PROGRESS ON THE INFRARED STRUCTURE OF GAUGE THEORY AMPLITUDES

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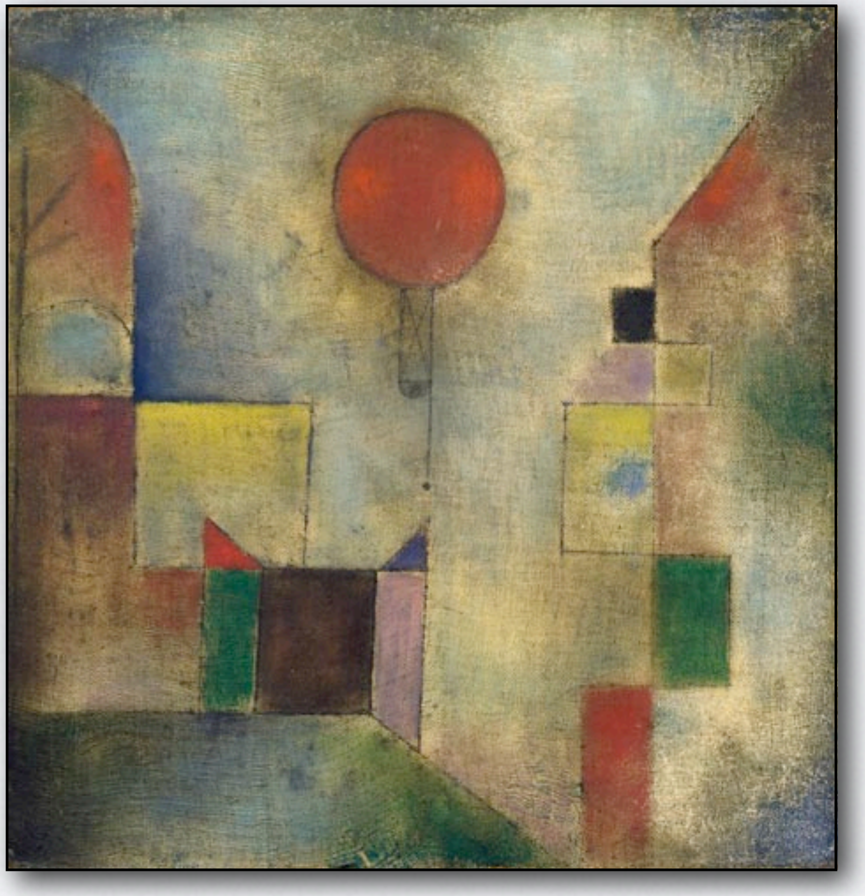
Outline

- On infrared divergences
- Dipoles and multipoles
- The high-energy limit
- Weaving multi-particle webs
- Loops and legs
- Outlook

ON INFRARED DIVERGENCES



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- The cancellation is incomplete.
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  - For exclusive final states: parton shower event generators, (N)LL accuracy.

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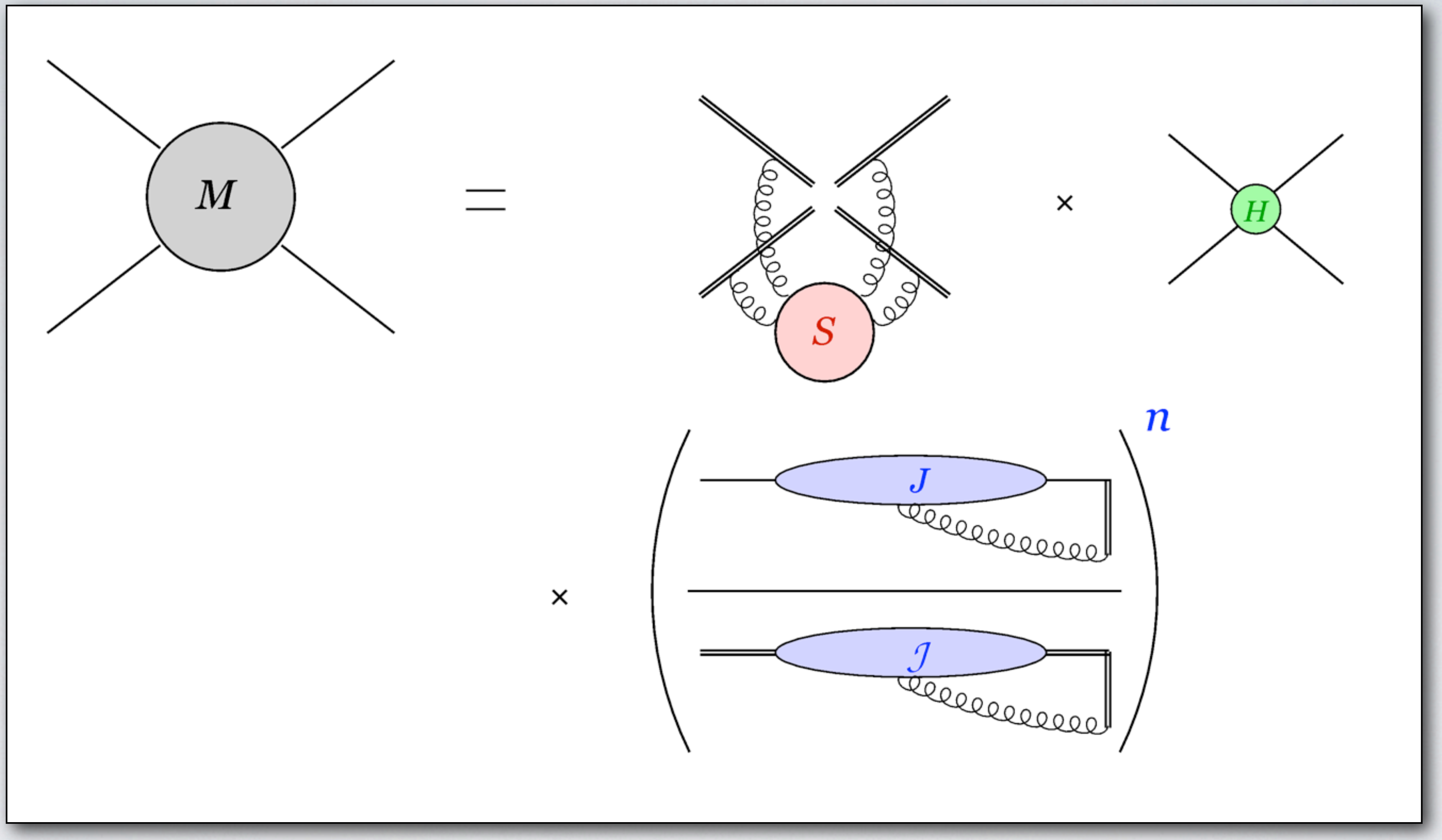
- There is actual (non-perturbative) physics in the IR.
  - We understand infrared radiation to all orders in any gauge theory.
  - Power-suppressed non-perturbative corrections to QCD cross sections can be modeled.
  - Links to the strong coupling regime can be established for SUSY gauge theories.
    - ➡ N = 4 Super Yang-Mills planar amplitudes: ABDKS ansatz.
    - ➡ Non planar amplitudes: awaiting string theory input.

Factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

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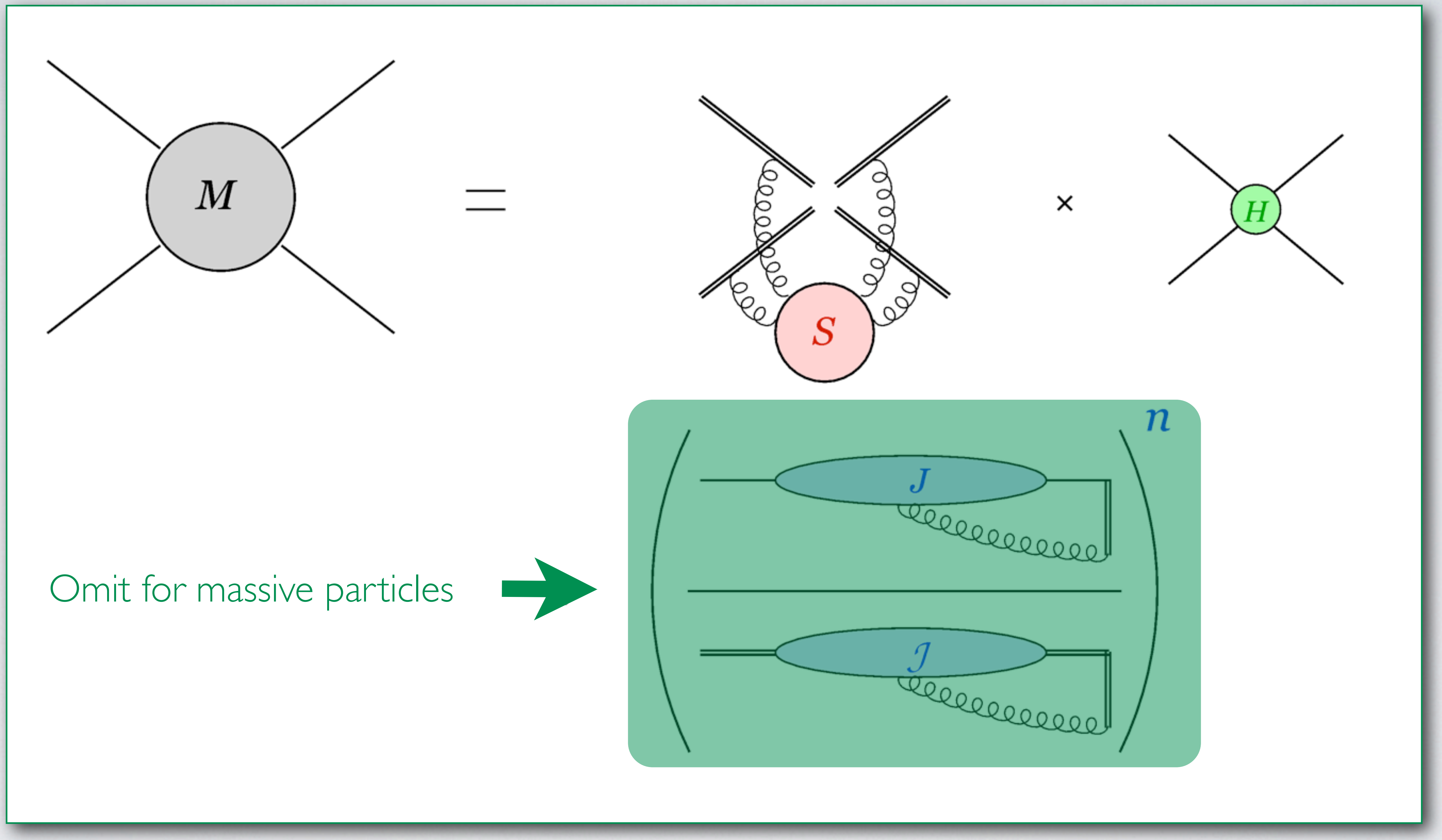


Omit for massive particles

A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

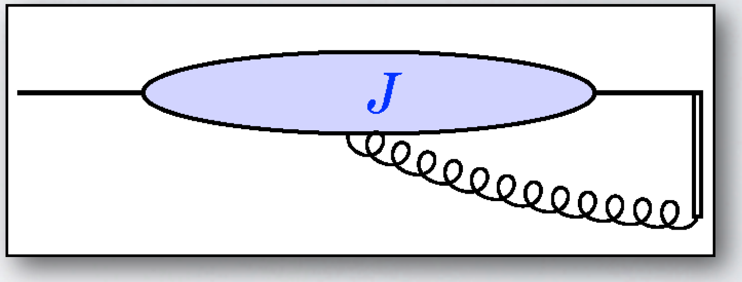
Operator Definitions

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right],$$

Here we introduced dimensionless four-velocities $p_i^\mu = Q\,\beta_i^\mu$, (for massless particles $\beta_i^2 = 0$), and factorization vectors n_i^μ, $n_i^2 \neq 0$ to define, if needed, the jets.

