# ON THE INFRARED STRUCTURE OF PERTURBATIVE GAUGE THEORIES

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Les Houches - 12-13/06/14

# Outline

- Bugs and features of perturbation theory
- A first look at infrared enhancements
- Factorization, evolution, summation
- From form factors to planar amplitudes
- Taming color exchanges
- Weaving multi-particle webs
- Outlook

# BUGS AND FEATURES OF PERTURBATION THEORY

# The bugs in PT

$$\mathcal{M}(Q,\alpha) = \mathcal{M}_0\left[1+\frac{\alpha}{\pi}C_1(Q)+\left(\frac{\alpha}{\pi}\right)^2 C_2(Q)+\ldots\right]$$

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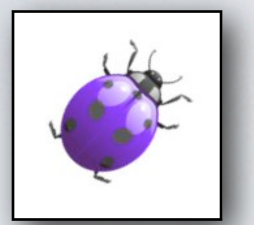

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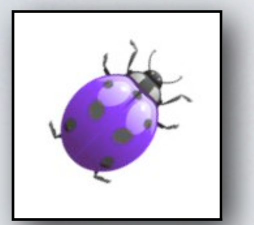

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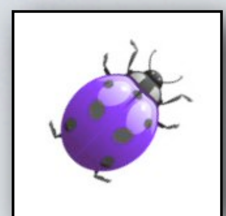

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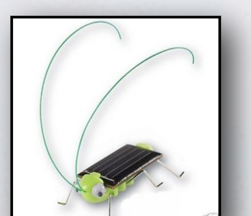

$$C_k\left(\frac{Q}{\mu},\frac{Q}{\mu_f}\right) \propto k! \quad\longrightarrow\quad \sum_k\left(\frac{\alpha}{\pi}\right)^k C_k \to \infty$$

$$\mathcal{M}\left(Q,\alpha\right) = \mathcal{M}_{\rm pert.}\left(Q,\alpha\right) + \mathcal{M}_{\rm non\,pert.}\left(Q,\alpha\right)$$

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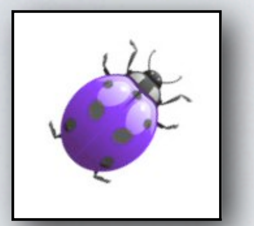

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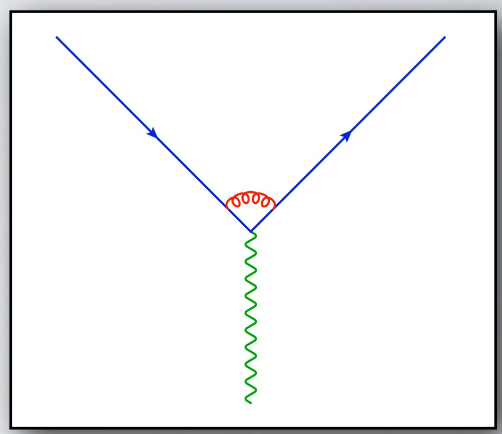
- Our mistake: control of high energy, short distances.
- Fix: locality, effective couplings, UV completion



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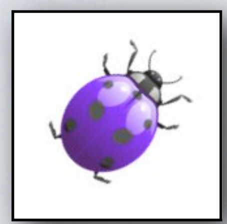

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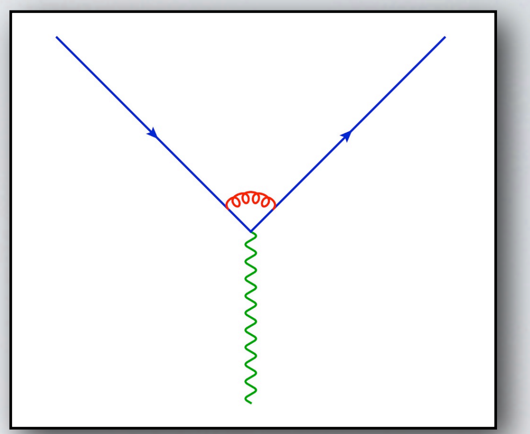

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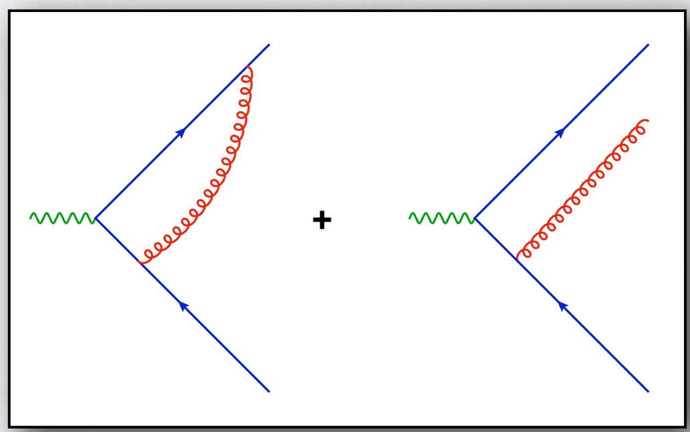

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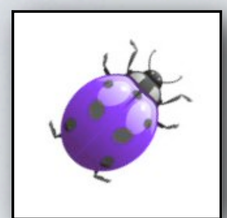

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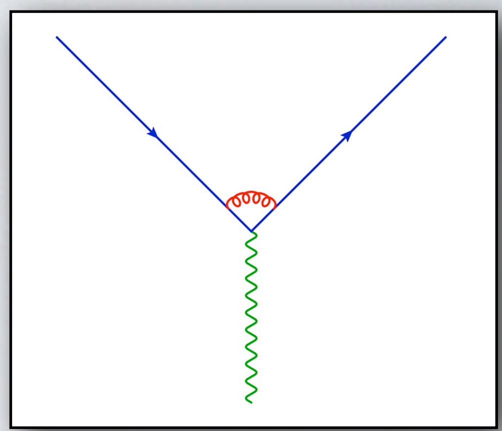

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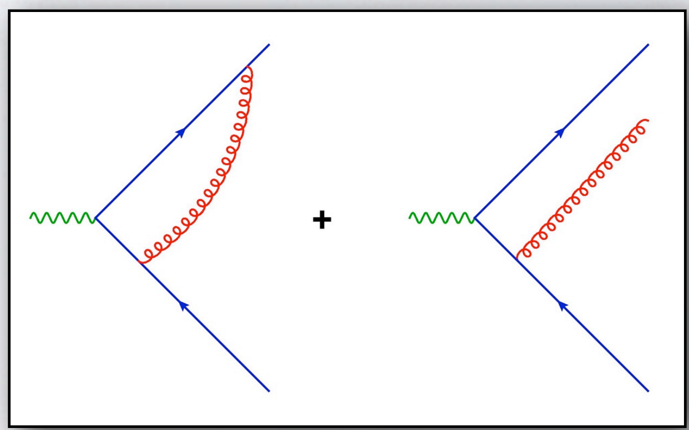
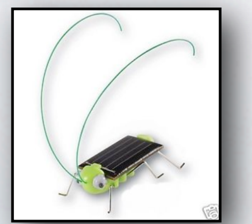
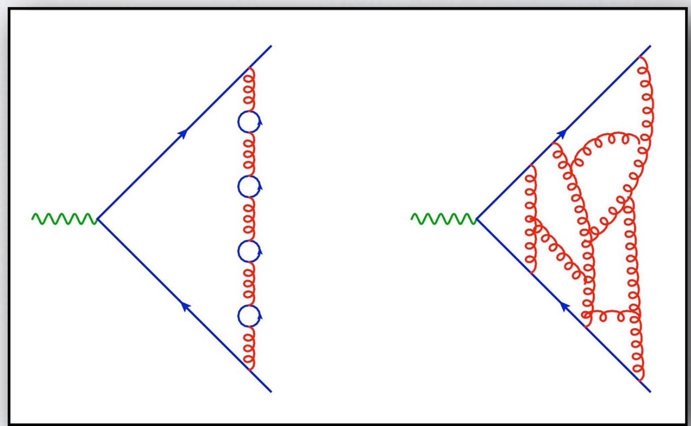


- Vacuum state, operator product expansion.
- Our mistake: neglected operators, solutions.
- Fix: include non-perturbative contributions.



# Features, not bugs

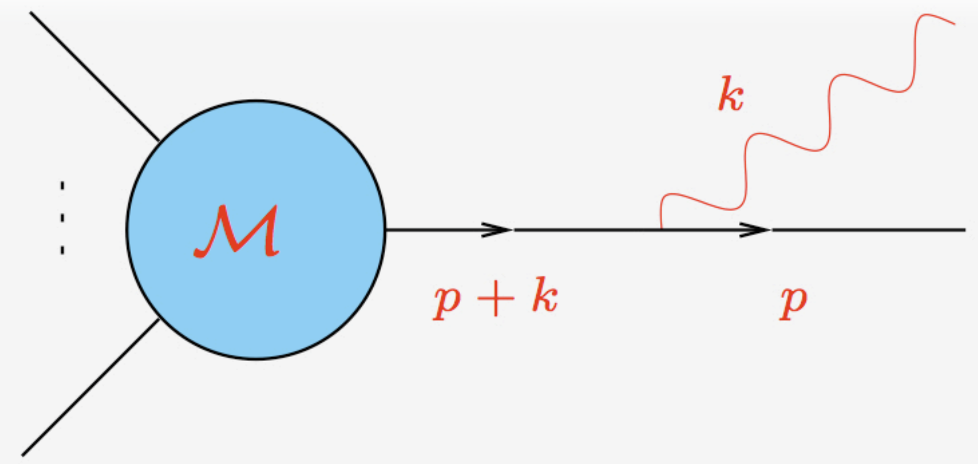
- Quantum mechanics does not destroy predictivity.
- Ultraviolet physics can be factorized and parametrized.
- Renormalization group predicts asymptotic behaviors.
- Local effective field theories.

