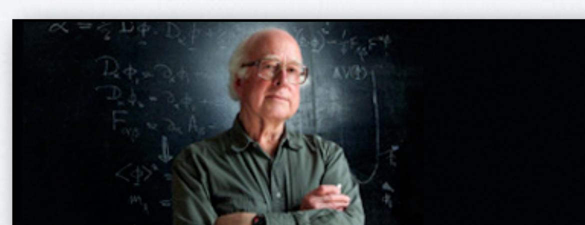
A BRIEF HISTORY OF NLP LOGARITHMS

Lorenzo Magnea

University of Torino - INFN Torino

HCTP - Edinburgh - 25/01/16



Outline

- Introduction
- Threshold resummations at leading power
- Gathering evidence beyond leading power
- Next-to-soft approximation
- A hard collinear problem
- Perspective

Outline

- Introduction
- Threshold resummations at leading power
- Gathering evidence beyond leading power
- Next-to-soft approximation
- A hard collinear problem ← **For the full story, see tomorrow's talk by Leonardo Vernazza**
- Perspective

INTRODUCTION

Logarithms

- Multi-scale problems in renormalizable quantum field theories have perturbative corrections of the form $\alpha_s^n \log^k \left(Q_i^2/Q_j^2\right)$
  - Renormalization and factorization logs: $\alpha_s^n \log^n \left(Q^2/\mu^2\right)$
  - High-energy logs: $\alpha_s^n \log^{n-1} \left(s/t\right)$
  - Sudakov logs: $\alpha_s^n \log^{2n-1} \left(1-z\right)\,, \quad 1-z = W^2/Q^2,\, 1 - M^2/\hat{s}\,,\, Q_\perp^2/Q^2,\, \ldots$

- Logarithms encode process-independent features of perturbation theory. For Sudakov logs: the structure of infrared and collinear divergences.

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)$$

- Threshold (and p_T) logarithms arise for distributions with point support at Born level.
  - Leading power logarithms are then singular (finiteness needs virtual corrections).

More logarithms

- Threshold logarithms are associated with kinematic variables ξ that vanish at Born level and get corrections that are enhanced because phase space for real radiation is restricted near partonic threshold: examples are $1 - T$, $1 - M^2/\hat{s}$, $1 - x_{BJ}$.

- At leading power in the threshold variable ξ logarithms are directly related to soft and collinear divergences: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+p\epsilon}} = -\frac{1}{p\epsilon}\,\delta(\xi) + \left(\frac{1}{\xi}\right)_+ - p\epsilon\left(\frac{\log\xi}{\xi}\right)_+ + \ldots$$

Cancels virtual IR poles

Leading power threshold logs

- Beyond the leading power, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1}\left[c_{nm}^{(-1)}\left(\frac{\log^m\xi}{\xi}\right)_+ + c_n^{(\delta)}\,\delta(\xi) + c_{nm}^{(0)}\log^m\xi + \ldots\right]$$

Resummed to high accuracy

All-order structure in some cases

NLP threshold logs

- The structure of NLP logarithms may be understood to all orders: **new territory!**

LEADING POWER

Electroweak annihilation

We will focus on processes involving parton annihilation into electroweak final states (Drell-Yan, Higgs, di-boson final states): very well understood at LP, simpler at NLP.

- LP threshold resummation is based on factorization: the Mellin-space partonic cross section reads

$$\omega(N,\epsilon) = \left|H_{\rm DY}\right|^2 \psi(N,\epsilon)^2 U(N) + \mathcal{O}\left(\frac{1}{N}\right).$$

- Collinear poles can be subtracted with suitable parton distributions,

$$\widehat{\omega}_{\overline{\rm MS}}(N) \equiv \frac{\omega(N,\epsilon)}{\phi_{\overline{\rm MS}}(N,\epsilon)^2}$$

- Each factor in ω obeys evolution equations near threshold, leading to exponentiation.

$$\psi_R(N,\epsilon) = \exp\left\{\int_0^1 dz \, \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \, \kappa_\psi\left(\overline{\alpha}\left((1-y)^2 Q^2\right), \epsilon\right)\right\}.$$

- Real and virtual contributions can be treated separately.

G. Sterman / Hadronic cross sections

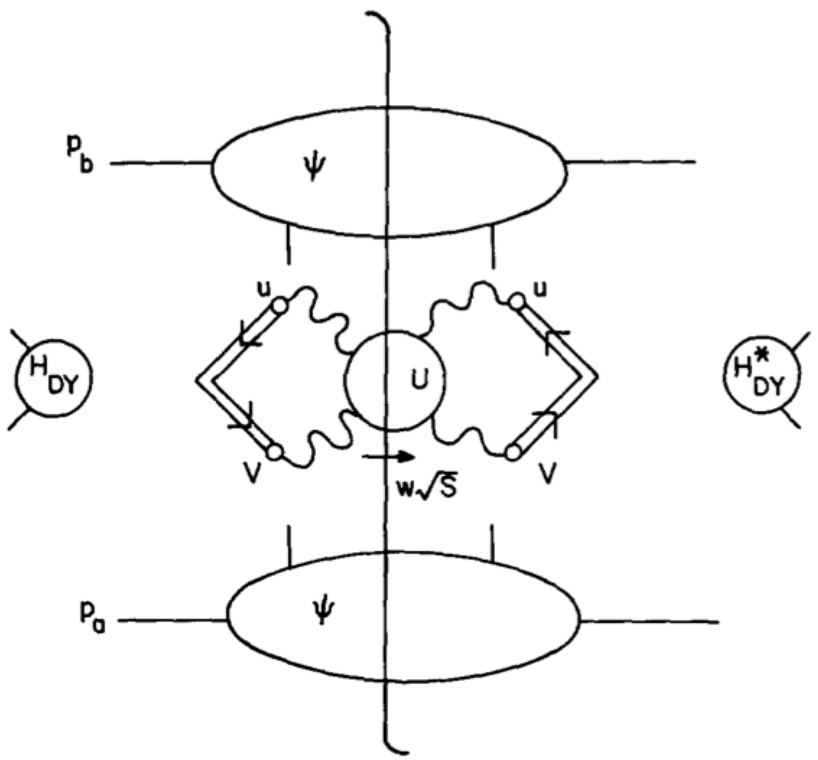


Fig 2.2 Graphical form of $z \to 1$ factorization for the DY process.

The original factorization near threshold

Color singlet hard scattering

A well-established formalism exists for distributions in processes that are electroweak at tree level (Gardi, Grunberg 07). For an observable r vanishing in the two-jet limit

$$\frac{d\sigma}{dr} = \delta(r)\big[1 + \mathcal{O}(\alpha_s)\big] + C_R \frac{\alpha_s}{\pi} \left\{ \left[-\frac{\log r}{r} + \frac{b_1 - d_1}{r} \right]_+ + \mathcal{O}(r^0) \right\} + \mathcal{O}(\alpha_s^2)$$

The Mellin (Laplace) transform, $\sigma(N) = \int_0^1 dr\,(1-r)^{N-1}\,\frac{d\sigma}{dr}$

exhibits log N singularities that can be organized in exponential form

$$\sigma\left(\alpha_s, N, Q^2\right) = H(\alpha_s)\,\mathcal{S}\left(\alpha_s, N, Q^2\right) + \mathcal{O}\left(1/N\right)$$

where the exponent of the `Sudakov factor' is in turn a Mellin transform

$$\mathcal{S}\left(\alpha_s, N, Q^2\right) = \exp\left\{ \int_0^1 \frac{dr}{r} \left[(1-r)^{N-1} - 1 \right] \mathcal{E}\left(\alpha_s, r, Q^2\right) \right\}$$

and the general form of the kernel is

$$\mathcal{E}\left(\alpha_s, r, Q^2\right) = \int_{r^2 Q^2}^{r Q^2} \frac{d\xi^2}{\xi^2}\, A\left(\alpha_s(\xi^2)\right) + B\left(\alpha_s(r Q^2)\right) + D\left(\alpha_s(r^2 Q^2)\right)$$

where A is the cusp anomalous dimension, and B and D have distinct physical characters.

