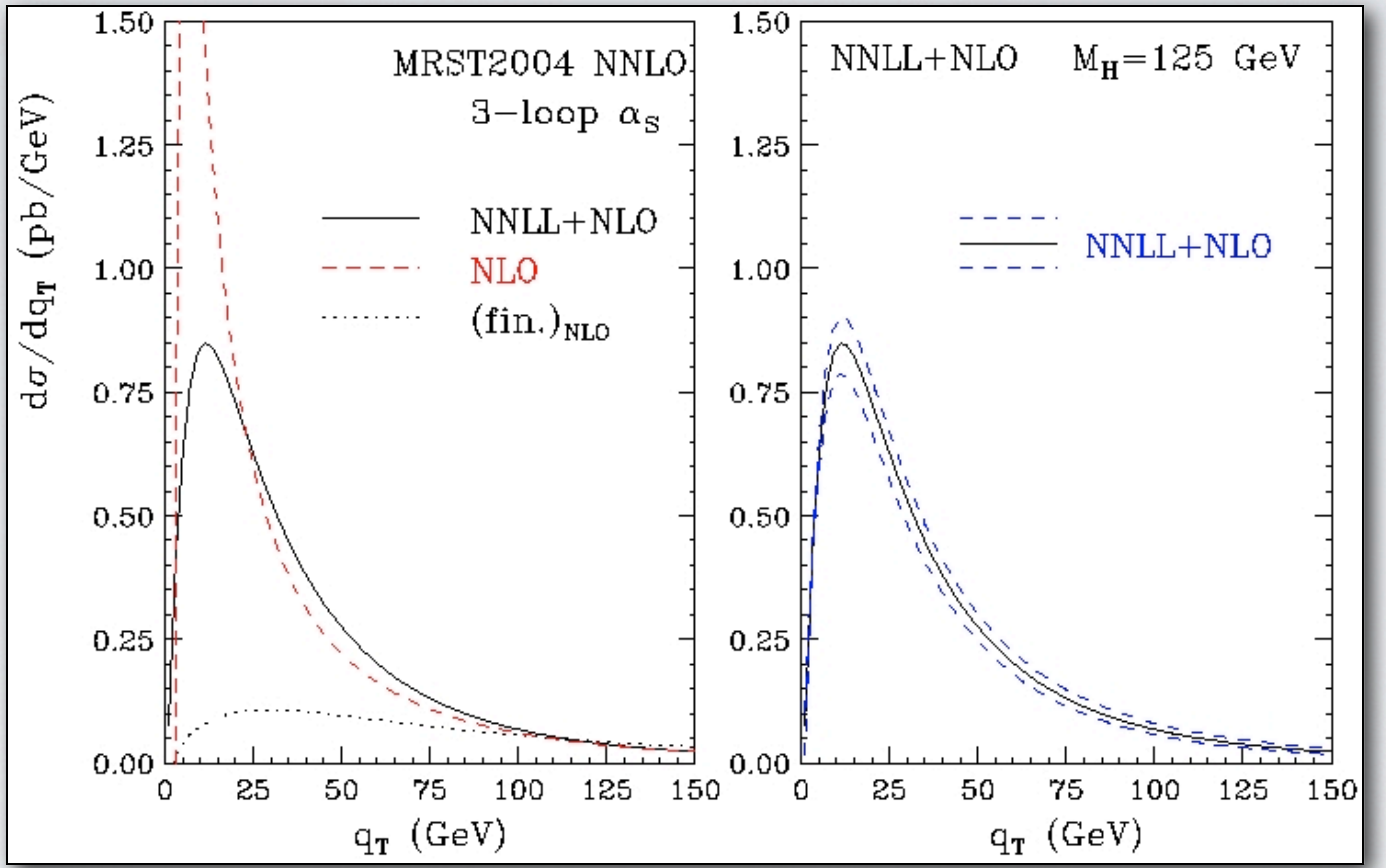


Note shift in the distribution due to non-perturbative corrections extrapolated from all-order resummed result

CDF data on Z production compared with QCD predictions at fixed order (dotted), with resummation (dashed), and with the inclusion of power corrections (solid).

Impact of resummation

Predictions for the Higgs boson q_T spectrum at LHC (M. Grazzini)



Predictions for the q_T spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.