- We do not need exact knowledge of asymptotic states.
- Infrared physics can be factorized and parametrized.
- Infrared and collinear logarithms can be resummed.
- Non-local effective field theories.

- Perturbation theory knows about its own limitations.
- Non-perturbative contributions can be systematically included.
- Power corrections to observables can be computed.
- Condensates, instantons, bound states.

# A FIRST LOOK

# Textbook theory ...




$$\rightarrow \quad - ig\overline{u}(p)\not\epsilon(k)t_a \frac{i(\not p + \not k)}{(p+k)^2 + i\varepsilon}\mathcal{M}\,,$$

Singularities arise only when propagators go on shell

$$2p \cdot k = 2p_0 k_0 (1 - \cos\theta_{pk}) = 0\,,$$
$$\rightarrow k_0 = 0\ (IR); \quad \cos\theta_{pk} = 1\,.$$

➡ Emission is not suppressed at long distances

➡ Isolated charged particles are not true asymptotic states of unbroken gauge theories

- A serious problem: the S matrix does not exist in the usual Fock space
- Possible solutions: construct finite transition probabilities (KLN theorem)
  construct better asymptotic states (coherent states)
- Long-distance singularities obey a pattern of exponentiation

$$\mathcal{M} = \mathcal{M}_0 \left[1 - \kappa\,\frac{\alpha}{\pi}\,\frac{1}{\epsilon} + \ldots\right] \;\Rightarrow\; \mathcal{M} = \mathcal{M}_0 \exp\left[-\kappa\,\frac{\alpha}{\pi}\,\frac{1}{\epsilon} + \ldots\right]$$

# ... and Practice

Why worry about stuff that cancels in physical observables?

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Why worry about stuff that cancels in physical observables?

- You still have to actually cancel it.
  - Cancellation must be performed analytically before numerical integrations.
  - One needs local counterterms for matrix elements in all singular regions.
  - State of the art: NLO multileg, NNLO for a few processes.
- The cancellation is incomplete.
  - Singularities leave behind finite but potentially large logarithms.
  - For inclusive observables: analytic resummation to high logarithmic accuracy.
  - For exclusive final states: parton shower event generators, (N)LL accuracy.

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)$$

- There is actual (non-perturbative) physics in the IR.
  - We understand infrared radiation to all orders in any gauge theory.
  - Power-suppressed non-perturbative corrections to QCD cross sections can be modeled.
  - Links to the strong coupling regime can be established for SUSY gauge theories.
    - ➡ N = 4 Super Yang-Mills planar amplitudes: ABDKS ansatz.
    - ➡ Non planar amplitudes: awaiting string theory input.

# Impact of resummation

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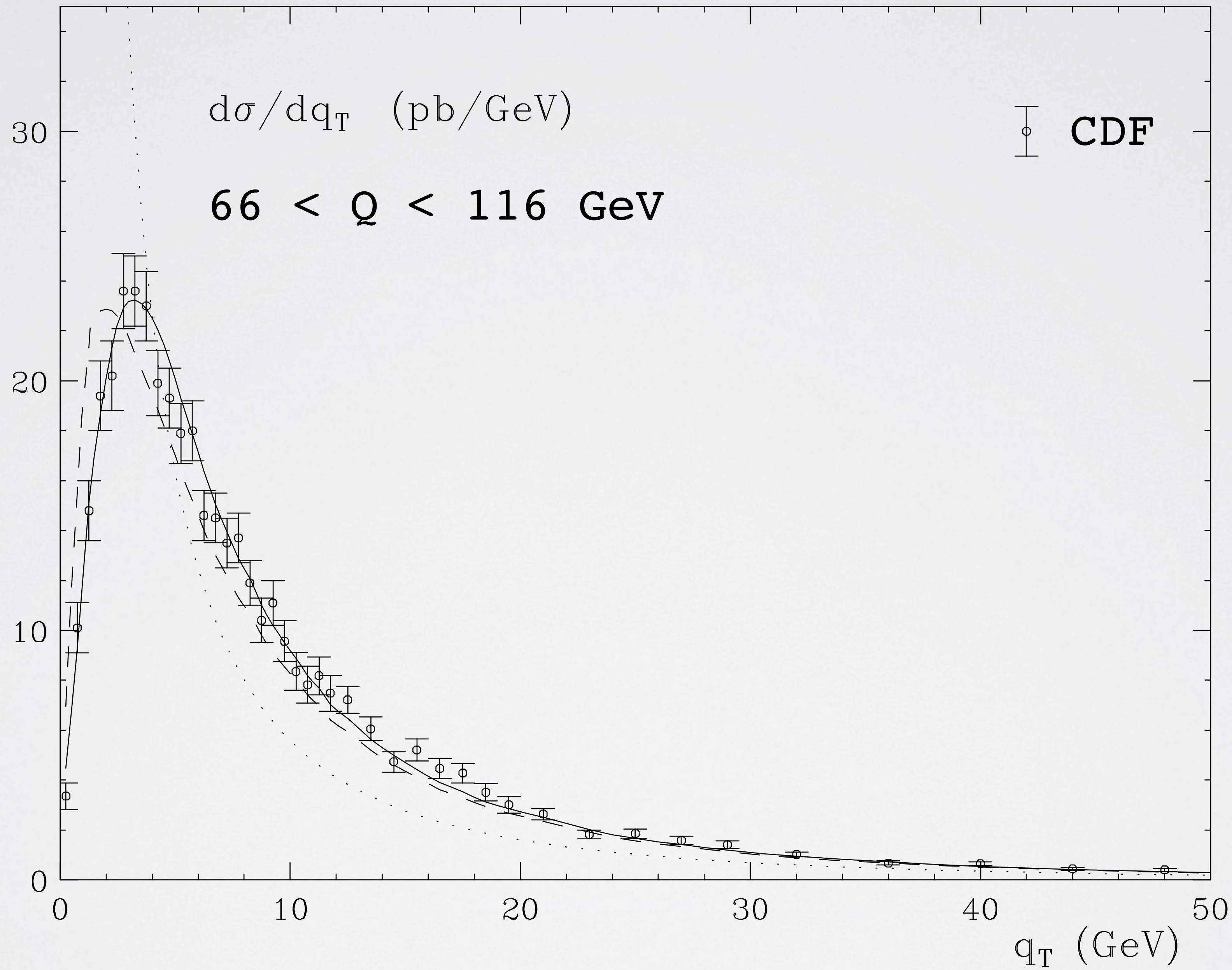
Z-boson $q_T$ spectrum at Tevatron (A. Kulesza et al.)




CDF data on $Z$ production compared with QCD predictions at fixed order (dotted), with resummation (dashed), and with the inclusion of power corrections (solid).

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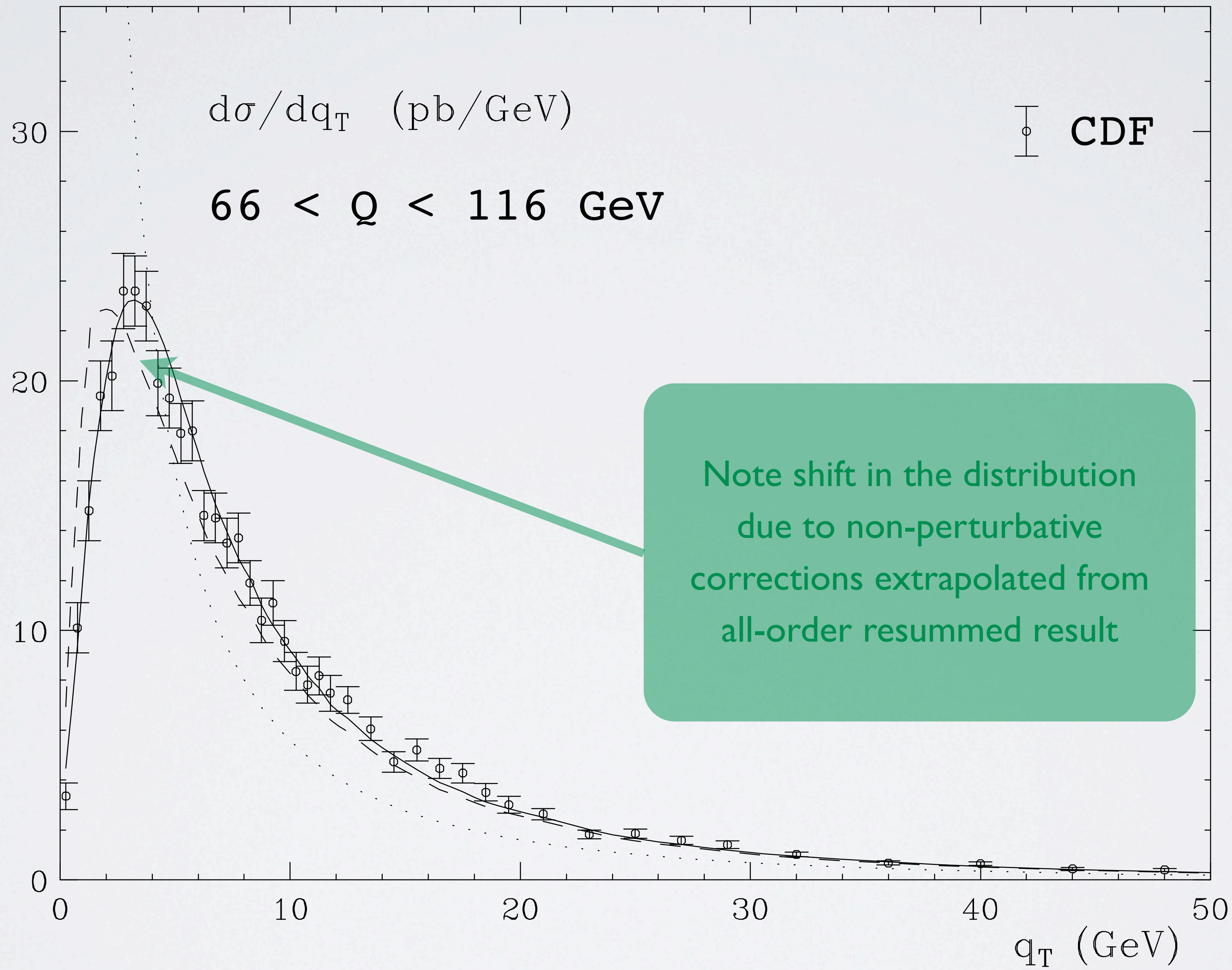