The LP perturbative exponent

A classic way to organize Sudakov logarithms is in terms of the Mellin (Laplace) transform of the momentum space cross section (Catani et al. 93),

$$d\sigma(\alpha_s, N) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=0}^{2n} c_{nk} \log^k N + \mathcal{O}(1/N)$$
$$= H(\alpha_s) \exp\left[\log N\, g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s\, g_3(\alpha_s \log N) + \ldots\right] + \mathcal{O}(1/N)$$

This displays the main features of LP Sudakov resummation

- Predictive: a k-loop calculation determines g_k and thus a whole tower of logarithms to all orders in perturbation theory.
- Effective:
  - the range of applicability of perturbation theory is extended (finite order: $\alpha_s \log^2 N$ small. NLL resummed: α_s small);
  - the renormalization scale dependence is naturally reduced.
- Theoretically interesting: resummation ambiguities related to the Landau pole give access to non-perturbative power-suppressed corrections.
- Well understood:
  - NLL Sudakov resummations exist for most inclusive observables at hadron colliders, NNLL and approximate N^3LL in simple cases.

Non-logarithms

Delta-function terms arise from virtual corrections and phase space integration. They yield constants in Mellin space ("π^2") which can be controlled and "exponentiate" for simple processes (Parisi 80; Sterman 87; Eynck, Laenen, LM 03; Ahrens, Becher, Neubert, Yang 08),

- For EW annihilation, virtual terms reconstruct the full form factor.
- In dimensional regularization, each term exponentiates with no prefactor.

$$\begin{aligned}\omega(N,\epsilon) &= |H_{\mathrm{DY}}\,\mathcal{R}(\epsilon)\sqrt{U_V(\epsilon)}|^2\,\psi_R(N,\epsilon)^2\,U_R(N) + \mathcal{O}\left(\frac{1}{N}\right)\\ &= |\Gamma(Q^2,\epsilon)|^2\,\psi_R(N,\epsilon)^2\,U_R(N,\epsilon) + \mathcal{O}\left(\frac{1}{N}\right),\end{aligned}$$

- Real and virtual factors are separately finite.

$$\widehat{\omega}_{\overline{\mathrm{MS}}}(N) \equiv \frac{\omega(N,\epsilon)}{\phi_{\overline{\mathrm{MS}}}(N,\epsilon)^2} = \left(\frac{|\Gamma(Q^2,\epsilon)|^2}{\phi_V(\epsilon)^2}\right)\left(\frac{\psi_R(N,\epsilon)^2\,U_R(N,\epsilon)}{\phi_R(N,\epsilon)^2}\right) + \mathcal{O}\left(\frac{1}{N}\right)$$

- An improved resummation formula can be written for DY, DIS and Higgs total rates: all constants are defined in the exponent.

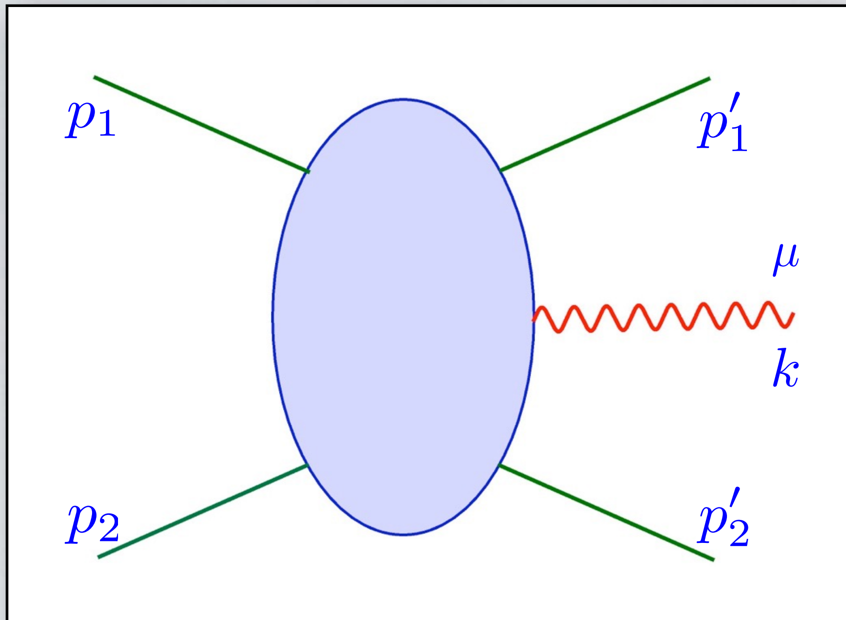
$$\begin{aligned}\widehat{\omega}_{\overline{\mathrm{MS}}}(N) = &\left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2\left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_V(\epsilon)}\right)^2 \exp\left[F_{\overline{\mathrm{MS}}}(\alpha_s)\right]\times\\ &\times\exp\left[\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left\{2\int_{Q^2}^{(1-z)^2Q^2}\frac{d\mu^2}{\mu^2}A\left(\alpha_s(\mu^2)\right)+\right.\right.\\ &\left.\left.+D\left(\alpha_s\left((1-z)^2Q^2\right)\right)\right\}\right]+\mathcal{O}\left(\frac{1}{N}\right).\end{aligned}$$

Less predictive than conventional resummation: the exponent receives corrections order by order. Empirically, exponentiated lower-order constants provide much of the exact result.

GATHERING EVIDENCE

The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)



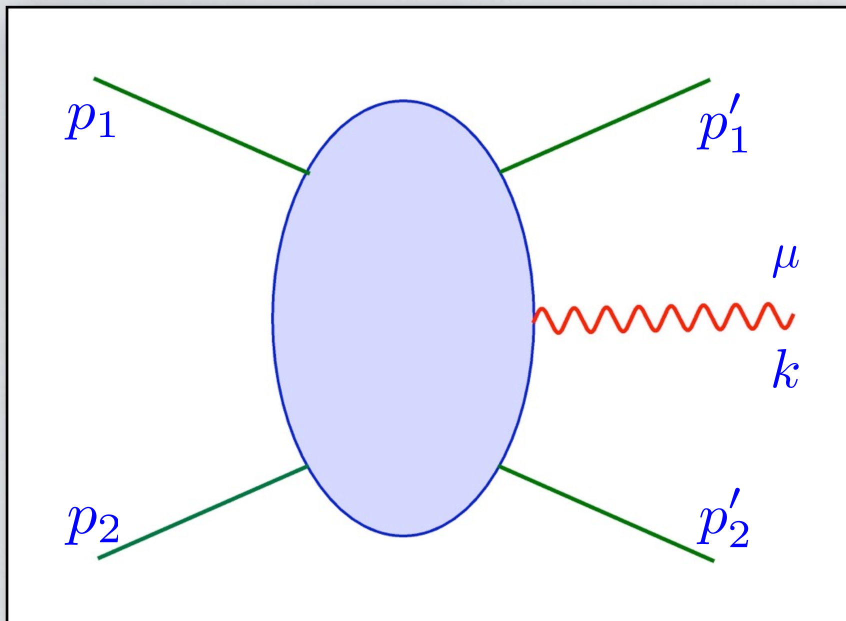
A radiative matrix element

$$M_\mu = e\left(\frac{p_{1\mu}'}{p_1'\cdot k} - \frac{p_{1\mu}}{p_1\cdot k}\right) T(\nu,\Delta) + e\left(\frac{p_{1\mu}' p_2'\cdot k}{p_1'\cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_2\cdot k}{p_1\cdot k} - p_{2\mu}\right)\frac{\partial T(\nu,\Delta)}{\partial \nu} + O(k), \qquad (1.7)$$