Impact of resummation

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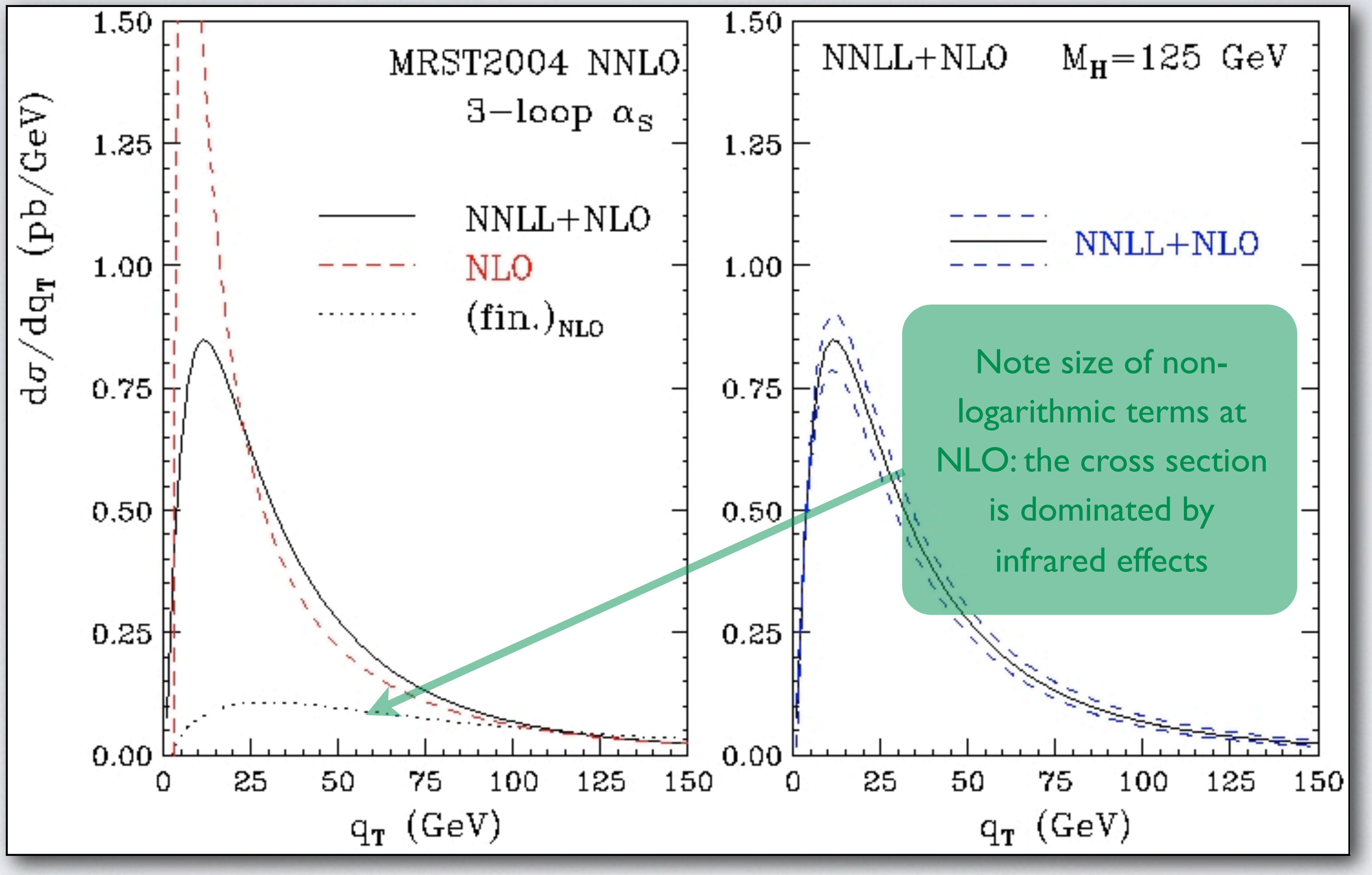

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TOOLS

Dimensional regularization

Exponentiation of infrared poles requires solving d-dimensional evolution equations. The running coupling in d = 4 - 2 ε obeys

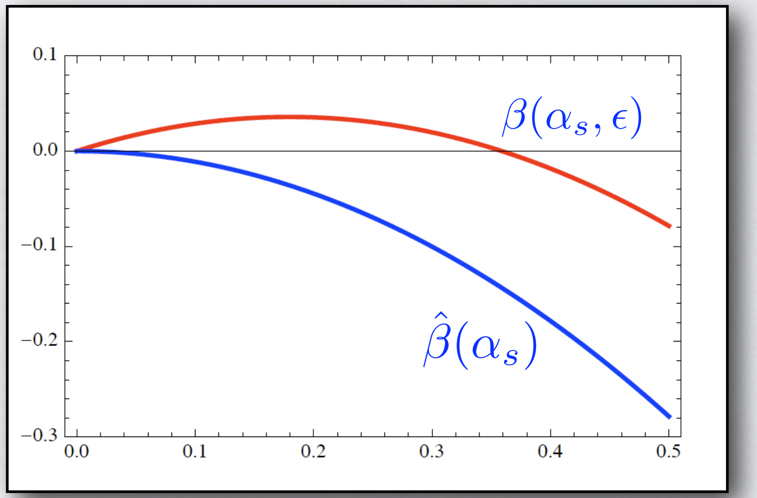
$$\mu \frac{\partial \overline{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \overline{\alpha}) = -2\,\epsilon\,\overline{\alpha} + \hat{\beta}(\overline{\alpha}) \quad , \quad \hat{\beta}(\overline{\alpha}) = -\frac{\overline{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\overline{\alpha}}{\pi}\right)^n \, .$$

The one-loop solution is

$$\overline{\alpha}\left(\mu^2, \epsilon\right) = \alpha_s(\mu_0^2) \left[\left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon} - \frac{1}{\epsilon}\left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon}\right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} \, .$$

The β function develops an IR-free fixed point, so that the coupling vanishes at μ = 0 for fixed ε < 0. The Landau pole is at

$$\mu^2 = \Lambda^2 \equiv Q^2 \left(1 + \frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)}\right)^{-1/\epsilon} \, .$$



For negative ε the beta function develops a second zero, O(ε) from the origin.

- ➡ Integrations over the scale of the coupling can be analytically performed.
- ➡ All infrared and collinear poles arise by integration over the scale of the running coupling.