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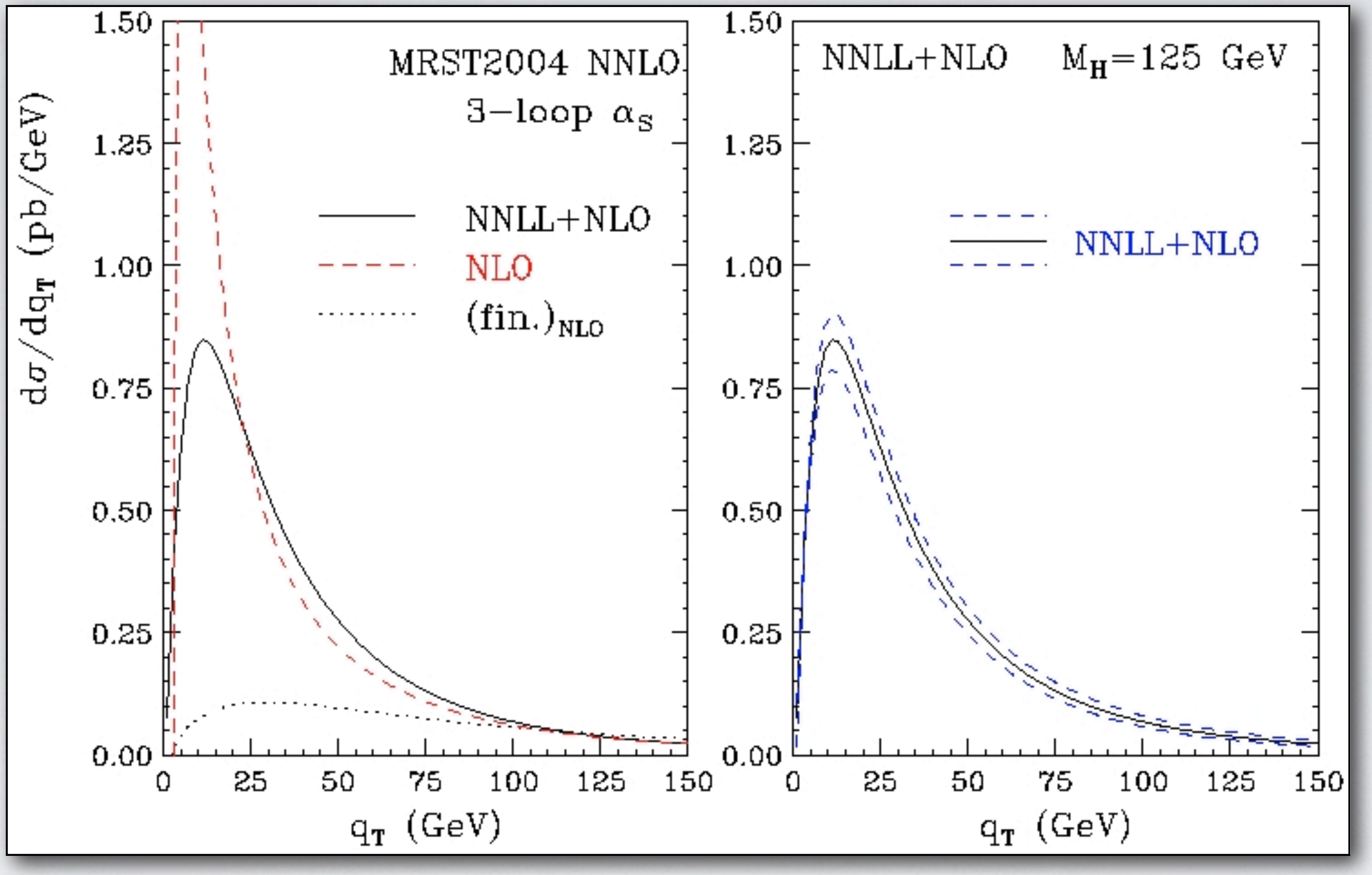
# Impact of resummation

Predictions for the Higgs boson $q_T$ spectrum at LHC (M. Grazzini)




Predictions for the $q_T$ spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.

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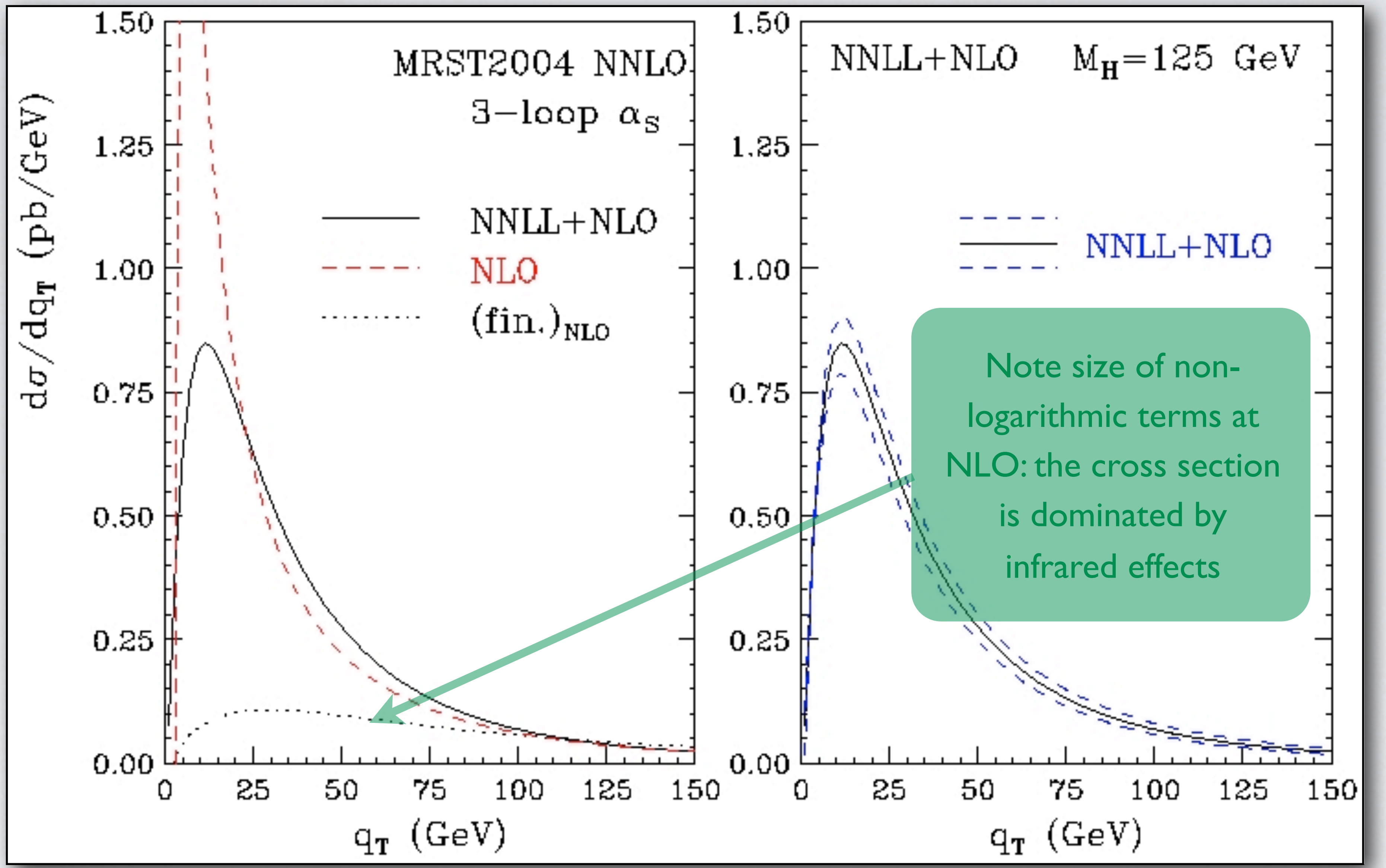