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 \,|\Phi_n(\infty, 0)\, \psi(0)\,| p \rangle\,.$$



where Φ_n is the Wilson line operator along the direction n^μ,

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[i g \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A(\lambda n)\right].$$

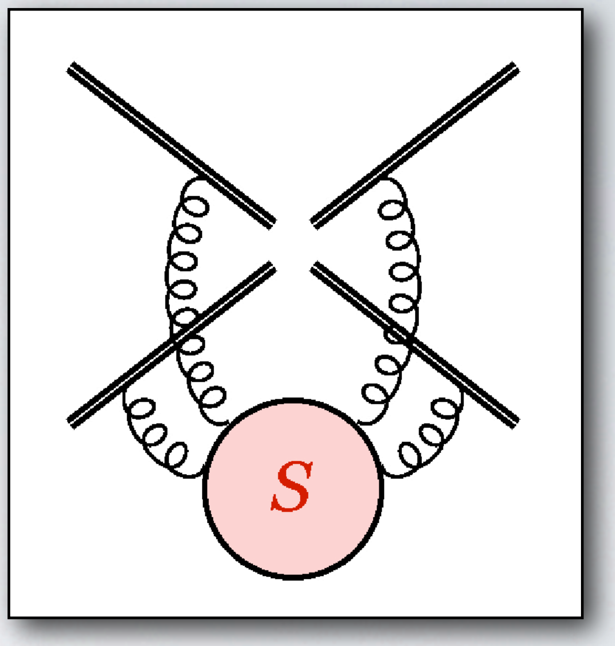
Note: Wilson lines represent fast or very massive particles, not recoiling against soft radiation.

The vectors n^μ:

- Ensure gauge invariance of the jets.
- Separate collinear gluons from wide-angle soft ones.
- Replace other hard partons with a collinear-safe absorber.

Soft Matrices

The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.



$$\left(c_L\right)_{\{a_k\}} \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \epsilon\right) = \langle 0| \prod_{k=1}^{n} \left[\Phi_{\beta_k}\left(\infty, 0\right)\right]_{a_k}^{\ b_k} |0\rangle \left(c_K\right)_{\{b_k\}} ,$$

The soft function S obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \epsilon\right) = - \mathcal{S}_{LJ}\left(\beta_i \cdot \beta_j, \epsilon\right) \Gamma^{\mathcal{S}}_{JK}\left(\beta_i \cdot \beta_j, \epsilon\right)$$

NOTE: Γ^S is singular for massless theories, due to overlapping UV and collinear poles.

S is a pure counterterm. In dimensional regularization, using $\alpha_s(\mu^2 = 0, \varepsilon < 0) = 0$,

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp\left[-\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon\right)\right] .$$

The determination of the soft anomalous dimension matrix Γ^S is the keystone of the resummation program for multiparton amplitudes and cross sections.

- It governs the interplay of color exchange with kinematics in multiparton processes.
- It is the only source of multiparton correlations for singular contributions.
- Collinear effects are `color singlet’ and can be extracted from two-parton scatterings.

DIPOLES AND MULTIPOLES

The Dipole Formula

For massless partons, the soft anomalous dimension matrix obeys an exact equation based on a `conformal anomaly', which correlates color exchange with kinematics.

The simplest solution to this equation is a sum over color dipoles (Becher, Neubert; Gardi, LM, 09). It gives an ansatz for the all-order singularity structure of all multiparton fixed-angle massless scattering amplitudes: the **dipole formula**.

- All soft and collinear singularities can be collected in a multiplicative operator Z

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

- Z contains both soft singularities from S, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d\ln\mu} Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = - Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) .$$

The matrix Γ has a surprisingly simple dipole structure. It reads

$$\boxed{\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right) \sum_{j\neq i} \ln\left(\frac{-2\,p_i\cdot p_j}{\mu^2}\right) \mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n} \gamma_{J_i}\left(\alpha_s(\mu^2)\right) .}$$

Note that all singularities are generated by integration over the scale of the coupling.

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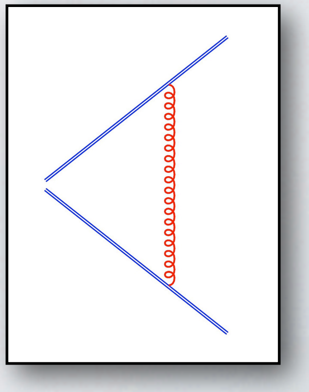

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Features of the dipole formula

- All known results for IR divergences of massless gauge theory amplitudes are recovered.
- The absence of multiparton correlations implies remarkable diagrammatic cancellations.
- The color matrix structure is fixed at one loop: path-ordering is not needed.
- All divergences are determined by a handful of anomalous dimensions.
- The cusp anomalous dimension plays a very special role: a universal IR coupling.

Can this be the definitive answer for IR divergences in massless non-abelian gauge theories?

- There are precisely two sources of possible corrections.
  - Quadrupole correlations may enter starting at three loops: they must be tightly constrained functions of conformal cross ratios of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) \,, \qquad \rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

  - The cusp anomalous dimension may violate Casimir scaling beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \, \widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)$$

  - The functional form of Δ is further constrained by: collinear limits, Bose symmetry, bounds on weights, high-energy constraints. (Becher, Neubert; Dixon, Gardi, LM, 09).
  - Recent evidence points to a non-vanishing Δ at four loops (Caron-Huot).

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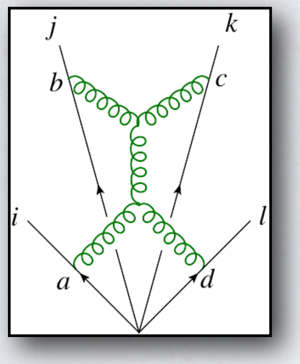

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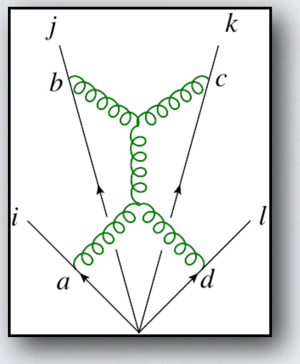

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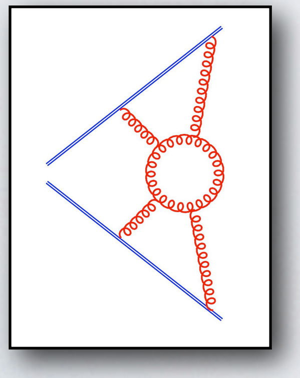

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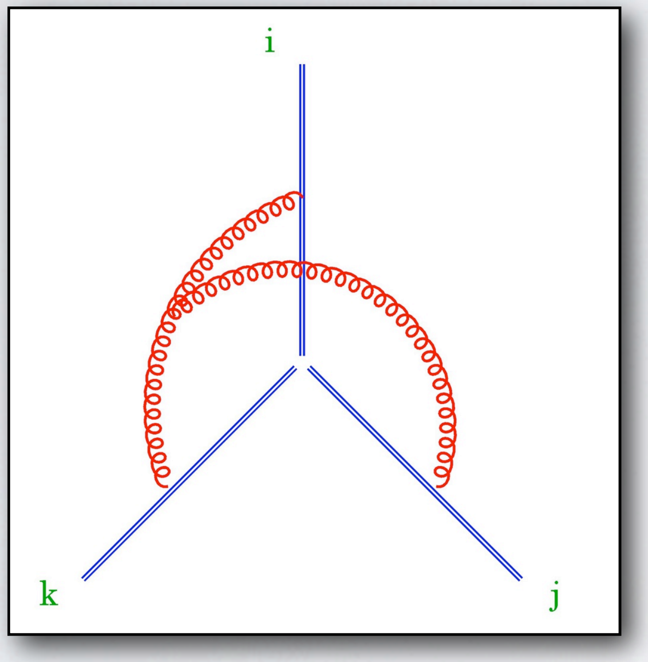
Ferroglia, Neubert, Pecjak, Yang 2009;
Kidonakis 2009; Mitov, Sterman, Sung 2010;
Chien, Schwartz, Simons-Duffin, Stewart 2011;

Tripoles at two loops

For massive partons, the conformal invariance of semi-infinite Wilson line correlators is not anomalous, and the soft anomalous dimension is unconstrained.

An explicit calculation shows that tripole color correlations arise (Ferroglia et al. 09). They are expressed in terms if the cusp angles between massive partons



The connected two-loop tripole diagram

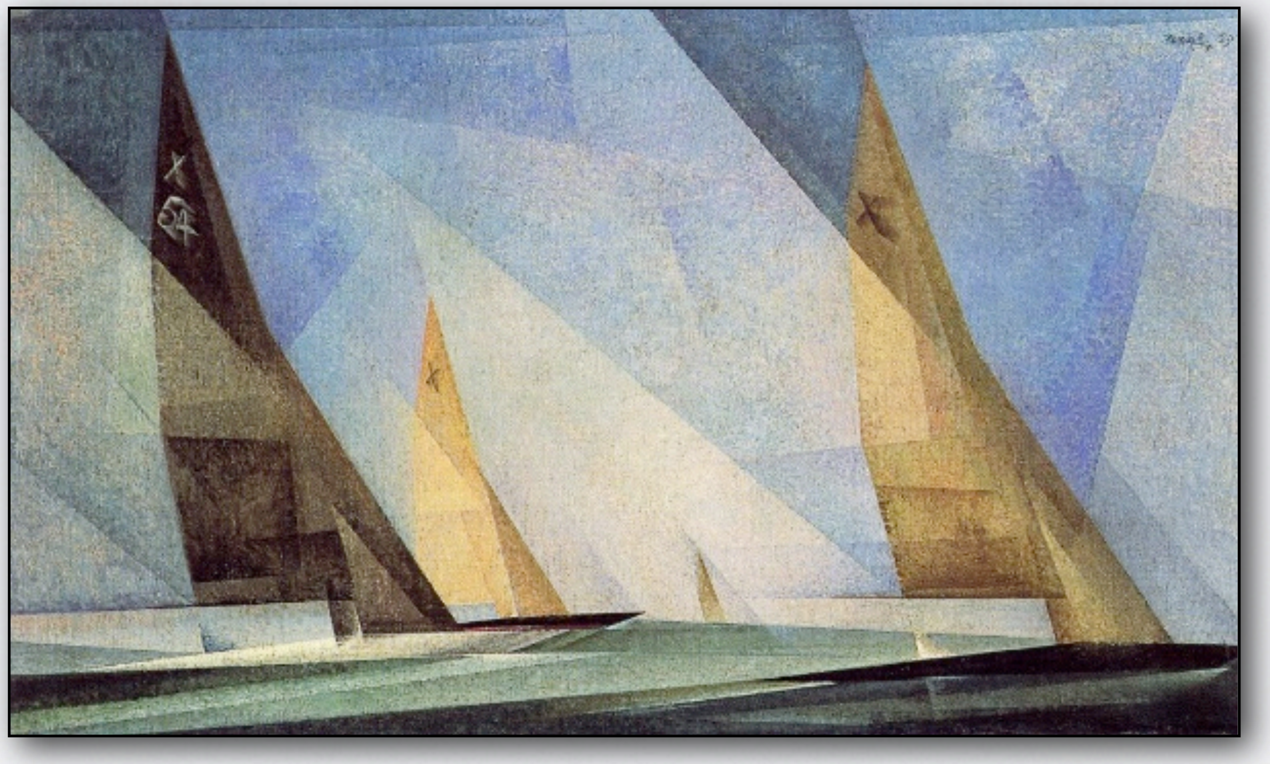
$$\begin{aligned}\gamma_{ij} &= \frac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2\,\beta_j^2}} \equiv -\alpha_{ij} - \frac{1}{\alpha_{ij}} \\ \xi_{ij} &= \cosh^{-1}\left(-\frac{\gamma_{ij}}{2}\right) = \ln\left(\alpha_{ij}\right) \\ \Gamma^{(1)}_{\rm cusp}\left(\xi\right) &= \xi \coth\left(\xi\right)\, \gamma_K^{(1)} = -\frac{1+\alpha^2}{1-\alpha^2} \ln\left(\alpha\right)\, \gamma_K^{(1)}\end{aligned}$$

$$\begin{aligned}\Gamma^{(2)}_{\rm trip}\left(\xi_{mn}\right) &= i f_{abc} \sum_{ijk} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c\; \mathcal{F}^{(2)}\left(\xi_{ij}, \xi_{jk}, \xi_{ki}\right) \\ \mathcal{F}^{(2)}\left(\xi_{ij}, \xi_{jk}, \xi_{ki}\right) &= \frac{4}{3} \sum_{ijk} \epsilon_{ijk}\; g\left(\xi_{ij}\right) \xi_{jk} \coth\left(\xi_{jk}\right)\end{aligned}$$

Note:

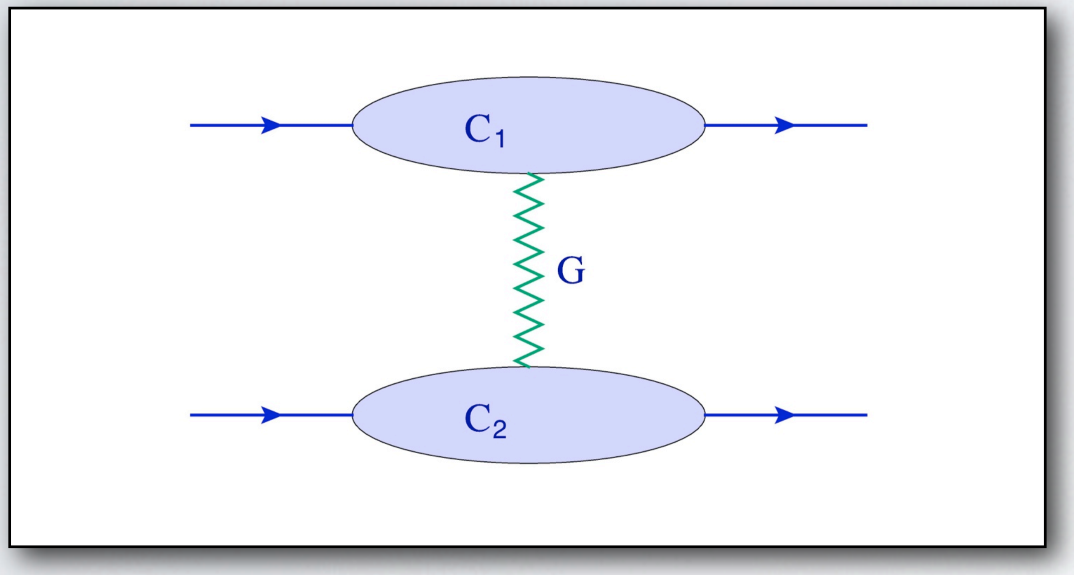
- Non-trivial factorized form in terms of cusp angles
- Non-trivial massless limit, the connected tripole cancels against 'planar' graphs.

