THRESHOLD LOGARITHMS BEYOND LEADING POWER

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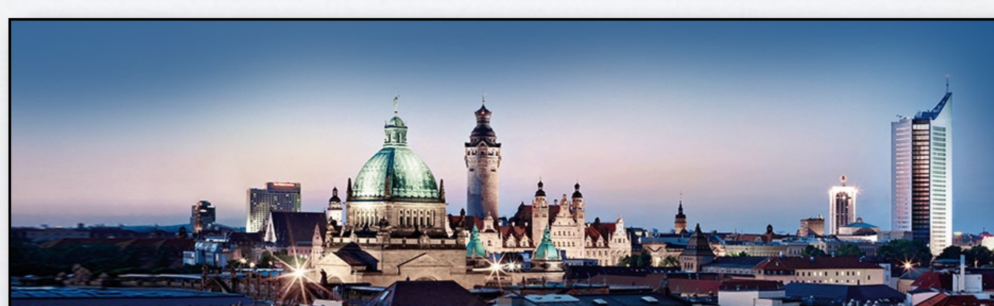

Outline

- Introduction
- Threshold resummations from LP to NLP
- Next-to-soft approximation
- Hard collinear issues
- Non-abelian intricacies
- Outlook

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INTRODUCTION

Factorization

We will focus on processes involving parton annihilation into electroweak final states (Drell-Yan, Higgs, di-boson final states): very well understood at LP, simpler at NLP.

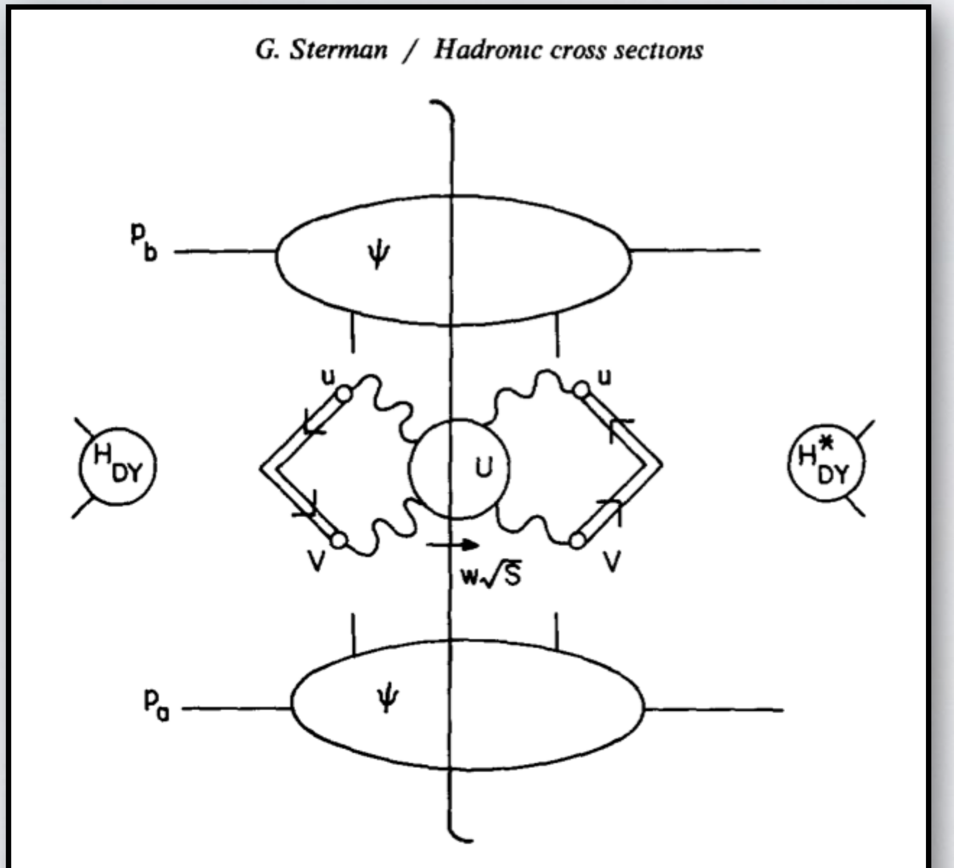


Fig 2.2 Graphical form of $z \to 1$ factorization for the DY process.

The original factorization near threshold

- LP threshold resummation is based on factorization: the Mellin-space partonic cross section reads

$$\omega(N, \epsilon) = |H_{\rm DY}|^2 \, \psi(N, \epsilon)^2 U(N) + \mathcal{O}\left(\frac{1}{N}\right) .$$

- Collinear poles can be subtracted with suitable parton distributions,

$$\widehat{\omega}_{\overline{\rm MS}} \, (N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\overline{\rm MS}} \, (N, \epsilon)^2}$$

- Each factor in ω obeys evolution equations near threshold, leading to exponentiation.

- Real and virtual contributions can be treated separately.

$$\psi_R(N, \epsilon) = \exp \left\{ \int_0^1 dz \, \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \, \kappa_\psi \left(\overline{\alpha} \left((1-y)^2 Q^2 \right), \epsilon \right) \right\} .$$

Exponentiation

A well-established formalism exists for distributions in processes that are electroweak at tree level (Gardi, Grunberg 07). For an observable r vanishing in the two-jet limit

$$\frac{d\sigma}{dr} = \delta(r)\left[1 + \mathcal{O}(\alpha_s)\right] + C_R \frac{\alpha_s}{\pi}\left\{\left[-\frac{\log r}{r} + \frac{b_1 - d_1}{r}\right]_+ + \mathcal{O}(r^0)\right\} + \mathcal{O}(\alpha_s^2)$$

The Mellin (Laplace) transform, $\sigma(N) = \int_0^1 dr\,(1-r)^{N-1}\,\frac{d\sigma}{dr}$

exhibits log N singularities that can be organized in exponential form

$$\sigma\left(\alpha_s, N, Q^2\right) = H(\alpha_s)\,\mathcal{S}\left(\alpha_s, N, Q^2\right) + \mathcal{O}\left(1/N\right)$$

where the exponent of the `Sudakov factor' is in turn a Mellin transform

$$\mathcal{S}\left(\alpha_s, N, Q^2\right) = \exp\left\{\int_0^1 \frac{dr}{r}\left[(1-r)^{N-1} - 1\right]\mathcal{E}\left(\alpha_s, r, Q^2\right)\right\}$$

and the general form of the kernel is

$$\mathcal{E}\left(\alpha_s, r, Q^2\right) = \int_{r^2Q^2}^{rQ^2} \frac{d\xi^2}{\xi^2}\,A\left(\alpha_s(\xi^2)\right) + B\left(\alpha_s(rQ^2)\right) + D\left(\alpha_s(r^2Q^2)\right)$$

where A is the cusp anomalous dimension, and B and D have distinct physical characters.

Resummation

A classic way to organize threshold logarithms in terms of the Mellin (Laplace) transform of the momentum space cross section (Catani et al. 93) is to write

$$
\begin{aligned}
d\sigma(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=0}^{2n} c_{nk} \log^k N + \mathcal{O}(1/N) \\
&= H(\alpha_s) \exp\Big[\log N\, g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s\, g_3(\alpha_s \log N) + \ldots\Big] + \mathcal{O}(1/N)
\end{aligned}
$$

This displays the main features of threshold resummation

- Predictive: a k-loop calculation determines g_k and thus a whole tower of logarithms to all orders in perturbation theory.
- Effective:
  - the range of applicability of perturbation theory is extended (finite order: $\alpha_s \log^2 N$ small. NLL resummed: α_s small);
  - the renormalization scale dependence is naturally reduced.
- Theoretically interesting: resummation ambiguities related to the Landau pole give access to non-perturbative power-suppressed corrections.
- Well understood:
  - NLL Sudakov resummations exist for most inclusive observables at hadron colliders, NNLL and approximate N^3LL in simple cases.

FROM LEADING
TO NEXT-TO-LEADING POWER

FROM LEADING TO NEXT-TO-LEADING POWER

More logarithms

- Threshold logarithms are associated with kinematic variables ξ that vanish at Born level and get corrections that are enhanced because phase space for real radiation is restricted near partonic threshold: examples are $1 - T$, $1 - M^2/\hat{s}$, $1 - x_{BJ}$.