Low's original expression for the radiative matrix element

The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)



A radiative matrix element

Eikonal approximation

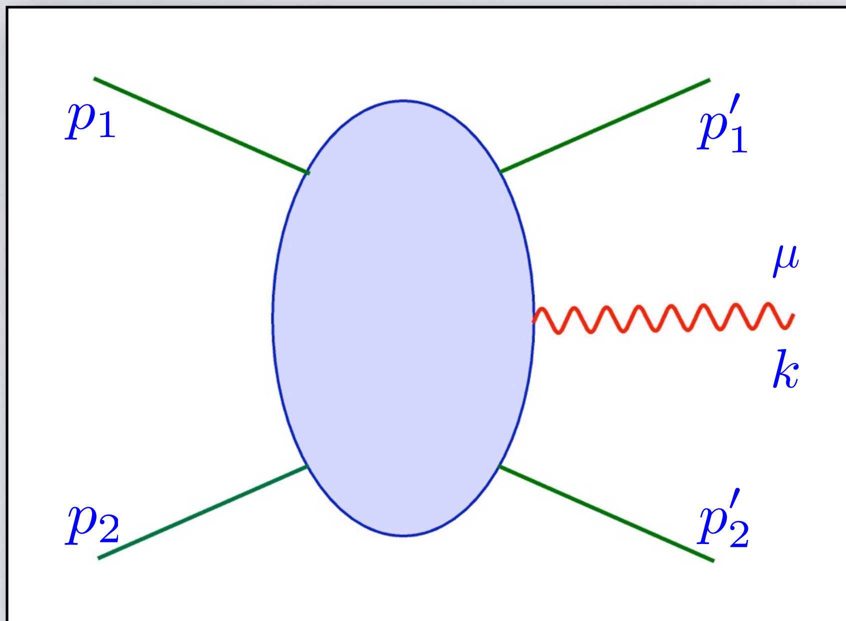
$$M_\mu = e\left(\frac{p_{1\mu}'}{p_1'\cdot k} - \frac{p_{1\mu}}{p_1\cdot k}\right) T(\nu,\Delta)$$
$$+ e\left(\frac{p_{1\mu}' p_2'\cdot k}{p_1'\cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_2\cdot k}{p_1\cdot k} - p_{2\mu}\right)\frac{\partial T(\nu,\Delta)}{\partial \nu} + O(k), \qquad (1.7)$$

Low's original expression for the radiative matrix element

Next-to-eikonal contribution

The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)



A radiative matrix element

Eikonal approximation

$$M_\mu = e\left(\frac{p_{1\mu}'}{p_1'\cdot k} - \frac{p_{1\mu}}{p_1\cdot k}\right)T(\nu,\Delta)$$
$$+ e\left(\frac{p_{1\mu}' p_2'\cdot k}{p_1'\cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_2\cdot k}{p_1\cdot k} - p_{2\mu}\right)\frac{\partial T(\nu,\Delta)}{\partial \nu} + O(k), \qquad (1.7)$$

Low's original expression for the radiative matrix element

Next-to-eikonal contribution

The radiative matrix element for the emission of a (next-to-) soft photon is determined by the Born amplitude T and its first derivative w.r.t. external momenta.

- Low's result established for a single charged scalar particle, follows from gauge invariance.
- It generalizes the well known properties of soft emissions in the eikonal approximation.
- The theorem was extended by (Burnett, Kroll 68) to particles with spin.
- The LBK theorem applies to massive particles and uses the mass as a collinear cutoff.
- It was extended to massless particles by (Del Duca 90), as discussed below.

Modified DGLAP

An important source of known NLP logarithms is the DGLAP anomalous dimension. Non-trivial connections between LP and NLP logarithms in DGLAP were uncovered (Moch, Vermaseren, Vogt 08) and made systematic (Dokshitzer, Marchesini, Salam 08).

Conventional DGLAP for a quark distribution reads

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}, \mu^2\right) P_{qq}(z, \alpha_s(\mu^2)).$$

$\longrightarrow$

$$\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2),$$

The large-N behavior of the anomalous dimension is single-logarithmic in the MS scheme. NLP terms suppressed by N are related to LP

$$\gamma_N(\alpha_s) = -A(\alpha_s) \ln \bar{N} + B_\delta(\alpha_s) - C_\gamma(\alpha_s) \frac{\ln \bar{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

MVV relations

$$C_1 = 0, \qquad C_2 = 4 C_F A_1, \qquad C_3 = 8 C_F A_2. \qquad (3.12)$$

Especially the relation for C_3 is very suggestive and seems to call for a structural explanation.

These relation extend to the function D, and recursively to all orders: modified splitting functions can be defined which vanish at large x beyond one loop.

Modified DGLAP

DMS, with refinements implied by (Basso, Korchemsky 06) propose to modify DGLAP as

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi\left(\frac{x}{z}, z^\sigma \mu^2\right) \mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right).$$

applying to both PDF's and fragmentation, with $\sigma = \pm 1$ respectively, and the same kernel (Gribov-Lipatov reciprocity). The resulting kernel P and is claimed to vanish as $z \to 1$ beyond one loop in the "physical" MC scheme where $\alpha_s = \gamma_{cusp}$. Therefore

$$\mathcal{P}(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s)\delta(1-z) + \mathcal{O}(1-z).$$

The modified equation cannot be diagonalized by Mellin transform: it must be solved by iteration, using a formal translation operator

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} e^{-\ln z \left(\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2}\right)} \times \psi\left(\frac{x}{z}, \mu^2\right) \mathcal{P}(z, \alpha_s(\mu^2)),$$

In practice, this procedure constructs a modified kernel where high-order terms are generated by shifts of lower-orders

An educated guess

Available NLP information can be combined in an ansatz for generalized threshold resummation applicable to EW annihilation processes and DIS (Laenen, LM, Stavenga 08).

$$
\begin{aligned}
\ln\left[\hat{\omega}(N)\right] = {} & \mathcal{F}_{\rm DY}\left(\alpha_s(Q^2)\right) \\
& + \int_0^1 dz\, z^{N-1} \left\{ \frac{1}{1-z} D\left[\alpha_s\left(\frac{(1-z)^2 Q^2}{z}\right)\right] \right. \\
& \left. + 2 \int_{Q^2}^{(1-z)^2 Q^2/z} \frac{dq^2}{q^2} P_s\left[z, \alpha_s(q^2)\right] \right\}_+ ,
\end{aligned}
$$

Exponentiation of constants

Refinement of phase space

DMS kernel

$$\frac{2}{1-z} \longrightarrow \frac{2\,z}{1-z}$$

This expression, and similar ones for DIS and Higgs production via gluon fusion, incorporate

- The exponentiation of N-independent terms.
- A treatment of phase space consistent up to O(1-z), including running coupling effects.
- The DMS modification of the DGLAP kernel, including the NLP term in the LO kernel.
- Note: DMS brings to the exponent a C_F^2 contribution crucial to fit two-loop NLP logs.