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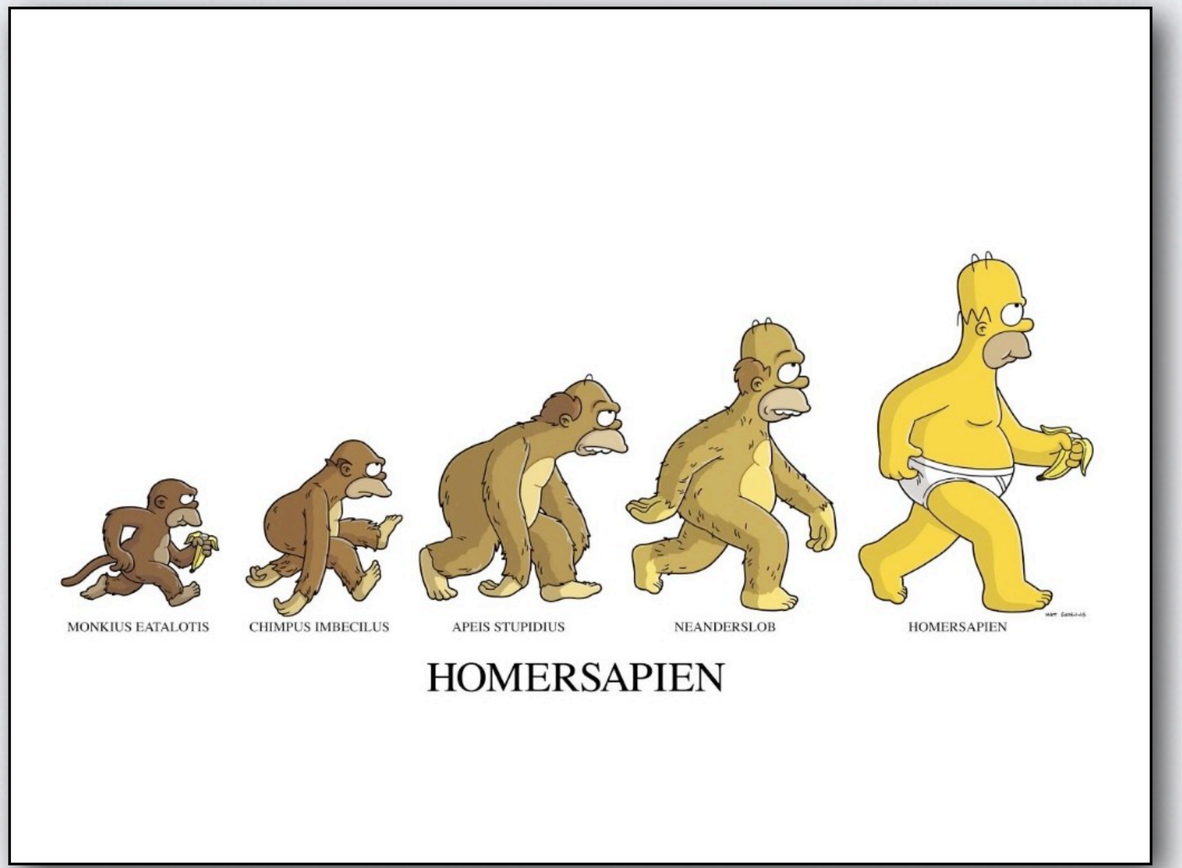
# FACTORIZATION
# EVOLUTION
# SUMMATION



# Ultraviolet factorization

All factorizations separating dynamics at different energy scales lead to resummation of logarithms of the ratio of scales.

Renormalization is a textbook example.

- Renormalization factorizes cutoff dependence.

$$G_0^{(n)}\left(p_i, \Lambda, g_0\right) = \prod_{i=1}^{n} Z_i^{1/2}\left(\Lambda/\mu, g(\mu)\right)\ G_R^{(n)}\left(p_i, \mu, g(\mu)\right)$$

- Factorization requires the introduction of an arbitrarily chosen scale $\mu$.

- Results must be independent of the arbitrary choice of $\mu$.

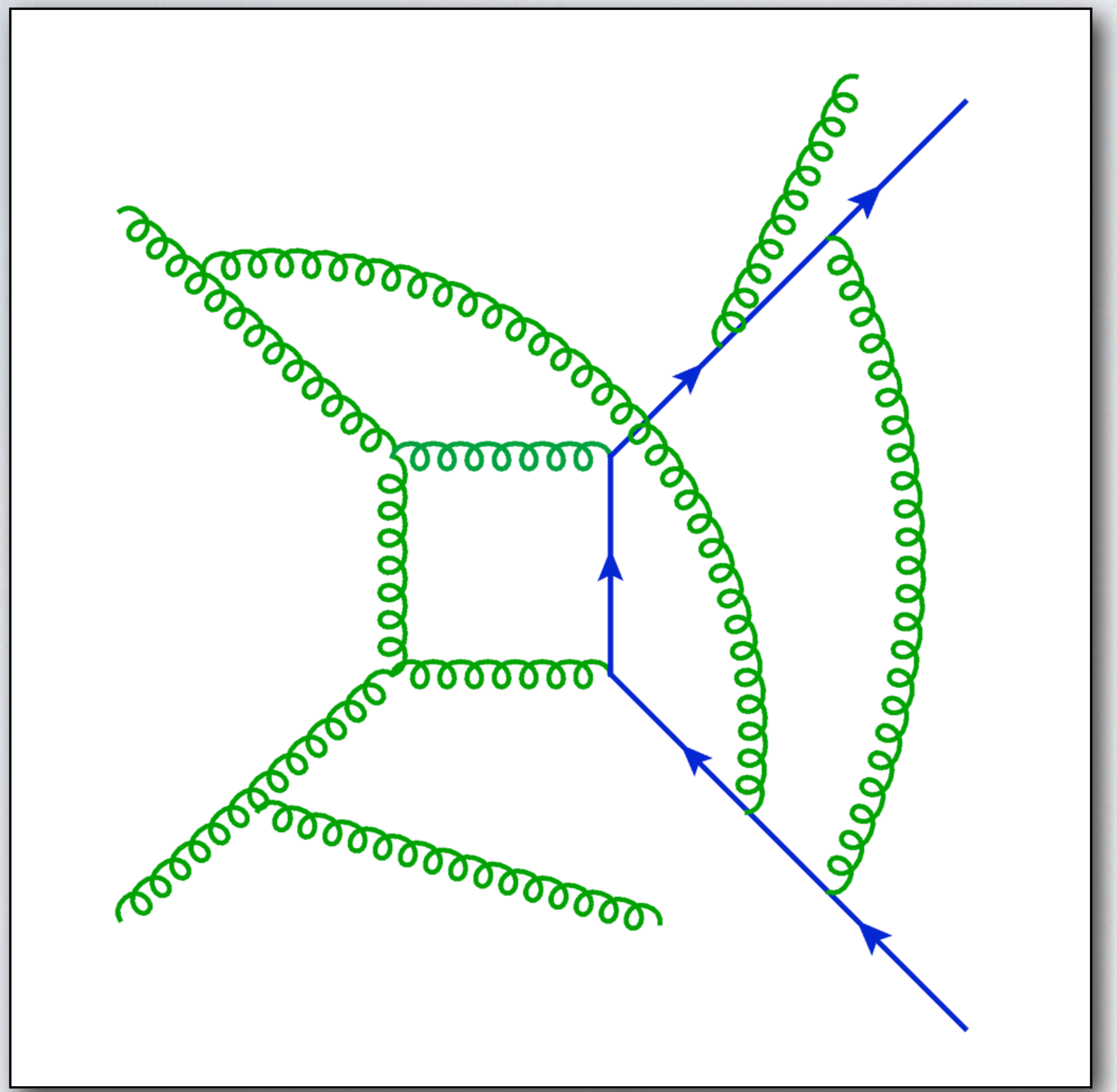
$$\frac{dG_0^{(n)}}{d\mu} = 0 \quad \rightarrow \quad \frac{d\log G_R^{(n)}}{d\log\mu} = -\sum_{i=1}^{n} \gamma_i\left(g(\mu)\right) \ .$$

- The simple functional dependence of the factors is dictated by separation of variables.

- Proving factorization is the difficult step: it requires all-order diagrammatic analyses. Evolution equations follow automatically.

- Solving RG evolution resums logarithms of $Q^2/\mu^2$ into $\alpha_s(\mu^2)$.