- At leading power in the threshold variable ξ logarithms are directly related to soft and collinear divergences: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+p\epsilon}} = -\frac{1}{p\epsilon}\,\delta(\xi) + \left(\frac{1}{\xi}\right)_+ - p\epsilon\left(\frac{\log\xi}{\xi}\right)_+ + \dots$$

Cancels virtual IR poles

Leading power threshold logs

- Beyond the leading power, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1}\left[c_{nm}^{(-1)}\left(\frac{\log^m\xi}{\xi}\right)_+ + c_n^{(\delta)}\,\delta(\xi) + c_{nm}^{(0)}\log^m\xi + \dots\right]$$

Resummed to high accuracy

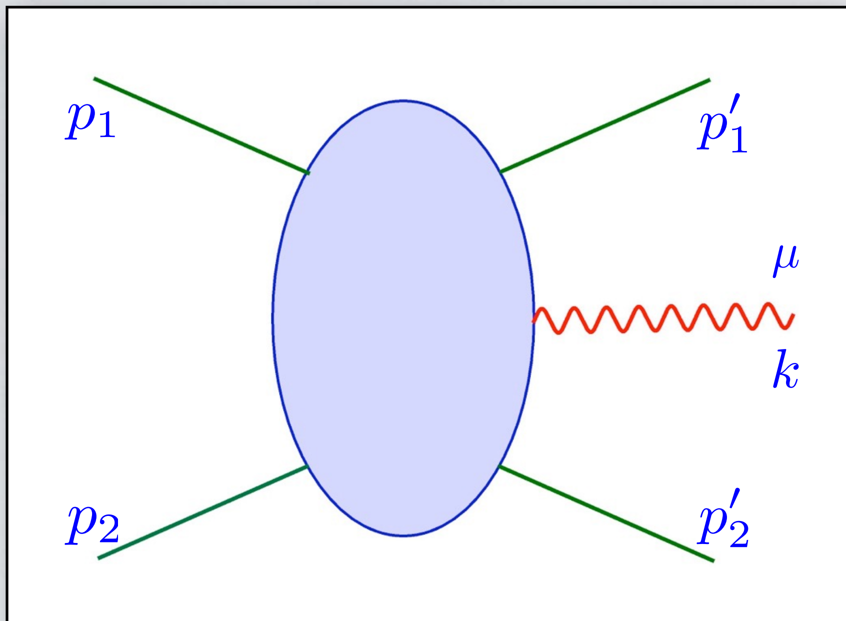
All-order structure in some cases

NLP threshold logs

- The structure of NLP threshold logarithms may be understood to all orders.

The LBKD Theorem

The earliest evidence that infrared effects can be controlled at NLP is Low's theorem (Low 58)



A radiative matrix element

$$M_\mu = e\left(\frac{p_{1\mu}'}{p_1'\cdot k} - \frac{p_{1\mu}}{p_1\cdot k}\right) T(\nu,\Delta) + e\left(\frac{p_{1\mu}' p_2'\cdot k}{p_1'\cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_2\cdot k}{p_1\cdot k} - p_{2\mu}\right)\frac{\partial T(\nu,\Delta)}{\partial \nu} + O(k),$$

Low's original expression for the radiative matrix element

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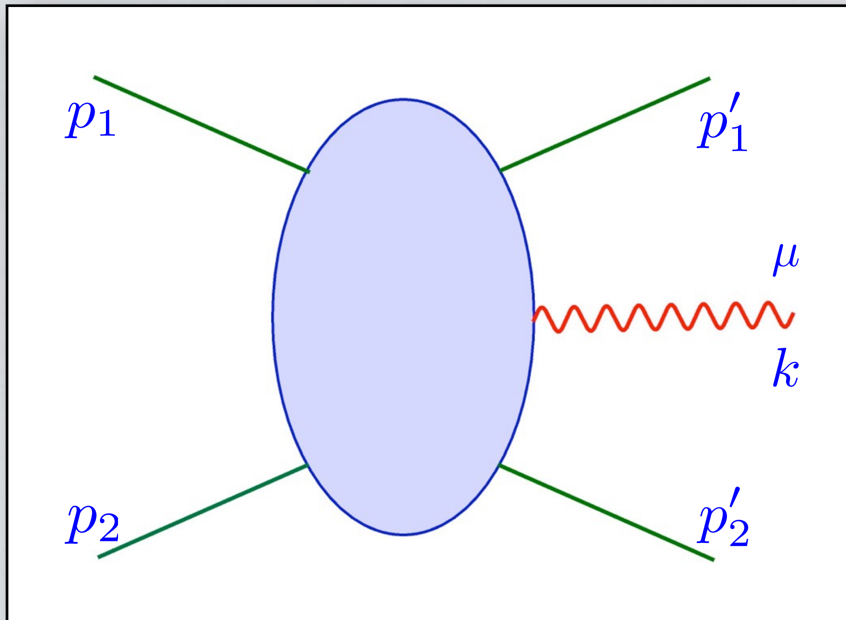

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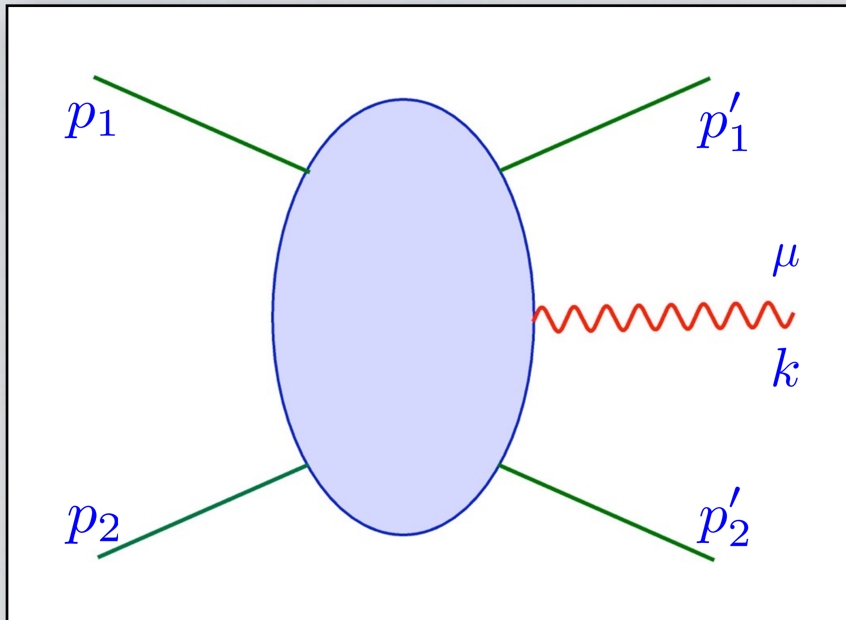

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Low's original expression for the radiative matrix element

Next-to-eikonal contribution

The radiative matrix element for the emission of a (next-to-) soft photon is determined by the Born amplitude T and its first derivative with respect to external momenta.

- Low's result established for a single charged scalar particle, follows from gauge invariance.
- It generalizes the well known properties of soft emissions in the eikonal approximation.
- The theorem was extended by (Burnett, Kroll 68) to particles with spin.
- The LBK theorem applies to massive particles and uses the mass as a collinear cutoff.
- It was extended to massless particles by (Del Duca 90), as discussed below.

Modified DGLAP

An important source of known NLP logarithms is the DGLAP anomalous dimension. Non-trivial connections between LP and NLP logarithms in DGLAP were uncovered (Moch, Vermaseren, Vogt 08) and made systematic (Dokshitzer, Marchesini, Salam 08).

Conventional DGLAP for a quark distribution reads

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}, \mu^2\right) P_{qq}(z, \alpha_s(\mu^2)).$$

$\longrightarrow$

$$\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2),$$

The large-N behavior of the anomalous dimension is single-logarithmic in the MS scheme. NLP terms suppressed by N are related to LP

$$\gamma_N(\alpha_s) = -A(\alpha_s) \ln \bar{N} + B_\delta(\alpha_s) - C_\gamma(\alpha_s) \frac{\ln \bar{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

MVV relations

$$C_1 = 0, \qquad C_2 = 4 C_F A_1, \qquad C_3 = 8 C_F A_2. \tag{3.12}$$

Especially the relation for C_3 is very suggestive and seems to call for a structural explanation.

These relation extend to the function D, and recursively to all orders: modified splitting functions can be defined which vanish at large x beyond one loop.

Modified DGLAP

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Modified DGLAP

DMS, with refinements implied by (Basso, Korchemsky 06) propose to modify DGLAP as

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi\left(\frac{x}{z}, z^\sigma \mu^2\right) \mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right).$$

applying to both PDF's and fragmentation, with $\sigma = \pm 1$ respectively, and the same kernel (Gribov-Lipatov reciprocity). The resulting kernel P and is claimed to vanish as $z \to 1$ beyond one loop in the "physical" MC scheme where $\alpha_s = \gamma_{cusp}$. Therefore

$$\mathcal{P}(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s)\delta(1-z) + \mathcal{O}(1-z).$$

The modified equation cannot be diagonalized by Mellin transform: it must be solved by iteration, using a formal translation operator

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} e^{-\ln z \left(\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2}\right)} \times \psi\left(\frac{x}{z}, \mu^2\right) \mathcal{P}(z, \alpha_s(\mu^2)),$$

In practice, this procedure constructs a modified kernel where high-order terms are generated by shifts of lower-orders

Towards systematics

The problem of NLP threshold logarithms has been of interest for a long time, and several different approaches have been proposed. Recent years have seen a resurgence of interest, both from a theoretical point of view and for phenomenology.