THE HIGH-ENERGY LIMIT



Regge factorization

- In perturbative QCD the high-energy limit is governed by t-channel gluon exchange.
- In the t/s → 0 limit gluons in the t-channel `Reggeize' with a computable Regge trajectory.
- The amplitude factorizes with universal t-channel exchange connecting two impact factors.



Quark-quark scattering: the t-channel gluon Reggeizes

- Large logarithms of s/t are generated by a simple replacement of the t-channel propagator,

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

- The Regge trajectory has a perturbative expansion, with IR divergent coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right)$$

- The gluon has been shown to Reggeize at NLL, the two-loop Regge trajectory is known.

$$\alpha^{(1)} = C_A \frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A \frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A \left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)} \frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

- Gluon reggeization is exact in the planar limit for the N=4 SYM four-point amplitude.

Regge master formula

- Regge factorization, under the assumption of `only poles' in the J plane, and including crossing information, leads to a `master formula' for color-octet t-channel exchange.

$$\mathcal{M}^{[8]}_{ab}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s,\epsilon\right) = 2\pi\alpha_s\, H^{(0),[8]}_{ab}\left[C_a\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right)A_+\left(\frac{s}{t},\alpha_s,\epsilon\right)C_b\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right)\right.$$

$$\left.+\,\kappa\, C_a\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right)A_-\left(\frac{s}{t},\alpha_s,\epsilon\right)C_b\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right) + \mathcal{R}^{[8]}_{ab}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s,\epsilon\right) + \mathcal{O}\left(\frac{t}{s}\right)\right],$$

- Here the signature factor $\kappa = \frac{4-N_c^2}{N_c^2}$ for quarks, while $\kappa = 1$ for gluons.

- The Regge trajectory appears in the (anti)symmetrized factors

$$A_\pm\left(\frac{s}{t},\alpha_s,\epsilon\right) = \left(\frac{-s}{-t}\right)^{\alpha(t)} \pm \left(\frac{s}{-t}\right)^{\alpha(t)}$$

- Regge factorization is proved only at LL and at NLL for the real part of the amplitude, but it is valid for finite terms as well.

- We have introduced a color-octet non-factorizing remainder function $\mathcal{R}^{[8]}_{ab}$.

- The remainder function starts at NNLL for the real part, but could in principle have NLL imaginary parts. They will turn out to vanish.

Del Duca, Duhr, Gardi, LM, White 2011
Del Duca, Falcioni, LM, Vernazza 2013

Infrared master formula

- We consider quark and gluon four-point amplitudes in QCD. Soft-collinear factorization leads to a `master formula' for these amplitudes valid to leading power in t/s.

$$\mathcal{M}\left(\frac{s}{t},\alpha_s\right) = \left[\prod_{i=1}^{4}\mathcal{Z}^{(i)}_{\mathbf{1},\mathbf{R}}\left(\frac{t}{\mu^2},\alpha_s\right)\right]\widetilde{\mathcal{Z}}_{\mathrm{S}}\left(\frac{s}{t},\alpha_s\right)\mathcal{H}\left(\frac{s}{t},\alpha_s\right)$$

$$\widetilde{\mathcal{Z}}_{\mathrm{S}}\left(\frac{s}{t},\alpha_s\right) = \exp\left\{K(\alpha_s)\left[\left(\log\left(\frac{s}{-t}\right)-i\,\frac{\pi}{2}(1+\kappa_{ab})\right)\mathbf{T}_t^2+i\,\frac{\pi}{2}\,\left(\mathbf{T}_s^2-\mathbf{T}_u^2+\kappa_{ab}\mathbf{T}_t^2\right)\right]\right\}$$

$$\mathcal{Z}^{(i)}_{\mathbf{1},\mathbf{R}}\left(\frac{t}{\mu^2},\alpha_s\right) = \exp\left\{\frac{1}{2}\left[K\left(\alpha_s\right)\log\left(\frac{-t}{\mu^2}\right)+D\left(\alpha_s\right)\right]\mathcal{C}^{(i)}+\,B^{(i)}\left(\alpha_s\right)\right\}$$

- We introduced `Mandelstam' color operators, and used color and momentum conservation

$$\begin{aligned}\mathbf{T}_s &= \mathbf{T}_1+\mathbf{T}_2 = -\left(\mathbf{T}_3+\mathbf{T}_4\right)\,,\\ \mathbf{T}_t &= \mathbf{T}_1+\mathbf{T}_3 = -\left(\mathbf{T}_2+\mathbf{T}_4\right)\,,\\ \mathbf{T}_u &= \mathbf{T}_1+\mathbf{T}_4 = -\left(\mathbf{T}_2+\mathbf{T}_3\right)\end{aligned}$$

$$\mathbf{T}_s^2+\mathbf{T}_t^2+\mathbf{T}_u^2 = \sum_{i=1}^{4}C_i$$

$$s+t+u=0$$

- Coupling dependence for leading logarithms is completely determined by the cusp anomalous dimension and by the β function, through the integral (Korchemsky 94-96)

$$K\left(\alpha_s(\mu^2),\epsilon\right) = -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2,\epsilon)\right) = \frac{\alpha_s}{\pi}\,\frac{\widehat{\gamma}_K^{(1)}}{4\epsilon}+\left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon}-\frac{b_0\widehat{\gamma}_K^{(1)}}{32\epsilon^2}\right)+\mathcal{O}\left(\alpha_s^3\right)$$

Del Duca, Duhr, Gardi, LM, White 2011

The rise and fall of Reggeization

- At leading logarithmic accuracy, only the t-channel contribution survives. If, at LO and at leading power in t/s, the scattering is dominated by t-channel exchange, then the hard function is an eigenstate of the color operator $\mathbf{T}_t^2$,

$$\mathbf{T}_t^2 \, \mathcal{H}^{gg\to gg} \xrightarrow{|t/s|\to 0} C_t \, \mathcal{H}_t^{gg\to gg}$$

- Leading-logarithmic Reggeization for arbitrary t-channel color representations follows, with the Regge trajectory given by the integral K times the appropriate Casimir eigenvalue.

$$\mathcal{M}^{gg\to gg} = \left(\frac{s}{-t}\right)^{C_A \, K\left(\alpha_s(\mu^2),\epsilon\right)} Z_{\mathbf{1}} \, \mathcal{H}_t^{gg\to gg}$$

- The infrared operator Z can be systematically expanded beyond LL, using the BCH formula. The real part of the amplitude Reggeizes at NLL for general t-channel exchanges.

- At NNLL Regge pole factorization generically breaks down.
  - At two loops, terms that are non-logarithmic and non-diagonal in a t-channel basis arise.
  - At three loops, the first Reggeization-breaking logarithms of s/t arise.

$$\mathcal{E}_0\left(\alpha_s,\epsilon\right) \equiv -\frac{1}{2}\pi^2 K^2\left(\alpha_s,\epsilon\right) \left(\mathbf{T}_s^2\right)^2 \qquad \mathcal{E}_1\left(\frac{s}{t},\alpha_s,\epsilon\right) \equiv -\frac{\pi^2}{3} K^3(\alpha_s,\epsilon) \ln\left(\frac{s}{-t}\right) \left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]$$

  - The color commutator terms are related to Glauber exchanges (Forshaw *et al.*, Catani *et al.*)

Finite order expansions

- To proceed, we expand all factors in powers of the coupling and of the high-energy logarithm.

$$\widetilde{\mathcal{Z}}\left(\frac{s}{t}, \alpha_s, \epsilon\right) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \left(\frac{\alpha_s}{\pi}\right)^n \log^i\left(\frac{s}{-t}\right) \widetilde{Z}^{(n),i}\left(\epsilon\right) ,$$

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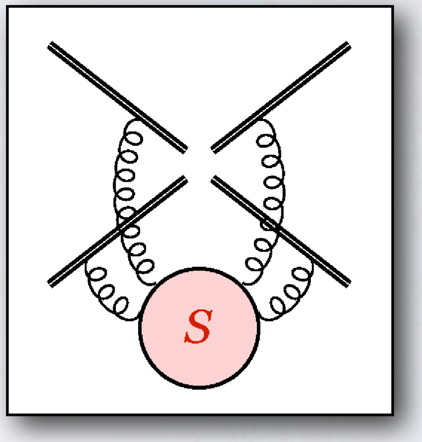
$$\begin{aligned} M^{(1),0} &= \left[Z_{1,\mathbf{R}}^{(1)} + i\pi K_1 \left(\mathbf{T}_s^2 - \frac{1}{2}\mathcal{C}_{\rm tot}\right)\right] H^{(0)} + H^{(1),0}, \\ M^{(1),1} &= K_1 \mathbf{T}_t^2 H^{(0)} + H^{(1),1} \, , \end{aligned}$$



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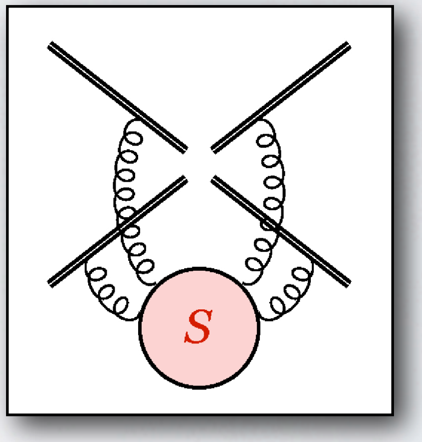
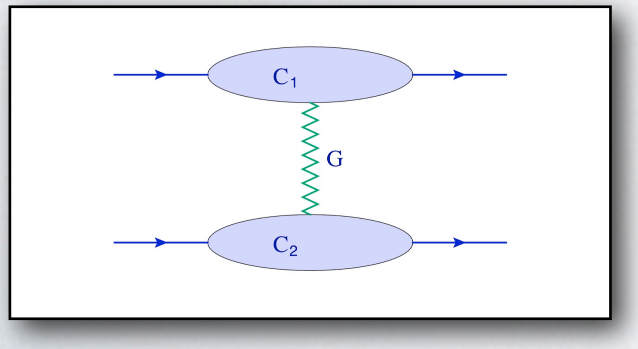


- For the octet component of the same matrix elements, Regge factorization yields



$$\begin{aligned} M_{ab}^{(1),0,[8]} &= \left[C_a^{(1)} + C_b^{(1)} - i\frac{\pi}{2}(1+\kappa)\alpha^{(1)}\right] H_{ab}^{(0),[8]}\,, \\ M_{ab}^{(1),1,[8]} &= \alpha^{(1)} H_{ab}^{(0),[8]}\,, \end{aligned}$$