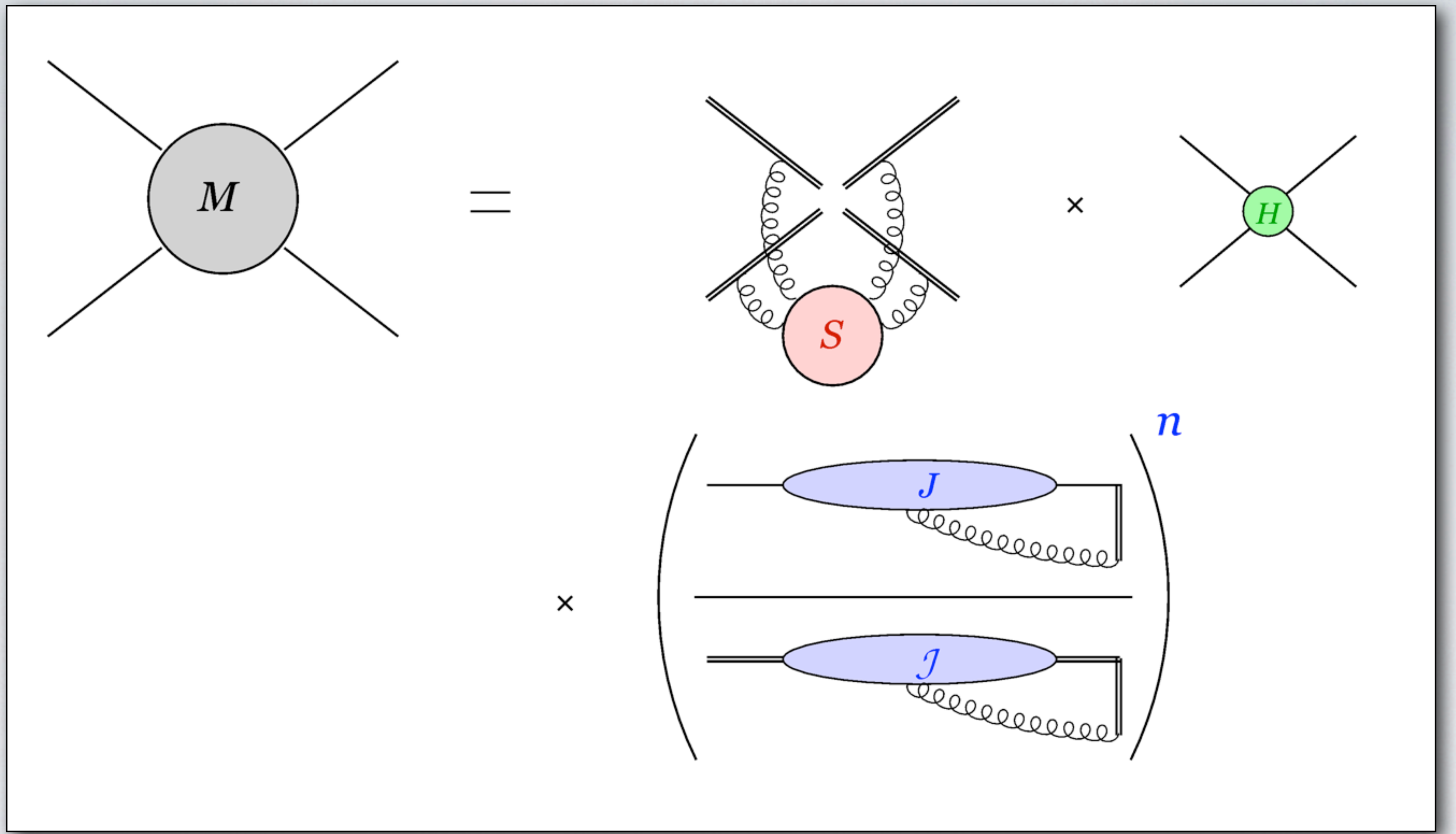
# Infrared factorization



A gauge theory Feynman diagram with potential soft and collinear enhancements

- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- For renormalized massless theories only soft and collinear regions give divergences.
- Soft and collinear emissions have universal features, common to all hard processes.
- Singular contributions can be studied to all orders in perturbation theory.
- Ward identities and power counting lead to decoupling of soft, collinear and hard factors.
- A soft-collinear factorization theorem for multi-particle matrix elements follows.

# Factorization: pictorial




A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

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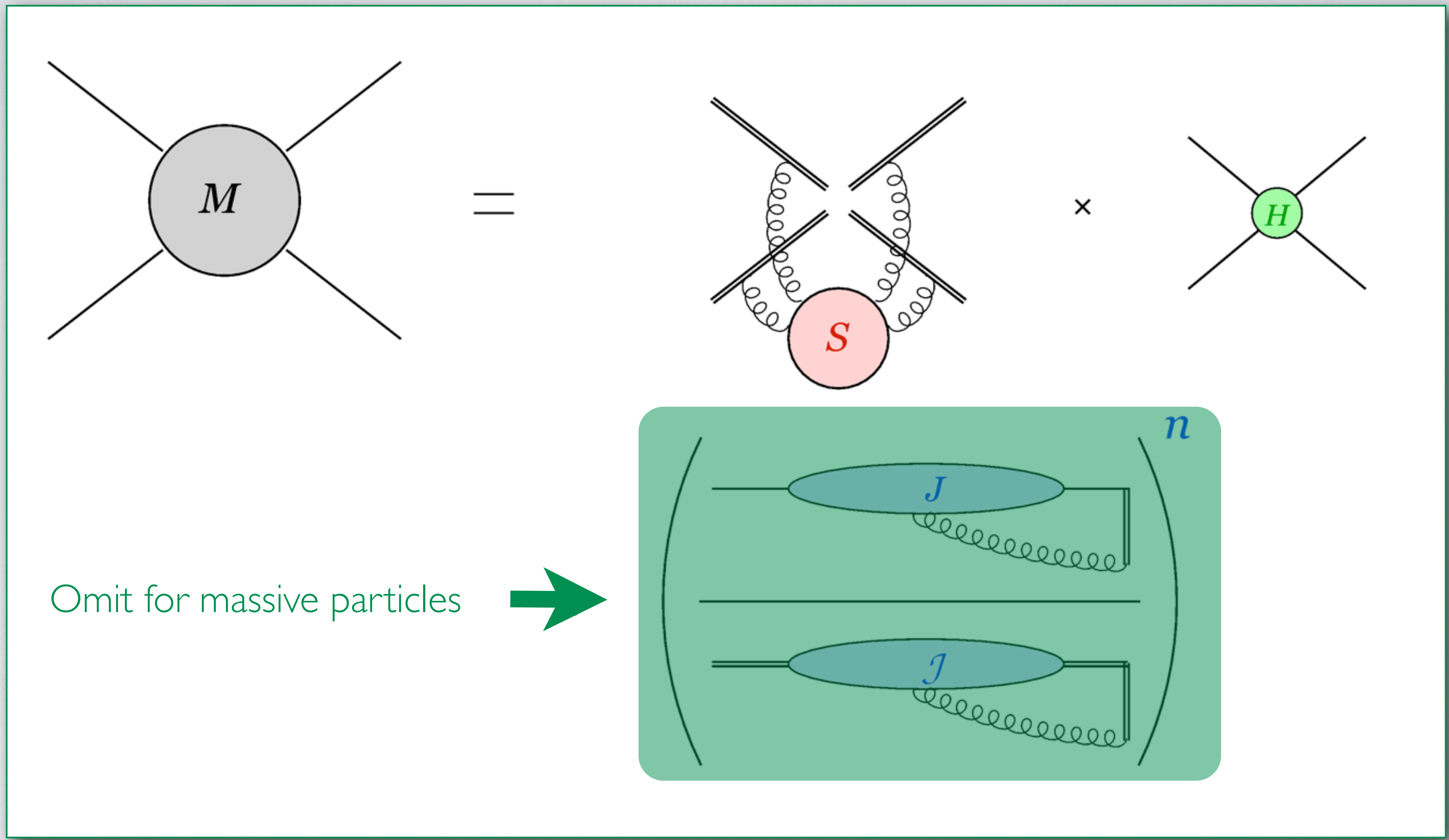
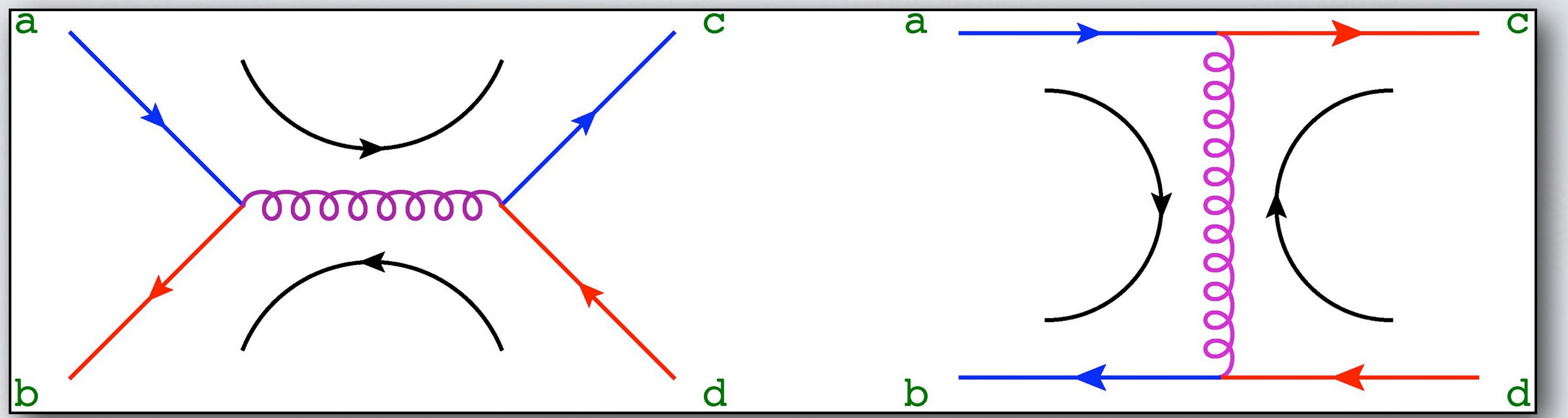



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

# Note: color flow

In order to understand the matrix structure of the soft function it is sufficient to consider the simple case of quark-antiquark scattering.