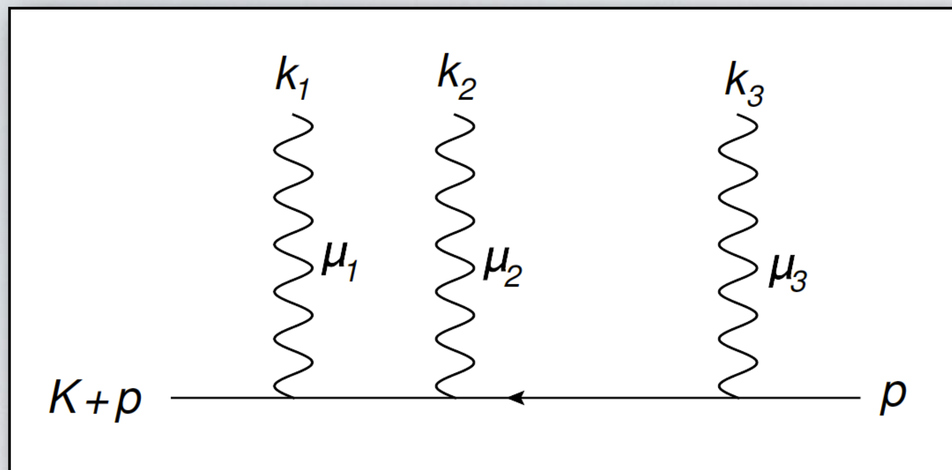
- Early attempts include a study of the impact of NLP logs on the Higgs cross section by Kraemer, Laenen, Spira (98); work on F_L by Akhoury and Sterman (99) (logs without plus distributions are however leading) and work by Grunberg *et al.* (07-09) on DIS.
- Important results can be obtained by using physical kernels (Vogt *et al.* 09-14) which are conjectured to be single-logarithmic at large z, which poses constraints on their factorized expression. Note in particular a recent application to Higgs production by De Florian, Mazzitelli, Moch, Vogt (14).
- Useful approximations can be obtained by combining constraints from large N with high-energy constraints for $N \sim 1$ and analiticity (Ball, Bonvini, Forte, Marzani, Ridolfi, 13), together with phase space refinements.
- SCET techniques are well-suited to the problem: see thorough one-loop analysis in (Larkoski, Neill, Stewart, 15); see also (Kolodrubetz, Moult, Neill, Stewart, 16).
- A lot of recent formal work on the behavior of gauge and gravity scattering amplitudes beyond the eikonal limit was triggered by a link to asymptotic symmetries of the S matrix (many authors from A(ndy Strominger) to Z(vi Bern), 14-15).

ISOLDE

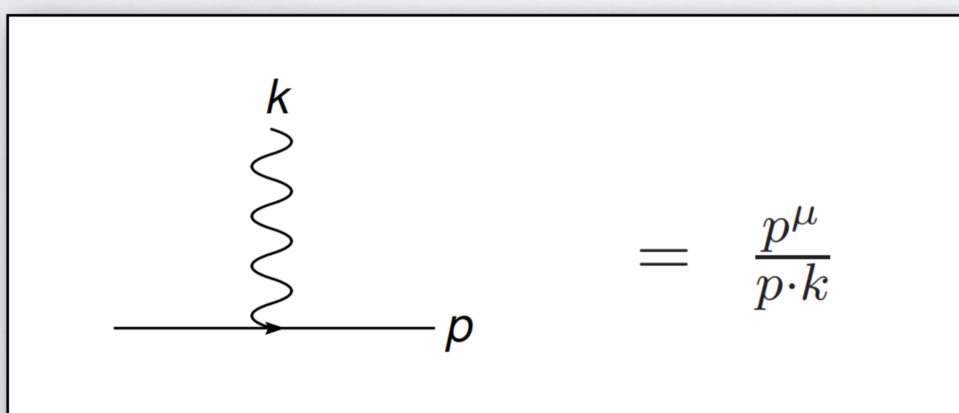
ISOLDE

Mild und leise wie er lächelt ...

On the soft approximation



A fast particle emitting soft photons



Eikonal Feynman rule

- Taking the soft approximation at leading power on emissions from an energetic (or very massive) particle yields a set of simplified Feynman rules.

- These rules correspond to emissions from a Wilson line oriented along the trajectory of the energetic particle, in the same color representation.

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n) \right] .$$

- The results do not depend on the energy and spin of the emitter, only on its direction and color charge.

- Physically, we are neglecting the recoil of the emitter: the only effect of interaction with soft radiation is that the emitter acquires a phase.

- The soft limit of a multi-particle amplitude is a correlator of Wilson lines

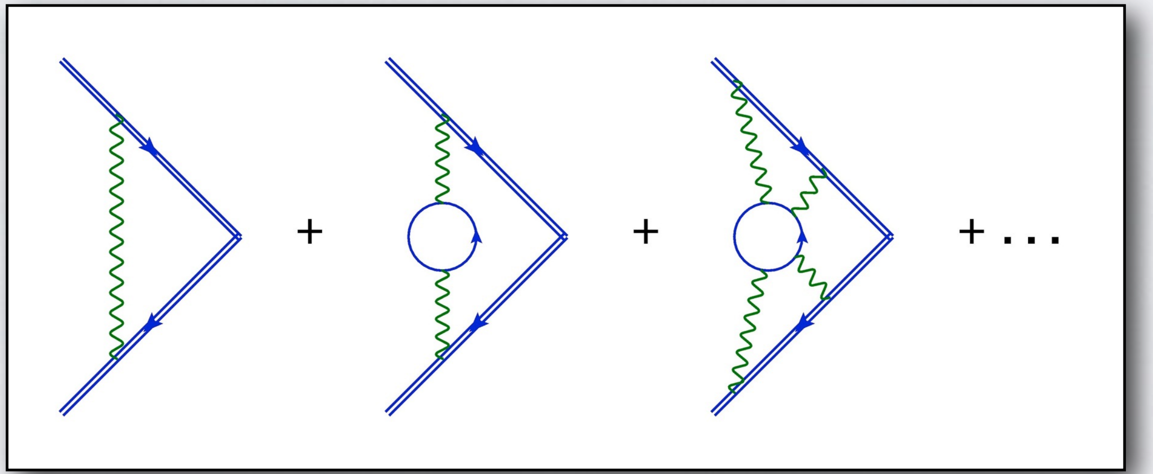
Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in exponential form.

$$S_n \equiv \langle 0 | \, \Phi_1 \otimes \ldots \otimes \Phi_n \, | 0 \rangle = \exp \left(\omega_n \right)$$

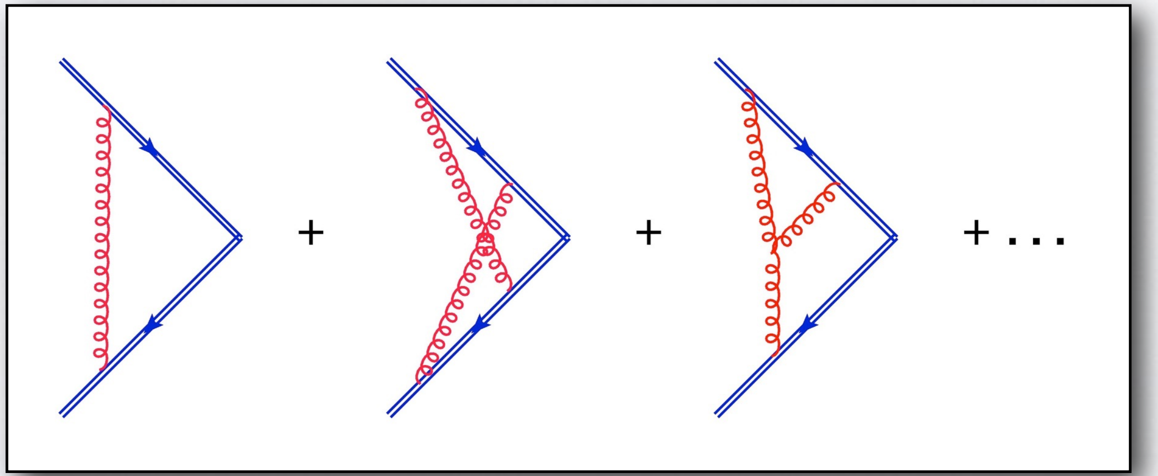
Diagrammatic rules exist to compute directly the logarithm of the correlators.

$$\omega_{2,\mathrm{QED}} =$$



Only connected photon subdiagrams contribute to the logarithm.

$$\omega_{2,\mathrm{QCD}} =$$



Only gluon subdiagrams which are two-eikonal irreducible contribute to the logarithm. They have modified color factors.