An educated guess: Drell-Yan

Parametrizing DY with

$$\widehat{\omega}(N) = \sum_{i=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \left[\sum_{m=0}^{2n} a_{nm} \ln^m \bar{N} + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m \bar{N}}{N}\right] + \mathcal{O}\left(\frac{\ln^p N}{N^2}\right),$$

Table 1
Comparison of exact and resummed 2-loop coefficients for the Drell–Yan cross section. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

	C_F^2		$C_A C_F$		$n_f C_F$	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

Table 2
Comparison of exact and resummed 2-loop coefficients for the DIS structure function. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

	C_F^2		$C_A C_F$		$n_f C_F$	
d_{23}	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
d_{22}	$\frac{39}{16}$	$\frac{55}{16}$	$\frac{11}{48}$	$\frac{11}{48}$	$-\frac{1}{24}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{1}{4}\zeta_2 - \frac{105}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$	$-\frac{107}{144}$	$-\frac{47}{144}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2 - \frac{431}{64}$	$-\frac{3}{4}\zeta_3 + \frac{53}{16}\zeta_2 - \frac{21}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2 - \frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2 - \frac{953}{1728}$	$\frac{1}{24}\zeta_2 - \frac{1699}{864}$	$-\frac{1}{8}\zeta_2 + \frac{73}{864}$

- Only one-loop NLP and DMS input has been used in the resummation formula.
- Leading NLP logarithms are reproduced exactly for all color structures at two loops.
- NLL and NNLL NLP logarithms are well approximated but not exact.
- Similar results hold for three-loop DIS, using two-loop information in the exponent.

Towards systematics

The problem of NLP threshold logarithms has been of interest for a long time, and several different approaches have been proposed. Recent years have seen a resurgence of interest, both from a theoretical point of view and for phenomenology.

- Early attempts include a study of the impact of NLP logs on the Higgs cross section by Kraemer, Laenen, Spira (98); work on F_L by Akhoury and Sterman (99) (logs without plus distributions are however leading) and work by Grunberg *et al.* (07-09) on DIS.
- Important results can be obtained by using physical kernels (Vogt *et al.* 09-14) which are conjectured to be single-logarithmic at large z, which poses constraints on their factorized expression. Note in particular a recent application to Higgs production by De Florian, Mazzitelli, Moch, Vogt (14).
- Useful approximations can be obtained by combining constraints from large N with high-energy constraints for $N \sim 1$ and analiticity (Ball, Bonvini, Forte, Marzani, Ridolfi, 13), together with phase space refinements.
- SCET techniques are well-suited to the problem: see thorough one-loop analysis in (Larkoski, Neill, Stewart,15), see also (Kolodrubetz, Moult, Neill, Stewart,16).
- A lot of recent formal work on the behavior of gauge and gravity scattering amplitudes beyond the eikonal limit was triggered by a link to asymptotic symmetries of the S matrix (many authors from A(ndy Strominger) to Z(vi Bern), 14-15).

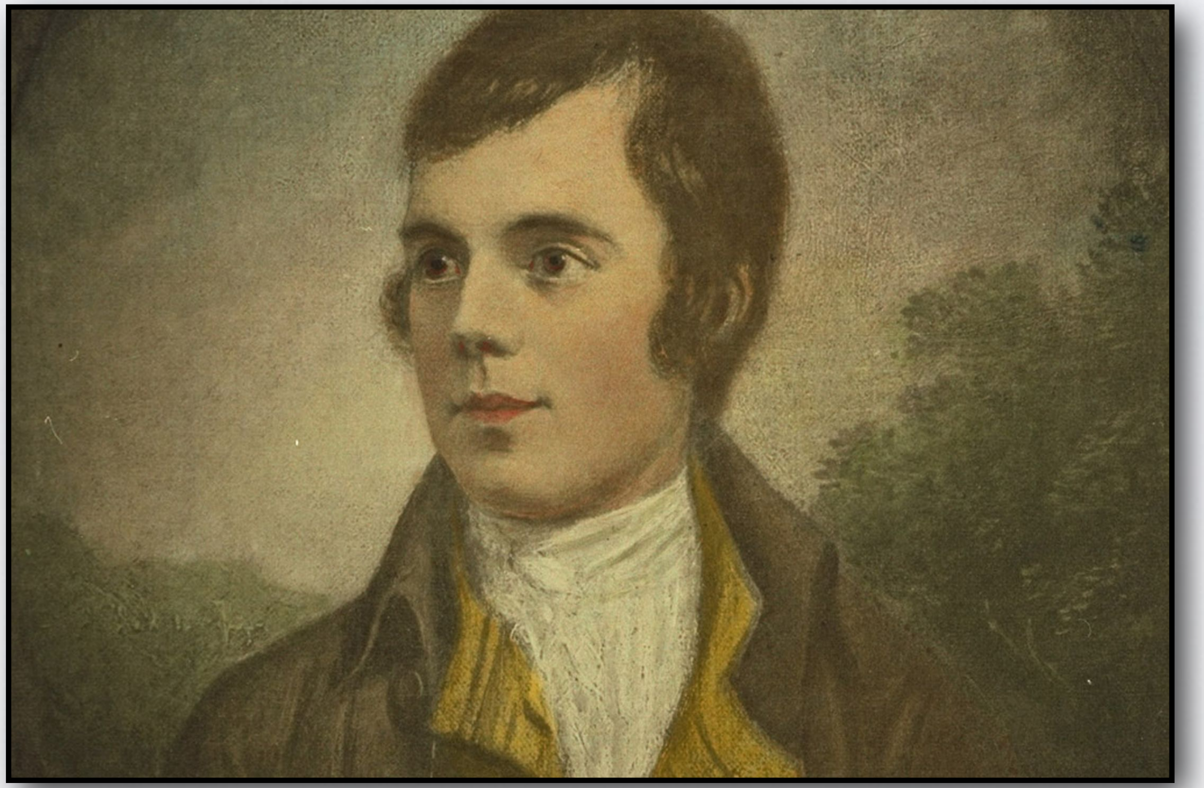
Towards systematics

The problem of NLP threshold logarithms has been of interest for a long time, and several different approaches have been proposed. Recent years have seen a resurgence of interest, both from a theoretical point of view and for phenomenology.