At tree level




Tree-level diagrams and color flows for quark-antiquark scattering

For this process only two color structures are possible. A basis in the space of available color tensors is

$$c^{(1)}_{abcd} = \delta_{ab}\delta_{cd}\,, \qquad c^{(2)}_{abcd} = \delta_{ac}\delta_{bd}$$

The matrix element is a vector in this space, and the Born cross section is

$$\mathcal{M}_{abcd} = \mathcal{M}_1\, c^{(1)}_{abcd} + \mathcal{M}_2\, c^{(2)}_{abcd} \longrightarrow \sum_{\text{color}} |\mathcal{M}|^2 = \sum_{J,L} \mathcal{M}_J\, \mathcal{M}^*_L \;\text{tr}\left[ c^{(J)}_{abcd} \left( c^{(L)}_{abcd} \right)^\dagger \right] \equiv \text{Tr}\Big[ HS \Big]_0$$

A virtual soft gluon will reshuffle color and mix the components of this vector

$$\text{QED}: \quad \mathcal{M}_{\text{div}} = S_{\text{div}} \mathcal{M}_{\text{Born}}\,; \qquad \text{QCD}: \quad [\mathcal{M}_{\text{div}}]_J = [S_{\text{div}}]_{JL}\, [\mathcal{M}_{\text{Born}}]_L$$

# Note: running coupling

Exponentiation of infrared poles requires solving d-dimensional evolution equations. The running coupling in $d = 4 - 2\,\epsilon$ obeys

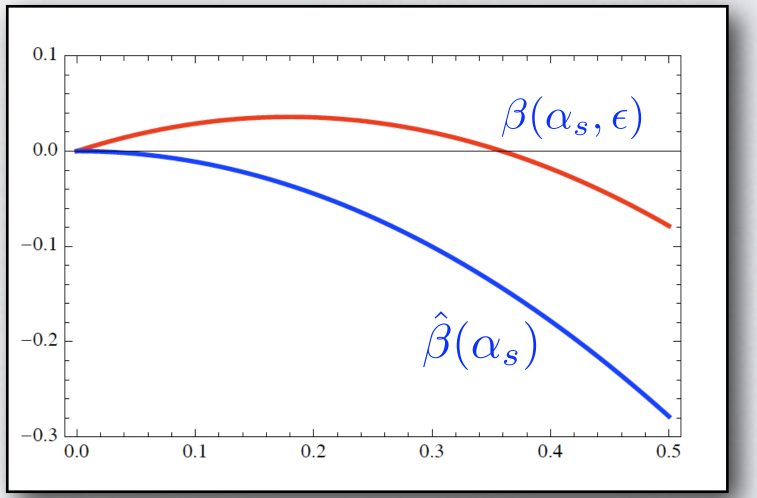
$$\mu \frac{\partial \overline{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \overline{\alpha}) = -2\,\epsilon\,\overline{\alpha} + \hat{\beta}(\overline{\alpha}) \quad , \quad \hat{\beta}(\overline{\alpha}) = - \frac{\overline{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left( \frac{\overline{\alpha}}{\pi} \right)^n .$$

The one-loop solution is

$$\overline{\alpha}\left(\mu^2, \epsilon\right) = \alpha_s(\mu_0^2) \left[ \left( \frac{\mu^2}{\mu_0^2} \right)^{\epsilon} - \frac{1}{\epsilon} \left( 1 - \left( \frac{\mu^2}{\mu_0^2} \right)^{\epsilon} \right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} .$$

The β function develops an IR-free fixed point, so that the coupling vanishes at μ = 0 for fixed ε < 0. The Landau pole is at

$$\mu^2 = \Lambda^2 \equiv Q^2 \left( 1 + \frac{4 \pi \epsilon}{b_0 \alpha_s(Q^2)} \right)^{-1/\epsilon} .$$




For negative **ε** the beta function develops a second zero, O(**ε**) from the origin.

- ➡ Integrations over the scale of the coupling can be analytically performed.
- ➡ All infrared and collinear poles arise by integration over the scale of the running coupling.

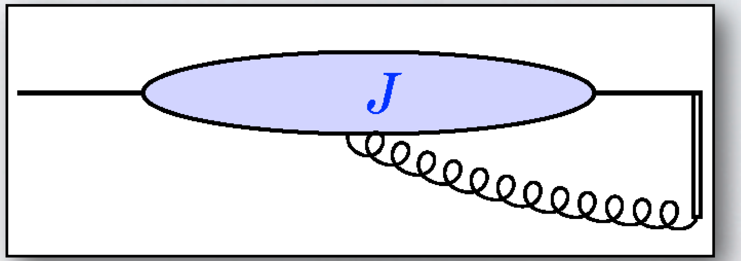
# Factorization: operators

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) \;=\; \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \, H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n} \left[ J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right] ,$$

Here we introduced dimensionless four-velocities $\beta_i^\mu = Q\, p_i^\mu$, $\beta_i^2 = 0$, and factorization vectors $n_i^\mu$, $n_i^2 \neq 0$ to define the jets,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) \;=\; \langle 0 \,|\Phi_n(\infty, 0)\, \psi(0)\, |p\rangle \,.$$




where $\Phi_n$ is the Wilson line operator along the direction $n^\mu$ ,

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n)\right] \,.$$

Note: Wilson lines represent fast particles, not recoiling against soft radiation

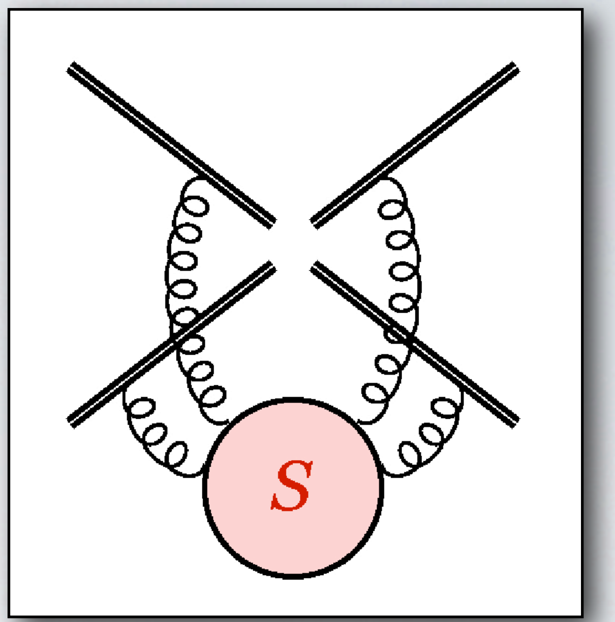
The vectors $n^\mu$ :

- Ensure gauge invariance of the jets.
- Separate collinear gluons from wide-angle soft ones.
- Replace other hard partons with a collinear-safe absorber.

# Soft Matrices

The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\left(c_L\right)_{\{a_k\}} \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \epsilon\right) = \langle 0 | \prod_{k=1}^{n} \left[\Phi_{\beta_k}\left(\infty, 0\right)\right]_{a_k}^{b_k} | 0 \rangle \left(c_K\right)_{\{b_k\}} ,$$




The soft function S obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \epsilon\right) = - \mathcal{S}_{LJ}\left(\beta_i \cdot \beta_j, \epsilon\right) \Gamma^{\mathcal{S}}_{JK}\left(\beta_i \cdot \beta_j, \epsilon\right)$$

NOTE: $\Gamma^S$ is singular for massless theories, due to overlapping UV and collinear poles.

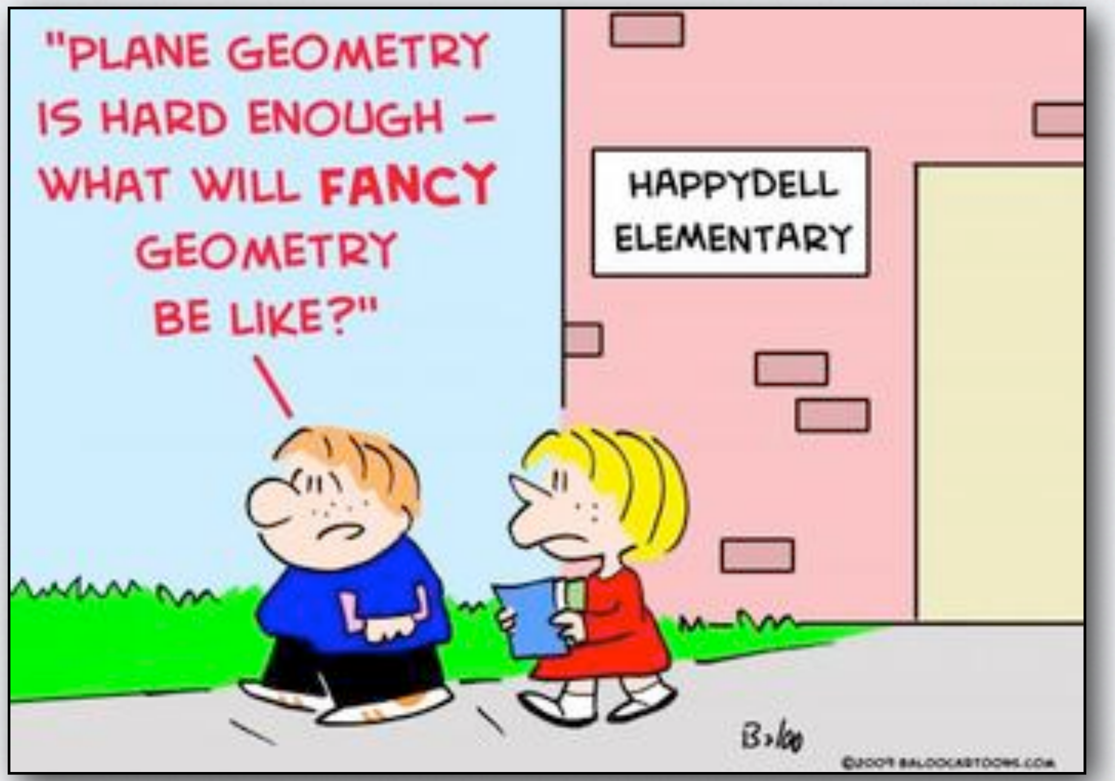
S is a pure counterterm. In dimensional regularization, using $\alpha_s(\mu^2 = 0, \varepsilon < 0) = 0$ ,

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp\left[-\frac{1}{2}\int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon\right)\right] .$$

The determination of the soft anomalous dimension matrix $\Gamma^S$ is the keystone of the resummation program for multiparton amplitudes and cross sections.