For eikonal form factors, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

Beyond the eikonal

The soft expansion can be organized beyond leading power using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic idea is simple, but the combinatorics cumbersome. For spinors

$$\frac{\not p + \not k}{2p\cdot k + k^2}\,\gamma^\mu u(p) \;=\; \left[\frac{p^\mu}{p\cdot k} + \frac{\not k\,\gamma^\mu}{2p\cdot k} - k^2\frac{p^\mu}{2(p\cdot k)^2}\right] u(p) + \mathcal{O}(k)$$

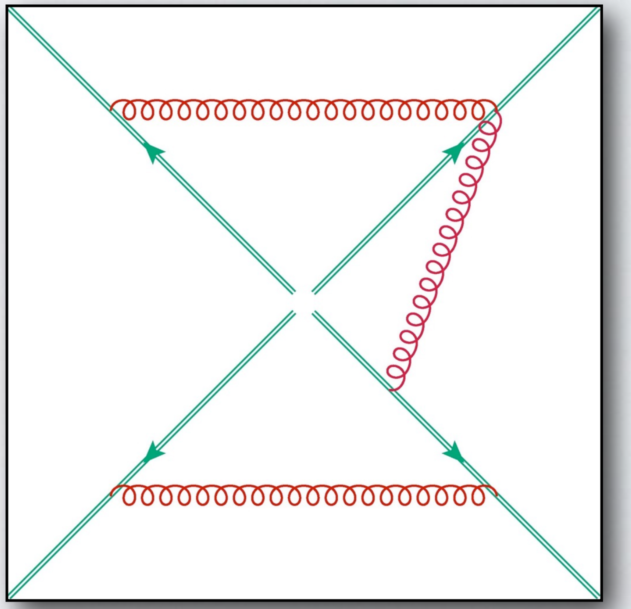
Eikonal — NE, spin-dependent — NE, spin-independent

- A class of factorizable contributions exponentiate via NE webs

$$\mathcal{M} = \mathcal{M}_0 \,\exp\left[\sum_{D_{\rm eik}} \tilde{C}(D_{\rm eik})\,\mathcal{F}(D_{\rm eik}) + \sum_{D_{\rm NE}} \tilde{C}(D_{\rm NE})\,\mathcal{F}(D_{\rm NE})\right] .$$

- Feynman rules exist for the NE exponent, including "seagull" vertices.

$$\mathcal{M} \;=\; \mathcal{M}_0 \,\exp\left[\mathcal{M}_{\rm eik} + \mathcal{M}_{\rm NE}\right]\,\left(1 + \mathcal{M}_r\right) + \mathcal{O}\left({\rm NNE}\right) \;.$$



A next-to-eikonal web

- Non-factorizable contributions involve single gluon emission from inside the hard function, and must be studied using LBDK's theorem.

Next-to-eikonal lines

Exponentiating next-to-soft contribution can be organised into next-to-eikonal lines (Laenen, Stavenga, White 08), which generate the appropriate Feynman rules. In momentum space

$$\tilde{F}(\beta) = \exp\left[\int \frac{d^dk}{(2\pi)^d} \tilde{A}_\mu(k) \left(-\frac{\beta^\mu}{\beta\cdot k} + \frac{k^\mu}{2\beta\cdot k} - k^2 \frac{\beta^\mu}{2(\beta\cdot k)^2} - \frac{ik_\nu \Sigma^{\nu\mu}}{p\cdot k}\right)\right.$$
$$+ \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dl}{(2\pi)^d} \tilde{A}_\mu(k)\tilde{A}_\nu(l) \left(\frac{\eta^{\mu\nu}}{2\beta\cdot(k+l)} - \frac{\beta^\nu l^\mu \beta\cdot k + \beta^\mu k^\nu \beta\cdot l}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}\right.$$
$$\left.\left. + \frac{(k\cdot l)\beta^\mu\beta^\nu}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]} - \frac{\Sigma^{\mu\nu}}{2p\cdot k}\right)\right].$$

- One recognises the eikonal Feynman rule and the NE scalar propagator corrections.
- At NLP level spin dependence enters: Σ is the spin angular momentum operator.
- Seagull vertices, with spin dependence, arise from the cancellation of propagators.
- Complicated momentum-dependent two-gluon correlations arise at next-to-soft level.
- This expression applies to semi-infinite lines, it is easily generalised in coordinate space.

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$$+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\tilde{A}_\mu(k)\tilde{A}_\nu(l)\left(\frac{\eta^{\mu\nu}}{2\beta\cdot(k+l)}-\frac{\beta^\nu l^\mu\beta\cdot k+\beta^\mu k^\nu\beta\cdot l}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}\right.$$
$$\left.+\frac{(k\cdot l)\beta^\mu\beta^\nu}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}-\frac{\Sigma^{\mu\nu}}{2p\cdot k}\right)\Bigg].$$

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$$+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\tilde{A}_\mu(k)\tilde{A}_\nu(l)\left(\frac{\eta^{\mu\nu}}{2\beta\cdot(k+l)}-\frac{\beta^\nu l^\mu\beta\cdot k+\beta^\mu k^\nu\beta\cdot l}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}\right.$$

$$\left.\left.+\frac{(k\cdot l)\beta^\mu\beta^\nu}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}-\frac{\Sigma^{\mu\nu}}{2p\cdot k}\right)\right].$$

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NE, spin-independent

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$$+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\tilde{A}_\mu(k)\tilde{A}_\nu(l)\left(\frac{\eta^{\mu\nu}}{2\beta\cdot(k+l)} - \frac{\beta^\nu l^\mu \beta\cdot k + \beta^\mu k^\nu \beta\cdot l}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}\right.$$
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Seagulls

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- One recognises the eikonal Feynman rule and the NE scalar propagator corrections.
- At NLP level spin dependence enters: Σ is the spin angular momentum operator.
- Seagull vertices, with spin dependence, arise from the cancellation of propagators.
- Complicated momentum-dependent two-gluon correlations arise at next-to-soft level.
- This expression applies to semi-infinite lines, it is easily generalised in coordinate space.

Next-to-eikonal lines

Exponentiating next-to-soft contribution can be organised into next-to-eikonal lines (Laenen, Stavenga, White 08), which generate the appropriate Feynman rules. In momentum space

$$\tilde{F}(\beta) = \exp\left[\int \frac{d^dk}{(2\pi)^d}\tilde{A}_\mu(k)\left(-\frac{\beta^\mu}{\beta\cdot k}+\frac{k^\mu}{2\beta\cdot k}-k^2\frac{\beta^\mu}{2(\beta\cdot k)^2}-\frac{ik_\nu\Sigma^{\nu\mu}}{p\cdot k}\right)\right.$$
$$+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\tilde{A}_\mu(k)\tilde{A}_\nu(l)\left(\frac{\eta^{\mu\nu}}{2\beta\cdot(k+l)}-\frac{\beta^\nu l^\mu\beta\cdot k+\beta^\mu k^\nu\beta\cdot l}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}\right.$$
$$\left.\left.+\frac{(k\cdot l)\beta^\mu\beta^\nu}{2(\beta\cdot l)(\beta\cdot k)[\beta\cdot(k+l)]}-\frac{\Sigma^{\mu\nu}}{2p\cdot k}\right)\right].$$

Eikonal

Two-gluon correlations

Seagulls

NE, spin-independent

Two-gluon correlations

NE, spin-dependent

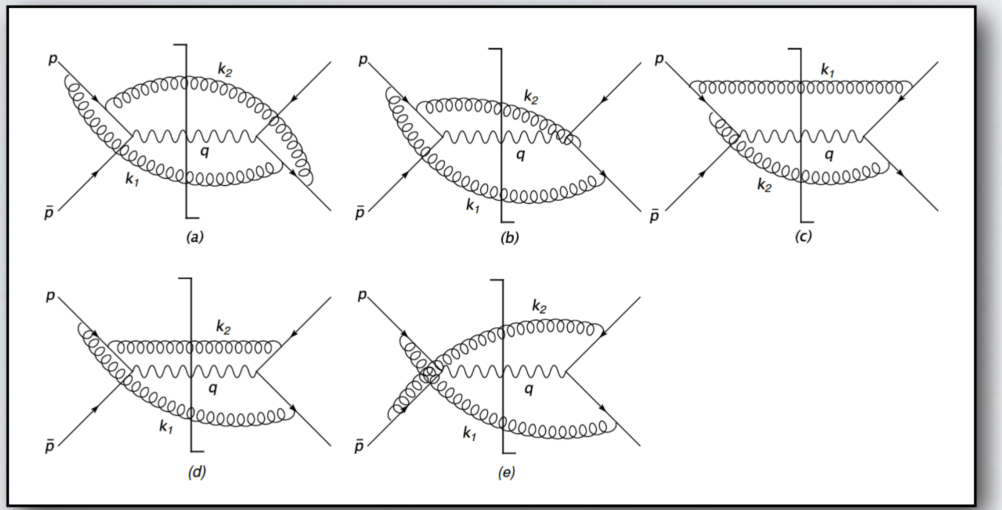
- One recognises the eikonal Feynman rule and the NE scalar propagator corrections.
- At NLP level spin dependence enters: Σ is the spin angular momentum operator.
- Seagull vertices, with spin dependence, arise from the cancellation of propagators.
- Complicated momentum-dependent two-gluon correlations arise at next-to-soft level.
- This expression applies to semi-infinite lines, it is easily generalised in coordinate space.

Double real two-loop Drell-Yan

Multiple real emission contributions to EW annihilation processes involve only factorizable contributions. NE Feynman rules can be tested this level.

Defining the Drell-Yan K-factor as

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz} \,,$$



Real emission Feynman diagrams for the abelian part of the NNLO K-factor.

As a test, we (re)computed the C_F^2 part of K at NNLO from ordinary Feynman diagrams, and then using NE Feynman rules, finding complete agreement. As expected, plus distributions arise from the eikonal approximation.