Two loops: universality breaking

- At two loops and at NNLL the impact factors display universality breaking

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C_a^{(2)} \;=\;& \frac{1}{2} Z^{(2)}_{1,\mathbf{R},aa} - \frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},aa}\right)^2 + \frac{1}{4} Z^{(1)}_{1,\mathbf{R},aa}\,\mathrm{Re}\left[\hat{H}^{(1),0,[8]}_{aa}\right] - \frac{1}{2} R^{(2),0,[8]}_{aa} \\
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\Delta_{(2),0,[8]} &= M^{(2),0,[8]}_{qg} - \left[C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4}(1+\kappa)\left(\alpha^{(1)}\right)^2\right] H^{(0),[8]}_{qg} \\
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$$

we find that

$$
\Delta_{(2),0,[8]} = \frac{\pi^2 K_1^2}{2} \left[\frac{3}{2} \left(\frac{N_c^2+1}{N_c^2}\right)\right] = \frac{\pi^2}{\epsilon^2} \frac{3}{16} \left(\frac{N_c^2+1}{N_c^2}\right) \qquad \checkmark
$$

Three loops: factorization breaking

- The real single-logarithmic contribution at NNLL should give the Regge trajectory at three loops. As expected, fitting the coefficient we find a non-universal result. Using

$$\left[\left(\mathbf{T}_t^2\left(\mathbf{T}_s^2\right)^2+\mathbf{T}_s^2\mathbf{T}_t^2\mathbf{T}_s^2+\left(\mathbf{T}_s\right)^2\mathbf{T}_t^2\right)H^{(0)}\right]^{[8]}=\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_s^2\right)_{[8],[n]}\right|^2H^{(0),[8]}$$

we find

$$\begin{aligned}\alpha_{\rm fit}^{(3)} \;&=\; C_AK_3+\frac{\pi^2K_1^3}{2}\left[\mathcal{C}_{{\rm tot},ij}N_c\left(\mathbf{T}_{s,ij}^2\right)_{[8],[8]}-\frac{\mathcal{C}_{{\rm tot},ij}^2N_c}{4}+\frac{1+\kappa}{2}N_c^3\right.\\&\left.-\frac{1}{3}\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_{s,ij}^2\right)_{[8],n}\right|^2\right]-R^{(3),1,[8]}+\mathcal{O}\left(\epsilon^{-2}\right)\end{aligned}$$

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$$\alpha^{(3)}_{\text{fit}} = \boxed{C_A K_3} + \boxed{\begin{aligned}&\frac{\pi^2 K_1^3}{2}\Big[\mathcal{C}_{\text{tot},ij}N_c\left(\mathbf{T}^2_{s,ij}\right)_{[8],[8]}-\frac{\mathcal{C}^2_{\text{tot},ij}N_c}{4}+\frac{1+\kappa}{2}N_c^3\\&-\frac{1}{3}\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}^2_{s,ij}\right)_{[8],n}\right|^2\Big]\end{aligned}} - R^{(3),1,[8]}+\mathcal{O}\left(\epsilon^{-2}\right)$$

universal

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universal

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- We define then a single-logarithmic three-loop non-factorizing remainder as

$$\begin{aligned}\widetilde{R}_{ij}^{(3),1,[8]} &= \pi^2 K_1^3\left[\mathcal{C}_{\text{tot},ij}N_c\left(\mathbf{T}_{s,ij}^2\right)_{[8],[8]} - \frac{\mathcal{C}_{\text{tot},ij}^2 N_c}{4}\right. \\ &\qquad \left. + \frac{1+\kappa}{2}N_c^3 - \frac{1}{3}\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_{s,ij}^2\right)_{[8],n}\right|^2\right] + \mathcal{O}\left(\epsilon^{-2}\right)\end{aligned}$$

- In the context of Regge theory the contributions could source a new resummation associated with Regge cuts in the J plane.

Del Duca, Falcioni, LM, Vernazza 2013-2014

Three loops and beyond

- Explicitly, the non-factorizing remainders at three loops for qq, gg and qg amplitudes are

$$\begin{aligned} \widetilde{R}_{qq}^{(3),1,[8]} &= \frac{2}{3}\pi^2 K_1^3 \, \frac{2N_c^2-5}{N_c} = \frac{\pi^2}{\epsilon^3}\,\frac{2N_c^2-5}{12N_c}\,, \\ \widetilde{R}_{gg}^{(3),1,[8]} &= -\frac{16}{3}\pi^2 K_1^3\, N_c = -\frac{\pi^2}{\epsilon^3}\,\frac{2}{3}\,N_c \\ \widetilde{R}_{qg}^{(3),1,[8]} &= -\frac{1}{3}\pi^2 K_1^3 N_c = -\frac{\pi^2}{\epsilon^3}\,\frac{N_c}{24}\,. \end{aligned}$$

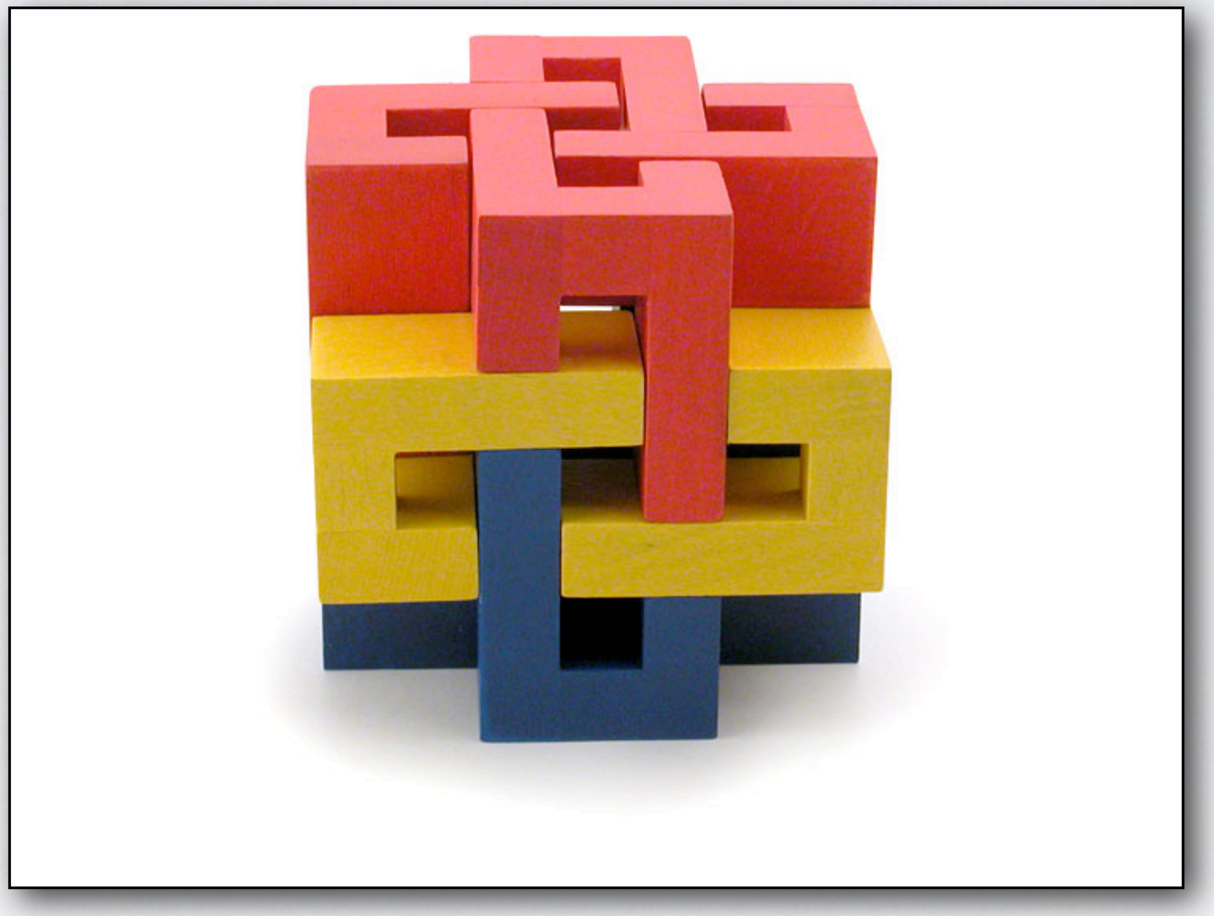
Note that all remainders are subleading in N_c as they must.

- Furthermore, one can prove a sequence of all-order identities for the hard parts

$$\begin{aligned} \mathrm{Im}(\hat{H}^{(n),n,[8]}) &= 0 \\ \mathrm{Re}(\hat{H}^{(n),n,[8]}) &= \frac{1}{n!}\left(\hat{H}^{(1),1,[8]}\right)^n = O(\epsilon^n) \\ \mathrm{Im}(\hat{H}^{(n),n-1,[8]}) &= -\pi\frac{1+\kappa}{2}\left(n\hat{H}^{(n),n,[8]}\right) = O(\epsilon^n) \\ \mathrm{Re}(\hat{H}^{(n),n-1,[8]}) &= \mathrm{Re}(\hat{H}^{(2),1})\hat{H}^{(n-2),n-2} + (2-n)\mathrm{Re}(\hat{H}^{(1),0,[8]})\hat{H}^{(n-1),n-1} \\ &= \mathcal{O}(\epsilon^{n-2}) \end{aligned}$$

proving a `strong vanishing' of the hard part up to NLL.

WEAVING MULTI-PARTICLE WEBS



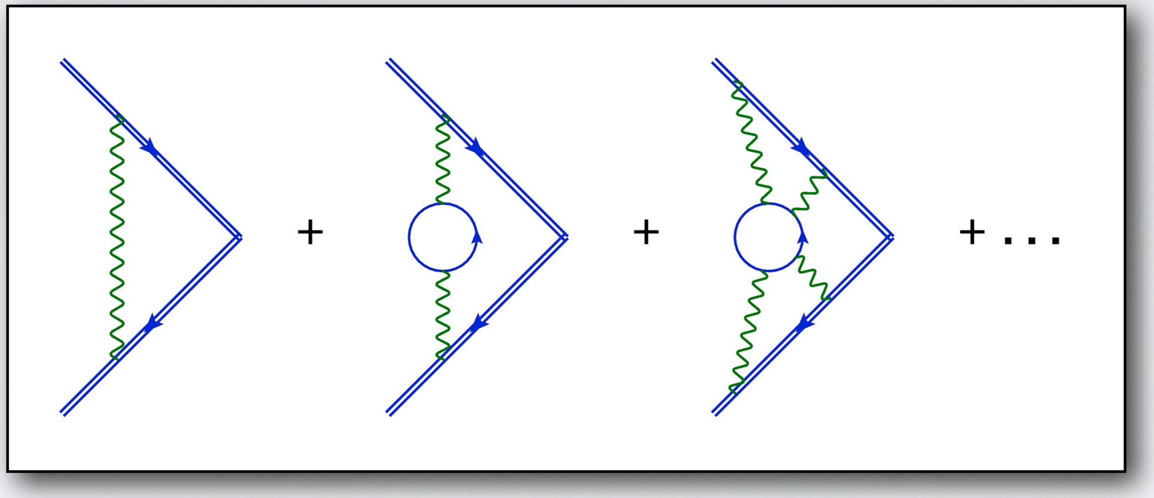
Eikonal exponentiation

All correlators of Wilson lines, regardless of shape, resum in exponential form.

$$S_n \equiv \langle 0 | \, \Phi_1 \otimes \ldots \otimes \Phi_n \, | 0 \rangle = \exp\left(\omega_n \right)$$

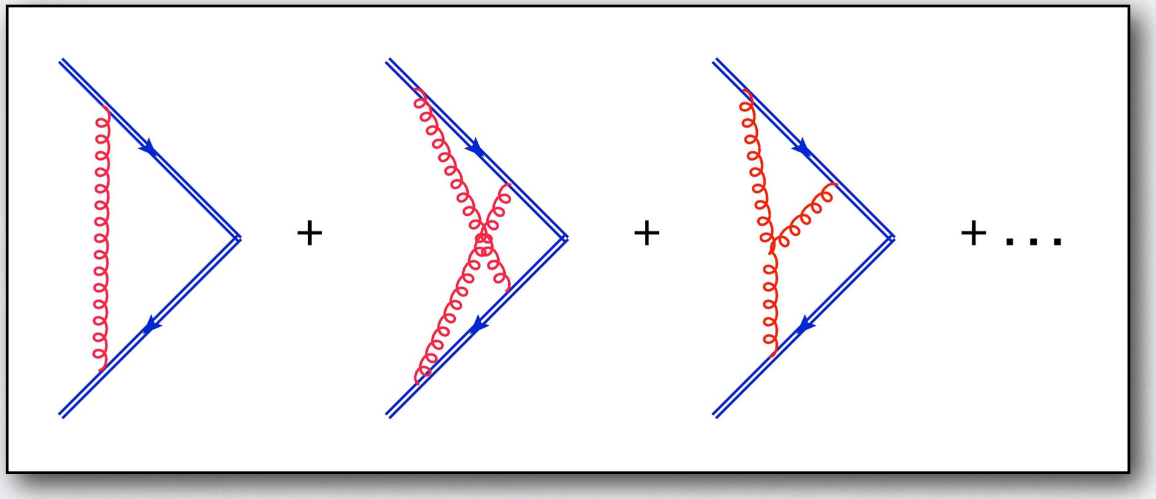
Diagrammatic rules exist to compute directly the logarithm of the correlators.

$$\omega_{2,\mathrm{QED}} =$$



Only connected photon subdiagrams contribute to the logarithm.

$$\omega_{2,\mathrm{QCD}} =$$



Only gluon subdiagrams which are two-eikonal irreducible contribute to the logarithm. They have modified color factors.