- It governs the interplay of color exchange with kinematics in multiparton processes.
- It is the only source of multiparton correlations for singular contributions.
- Collinear effects are `color singlet' and can be extracted from two-parton scatterings.

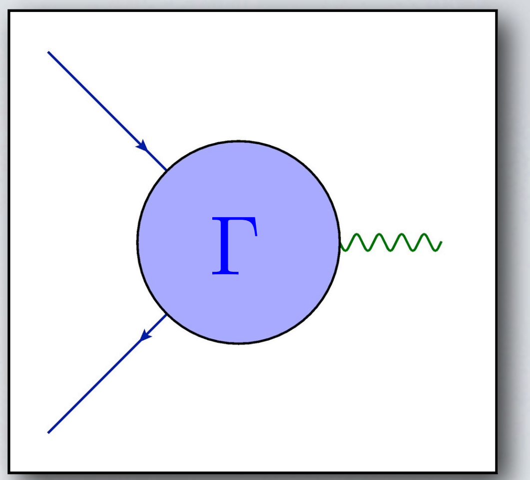
# FROM FORM FACTORS TO PLANAR AMPLITUDES



# Gauge theory form factors

Form factors are matrix elements of conserved currents. For example for a massless Dirac fermion

$$\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_\mu(0) | p_1, p_2 \rangle = \overline{v}(p_2) \gamma_\mu u(p_1) \, \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \, .$$




Form factors obey soft-collinear factorization with trivial color structure.

In dimensional regularization, the $Q^2$ dependence is fully determined by evolution (Sterman, LM).

$$\Gamma\left(Q^2, \epsilon\right) = \exp\left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[ G\Big(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\Big) - \frac{1}{2} \gamma_K\Big(\overline{\alpha}\left(\xi^2, \epsilon\right)\Big) \log\left(\frac{-Q^2}{\xi^2}\right) \right] \right\} .$$

**Tools of the trade**:

- The d-dimensional running coupling, vanishing at $Q^2 = 0$ for $\epsilon < 0$, provides the boundary value.
- The cusp anomalous dimension $\gamma_K$, governing the UV singularity of a cusped Wilson line. Up to three loops it is proportional to the Casimir eigenvalue of the relevant representation (Casimir scaling):

$$\gamma_K^{[i]}(\alpha_s) = C_2^{[i]} \, \widehat{\gamma}_K(\alpha_s) + \mathcal{O}(\alpha_s^4)$$

- The collinear anomalous dimension G, generating subleading collinear poles.

# Gauge theory form factors

The exponentiation is non trivial: only poles up to $(1/\varepsilon)^{n+1}$ appear in the exponent at $n$ loops.

- All poles are generated by the integration over the scale of the d-dimensional coupling.
- All poles beyond $(1/\varepsilon)^2$ are due to the running of the four-dimensional coupling.

In a conformal gauge theory (regulated by $\varepsilon < 0$) all integrations are trivial.

$$\log\left[\Gamma\left(Q^2,\epsilon\right)\right] = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \mathrm{e}^{-\mathrm{i}\pi n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2}+\frac{G^{(n)}(\epsilon)}{n\epsilon}\right].$$

Exact results can be derived in the conformal case (Dixon, Sterman, LM):

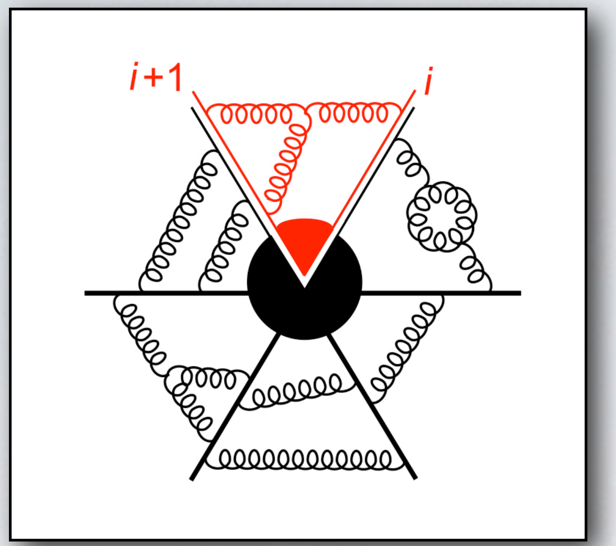
$$\boxed{\lim_{\epsilon\to 0}\left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 = \exp\left[\frac{\pi^2}{4}\,\gamma_K\left(\alpha_s\right)\right].}$$

$$\boxed{G(\alpha_s,\epsilon) = 2\,B_\delta\left(\alpha_s\right) + G_{\rm eik}\left(\alpha_s\right)\,,}$$

- The analytic continuation of the form factor is governed by the cusp anomalous dimension.
- The collinear anomalous dimension has a spin-independent part determined by a Wilson line (eikonal) form factor. Spin enters only through the DGLAP kernel B.
- These results can be checked at strong coupling using AdS/CFT (Alday, Maldacena).

# Exact results for planar amplitudes

All infrared divergences of planar gauge theory amplitudes are determined by the form factors.




Wedges for planar amplitudes

- In the **planar** limit, gluon exchanges are confined to wedges.
- Only one color structure (single trace) survives in the planar limit.
- The soft matrix is proportional to the identity in color space.

- In a **conformal** theory S-matrix elements do not exist ...
- Regularization breaks conformal invariance and may be expected to determine the structure of scattering amplitudes.

- Indeed, in planar N = 4 Super Yang-Mills theory the results for IR divergences are largely inherited by finite parts.
- Two- and three-loop results suggested the `ABDKS' ansatz

$$\mathcal{M}_n = \exp\left\{\sum_{k=1}^{\infty}\left(\frac{N_c\alpha_s}{2\pi}\right)^k\left[f^{(k)}(\epsilon)\, M_n^{(1)}(k\,\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\right]\right\}$$

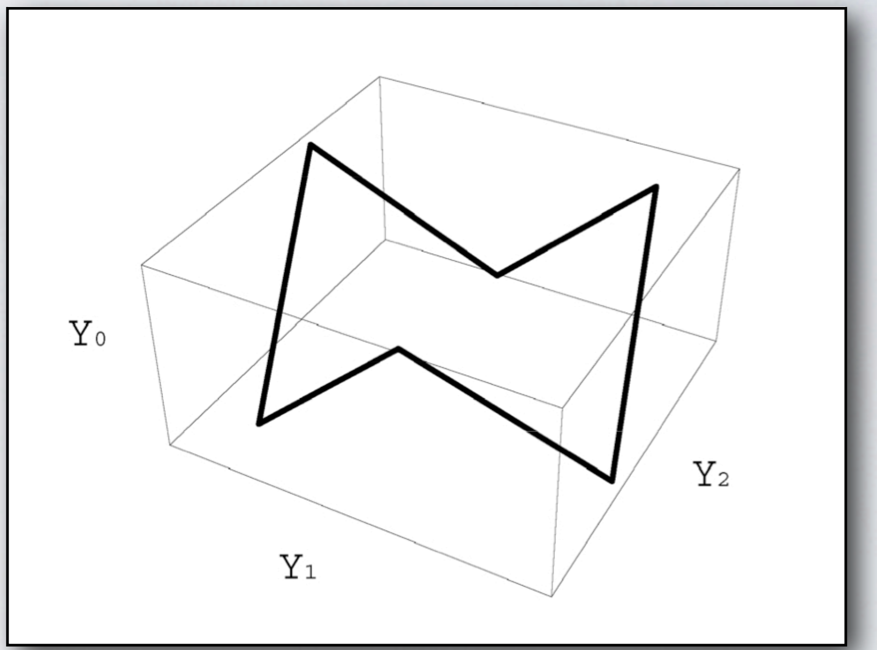
- The ansatz holds for four- and five-point planar amplitudes: they are `exactly solved', using a dual superconformal invariance of planar amplitudes (Korchemsky *et al.*).
- At n > 5 points, a remainder function of conformal cross ratios of momentum invariants arises: it gives the `true' four-dimensional dynamical content of the planar theory.