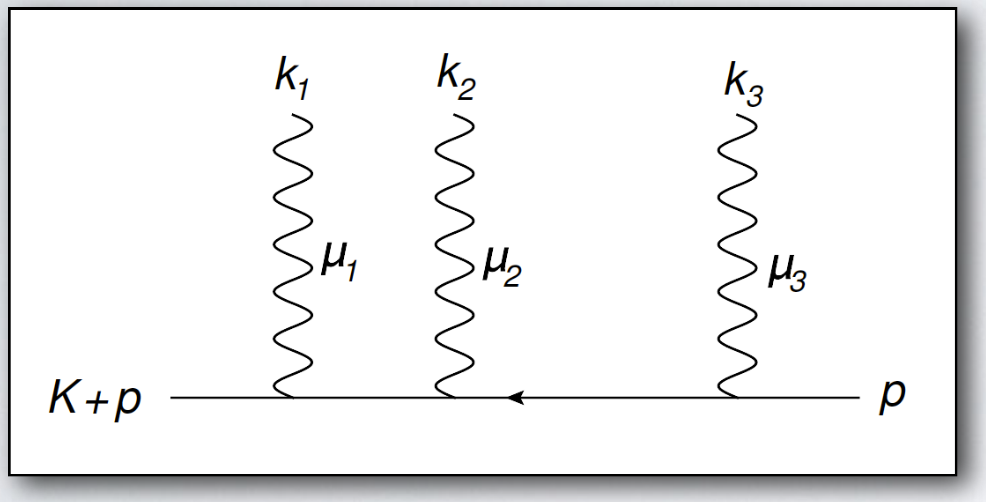
- Early attempts include a study of the impact of NLP logs on the Higgs cross section by Kraemer, Laenen, Spira (98); work on F_L by Akhoury and Sterman (99) (logs without plus distributions are however leading) and work by Grunberg *et al.* (07-09) on DIS.
- Important results can be obtained by using physical kernels (Vogt *et al.* 09-14) which are conjectured to be single-logarithmic at large z, which poses constraints on their factorized expression. Note in particular a recent application to Higgs production by De Florian, Mazzitelli, Moch, Vogt (14).
- Useful approximations can be obtained by combining constraints from large N with high-energy constraints for $N \sim 1$ and analiticity (Ball, Bonvini, Forte, Marzani, Ridolfi, 13), together with phase space refinements.
- SCET techniques are well-suited to the problem: see thorough one-loop analysis in (Larkoski, Neill, Stewart, 15), see also (Kolodrubetz, Moult, Neill, Stewart, 16).
- A lot of recent formal work on the behavior of gauge and gravity scattering amplitudes beyond the eikonal limit was triggered by a link to asymptotic symmetries of the S matrix (many authors from A(ndy Strominger) to Z(vi Bern), 14-15).

Thursday talk by Chris White

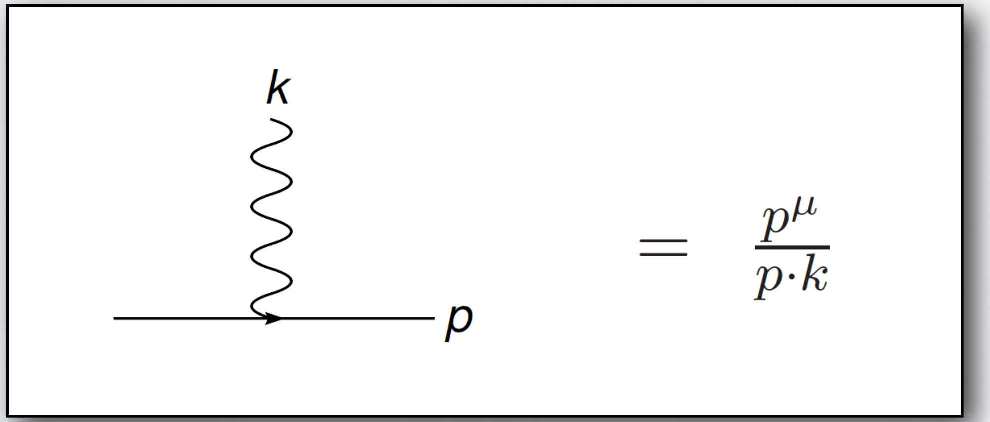
... *WE'LL TAK A CUP O' KINDNESS YET* ...



On the eikonal approximation



A fast particle emitting soft photons



Eikonal Feynman rule

- Taking the soft approximation at leading power on emissions from an energetic (or very massive) particle yields a set of simplified Feynman rules.

- These rules correspond to emissions from a Wilson line oriented along the trajectory of the energetic particle, in the same color irrep.

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A(\lambda n) \right] .$$

- The results do not depend on the energy and spin of the emitter, only on its direction and color charge.

- Physically, we are neglecting the recoil of the emitter: the only effect of interaction with soft radiation is that the emitter acquires a phase.

- The soft limit of a multi-particle amplitude is a correlator of Wilson lines

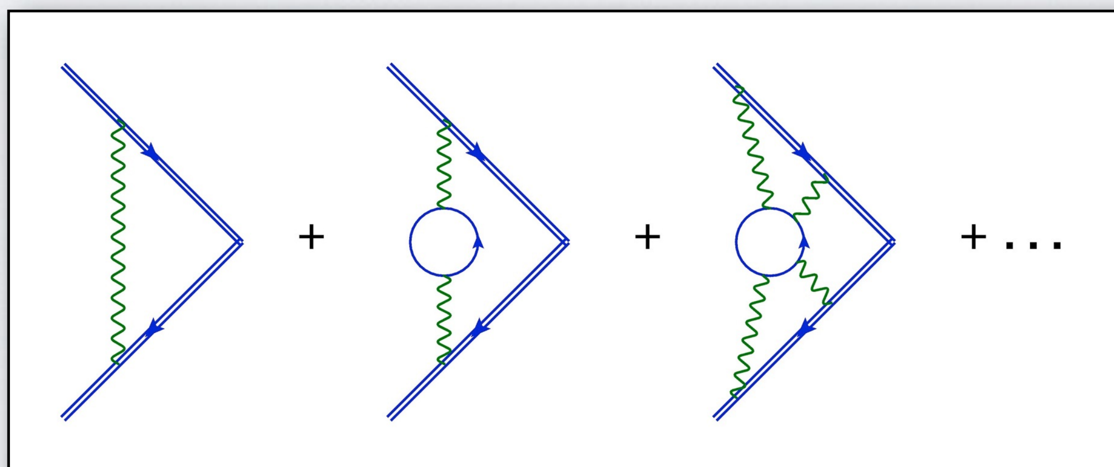
Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in exponential form.

$$S_n \equiv \langle 0 | \, \Phi_1 \otimes \ldots \otimes \Phi_n \, | 0 \rangle = \exp \left(\omega_n \right)$$

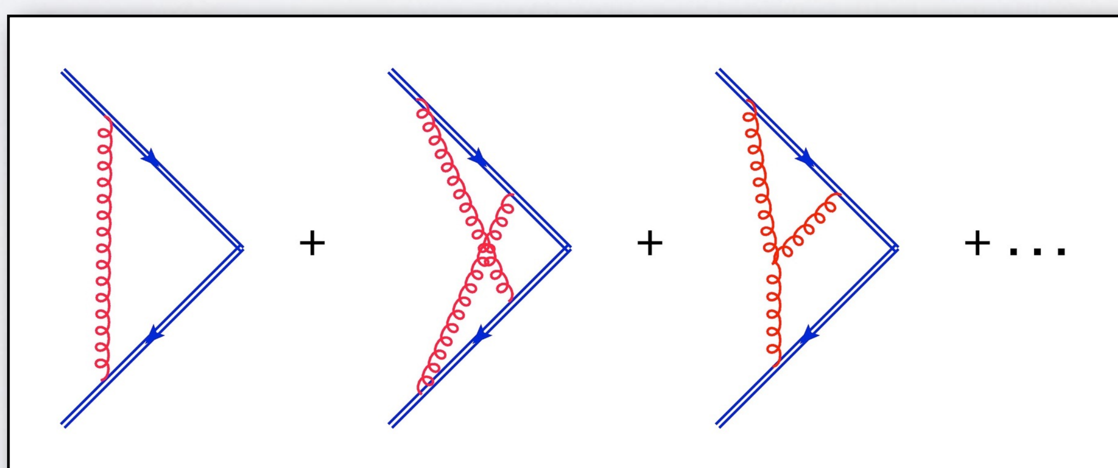
Diagrammatic rules exist to compute directly the logarithm of the correlators.

$$\omega_{2,\mathrm{QED}} =$$



Only connected photon subdiagrams contribute to the logarithm.

$$\omega_{2,\mathrm{QCD}} =$$



Only gluon subdiagrams which are two-eikonal irreducible contribute to the logarithm. They have modified color factors.