Next-to-eikonal terms arise from single-gluon corrections: seagull-type contributions vanish for the inclusive cross section.

$$\begin{aligned} K^{(2)}_{\rm NE}(z) = \left(\frac{\alpha_s}{4\pi} C_F\right)^2 \Bigg[& -\frac{32}{\epsilon^3}\mathcal{D}_0(z) + \frac{128}{\epsilon^2}\mathcal{D}_1(z) - \frac{128}{\epsilon^2}\log(1-z) \\ & - \frac{256}{\epsilon}\mathcal{D}_2(z) + \frac{256}{\epsilon}\log^2(1-z) - \frac{320}{\epsilon}\log(1-z) \\ & + \frac{1024}{3}\mathcal{D}_3(z) - \frac{1024}{3}\log^3(1-z) + 640\log^2(1-z) \Bigg] \,, \end{aligned}$$

The abelian part of the NNLO K-factor from real emission, omitting constants

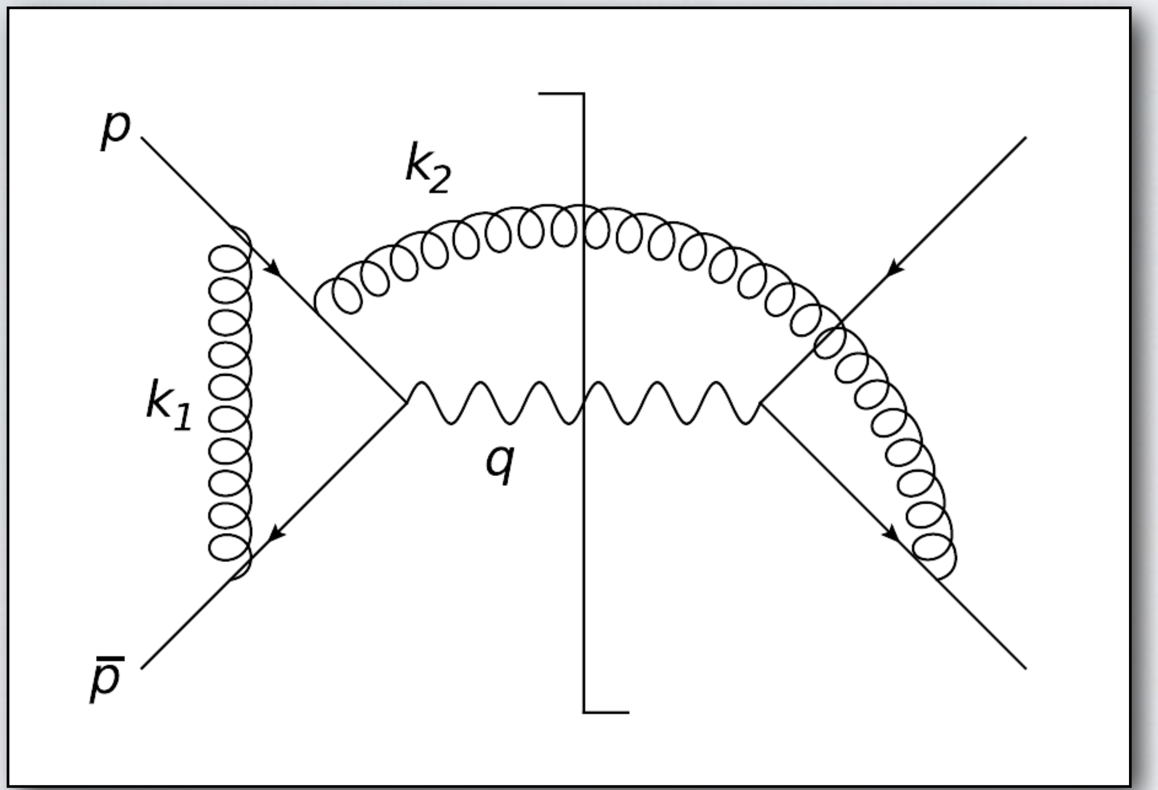
BRÜNNHILDE

BRÜNNHILDE

Starke Scheite schichtet mir dort …

A collinear problem

Non-factorizable contributions start at NNLO. For massive particles they can be traced to the original LBK theorem. For massless particles a new contribution to NLP logs emerges.



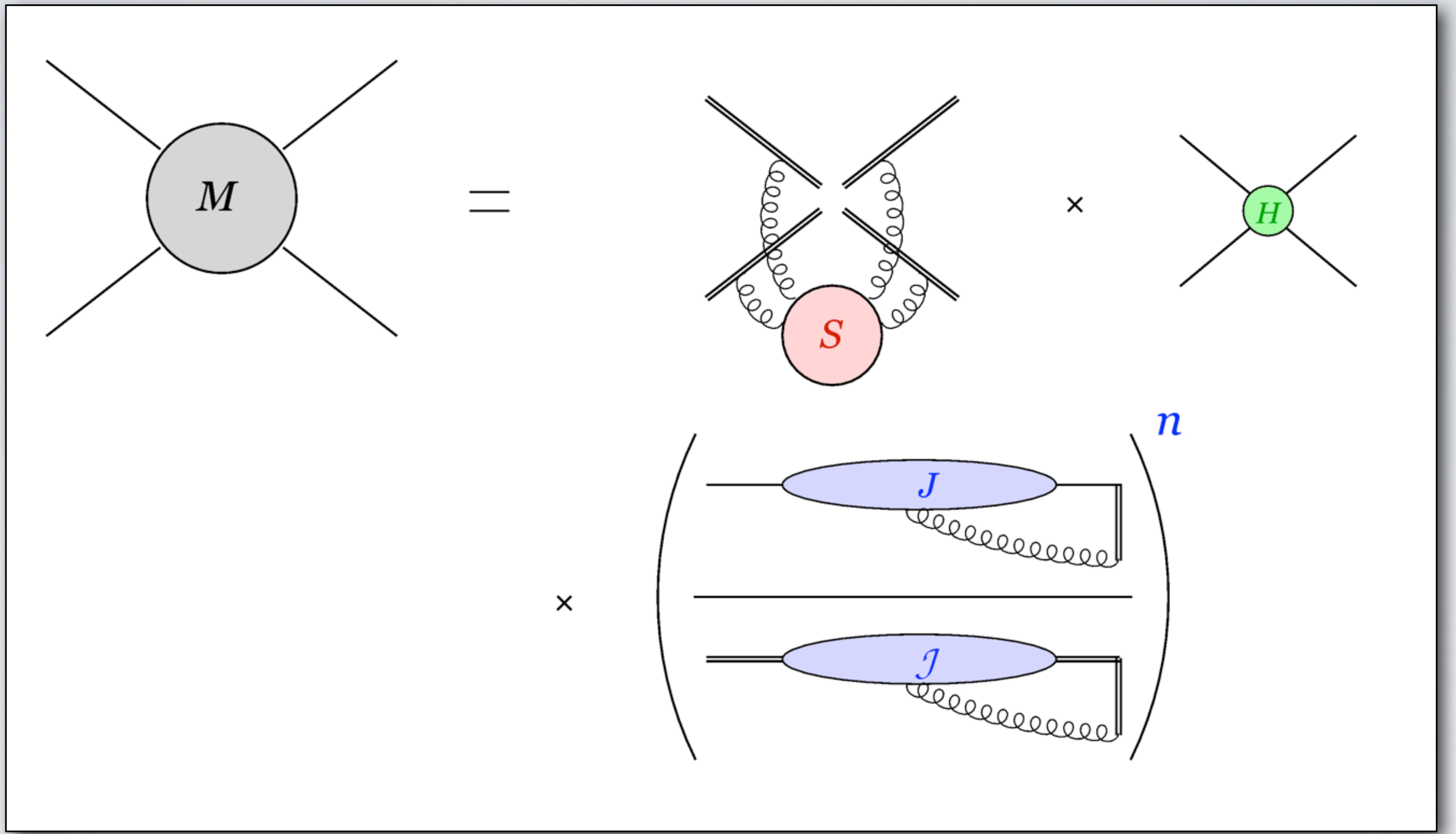
A Feynman diagram containing a collinear enhancement

- Gluon k_2 is always (next-to) soft for EW annihilation near threshold.
- When k_1 is (next-to) soft all logs are captured by NE rules.
- Contributions with k_1 hard and collinear are missed by the soft expansion.
- The collinear pole interferes with soft emission and generates NLP logs.
- The problem first arises at NNLO

- These contributions are missed by the LBK theorem: it applies to an expansion in E_k/m.
- They can be analized using the method of regions: the relevant factor is $(p \cdot k_2)^{-\epsilon}/\epsilon$.
- They cause the breakdown of next-to-soft theorems for amplitudes beyond tree level.

  ➡ the soft expansion and the limit $\epsilon \to 0$ do not commute.
- They require an extension of LBK to $m^2/Q < E_k < m$. It was provided by Del Duca (90).