For eikonal form factors, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

Multiparticle webs

The concept of web generalizes non-trivially to the case of multiple Wilson lines. (Gardi, Smillie, White, *et al*).

A **web** is a set of diagrams which differ only by the order of the gluon attachments on each Wilson line. They are weighted by modified color factors.

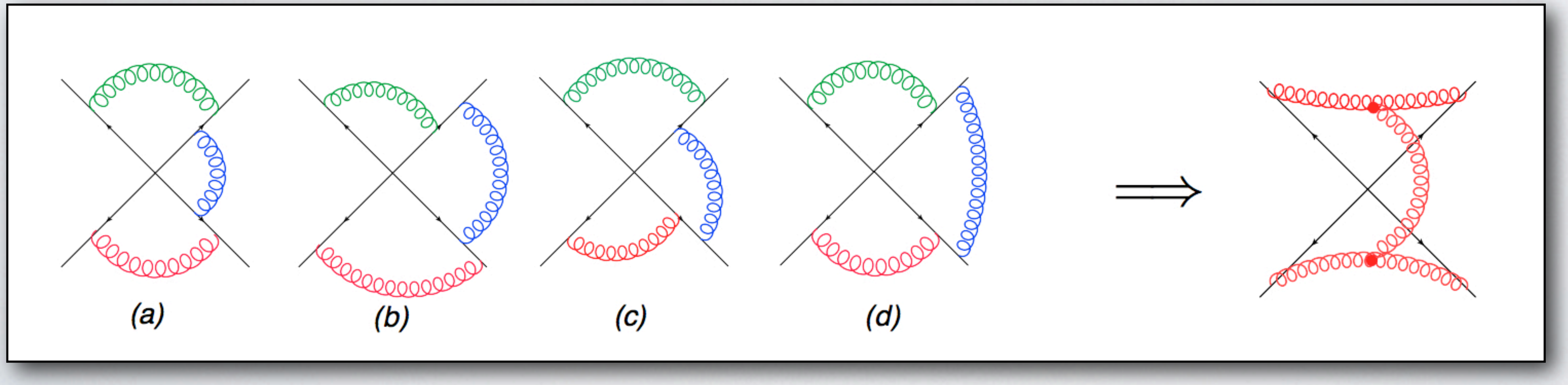
Writing each diagram as the product of its natural color factor and a kinematic factor

$$D = C(D)\mathcal{F}(D)$$

a web W can be expressed as a sum of diagrams in terms of a web mixing matrix R

$$W = \sum_D \widetilde{C}(D)\mathcal{F}(D) = \sum_{D,D'} C(D')R(D',D)\mathcal{F}(D)$$

The non-abelian exponentiation theorem holds: each web has the color factor of a fully connected gluon subdiagram (Gardi, Smillie, White).



Gardi, Smillie, White 2010-2012;
Mitov, Sterman, Sung 2010

Computing webs

Bare Wilson-line correlators vanish beyond tree level in dimensional regularization: they are given by scale-less integrals. We require renormalized correlators, which depend on the Minkowsky angles between the Wilson lines.

$$S_{\rm ren}\left(\gamma_{ij},\alpha_s,\epsilon\right) \, = \, S_{\rm bare}\left(\gamma_{ij},\alpha_s,\epsilon\right) Z\left(\gamma_{ij},\alpha_s,\epsilon\right) \, = \, Z\left(\gamma_{ij},\alpha_s,\epsilon\right) \, , \qquad \gamma_{ij} \, = \, \frac{2\beta_i\cdot\beta_j}{\sqrt{\beta_i^2\beta_j^2}}$$

To compute the counterterm Z we make use of an auxiliary, IR-regularized correlator

$$\begin{aligned} \widehat{S}_{\rm ren}\left(\gamma_{ij},\alpha_s,\epsilon,m\right) \quad &= \quad \widehat{S}_{\rm bare}\left(\gamma_{ij},\alpha_s,\epsilon,m\right) Z\left(\gamma_{ij},\alpha_s,\epsilon\right) \\ &\equiv \quad \exp\left(\omega\right)\,\exp\left(\zeta\right) \, = \, \exp\left\{\omega+\zeta+\frac{1}{2}\left[\omega,\zeta\right]+\ldots\right\} \end{aligned}$$

The expression of Z in terms of the anomalous dimension Γ follows from RG arguments

$$Z \, = \, \exp\left[\frac{\alpha_s}{\pi}\frac{1}{2\epsilon}\Gamma^{(1)}+\left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{1}{4\epsilon}\Gamma^{(2)}-\frac{b_0}{4\epsilon^2}\Gamma^{(1)}\right)+\left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{1}{6\epsilon}\Gamma^{(3)}+\frac{1}{48\epsilon^2}\left[\Gamma^{(1)},\Gamma^{(2)}\right]+\ldots\right)\right]$$

Combining informations one can get Γ directly from the logarithm of the regularized S

$$\begin{aligned} \Gamma^{(1)} \quad &= \quad -2\,\omega^{(1,-1)} \\ \Gamma^{(2)} \quad &= \quad -4\,\omega^{(2,-1)}-2\left[\omega^{(1,-1)},\omega^{(1,0)}\right] \end{aligned} \qquad \omega=\sum_{n=1}^{\infty}\sum_{k=-n}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n\epsilon^k\;\omega^{(n,k)}$$

Computing regularized webs is a game of combinatorics and renormalization theory.

Three-loop progress

The computation of the three-loop multi-particle soft anomalous dimension is under way (see earlier talk by Einan Gardi).

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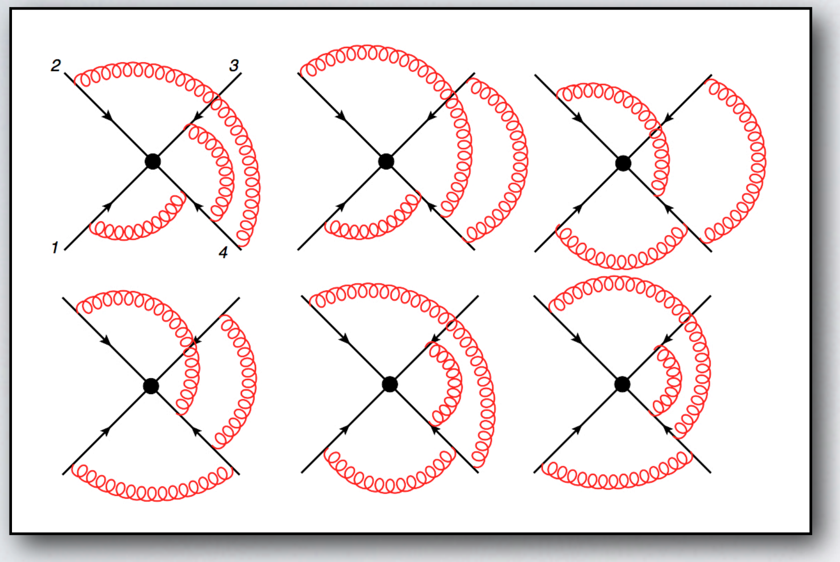

(1113) web

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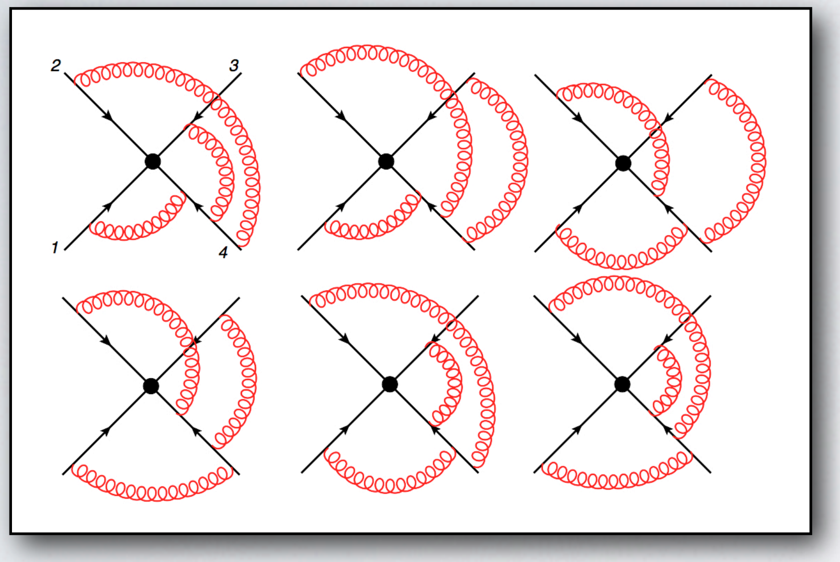

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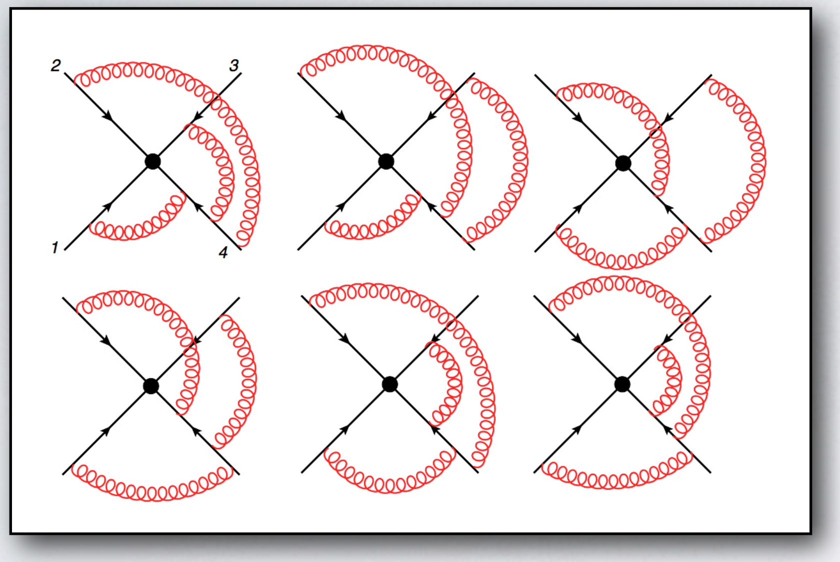

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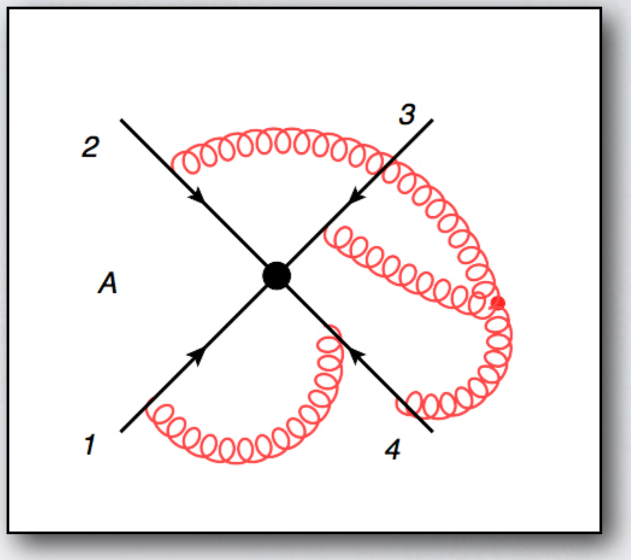

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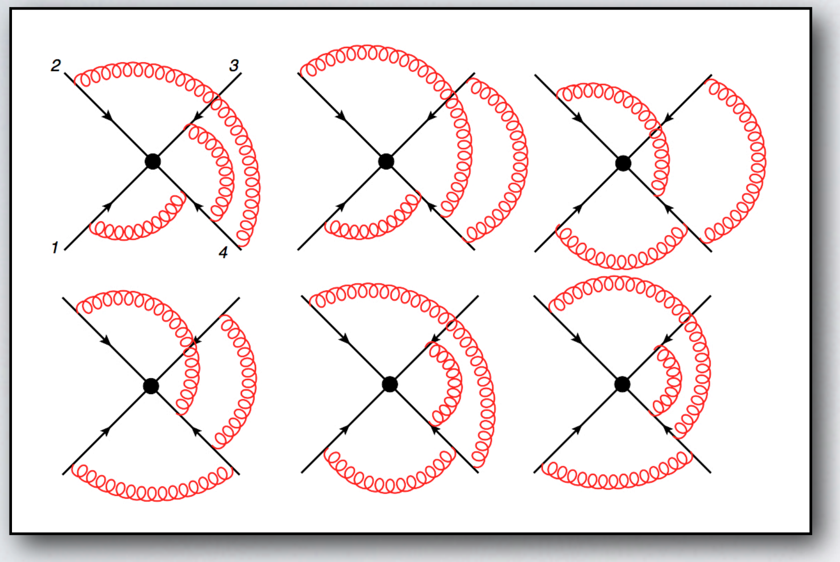