Factorization

All factorizations separating dynamics at different energy scales lead to resummation of logarithms of the ratio of scales.

Renormalization is a textbook example.

- Renormalization factorizes cutoff dependence.

$$G_0^{(n)}\left(p_i, \Lambda, g_0\right) = \prod_{i=1}^{n} Z_i^{1/2}\left(\Lambda/\mu, g(\mu)\right)\; G_R^{(n)}\left(p_i, \mu, g(\mu)\right)$$

- Factorization requires the introduction of an arbitrarily chosen scale μ.

- Results must be independent of the arbitrary choice of μ.

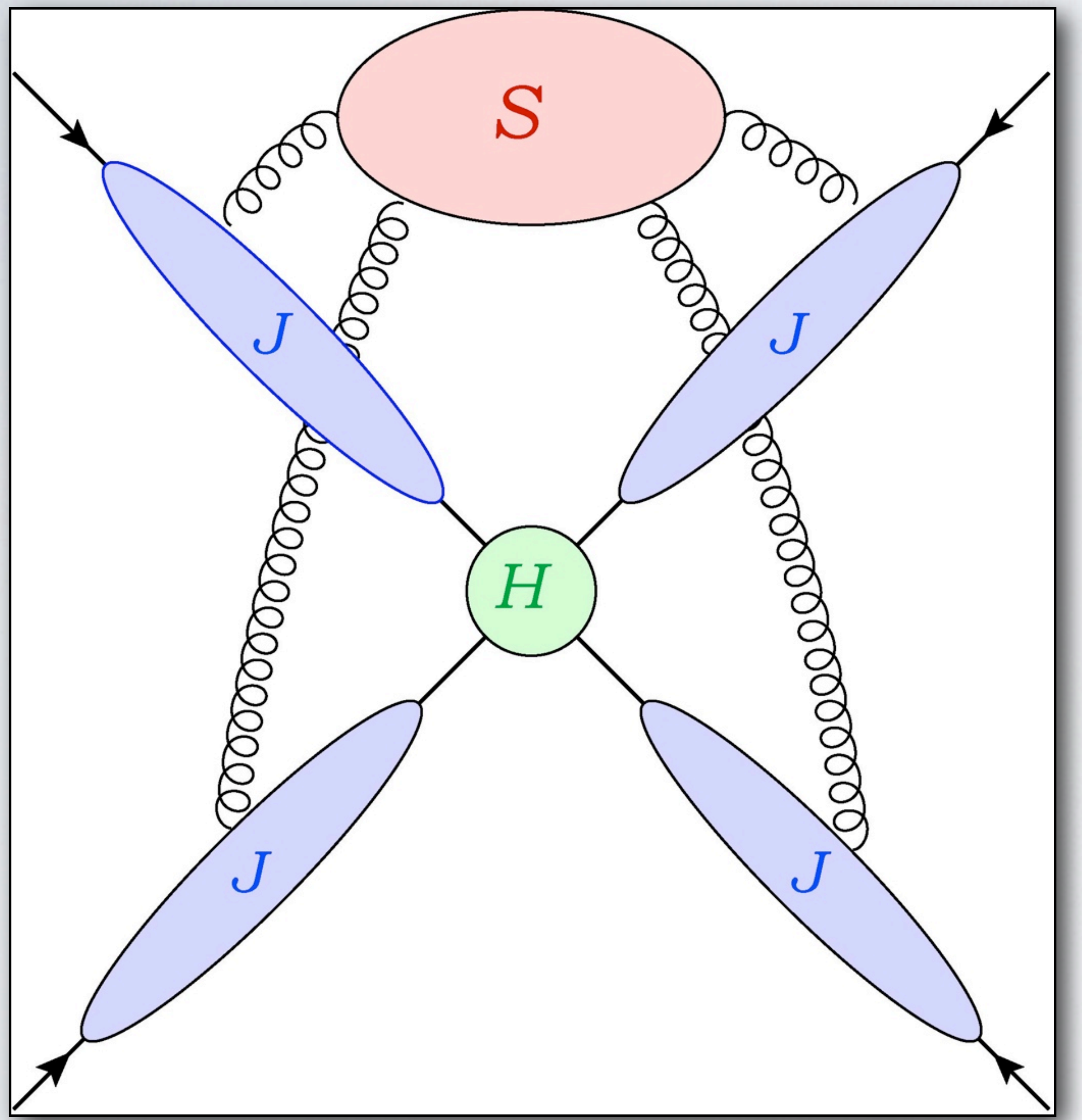
$$\frac{dG_0^{(n)}}{d\mu} = 0 \quad \rightarrow \quad \frac{d\log G_R^{(n)}}{d\log\mu} = -\sum_{i=1}^{n} \gamma_i\left(g(\mu)\right) \; .$$

- The simple functional dependence of the factors is dictated by separation of variables.

- Proving factorization is the difficult step: it requires all-order diagrammatic analyses. Evolution equations follow automatically.

- Solving RG evolution resums logarithms of Q^2/μ^2 into $\alpha_s(\mu^2)$.