LP factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

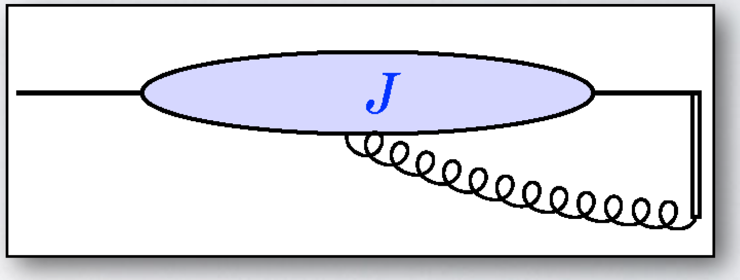
LP factorization: operators

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right],$$

Here we introduced dimensionless four-velocities $\beta_i^\mu = Q\, p_i^\mu$, $\beta_i^2 = 0$, and factorization vectors n_i^μ, $n_i^2 \neq 0$ to define the jets,

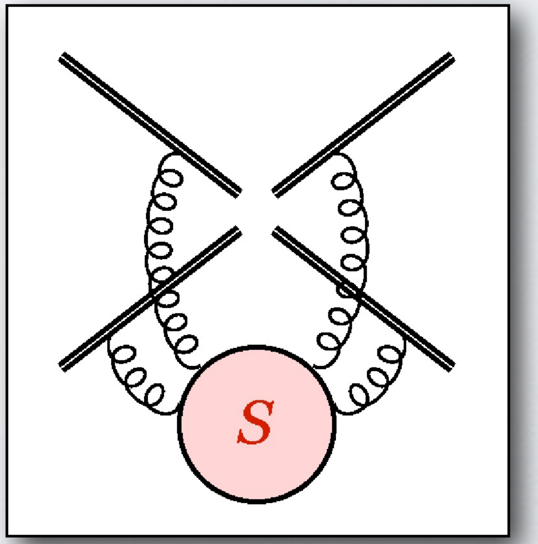
$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 \,|\, \Phi_n(\infty, 0)\, \psi(0)\, | p \rangle\,.$$



where Φ_n is the Wilson line operator along the direction n^μ.

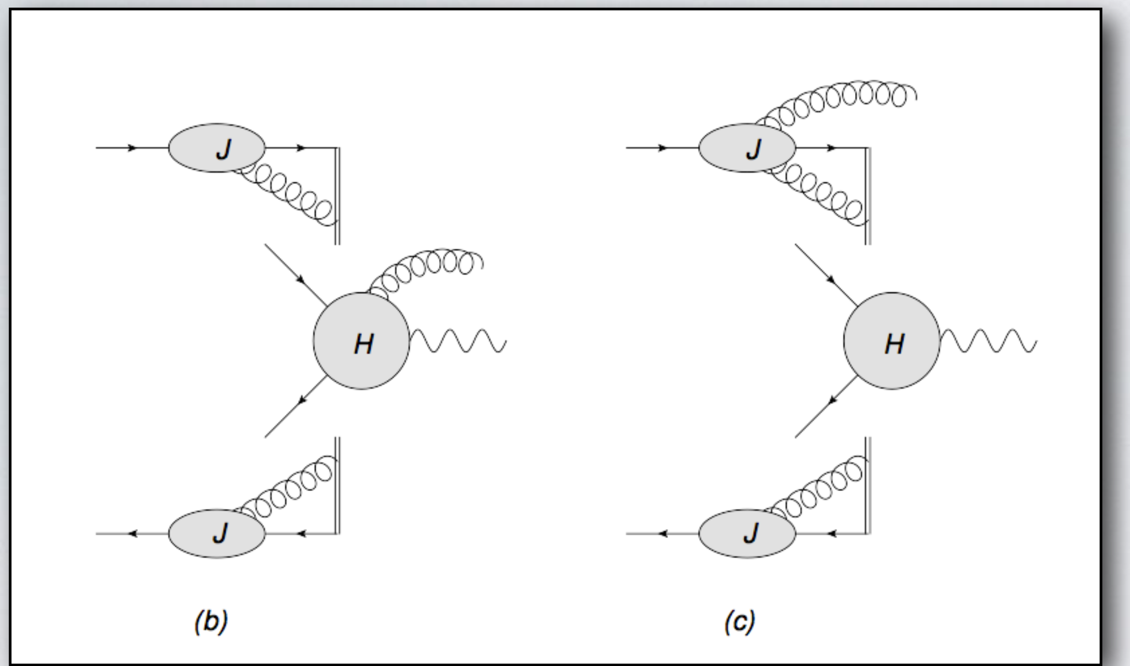
The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$(c_L)_{\{a_k\}}\, \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \epsilon\right) = \langle 0 | \prod_{k=1}^{n}\left[\Phi_{\beta_k}(\infty, 0)\right]_{a_k}^{\;b_k} |0\rangle \, (c_K)_{\{b_k\}},$$



where the c_L are a basis of color tensors for the process at hand.

NLP factorization: a new jet



Factorized contributions to the radiative amplitude

Soft radiation can arise either from the jets or from the hard function

$$\mathcal{A}_\mu \, \epsilon^\mu(k) \;=\; \mathcal{A}^J_\mu \, \epsilon^\mu(k) + \mathcal{A}^H_\mu \, \epsilon^\mu(k) \,,$$

The amplitude for emission from the jets can be precisely defined in terms of a new jet function

$$\mathcal{A}^J_\mu \;=\; \sum_{i=1}^{2} H(p_i - k; p_j, n_j) \, J_\mu(p_i, k, n_i) \prod_{j \neq i} J(p_j, n_j) \;\equiv\; \sum_{i=1}^{2} \mathcal{A}^{J_i}_\mu \,.$$

$$J_\mu \left(p, n, k, \alpha_s(\mu^2), \epsilon \right) u(p) \;=\; \int d^d y \; \mathrm{e}^{-\mathrm{i}(p-k)\cdot y} \; \langle 0 \, | \, \Phi_n(y, \infty) \, \psi(y) \, j_\mu(0) \, | \, p \rangle \;,$$

defines the radiative jet.

- At tree level the radiative jet displays the expected dependence on spin.
- Dependence on the gauge vector n^μ starts at loop level: simplifications arise for $n^2 = 0$.

$$J^{\nu(0)}\left(p, n, k\right) \;=\; \frac{\not{k}\gamma^\nu}{2p\cdot k} - \frac{p^\nu}{p\cdot k}$$
$$= -\,\frac{p^\nu}{p\cdot k} + \frac{k^\nu}{2p\cdot k} - \frac{\mathrm{i}\,k_\alpha \Sigma^{\alpha\mu}}{2p\cdot k} \,.$$

Beyond Low's theorem

A slightly modified version of Del Duca's result gives the radiative amplitude in terms of the non-radiative one, its derivatives, and the two "jet" functions.

$$\mathcal{A}^\mu(p_j,k) = \sum_{i=1}^{2}\Bigg\{ q_i \left(\frac{(2p_i-k)^\mu}{2p_i\cdot k - k^2} + G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu}\right) + G_i^{\nu\mu}\left[\frac{J_\nu(p_i,k,n_i)}{J(p_i,n_i)} - q_i\frac{\partial}{\partial p_i^\nu}\Big(\ln J(p_i,n_i)\Big)\right]\Bigg\}\mathcal{A}(p_i;p_j)\,.$$

The tensors $G^{\mu\nu}$ project out the eikonal contribution present in the first term.

$$\eta^{\mu\nu} = G^{\mu\nu} + K^{\mu\nu}\,, \qquad K^{\mu\nu}(p;k) = \frac{(2p-k)^\nu}{2p\cdot k - k^2}\,k^\mu\,,$$

The factorized expression for the radiative amplitude can be simplified.

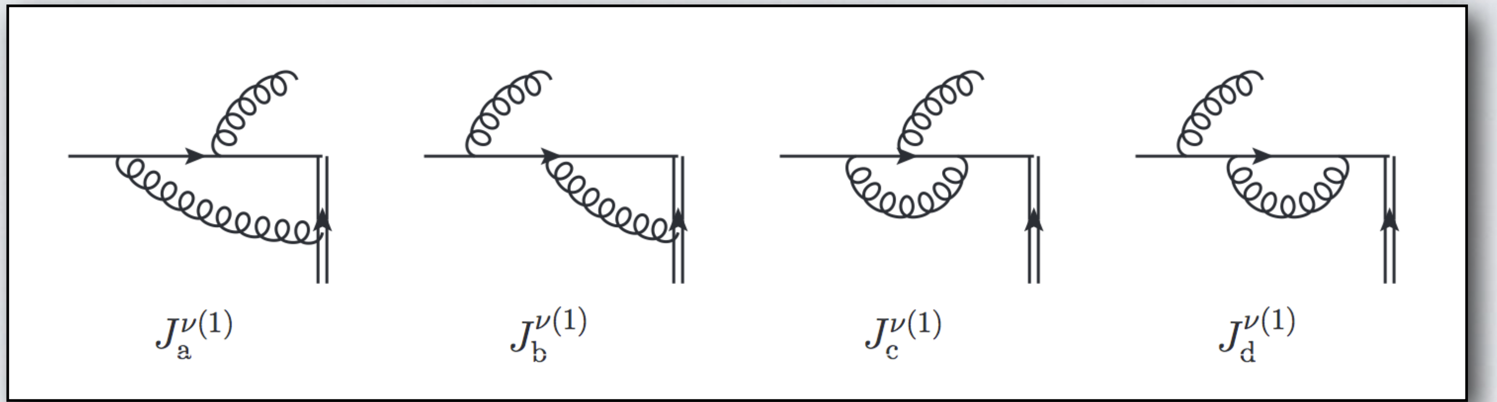
- The jet factor is RG invariant: it can be computed in bare perturbation theory.
- With this choice one can use that $J(p,n) = 1$ for $n^2 = 0$: it is a pure counterterm.
- The choice of reference vectors is then physically motivated (and confirmed by a complete analysis using the method of regions): we take $n_1 = p_2$ and $n_2 = p_1$.

$$\mathcal{A}^\mu(p_j,k) = \sum_{i=1}^{2}\left(q_i\,\frac{(2p_i-k)^\mu}{2p_i\cdot k - k^2} + q_i\,G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu} + G_i^{\nu\mu}J_\nu(p_i,k)\right)\mathcal{A}(p_i;p_j)\,.$$

For general amplitudes, a full subtraction of the residual n dependence should be aimed at.