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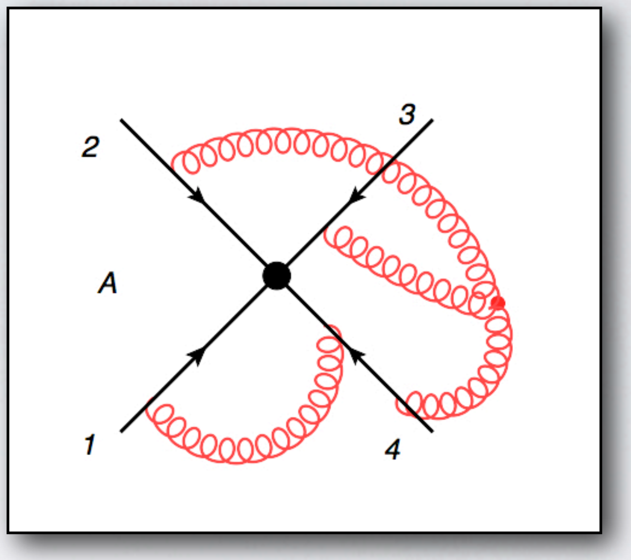

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In progress

(Falcioni, Gardi, Harley, LM, White**),**

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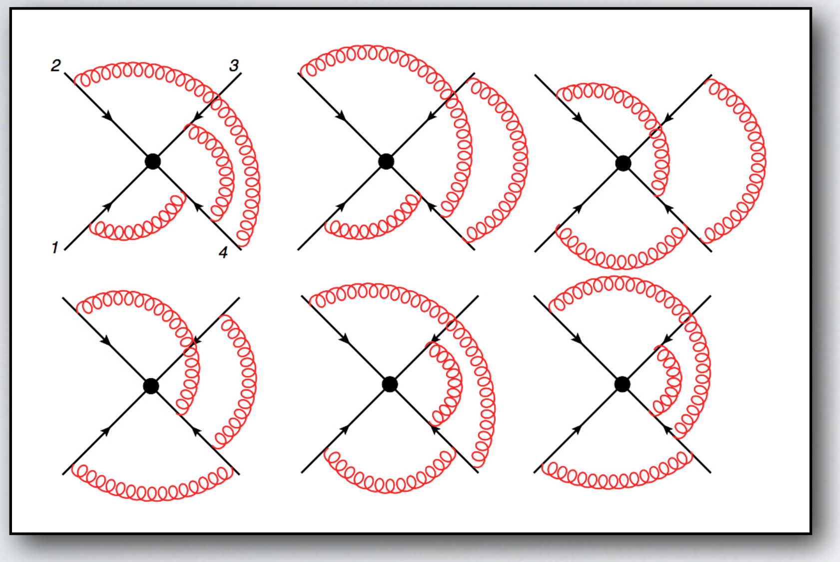

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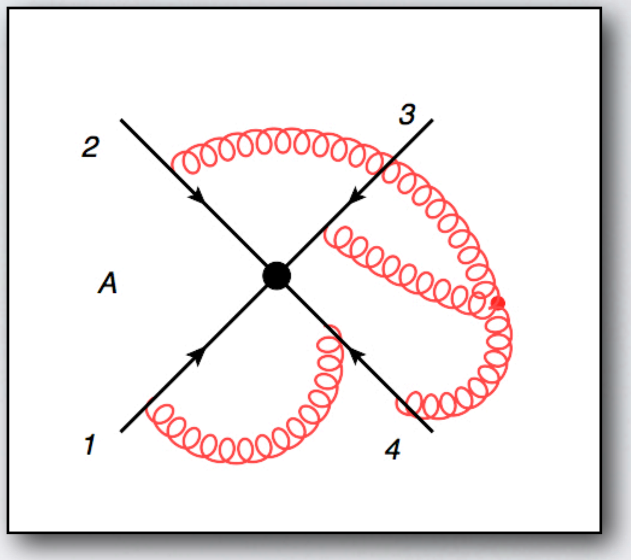

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(Falcioni, Gardi,
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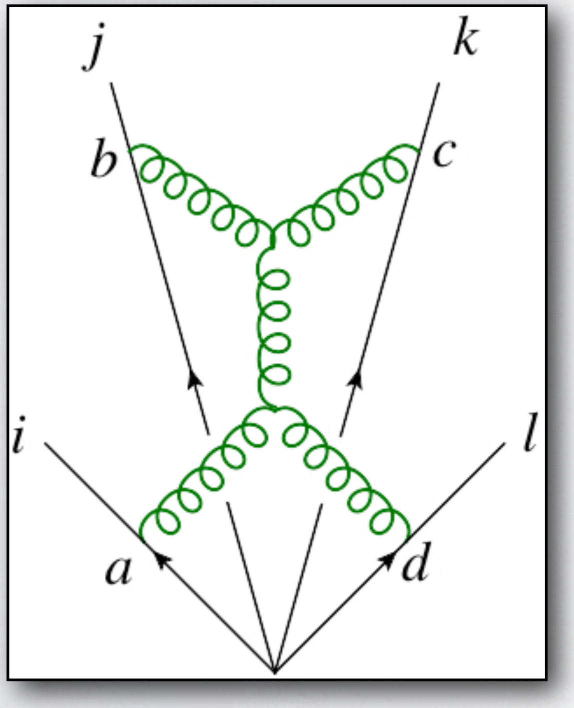
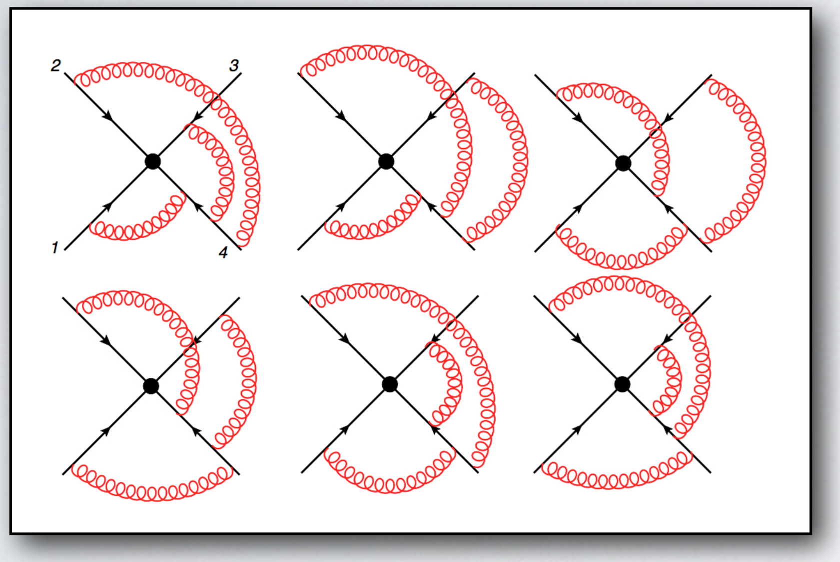


(1111) **web**

Three-loop progress

The computation of the three-loop multi-particle soft anomalous dimension is under way (see earlier talk by Einan Gardi).



✓ (Gardi)

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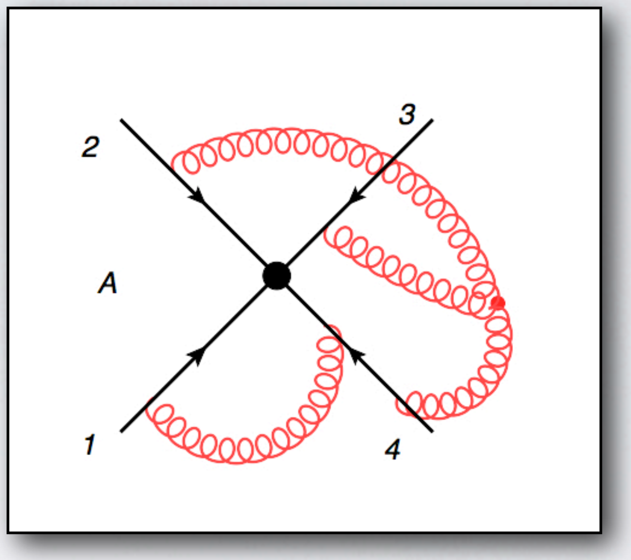
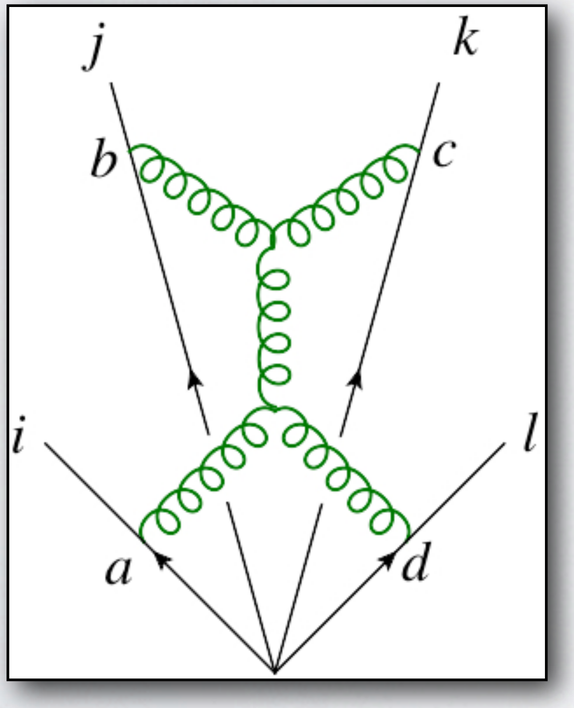


In progress
(Falcioni, Gardi, Harley, LM, White),

(1112) **web**



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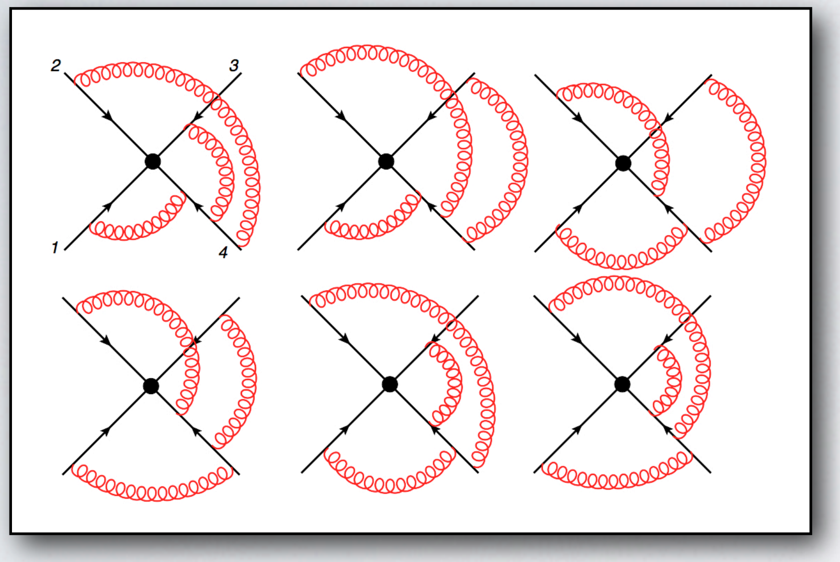

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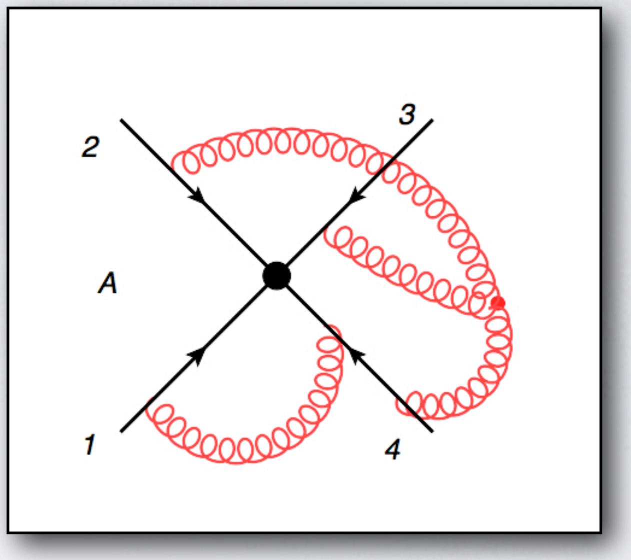