Sudakov Factorization

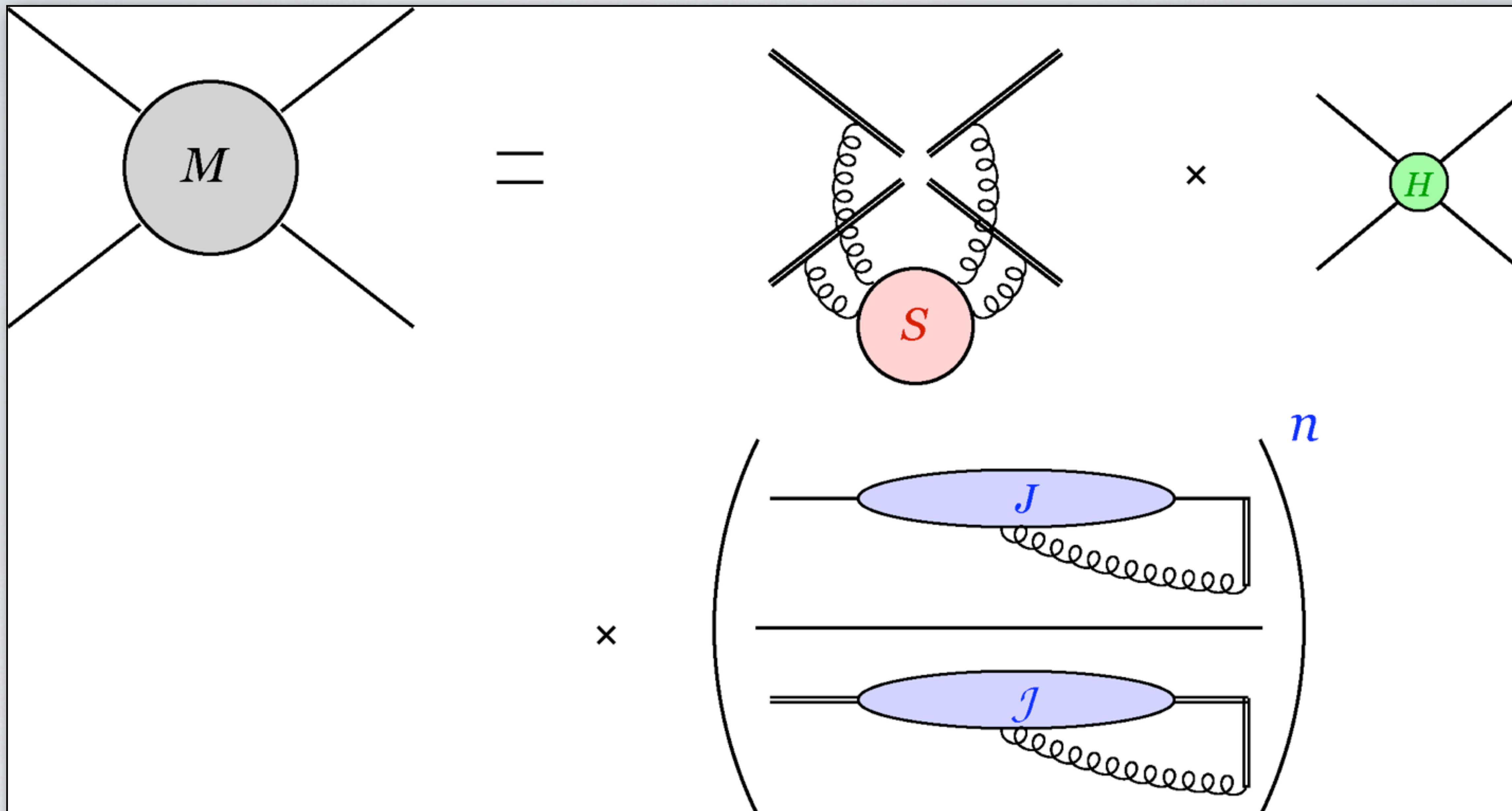


Leading integration regions in loop momentum space for Sudakov factorization

- Divergences arise in fixed-angle amplitudes from leading regions in loop momentum space.
- Soft gluons factorize both form hard (easy) and from collinear (intricate) virtual exchanges.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: S is proportional to the identity.
- In the planar limit S can be reabsorbed defining jets as square roots of elementary form factors.
- Beyond the planar limit S is determined by an anomalous dimension matrix Γ_S.
- Phenomenological applications to jet and heavy quark production at hadron colliders .

MULTICOLORED AMPLITUDES

Factorization: pictorial



A pictorial representation of Sudakov factorization for fixed-angle scattering amplitudes

Operator Definitions

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right],$$

We introduced factorization vectors n_i^μ, $n_i^2 \neq 0$ to define the jets,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 \,|\, \Phi_n(\infty, 0)\, \psi(0) \,|\, p\rangle \,.$$

where Φ_n is the Wilson line operator along the direction n^μ,

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[\mathrm{i} g \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n)\right] \,.$$

The vectors n^μ:

- Ensure gauge invariance of the jets.
- Separate collinear gluons from wide-angle soft ones.
- Replace other hard partons with a collinear-safe absorber.