The one-loop radiative jet

To achieve NNLO accuracy at NLP, we need the radiative jet function at one loop.
As a test, we compute the C_F^2 contributions. They simplify considerably for $n^2 = 0$.



Abelian-like Feynman diagrams for the bare one-loop radiative jet

The result has a simple structure, with characteristic scale and spin dependence

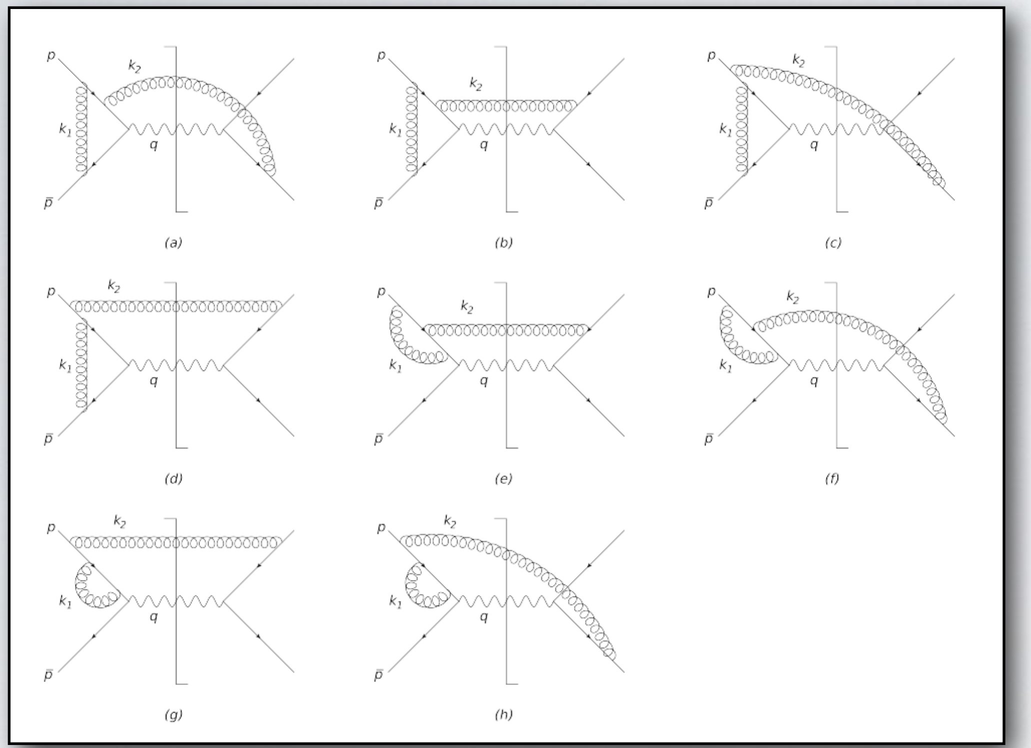
$$J^{\nu(1)}(p,n,k\,;\epsilon) = (2p\cdot k)^{-\epsilon}\Bigg[\left(\frac{2}{\epsilon}+4+8\epsilon\right)\left(\frac{n\cdot k}{p\cdot k}\frac{p^\nu}{p\cdot n}-\frac{n^\nu}{p\cdot n}\right)-(1+2\epsilon)\,\frac{i\,k_\alpha\Sigma^{\alpha\nu}}{p\cdot k}$$
$$+\left(\frac{1}{\epsilon}-\frac{1}{2}-3\epsilon\right)\frac{k^\nu}{p\cdot k}+(1+3\epsilon)\left(\frac{\gamma^\nu\,\not{n}}{p\cdot n}-\frac{p^\nu\,\not{k}\,\not{n}}{p\cdot k\,p\cdot n}\right)\Bigg]+\mathcal{O}(\epsilon^2,k)\,,$$

As a test, it obeys the Ward identity

$$k^\mu\,J_\mu\left(p,n,k,\alpha_s(\mu^2),\epsilon\right) \;=\; q\,J\left(p,n,\alpha_s(\mu^2),\epsilon\right)\;,$$

Real-virtual two-loop Drell-Yan

Real-virtual corrections to EW annihilation processes involve non-factorizable contributions. NE rules cannot reproduce the perturbative result at NLP, due to collinear interference.



Real-virtual Feynman diagrams for the abelian part of the NNLO K-factor.

As a test of the LBDK factorization, we computed the C_F^2 part of the real-virtual K-factor at NNLO from ordinary Feynman diagrams, and then using the radiative amplitude integrated over phase space. As expected, plus distributions arise from the eikonal approximation, fully determined by the dressed non-radiative amplitude. Derivative terms and the projected radiative jet contribute at NLP.

- All NLP terms are correctly reproduced, including those with no logarithms.
- The radiative jet reproduces exactly the NLP collinear contribution derived by the method of regions.

$$
\begin{aligned}
K^{(2)}_{\rm rv}(z) = \left(\frac{\alpha_s}{4\pi} C_F\right)^2 \Bigg\{ & \frac{32}{\epsilon^3}\Big[\mathcal{D}_0(z) - 1\Big] + \frac{16}{\epsilon^2}\Big[-4\mathcal{D}_1(z) + 3\mathcal{D}_0(z) + 4L(z) - 6\Big] \\
& + \frac{4}{\epsilon}\Big[16\mathcal{D}_2(z) - 24\mathcal{D}_1(z) + 32\mathcal{D}_0(z) - 16L^2(z) + 52L(z) - 49\Big] \\
& - \frac{128}{3}\mathcal{D}_3(z) + 96\mathcal{D}_2(z) - 256\mathcal{D}_1(z) + 256\mathcal{D}_0(z) \\
& + \frac{128}{3}L^3(z) - 232L^2(z) + 412L(z) - 408 \Bigg\},
\end{aligned}
$$

The abelian part of the NNLO K-factor from real-virtual diagrams, omitting constants

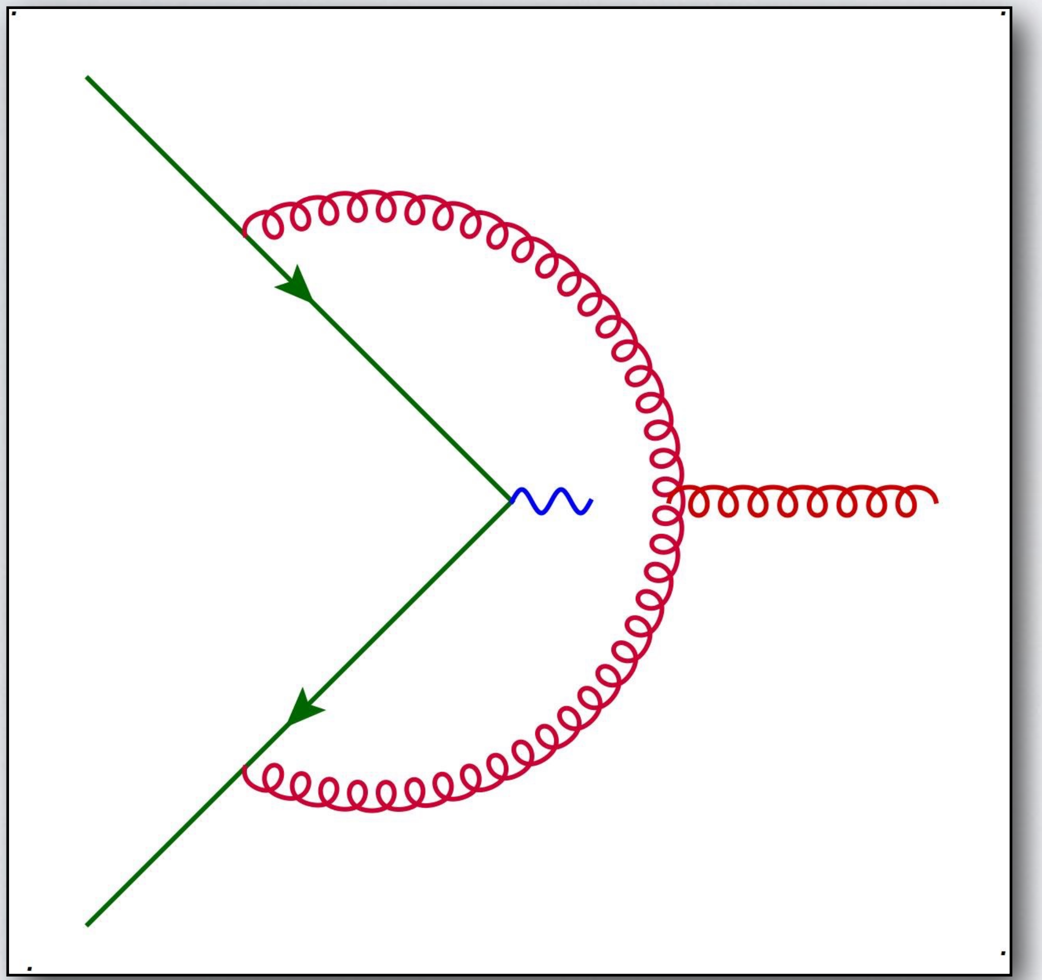
REGENBOGEN

REGENBOGEN

Zur Burg führt die Brücke …

Non-abelian troubles

The quasi-abelian factorization suffers from non-trivial limitations. They are exemplified by a single diagram contributing to the real-virtual part of the NNLO cross section.