For eikonal form factors, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

Multiparticle webs

The concept of web generalizes non-trivially to the case of multiple Wilson lines. (Gardi, Smillie, White, *et al*).

A **web** is a set of diagrams which differ only by the order of the gluon attachments on each Wilson line. They are weighted by modified color factors.

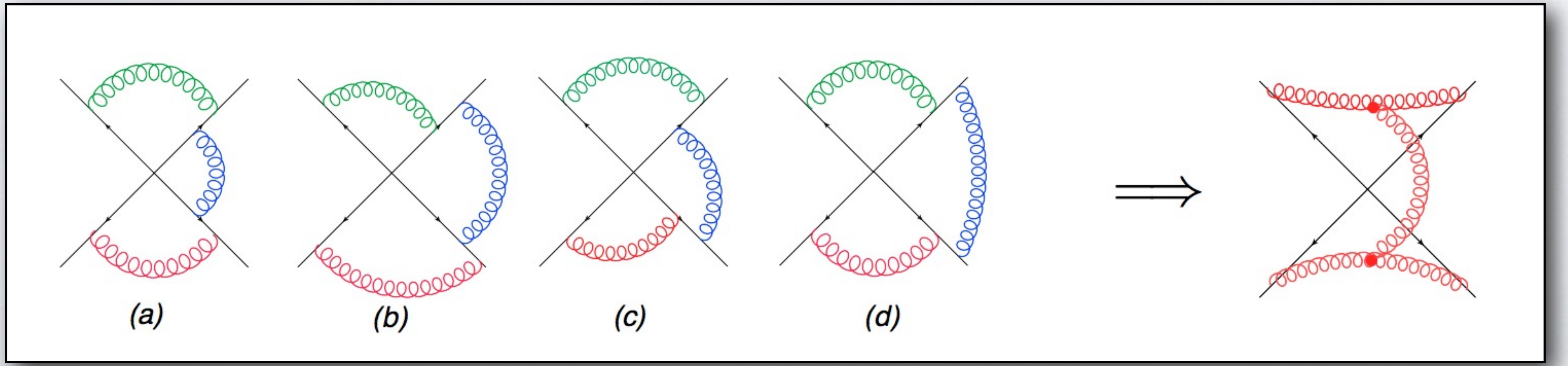
Writing each diagram as the product of its natural color factor and a kinematic factor

$$D = C(D)\mathcal{F}(D)$$

a web W can be expressed as a sum of diagrams in terms of a web mixing matrix R

$$W = \sum_{D} \widetilde{C}(D)\mathcal{F}(D) = \sum_{D,D'} C(D')R(D',D)\mathcal{F}(D)$$

The non-abelian exponentiation theorem holds: each web has the color factor of a fully connected gluon subdiagram (Gardi, Smillie, White).



Beyond the eikonal

The soft expansion can be organized beyond leading power using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic idea is simple, but the combinatorics cumbersome. For spinors

$$\frac{\not p + \not k}{2p \cdot k + k^2}\, \gamma^\mu u(p) \;=\; \left[\frac{p^\mu}{p \cdot k} + \frac{\not k\, \gamma^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} \right] u(p) + \mathcal{O}(k)$$

Eikonal   Spin-dependent   Spin-independent

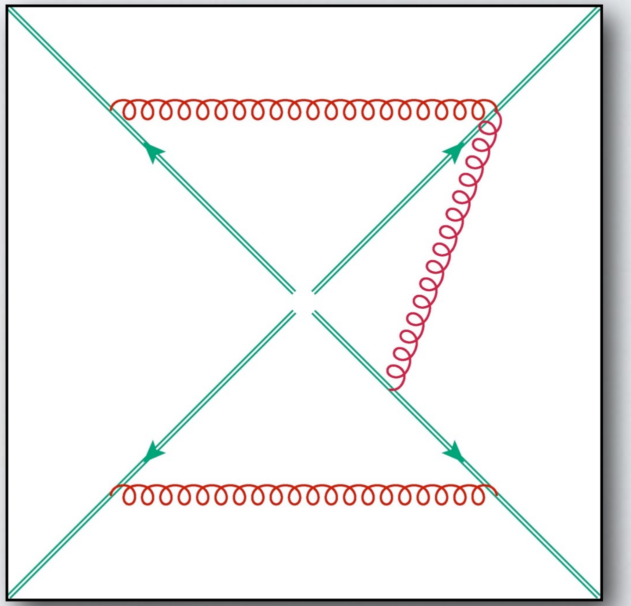
- A class of factorizable contributions exponentiate via NE webs

$$\mathcal{M} = \mathcal{M}_0 \, \exp \left[\sum_{D_{\rm eik}} \tilde{C}(D_{\rm eik})\, \mathcal{F}(D_{\rm eik}) + \sum_{D_{\rm NE}} \tilde{C}(D_{\rm NE})\, \mathcal{F}(D_{\rm NE}) \right] .$$

- Feynman rules exist for the NE exponent, including "seagull" vertices.

$$\mathcal{M} \;=\; \mathcal{M}_0 \, \exp \left[\mathcal{M}_{\rm eik} + \mathcal{M}_{\rm NE} \right] \left(1 + \mathcal{M}_r \right) + \mathcal{O}\left({\rm NNE} \right) \, .$$

- Non-factorizable contributions involve single gluon emission from inside the hard function, and must be studied using LBDK's theorem.



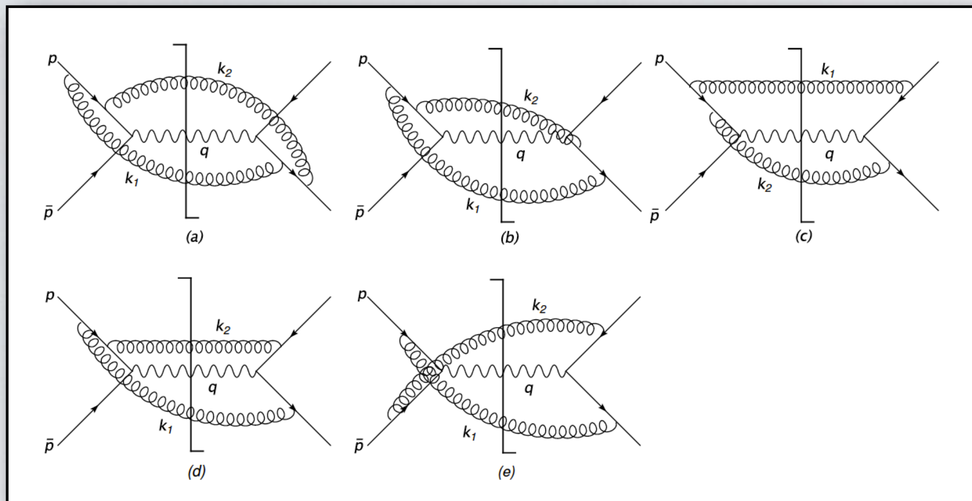
A next-to-eikonal web

Double real two-loop Drell-Yan

Multiple real emission contributions to EW annihilation processes involve only factorizable contributions. NE Feynman rules can be tested this level.