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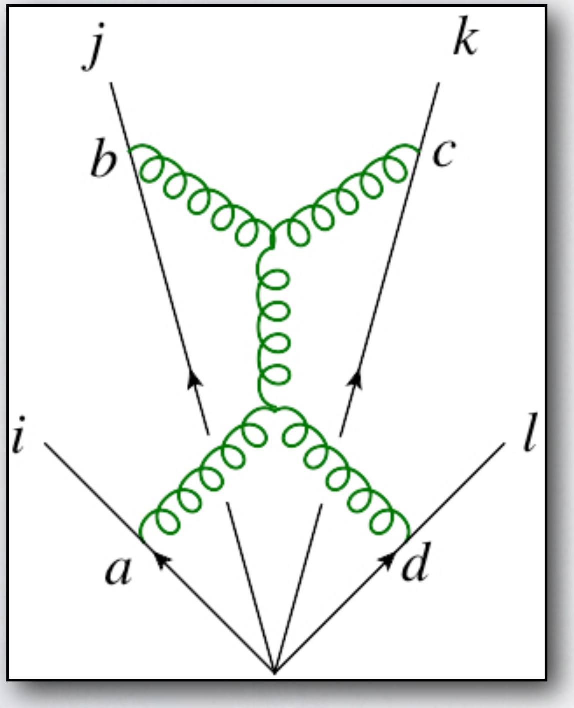


(1111) web

Progress in the massless limit (Almelid, Duhr, Gardi),

LOOPS AND LEGS

Gardi 2013; Henn, Huber 2013;
Gardi, Falcioni, Harley, LM, White 2014

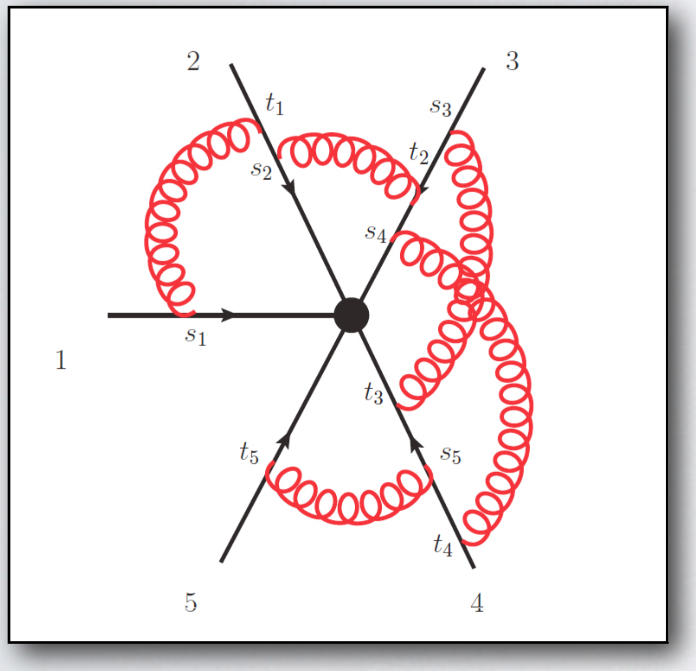
Multiple Gluon Exchange Webs

Multiple Gluon Exchange Webs (MGEWs) arise from a path integral weighted with the free part of the quantum YM action

$$\mathcal{S}\left(\gamma_{ij},\alpha_s(\mu),\epsilon,\frac{m}{\mu}\right)\bigg|_{\rm MGEW} \equiv \int [DA]\;\Phi^{(m)}_{\beta_1}\otimes\Phi^{(m)}_{\beta_2}\otimes\ldots\otimes\Phi^{(m)}_{\beta_L}\;\exp\Big\{{\rm i}S_0[A]\Big\}\,,$$

A general integral representation for diagrams *D* contributing to MGEWs can be written down to all orders, starting from a coordinate space representation of the Wilson lines

$$\mathcal{F}^{(n)}\left(D\right) \;=\; \kappa^n\;\Gamma(2n\epsilon)\int_0^1\prod_{k=1}^{n}\Big[dx_k\,\gamma_k\,P_\epsilon\left(x_k,\gamma_k\right)\Big]\;\phi^{(n)}_D\left(x_i;\epsilon\right)$$



A five-loop MGEW diagram

The variables x_k measure collinearity to Wilson lines, the overall UV pole is extracted, the coordinate-space gluon propagators give

$$P_\epsilon\left(x,\gamma\right)\;\equiv\;\left[x^2+(1-x)^2-\gamma\,x(1-x)\right]^{-1+\epsilon}$$

Non-abelian information is encoded in the order of gluon attachments on each Wilson line, through the kernel

$$\phi^{(n)}_D\left(x_i;\epsilon\right)\;=\;\int_0^1\prod_{k=1}^{n-1}dy_k\;\Big[\left(1-y_k\right)^{-1+2\epsilon}\;y_k^{-1+2k\epsilon}\Big]\;\Theta_D\big[\left\{x_k,y_k\right\}\big]$$

Replacing the set of ϑ functions by unity one recovers abelian exponentiation.

Gardi 2013

Subtracted Webs

Individual diagrams contain multiple UV poles and give uniform weight results. One must: combine them into webs (where leading poles cancel); subtract subdivergences via commutator counterterms; organize the result in a color basis. In the “right” variables

$$\gamma \to -\alpha - \frac{1}{\alpha}\,, \qquad\qquad P_\epsilon\left(x,\gamma\right) \to p_\epsilon\left(x,\alpha\right)$$

$$p_\epsilon\left(x,\alpha\right) \;=\; -\left(\alpha + \frac{1}{\alpha}\right)\left[q\left(x,\alpha\right)\right]^{-1+\epsilon}\,, \qquad q\left(x,\alpha\right) \;=\; x^2 + (1-x)^2 + \left(\alpha + \frac{1}{\alpha}\right) x(1-x)$$

Expansion in powers of ϵ will generate logarithms of $q(x,\alpha)$. Assembling the results

$$\overline{\omega}^{(n,-1)}_{a_1,\ldots,a_L} \;=\; \sum_A C^{(A)}_{a_1,\ldots,a_L}\, F^{(n)}_A\left(\alpha_1,\ldots\alpha_n\right)$$

$$F^{(n)}_A\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right) \;=\; \int_0^1 \left[\prod_{k=1}^{n} dx_k\, p_0(x_k,\alpha_k)\right] \mathcal{G}_n\Big(x_1,\ldots,x_n; q(x_1,\alpha_1),\ldots q(x_n,\alpha_n)\Big)$$

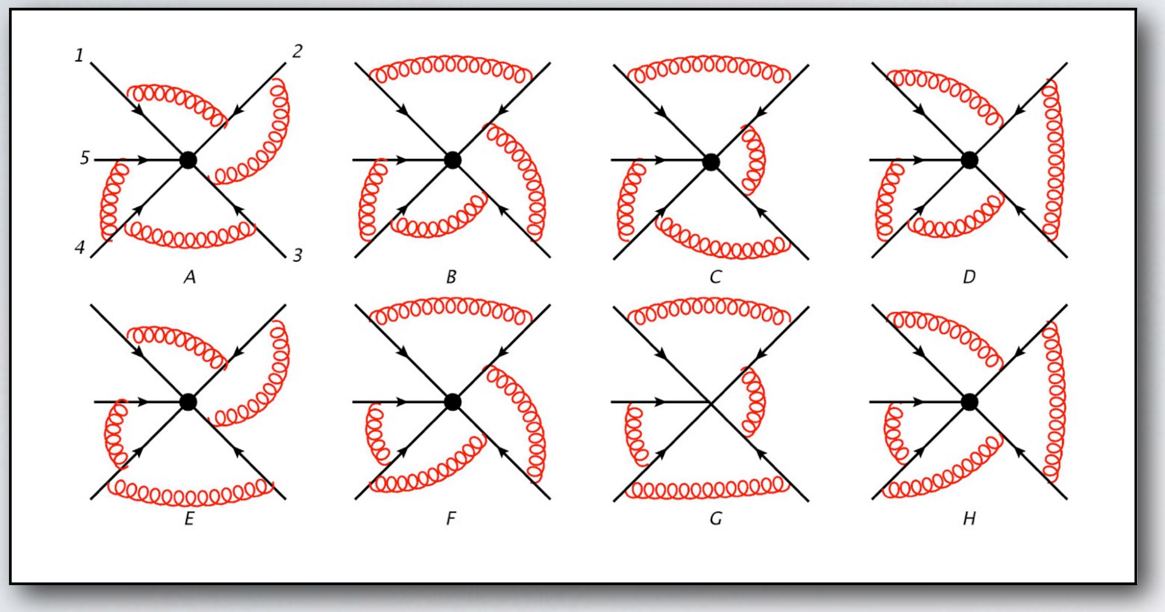
A lot of experimental and conceptual evidence is accumulating favoring the following

- Factorization conjecture: the subtracted MGEW kernel $\mathcal{G}$ is a sum of products of logarithmic functions of individual cusp variables of uniform weight n-1.
- Alphabet conjecture: the Symbol of all subtracted MGEW kinematic coefficients F_A is restricted to the letters

$$\left\{\alpha_k,\ \eta_k \equiv \alpha_k/\left(1-\alpha_k^2\right)\right\}$$

Four loops, five legs

The explicit evaluation of all three-loop MGEWs is nearing completion. We can however push further into the L&L space ....



The four-loop five-leg MGEW with attachments 1-2-2-2-1

The four-loop, five-line MGEW 1-2-2-2-1

- Contributes to a single color structure of the four-loop anomalous dimension.
- It contains eight diagrams connected by a mirror symmetry.
- Needs an elaborate set of nested commutator counterterms.

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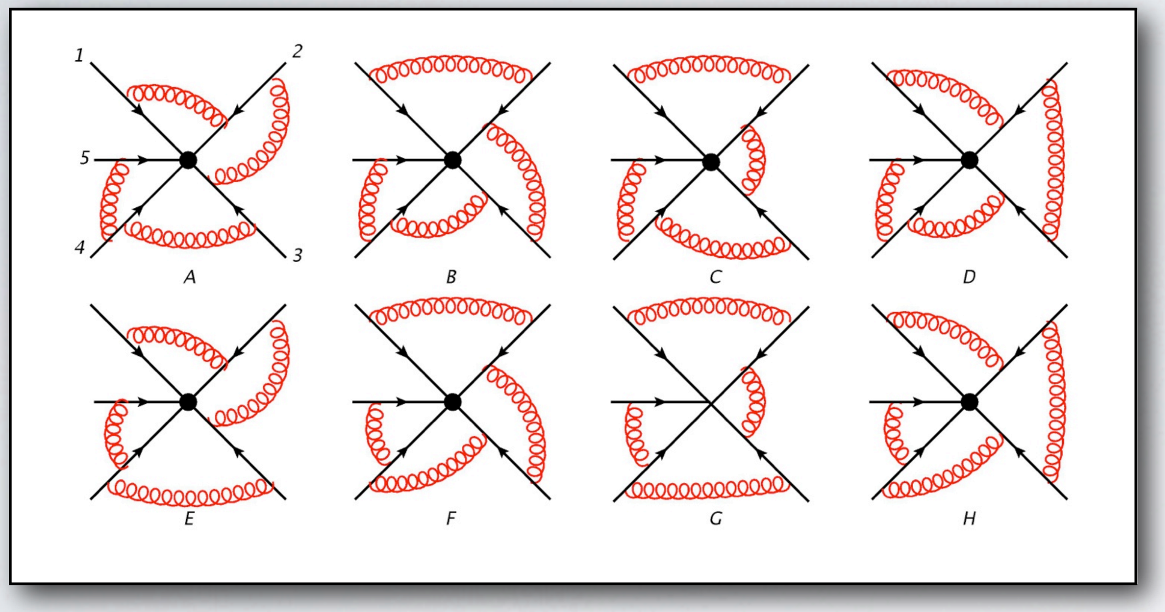