Eikonal functions

The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$(c_L)_{\{\alpha_k\}} \, \mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) = \sum_{\{\eta_k\}} \langle 0 | \prod_{i=1}^{n} \left[\Phi_{\beta_i}(\infty, 0)_{\alpha_k, \eta_k} \right] | 0 \rangle \, (c_K)_{\{\eta_k\}} \, ,$$

To avoid double counting, soft-collinear regions are subtracted dividing by eikonal jets $\mathcal{J}$.

$$\mathcal{J} \left(\frac{(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon \right) = \langle 0 | \Phi_n(\infty, 0) \, \Phi_\beta(0, -\infty) \, | 0 \rangle \, ,$$

- The eikonal functions S and $\mathcal{J}$ are pure counterterms in dimensional regularization.
  - ➡ Infrared poles are mapped to ultraviolet singularities.
- Functional dependence of jet and soft factors on the vectors n^μ_i is restricted by the classical invariance of Wilson lines under velocity rescalings, $n^\mu_i \to \kappa_i \, n^\mu_i$.
- Rescaling invariance for light-like velocities, $\beta_i^2 = 0$ is broken by quantum corrections.
  - ➡ UV counterterms contain collinear poles, corresponding to soft-collinear singularities.
- Double poles are determined by the cusp anomalous dimension $\gamma_K(\alpha_s)$.
  - ➡ $\gamma_K(\alpha_s)$ governs the renormalization of Wilson lines with light-like cusps.

Soft Matrices

The soft function S obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = - \mathcal{S}_{IJ} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \Gamma^{\mathcal{S}}_{JK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)$$

- Γ^S is singular due to overlapping UV and collinear poles.

S is a pure counterterm. In dimensional regularization, using $\alpha_s(\mu^2 = 0, \varepsilon < 0) = 0$, one finds

$$\mathcal{S} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp \left[-\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon\right) \right] .$$

Double poles cancel in the reduced soft function

$$\overline{\mathcal{S}}_{LK} \left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) = \frac{\mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{n} \mathcal{J}_i \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}$$

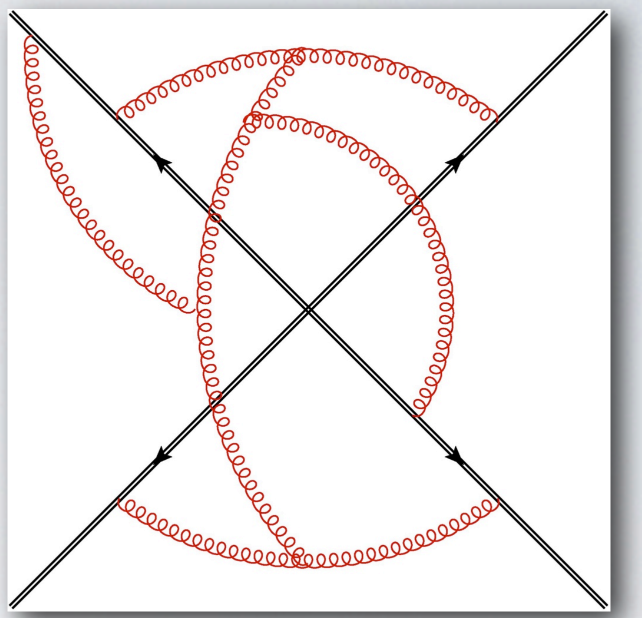
- The matrix $\overline{\mathcal{S}}$ must depend on rescaling invariant variables

$$\rho_{ij} \equiv \frac{n_i^2 \, n_j^2 \, (\beta_i \cdot \beta_j)^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2} .$$

- The anomalous dimension $\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$ for the evolution of $\overline{\mathcal{S}}$ is finite.

Surprising Simplicity

- The matrix Γ_S can be computed from the UV poles of S.
- Computations can be performed directly for the exponent: the relevant diagrams are called "webs".
- Γ_S appears highly complex at high orders.
- g-loop webs directly correlate color and kinematics of up to g+1 Wilson lines.



A web contributing to the soft anomalous dimension matrix

The two-loop calculation (M. Aybat, L. Dixon, G. Sterman) leads to a surprising result: for any number of light-like eikonal lines

$$\Gamma_{\mathcal{S}}^{(2)} = \frac{\kappa}{2}\,\Gamma_{\mathcal{S}}^{(1)} \qquad \kappa = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F C_F \, .$$

➡ No new kinematic dependence; no new matrix structure.

➡ κ is the two-loop coefficient of $\gamma_K(\alpha_s)$, rescaled by the appropriate quadratic Casimir,

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)} \left[2\,\frac{\alpha_s}{\pi} + \kappa \left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right) \right] .$$

Factorization Constraints

The kinematic dependence of eikonal functions is severely restricted by rescaling invariance.

- The classical symmetry of Wilson line correlators under $\beta_i \to \kappa_i \beta_i$ is violated only through the cusp anomaly.
  - ➡ For eikonal jets, no β_i dependence is possible at all except through the cusp
- In the reduced soft function, $S/\prod J$, the cusp anomaly cancels
  - ➡ The reduced soft function can depend on β_i only through rescaling-invariant combinations such as ρ_{ij} . For $n > 3$ hard partons, one may also construct

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

Consider the anomalous dimension matrix for the reduced soft function

$$\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s(\mu^2)\right) = \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}\left(\frac{(\beta_k \cdot n_k)^2}{n_k^2}, \alpha_s(\mu^2), \epsilon\right) .$$

Remarkably:

- Singular terms in Γ_S must be diagonal and proportional to γ_K
- Finite diagonal terms in Γ_S must conspire to construct ρ_{ij}'s.
- Off-diagonal terms in Γ_S must be finite, and must depend only on the cross-ratios ρ_{ijkl}

Factorization Constraints

The constraints can be formalized simply by using the chain rule: $\Gamma^{\overline{\mathcal{S}}}$ can depend on the factorization vectors n_i only through the eikonal jets, which are color diagonal.