A Feynman diagram contributing to all singular regions

- This diagram contributes to all relevant momentum regions.
- Non-abelian jet contributions when the virtual gluon is hard and (anti)collinear.

  Scales: $(2p \cdot k)^{-\epsilon} \qquad (2\bar{p} \cdot k)^{-\epsilon}$

- (Next-to-)soft contributions when the virtual gluon is soft.

  Scale: $\left(\frac{s\mu^2}{2p \cdot k \, 2\bar{p} \cdot k} \right)^{\epsilon}$

- The quasi-abelian radiative jet definition must be upgraded to an interacting final state gluon.
- New cases of double counting in the factorisation for (next-to-)soft-collinear regions.
- Introduce a radiative (next-to-)soft function and next-to-eikonal non-abelian radiative jets.

preliminary!

Matrix elements for factorisation

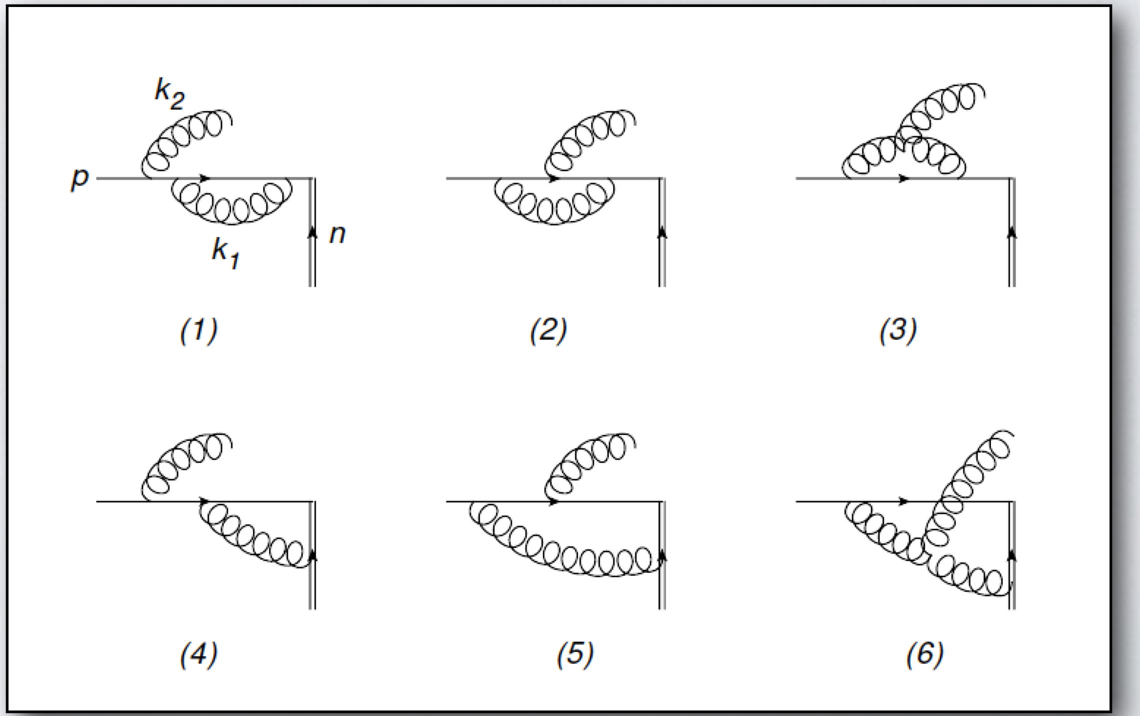
The necessary functions can be defined as matrix elements with single-particle physical states.

$$\epsilon^*_{(\lambda)}(k) \cdot J\,(p,k,n)\, u_{(s)}(p) \;=\; \langle k,\lambda|\, \Phi_n(0,\infty)\psi(0)\, |p,s\rangle \;,$$

$$\epsilon^*_{(\lambda)}(k) \cdot \mathcal{J}\,(p,k,n) \;=\; \langle k,\lambda|\, \Phi_n(0,\infty)\Phi_\beta(\infty,0)\, |0\rangle \;,$$

$$\epsilon^*_{(\lambda)}(k) \cdot W(\beta,\bar{\beta},k) \;=\; \langle k,\lambda|\, \Phi_\beta(0,\infty)\Phi_{\bar{\beta}}(\infty,0)\, |0\rangle \;.$$

- All functions are transverse (vanish when $\varepsilon \to k$).
- All functions can be understood as single particle contributions to cross-section-level quantities.
- All functions can naturally upgraded to next-to-soft level by replacing the Wilson lines Φ_β with F_β.
- Jet functions also involve emissions from the gauge Wilson line Φ_n but these vanish for $n^2 = 0$.
- Scale dependence matches the method of regions.



Diagrams contributing to the non-abelian radiative jet preliminary!

Non-abelian factorization

The quasi-abelian NLP factorisation formula for the real-virtual amplitude generalises to

$$\mathcal{A}^{\mu\,a}(p_j,k)\epsilon_\mu(k) = \epsilon_\mu(k)\sum_{i=1}^{2}\Bigg\{\left(\frac{1}{2}\,\overline{W}^{\mu\,a} + \mathbf{T}_i^a G_i^{\nu\mu}\,\frac{\partial}{\partial p_i^\nu}\right)\mathcal{A}(\{p_i\})$$
$$+\left(J^{\mu\,a}(p_i,k,n_i) - \tilde{\mathcal{J}}^{\mu\,a}(\beta_i,k,n_i) - \mathbf{T}_i^a\,G_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu}\frac{J(p_i,n_i)}{\tilde{\mathcal{J}}(\beta_i,n_i)}\right)$$
$$\times\,\tilde{\mathcal{H}}(p_j,n_j)\,\tilde{\mathcal{S}}(\beta_j,n_j)\prod_{j\neq i}\frac{J(p_j,n_j)}{\tilde{\mathcal{J}}(\beta_j,n_j)}\Bigg\},$$

- The first line contains all (next-to-)soft effects, corresponding to a non-abelian Low's theorem.
- The function W has been promoted to next-to-soft accuracy, and divided by the non-radiative soft function to reconstruct the non-radiative amplitude.
- Jet functions are fully subtracted with next-to-eikonal jets, leaving only hard collinear terms.
- The choice $n^2 = 0$ leads to a more transparent expression, easier to evaluate.

$$\mathcal{A}^{\mu\,a}(p_j,k)\epsilon_\mu(k) = \epsilon_\mu(k)\sum_{i=1}^{2}\left(\frac{1}{2}\,W^{\mu\,a} + \mathbf{T}_i^a G_i^{\nu\mu}\,\frac{\partial}{\partial p_i^\nu} + J^{\mu\,a}(p_i,k,n_i) - \tilde{\mathcal{J}}^{\mu\,a}(\beta_i,k,n_i)\right)\mathcal{A}(\{p_i\}).$$

All functions have been computed at one loop, and tested by reproducing the full non-abelian contribution to the two-loop real-virtual NNLO Drell-Yan cross section at NLP accuracy.

GÖTTERDÄMMERUNG

A Perspective

- Perturbation theory continues to display new and unexplored structures.
- Leading power threshold resummation is highly developed and provides some of the most precise predictions in perturbative QCD.
- Low's theorem is the first of many hints that NLP logs can be understood and organized.
- Different approaches catch a number of towers of NLP logs in simple processes.
- The next-to-soft approximation is well understood, using both diagrammatic and path integral approaches, even for multi-parton processes.
- Hard collinear emissions spoil Low's theorem: a new radiative jet function emerges.
- Non-abelian contributions introduce significant complications but can be handled.
- A complete treatment of NLP threshold logs is at hand.
- Much work to do to organize a true resummation formula, even for EW annihilation: we have a more intricate "factorization", we need to study and implement evolution equations.
- In order to achieve complete generality, we will need to include final state jets.

Recht so! Habt Dank! Ein wenig Rast …