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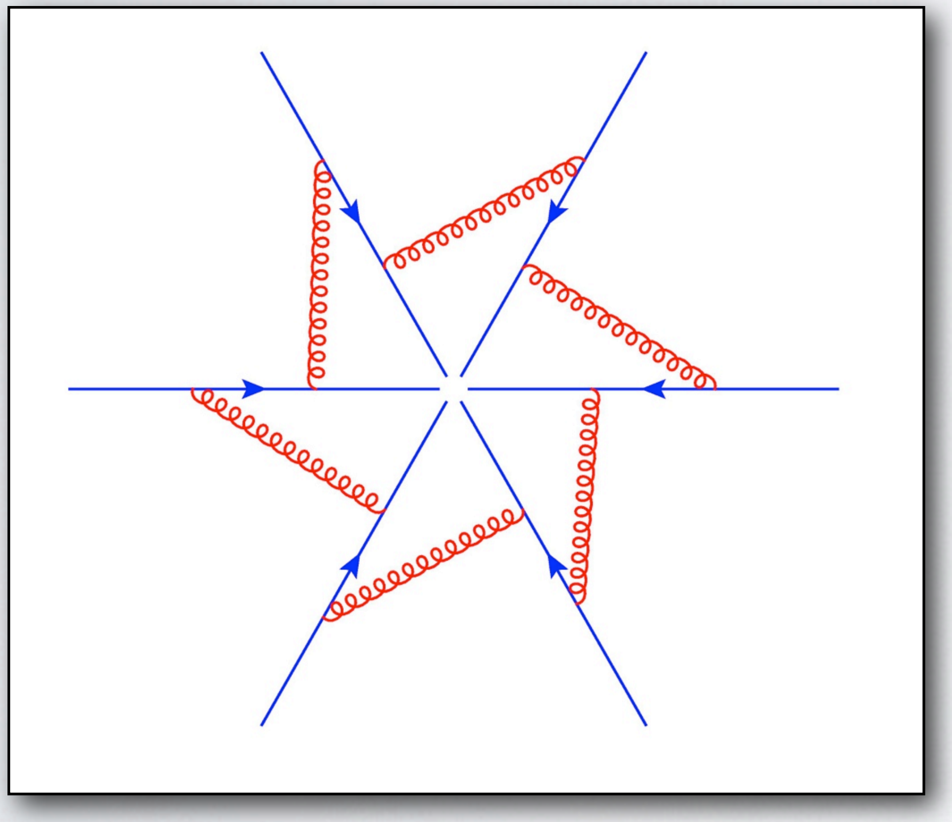
The result is a simple function of the logarithms $L_{ij} = \log\left(\frac{q(x_i,\alpha_{ij})}{x_i^2}\right)\,, \quad \Sigma_i = \log\left(\frac{x_i}{1-x_i}\right)$

$$
\begin{aligned}
\mathcal{G}_{(4)}\left(x_1,x_2,x_3,x_4\right) \quad = \quad & -\frac{1}{144}\Bigg\{L_{12}^3 - 3\,L_{23}^3 + 3\,L_{34}^3 - L_{45}^3 + 3\,L_{12}^2\Big[L_{23} + L_{34} - 3L_{45}\Big] \\
& + 3\,L_{23}^2\Big[L_{12} - 3\,L_{34} + 5\,L_{45}\Big] - 3\,L_{34}^2\Big[5\,L_{12} - 3\,L_{23} + L_{45}\Big] - 3\,L_{45}^2\Big[L_{23} + L_{34} - 3\,L_{12}\Big] \\
& + 6\,\Big[L_{12}L_{23}L_{34} - 3\,L_{12}L_{23}L_{45} + 3\,L_{12}L_{34}L_{45} - L_{23}L_{34}L_{45}\Big] \\
& + 24\,\Big[\Sigma_2^2\Big(L_{12} + L_{23} + L_{34} - 3\,L_{45}\Big) - \Sigma_3^2\Big(L_{23} + L_{34} + L_{45} - 3\,L_{12}\Big)\Big]\Bigg\}
\end{aligned}
$$

The Escher Staircase

Special diagrams contributing to MGEWs have special features, notably those which do not contain subdivergences. The most symmetric example is the "Escher Staircase", with kernel

$$\begin{aligned}\phi_{ES}^{(n)}\left(x_i;\epsilon\right) &= \int_0^\infty \prod_{k=1}^{n-1} d\xi_k \left[\xi_k^{-1+2k\epsilon}\left(1+\xi_k\right)^{-2(k+1)\epsilon}\right] \widehat{\Theta}_{ES}\Big[\left\{x_k,\xi_k\right\}\Big] \\ &= \int_{A_1}^{B_1} \frac{d\xi_1}{\xi_1} \int_{A_2(\xi_1)}^{B_2(\xi_1)} \frac{d\xi_2}{\xi_2} \ldots \int_{A_{n-1}(\xi_1,\ldots,\xi_{n-2})}^{B_{n-1}(\xi_1,\ldots,\xi_{n-2})} \frac{d\xi_{n-1}}{\xi_{n-1}} + \mathcal{O}\left(\epsilon\right)\end{aligned}$$



Escher Staircase with six loops and six legs

where $\xi_k = y_k/(1-y_k)$

- The integral has a *d log* form, with intricate limits.
- The nested theta functions can be made explicit

$$\begin{aligned}A_k\left(\xi_1,\ldots,\xi_{k-1}\right) &= \frac{x_{k+1}}{1-x_k}\left(1+\xi_{k-1}\right) \\ B_k\left(\xi_1,\ldots,\xi_{k-1}\right) &= \frac{\prod_{j=k+1}^{n}(1-x_j)}{\prod_{j=k+2}^{n+1} x_j} \prod_{j=1}^{k-1} \frac{1+\xi_j}{\xi_j}\end{aligned}$$

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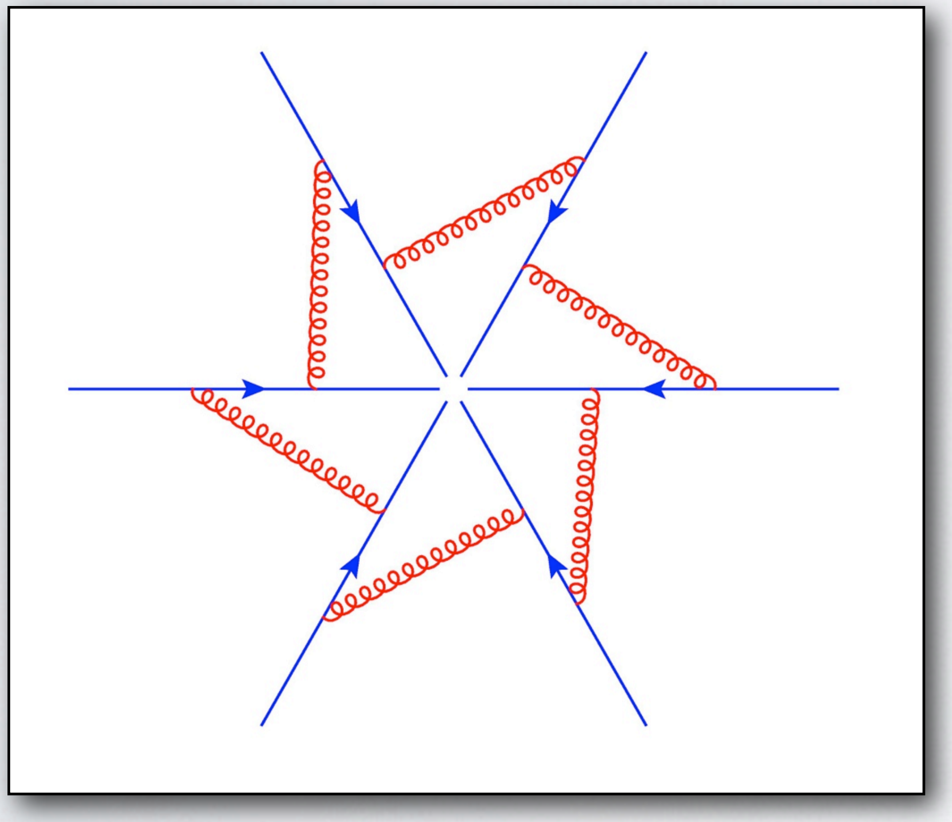
$$= \int_{A_1}^{B_1} \frac{d\xi_1}{\xi_1} \int_{A_2(\xi_1)}^{B_2(\xi_1)} \frac{d\xi_2}{\xi_2} \ldots \int_{A_{n-1}(\xi_1,\ldots,\xi_{n-2})}^{B_{n-1}(\xi_1,\ldots,\xi_{n-2})} \frac{d\xi_{n-1}}{\xi_{n-1}} + \mathcal{O}(\epsilon)$$



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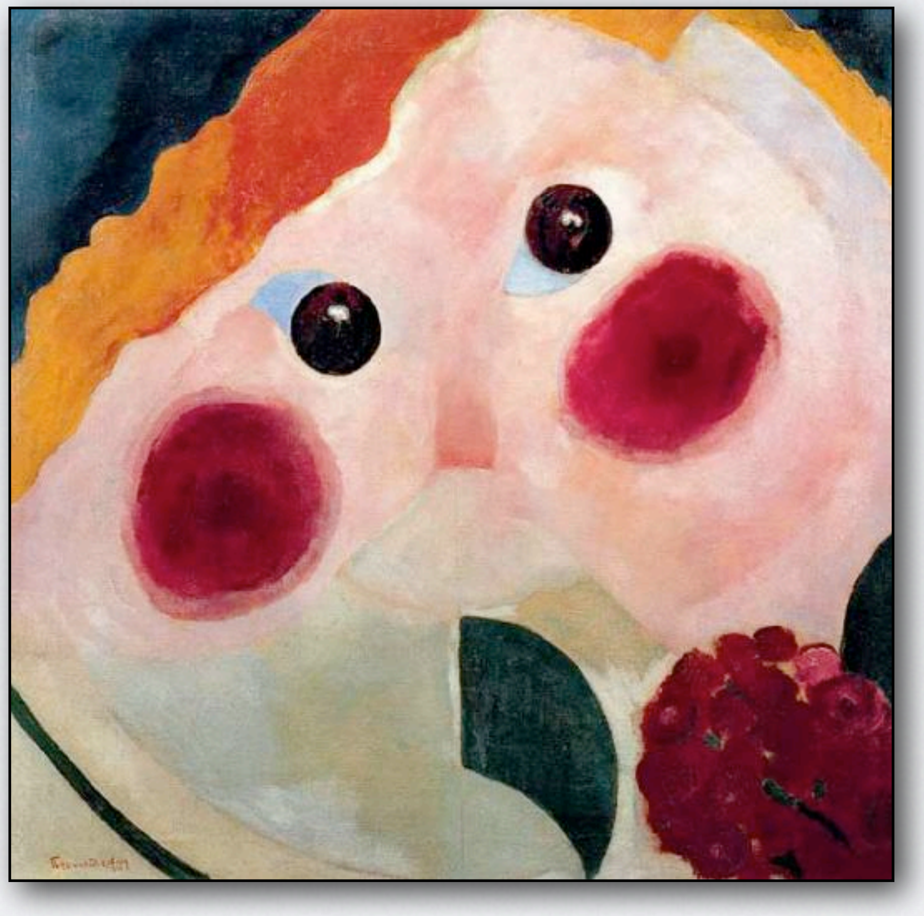
$$B_k(\xi_1,\ldots,\xi_{k-1}) = \frac{\prod_{j=k+1}^{n}(1-x_j)}{\prod_{j=k+2}^{n+1} x_j} \prod_{j=1}^{k-1} \frac{1+\xi_j}{\xi_j}$$

The result is remarkably simple!

$$\phi_{ES}^{(n)}(x_i;0) = \frac{1}{n!}\left[\log\left(\prod_{i=1}^{n}\frac{1-x_i}{x_i}\right)\right]^{n-1} \Theta^{(n)}(x_i)$$

OUTLOOK



Summary

- The past five years have seen rapid progress on the structure of infrared singularities of multi-particle gauge theory amplitudes.
- In the massless case, the dipole formula is being put to the test at three and four loops.
- The high-energy limit of the dipole formula provides insights into Reggeization and beyond, at least for divergent contributions to the amplitude.
- High-energy factorization generically breaks down at NNLL, with computable corrections, probably related to Regge cuts in the angular momentum plane.
- We recover from first principles the non-factorizing non-logarithmic two-loop remainder of Del Duca and Glover.
- We explicitly predict the leading non-factorizing high-energy logarithms at three loops for qq, gg and qg amplitudes in QCD.
- Similar results can be derived for multi-parton amplitudes in multi-Regge kinematics.
- We understand in detail the general structure of non-abelian eikonal exponentiation for multi-particle amplitudes.
- Special classes of diagrams can be computed to very high and in some cases to all orders.
- The computation of the full multi-particle soft anomalous dimension matrix at the three-loop order is feasible in a finite time.


THANK
MERCI
DANKE
GRAZIE
BEDANKEN
¡GRACIAS!
YOU