Defining $x_i \equiv (\beta_i \cdot n_i)^2 / n_i^2$, one finds

$$x_i \frac{\partial}{\partial x_i} \Gamma^{\overline{\mathcal{S}}}_{IJ} \left(\rho_{ij}, \alpha_s\right) = - \delta_{IJ} \; x_i \frac{\partial}{\partial x_i} \gamma_{\mathcal{J}} \left(x_i, \alpha_s, \epsilon\right) \; = \; - \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s\right) \delta_{IJ} \, .$$

This leads to a linear equation for the dependence of $\Gamma^{\overline{\mathcal{S}}}$ on its proper arguments, ρ_{ij}.

$$\boxed{\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{MN} \left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \, \gamma_K^{(i)} \left(\alpha_s\right) \, \delta_{MN} \qquad \forall i \, ,}$$

- The equation relates the kinematic dependence of Γ to γ_K, to all orders in perturbation theory
  ➡ and should remain true at strong coupling as well
- It correlates color and kinematics for any number of hard partons
- It admits a unique solution for amplitudes with up to three hard partons.
  ➡ For $n > 3$ hard partons, functions of ρ_{ijkl} solve the homogeneous equation.

RESULTS

Results for Form Factors

The simplest, well-known example is the Sudakov form factor

$$\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_\mu(0) | p_1, p_2 \rangle = \overline{v}(p_2) \gamma_\mu u(p_1) \; \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \; .$$

In dimensional regularization, it exponentiates exactly, including constant terms (G. Sterman, LM).

$$\Gamma\left(Q^2, \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\left[G\Big(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\Big) - \frac{1}{2}\,\gamma_K\Big(\overline{\alpha}\left(\xi^2, \epsilon\right)\Big)\,\log\left(\frac{-Q^2}{\xi^2}\right)\right]\right\} \, .$$

The exponentiation is non trivial: only poles up to $(1/\varepsilon)^{n+1}$ appear in the exponent at n loops.

- All poles are generated by the integration over the scale of the d-dimensional coupling.
- All poles beyond $(1/\varepsilon)^2$ are due to the running of the four-dimensional coupling.

In a conformal gauge theory (regulated by $\varepsilon < 0$) all integrations are trivial.

$$\log\left[\Gamma\left(Q^2, \epsilon\right)\right] = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \mathrm{e}^{-\mathrm{i}\pi n \epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right] \, .$$

All divergences of planar amplitudes are given by the form factor.
Exact results can be derived (L. Dixon, G. Sterman LM).

$$\lim_{\epsilon \to 0}\left|\frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)}\right|^2 = \exp\left[\frac{\pi^2}{4}\,\gamma_K\left(\alpha_s\right)\right] \, .$$

$$G(\alpha_s, \epsilon) = 2\,B_\delta\left(\alpha_s\right) + G_{\rm eik}\left(\alpha_s\right) \, ,$$

They can be checked at strong coupling using AdS/CFT (L. Alday, J. Maldacena).

The Dipole Formula

Up to three loops, the cusp anomalous dimension obeys Casimir scaling

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)}\,\widehat{\gamma}_K(\alpha_s)$$

Including possible terms that violate the scaling, which may appear beyond three loops, write

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\left[C^{(i)}\,\widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)\right] \qquad \forall i\,.$$

Concentrating on the scaling terms, and switching to the notation of color generators, where gluon insertions are represented by the color operators T_i, and $C^{(i)} = \mathbf{T}_i \cdot \mathbf{T}_i$, we get

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\text{Q.C.}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\,\mathrm{T}_i \cdot \mathrm{T}_i\,\widehat{\gamma}_K(\alpha_s)\,, \qquad \forall i$$

The solution is provided by the dipole formula (E. Gardi, LM; T. Becher, M. Neubert)

$$\Gamma^{\overline{\mathcal{S}}}_{\text{dip}}(\rho_{ij}, \alpha_s) = -\frac{1}{8}\,\widehat{\gamma}_K(\alpha_s) \sum_{j\neq i} \ln(\rho_{ij})\,\mathbf{T}_i \cdot \mathbf{T}_j + \frac{1}{2}\,\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_i \mathbf{T}_i \cdot \mathbf{T}_i\,,$$

As easily seen using gauge invariance, as embodied by the color identity $\sum_i T_i = 0$.

The dipole formula correlates color ad kinematics to all orders in perturbation theory in the simplest way: multiparton correlations are absent. All known results in massless gauge theories are of this form.

The Full Amplitude

It is possible to construct a dipole formula for the full amplitude, enforcing the cancellation of the dependence on the factorization vectors n_i through

$$\ln\left(\frac{(2p_i\cdot n_i)^2}{n_i^2}\right)+\ln\left(\frac{(2p_j\cdot n_j)^2}{n_j^2}\right)+\ln\left(\frac{(-\beta_i\cdot\beta_j)^2\, n_i^2 n_j^2}{2(\beta_i\cdot n_i)^2\, 2(\beta_j\cdot n_j)^2}\right)=2\ln\left(-2p_i\cdot p_j\right)\,.$$

Soft and collinear singularities can then be collected in a matrix Z

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right)=Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right)\,\mathcal{H}\left(\frac{p_i}{\mu},\frac{\mu_f}{\mu},\alpha_s(\mu^2),\epsilon\right)\,,$$

Z generates all singularities, and must satisfy its own matrix RG equation

$$\frac{d}{d\ln\mu}\,Z\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right)\;=\;-\,Z\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right)\,\Gamma\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right)\,.$$

The matrix Γ inherits the dipole structure from the soft matrix. It reads (T. Becher, M. Neubert)

$$\boxed{\Gamma_{\rm dip}\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right)=-\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{j\neq i}\ln\left(\frac{-\,2\,p_i\cdot p_j}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j+\sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s(\mu^2)\right)\,.}$$

Once again, all singularities are generated by integration over the scale of the coupling.