Defining the Drell-Yan K-factor as

$$K^{(n)}(z) \;=\; \frac{1}{\sigma^{(0)}}\frac{d\sigma^{(n)}(z)}{dz}\,,$$



Real emission Feynman diagrams for the abelian part of the NNLO K-factor.

As a test, we (re)computed the C_F^2 part of K at NNLO from ordinary Feynman diagrams, and then using NE Feynman rules, finding complete agreement. As expected, plus distributions arise from the eikonal approximation.

Next-to-eikonal terms arise from single-gluon corrections: seagull-type contributions vanish for the inclusive cross section.

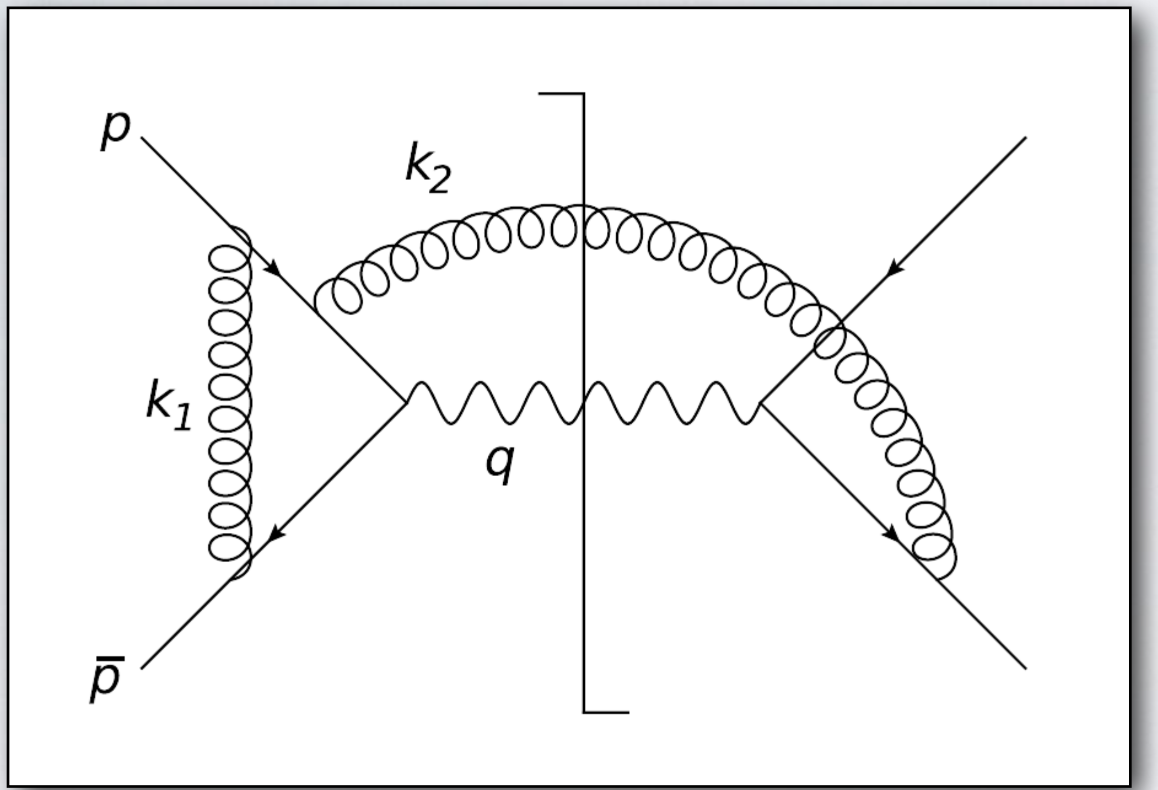
$$\begin{aligned} K^{(2)}_{\rm NE}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2 \Bigg[& -\frac{32}{\epsilon^3}\mathcal{D}_0(z) + \frac{128}{\epsilon^2}\mathcal{D}_1(z) - \frac{128}{\epsilon^2}\log(1-z) \\ & -\frac{256}{\epsilon}\mathcal{D}_2(z) + \frac{256}{\epsilon}\log^2(1-z) - \frac{320}{\epsilon}\log(1-z) \\ & +\frac{1024}{3}\mathcal{D}_3(z) - \frac{1024}{3}\log^3(1-z) + 640\log^2(1-z)\Bigg]\,, \end{aligned}$$

The abelian part of the NNLO K-factor from real emission, omitting constants

HARD COLLINEAR EMISSION

A collinear problem

Non-factorizable contributions start at NNLO. For massive particles they can be traced to the original LBK theorem. For massless particles a new contribution to NLP logs emerges.

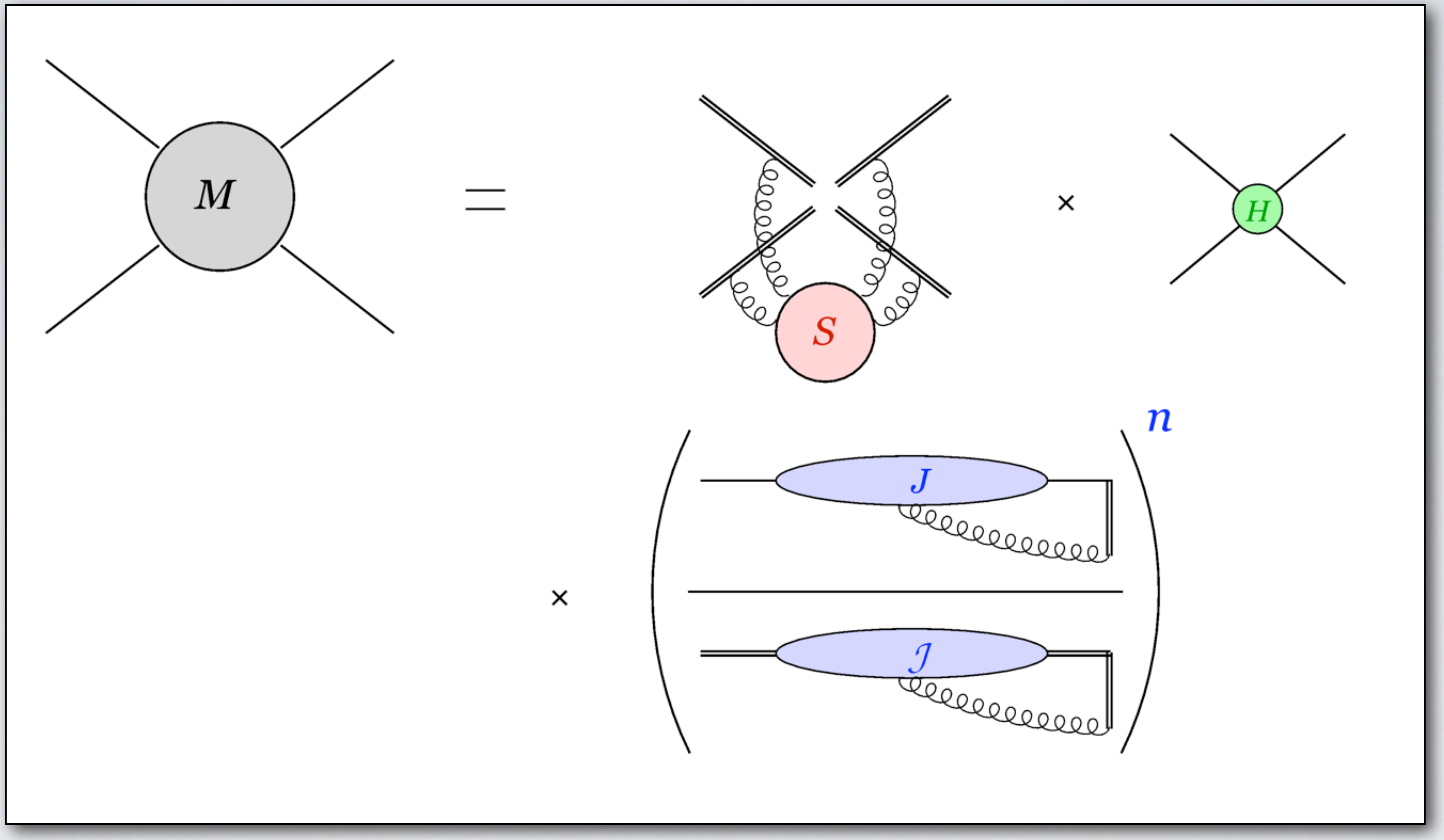


A Feynman diagram containing a collinear enhancement

- Gluon k_2 is always (next-to) soft for EW annihilation near threshold.
- When k_1 is (next-to) soft all logs are captured by NE rules.
- Contributions with k_1 hard and collinear are missed by the soft expansion.
- The collinear pole interferes with soft emission and generates NLP logs.
- The problem first arises at NNLO

- These contributions are missed by the LBK theorem: it applies to an expansion in E_k/m.
- They can be analized using the method of regions: the relevant factor is $(p \cdot k_2)^{-\varepsilon}/\varepsilon$.
- They cause the breakdown of next-to-soft theorems for amplitudes beyond tree level.
  ➡ the soft expansion and the limit $\varepsilon \to 0$ do not commute.
- They require an extension of LBK to $m^2/Q < E_k < m$. It was provided by Del Duca (90).

LP factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

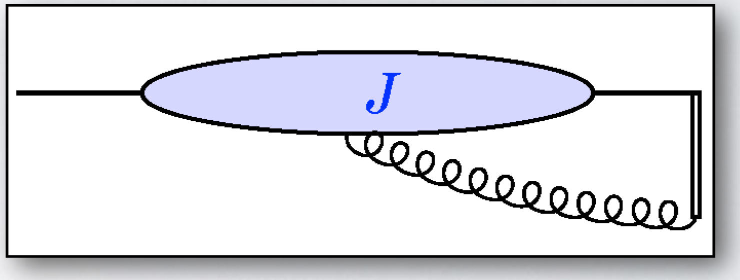
LP factorization: operators

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right],$$

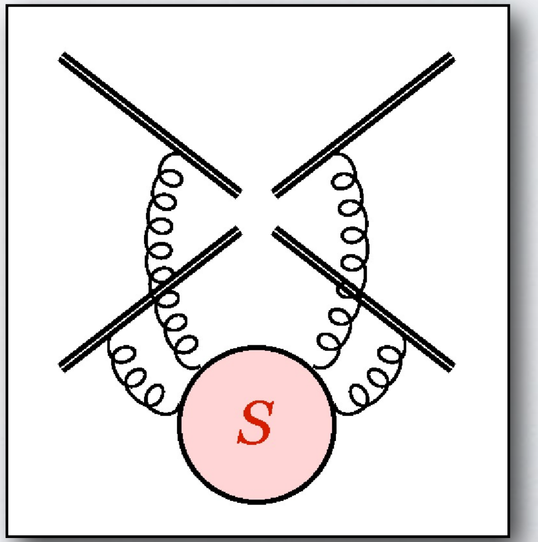
Here we introduced dimensionless four-velocities $\beta_i^\mu = Q\, p_i^\mu$, $\beta_i^2 = 0$, and factorization vectors n_i^μ, $n_i^2 \neq 0$ to define the jets,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 \,|\, \Phi_n(\infty, 0)\, \psi(0)\, |p\rangle\,.$$



where Φ_n is the Wilson line operator along the direction n^μ.

The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.



$$(c_L)_{\{a_k\}}\, \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \epsilon\right) = \langle 0 | \prod_{k=1}^{n} \left[\Phi_{\beta_k}(\infty, 0)\right]_{a_k}^{\;b_k} |0\rangle\, (c_K)_{\{b_k\}}\,,$$

where the c_L are a basis of color tensors for the process at hand.

Beyond Low's theorem

A slightly modified version of Del Duca's result gives the radiative amplitude in terms of the non-radiative one, its derivatives, and two "jet" functions.

$$\mathcal{A}^\mu(p_j,k) = \sum_{i=1}^{2} \left\{ q_i \left(\frac{(2p_i-k)^\mu}{2p_i\cdot k - k^2} + G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \right) + G_i^{\nu\mu} \left[\frac{J_\nu(p_i,k,n_i)}{J(p_i,n_i)} - q_i \frac{\partial}{\partial p_i^\nu} \Big(\ln J(p_i,n_i) \Big) \right] \right\} \mathcal{A}(p_i;p_j) \, .$$

The tensors $G^{\mu\nu}$ project out the eikonal contribution present in the first term.

$$J\Big(p,n,\alpha_s(\mu^2),\epsilon\Big) u(p) = \langle 0 \,|\, \Phi_n(\infty,0) \psi(0) \,|\, p \rangle \; ,$$

The jet of ordinary IR factorization.

$$J_\mu\left(p,n,k,\alpha_s(\mu^2),\epsilon\right) u(p) = \int d^d y \; \mathrm{e}^{-\mathrm{i}(p-k)\cdot y} \; \langle 0 \,|\; \Phi_n(y,\infty)\, \psi(y)\, j_\mu(0) \,|\, p \rangle \; ,$$

The radiative jet.

- At tree level the radiative jet displays the expected dependence on spin.
- Dependence on the gauge vector n^μ starts at loop level: simplifications arise for $n^2 = 0$.

$$J^{\nu(0)}(p,n,k) = \frac{\not{k}\gamma^\nu}{2p\cdot k} - \frac{p^\nu}{p\cdot k} = -\frac{p^\nu}{p\cdot k} + \frac{k^\nu}{2p\cdot k} - \frac{\mathrm{i}\, k_\alpha \Sigma^{\alpha\mu}}{2p\cdot k} \, .$$

PERSPECTIVE

A Perspective

- Perturbation theory continues to display new and unexplored structures.
- Leading power threshold resummation is highly developed and provides some of the most precise predictions in perturbative QCD.
- Mellin-space constants naturally reside in the exponent for simple processes.
- Low's theorem is the first of many hints that NLP logs can be understood and organized.
- Different approaches catch a number of towers of NLP logs in simple processes.
- The next-to-soft approximation is well understood, using both diagrammatic and path integral approaches, even for multi-parton processes.
- Hard collinear emissions spoil Low's theorem: a new radiative jet function emerges.
- A complete treatment of NLP threshold logs is at hand.
- Much work to do to organize a true resummation formula, even for EW annihilation: we have a more intricate "factorization", we must make sure to control double countings.
- In order to achieve complete generality, we will need to include final state jets.

THANK YOU!