Beyond the Minimal Solution

There are precisely two possible sources of corrections to the dipole formula

- The cusp anomalous dimension may violate Casimir scaling beyond three loops. This would add to Γ a contribution satisfying

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\rm H.C.} \left(\rho_{ij}, \alpha_s \right) = \frac{1}{4} \, \widetilde{\gamma}_K^{(i)} \left(\alpha_s \right) \, , \qquad \forall i \, .$$

- One may add to the dipole formula a solution to the homogeneous equation

$$\Gamma^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s \right) = \Gamma^{\overline{\mathcal{S}}}_{\rm dip} \left(\rho_{ij}, \alpha_s \right) \, + \, \Delta^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s \right) \, ,$$

where Δ must be a function of the conformal cross ratios ρ_{ijkl} ,

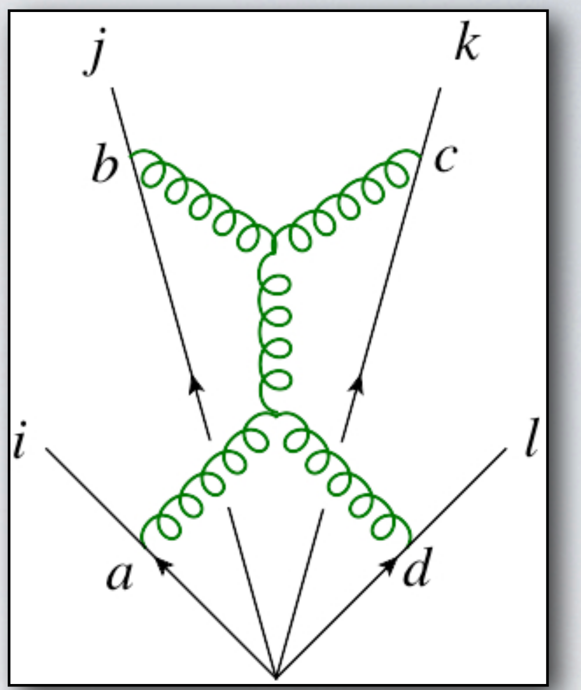
$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Delta^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s \right) = 0 \qquad \Leftrightarrow \qquad \Delta^{\overline{\mathcal{S}}} = \Delta^{\overline{\mathcal{S}}} \left(\rho_{ijkl}, \alpha_s \right) \, .$$

- Δ must directly correlate four partons: by the rules of eikonal exponentiation, it can start contributing at three loops.

- The functional form of Δ is further constrained by consistency in all collinear limits, Bose symmetry and transcendentality bounds.
(T. Becher, M. Neubert; L. Dixon, E. Gardi, LM)

Beyond the Minimal Solution

- Universality of collinear singularities as two momenta p_1 and p_2 become collinear forces the combination $\Delta_n(\rho_{ijkl}) - \Delta_{n-1}(\rho_{ijkl})$ to depend only on partons 1 and 2 in the collinear limit.
- The degree of transcendentality of the functions occurring in Δ at L loops is bound by $\tau < 2L$.
- Kinematic and color tensors in Δ must conspire to obey Bose symmetry



A three-loop web correlating four eikonal lines

As an example, a generic four-parton correlation takes the form

$$\Delta_4(\rho_{ijkl}) = \sum_i h^{(i)}_{abcd}\, \mathbf{T}^a_i\, \mathbf{T}^b_j\, \mathbf{T}^c_k\, \mathbf{T}^d_l\, \Delta^{(i)}_{4,\,\mathrm{kin}}(\rho_{ijkl}),$$

Considering for simplicity polynomial functions of $L_{ijkl} = \log(\rho_{ijkl})$, introducing three-loop color tensors, and enforcing Bose symmetry leads to a unique set of solutions

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}^a_1\mathbf{T}^b_2\mathbf{T}^c_3\mathbf{T}^d_4 \left[f_{ade}\, f_{cb}{}^e\, L^{h_1}_{1234} \left(L^{h_2}_{1423}\, L^{h_3}_{1342} - (-1)^{h_1+h_2+h_3}\, L^{h_2}_{1342}\, L^{h_3}_{1423} \right) + \mathrm{cycl.} \right].$$

Transcendentality and collinear consistency imply $h_1 + h_2 + h_3 \leq \tau \leq 5\,; \quad h_i \geq 1 \quad \forall i\,.$

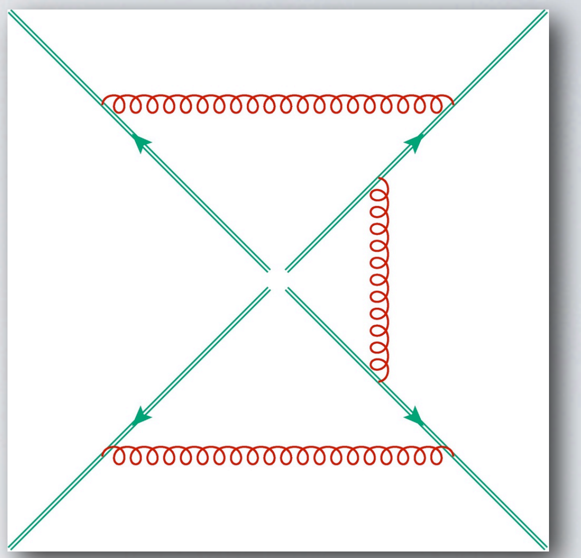
Functions that satisfy all constraints can be found, such as the above with $h_1 = 1, h_2 = h_3 = 2$. Quadrupole corrections to the dipole formula at three loops cannot be ruled out.

PERSPECTIVE

Recent Developments

- Understanding the structure of the multileg exponent.
(E. Gardi, E. Laenen, G. Stavenga, C. White; A. Mitov, G. Sterman, I. Sung)

$$\mathcal{Z} \equiv \int [\mathcal{D}A_s^\mu]\, e^{iS(A_s^\mu)} \left[\Phi^{(1)} \otimes \cdots \otimes \Phi^{(L)}\right] = \exp\left[\sum_D \tilde{C}(D)\, \mathcal{F}(D)\right].$$



Is this a web?

- The concept of web has been generalized to correlators of multiple Wilson lines using a "replica trick".
- The criterion of two-eikonal-line irreducibility is not sufficient, cancellation of UV subdivergences more intricate.
- An algorithm exists to compute directly the eikonal exponent

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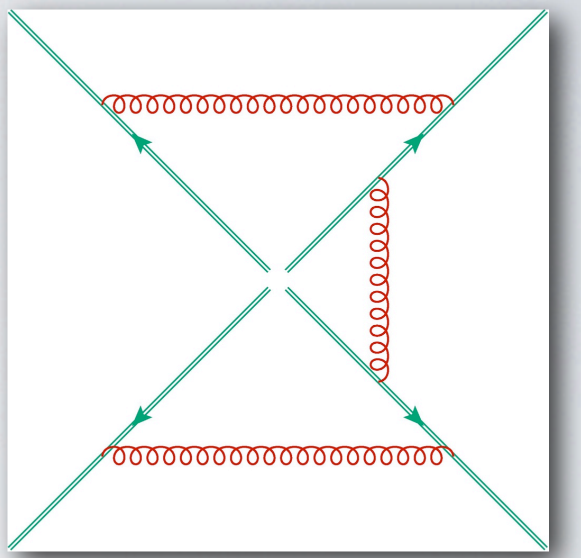

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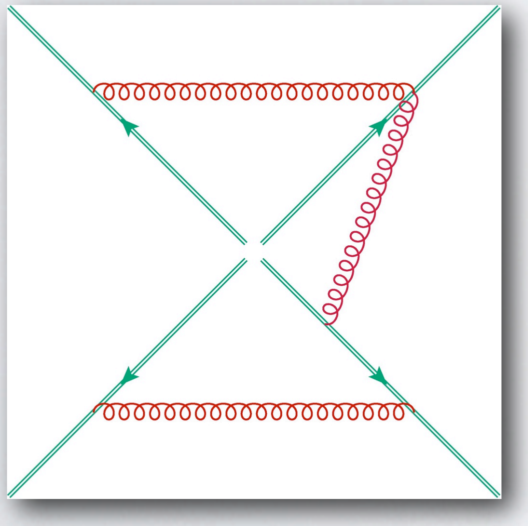

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- Extending exponentiation beyond the eikonal approximation (E. Laenen, LM, G. Stavenga, C. White)

$$\mathcal{M} = \mathcal{M}_0 \, \exp\left[\sum_{D_{\rm eik}} \tilde{C}(D_{\rm eik})\, \mathcal{F}(D_{\rm eik}) + \sum_{D_{\rm NE}} \tilde{C}(D_{\rm NE})\, \mathcal{F}(D_{\rm NE})\right] .$$

  - Phenomenological evidence that sub-eikonal logs exponentiate.
  - "Feynman rules" for the NE exponent, including "seagull" vertices.
  - Non-factorizable contribution studied using Low's theorem

Summary

- After $\mathcal{O}(10^2)$ years, soft and collinear singularities in gauge theories amplitudes are still a fertile field of study. A definitive solution may be at hand.
  - ✓ We are probing the all-order structure of the nonabelian exponent.
  - ✓ All-order results constrain, test and complement fixed-order calculations.
  - ✓ Understanding singularities has phenomenological applications through resummation.
- Factorization theorems $\Rightarrow$ Evolution equations $\Rightarrow$ Exponentiation.
- Dimensional continuation is the simplest and most elegant regulator.
  - ✓ Transparent mapping UV $\Rightarrow$ IR for `pure counterterm' functions.
- Remarkable simplifications in N = 4 SYM point to exact results.
- Factorization and velocity rescaling invariance severely constrain soft anomalous dimensions to all orders and for any number of legs.
- A simple dipole formula may encode all infrared singularites for any massless gauge theory, a natural generalization of the planar limit.
- The study of possible corrections to the dipole formula is under way.
- Next-to-eikonal contributions to amplitudes and cross sections can be organized.
- Applications to resummations, subtraction methods and parton showers are possible.

THANK YOU