# Exact results for planar amplitudes

- Remarkably, in N = 4 SYM planar amplitudes can be computed at strong coupling, via the AdS/CFT correspondence (Alday, Maldacena).
- The logarithm of the amplitude is the area of a minimal surface in AdS space, bounded by a polygonal Wilson loop, whose sides are determined by (light-like) external momenta.
- The area can be computed with purely geometrical methods.
- For the four-point function, in dimensional regularization,

$$\mathcal{M}_4 = \exp\left[ \mathrm{i}\, S_{\mathrm{div}} + \frac{\sqrt{\lambda}}{8\pi} \left( \log \frac{s}{t} \right)^2 + \widetilde{C} \right]$$

$$\mathrm{i}\, S_{\mathrm{div,s}} = \frac{\sqrt{\lambda}}{2\pi} \sqrt{\left( \frac{\mu^2}{-s} \right)^{\epsilon}} \left[ -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \frac{1 - \log 2}{2} \right]$$



Polygonal Wilson loop for strong coupling

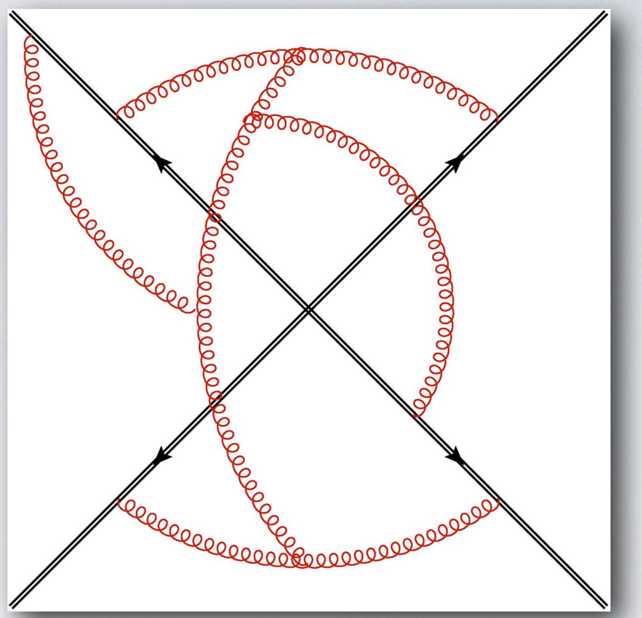
- This exactly matches the weak coupling ABDKS ansatz, and gives expression for the cusp and collinear anomalous dimensions at strong coupling.
- Integrability can be used to construct an exact equation (Beisert, Eden, Staudacher) satisfied by the (planar) cusp anomalous dimension, matching both weak and strong coupling results.
- The remainder function can also be determined at strong coupling: matching weak and strong coupling is subject of much current research.

# TAMING COLOR EXCHANGES

# TAMING COLOR EXCHANGES

# Surprising Simplicity

- The matrix $\Gamma_S$ can be computed from the UV poles of S.
- Computations can be performed directly for the exponent: relevant diagram sets are called "webs".
- $\Gamma_S$ appears highly complex at high orders.
- g-loop webs directly correlate color and kinematics of up to g+1 Wilson lines.



A diagram in a web contributing to the soft anomalous dimension matrix

The two-loop calculation (Aybat, Dixon, Sterman) leads to a surprising result: for any number of light-like eikonal lines

$$\Gamma_S^{(2)} = \frac{\kappa}{2}\,\Gamma_S^{(1)} \qquad \kappa = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F C_F \, .$$

- No new kinematic dependence; no new matrix structure.
- $\kappa$ is the two-loop coefficient of $\gamma_K(\alpha_s)$, rescaled by the appropriate quadratic Casimir,

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)} \left[ 2\,\frac{\alpha_s}{\pi} + \kappa \left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right) \right] \, .$$

# Properties of eikonal functions

- Eikonal functions like $\mathcal{S}$ and $\mathcal{J}$ are pure counterterms in dimensional regularization.
- The functional dependence on the vectors $n^\mu_i$ is restricted by the classical invariance of Wilson lines under velocity rescalings, $n^\mu_i \to \kappa_i \, n^\mu_i$.
- Rescaling invariance for light-like velocities, $\beta_i^2 = 0$ is broken by quantum corrections due to overlapping soft and collinear poles.
- This `collinear anomaly' is governed by the cusp anomalous dimension $\gamma_K(\alpha_s)$.

Eikonal jets $\mathcal{J}$, needed to avoid double counting of soft-collinear regions, are defined by

$$\mathcal{J}\left(\frac{(\beta\cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0 | \Phi_n(\infty, 0) \, \Phi_\beta(0, -\infty) \, | 0 \rangle \; ,$$

Double soft-collinear poles cancel in the reduced soft function

$$\overline{\mathcal{S}}_{LK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) = \frac{\mathcal{S}_{LK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{n} \mathcal{J}_i\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}$$

- In $\overline{\mathcal{S}}$ the anomaly must cancel. Thus it must depend on rescaling invariant variables.

$$\rho_{ij} \equiv \frac{n_i^2 \, n_j^2 \, (\beta_i\cdot\beta_j)^2}{(\beta_i\cdot n_i)^2(\beta_j\cdot n_j)^2} \, .$$

- The anomalous dimension $\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$ for the evolution of $\overline{\mathcal{S}}$ is finite.

# Factorization Constraints

The kinematic dependence of eikonal functions is severely restricted by rescaling invariance.

- The classical symmetry of Wilson line correlators under $\beta_i \to \kappa_i \, \beta_i$ is violated only through the cusp anomaly.
  - ➡ For eikonal jets, no $\beta_i$ dependence is possible at all except through the cusp
- In the reduced soft function, $S/\prod J$, the cusp anomaly cancels
  - ➡ The reduced soft function can depend on $\beta_i$ only through rescaling-invariant combinations such as $\rho_{ij}$. For $n > 3$ hard partons, one may also construct

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$

Consider the anomalous dimension matrix for the reduced soft function

$$\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s(\mu^2)\right) = \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}\left(\frac{(\beta_k \cdot n_k)^2}{n_k^2}, \alpha_s(\mu^2), \epsilon\right) .$$

Remarkably:

- Singular terms in $\Gamma_S$ must be diagonal and proportional to $\gamma_K$
- Finite diagonal terms in $\Gamma_S$ must conspire to construct $\rho_{ij}$'s.
- Off-diagonal terms in $\Gamma_S$ must be finite, and must depend only on the cross-ratios $\rho_{ijkl}$

# Factorization Constraints

The constraints can be formalized simply by using the chain rule: $\Gamma^{\overline{\mathcal{S}}}$ can depend on the factorization vectors $n_i$ only through the eikonal jets, which are color diagonal.

Defining $x_i \equiv (\beta_i \cdot n_i)^2 / n_i^2$ , one finds

$$x_i \frac{\partial}{\partial x_i} \Gamma^{\overline{\mathcal{S}}}_{IJ} \left( \rho_{ij}, \alpha_s \right) = - \delta_{IJ} \, x_i \frac{\partial}{\partial x_i} \gamma_{\mathcal{J}} \left( x_i, \alpha_s, \epsilon \right) \; = \; - \frac{1}{4} \gamma_K^{(i)} \left( \alpha_s \right) \delta_{IJ} \, .$$

This leads to a linear equation for the dependence of $\Gamma^{\overline{\mathcal{S}}}$ on its proper arguments, $\rho_{ij}$.

$$\boxed{\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{MN} \left( \rho_{ij}, \alpha_s \right) = \frac{1}{4} \, \gamma_K^{(i)} \left( \alpha_s \right) \, \delta_{MN} \qquad \forall i \, ,}$$

- The equation relates the kinematic dependence of $\Gamma$ to $\gamma_K$, to all orders in perturbation theory
  - and should remain true at strong coupling as well
- It correlates color and kinematics for any number of hard partons
- It admits a unique solution for amplitudes with up to three hard partons.
  - For $n > 3$ hard partons, functions of $\rho_{ijkl}$ solve the homogeneous equation.

# The Dipole Formula

We have found that, for massless partons, the soft anomalous dimension matrix obeys a set of exact equations that correlate color exchange with kinematics.

The simplest solution to these equations is a sum over color dipoles (Becher, Neubert; Gardi, LM, 09). It gives an ansatz for the all-order singularity structure of all multiparton fixed-angle massless scattering amplitudes: the dipole formula.

- All soft and collinear singularities can be collected in a multiplicative operator Z

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \, \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

- Z contains both soft singularities from S, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d \ln \mu} Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = - Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) .$$

The matrix Γ has a surprisingly simple dipole structure, the same as at one loop. It reads

$$\boxed{\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4} \widehat{\gamma}_K \left(\alpha_s(\mu^2)\right) \sum_{j \neq i} \ln\left(\frac{-2\, p_i \cdot p_j}{\mu^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^{n} \gamma_{J_i} \left(\alpha_s(\mu^2)\right) .}$$

Note that all singularities are again generated by integration over the scale of the coupling.

# Features of the dipole formula

- All known results for IR divergences of massless gauge theory amplitudes are recovered.
- The absence of multiparton correlations implies remarkable diagrammatic cancellations.
- The color matrix structure is fixed at one loop: path-ordering is not needed.
- The cusp anomalous dimension plays a very special role: a universal IR coupling.

Can this be the definitive answer for IR divergences in massless non-abelian gauge theories?

- There are precisely two sources of possible corrections.
  - Quadrupole correlations may enter starting at three loops: they must be tightly constrained functions of conformal cross ratios of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) \,, \qquad \rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

  - The cusp anomalous dimension may violate Casimir scaling beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \, \widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)$$

  - The functional form of $\Delta$ is further constrained by: collinear limits, Bose symmetry, bounds on weights, high-energy constraints. (Becher, Neubert; Dixon, Gardi, LM, 09).
  - A four-loop analysis indicates that Casimir scaling holds (Becher, Neubert, Vernazza).
  - Recent evidence for non-vanishing $\Delta$ at four loops from Regge limit (Caron-Huot).

# WEAVING MULTI-PARTICLE WEBS

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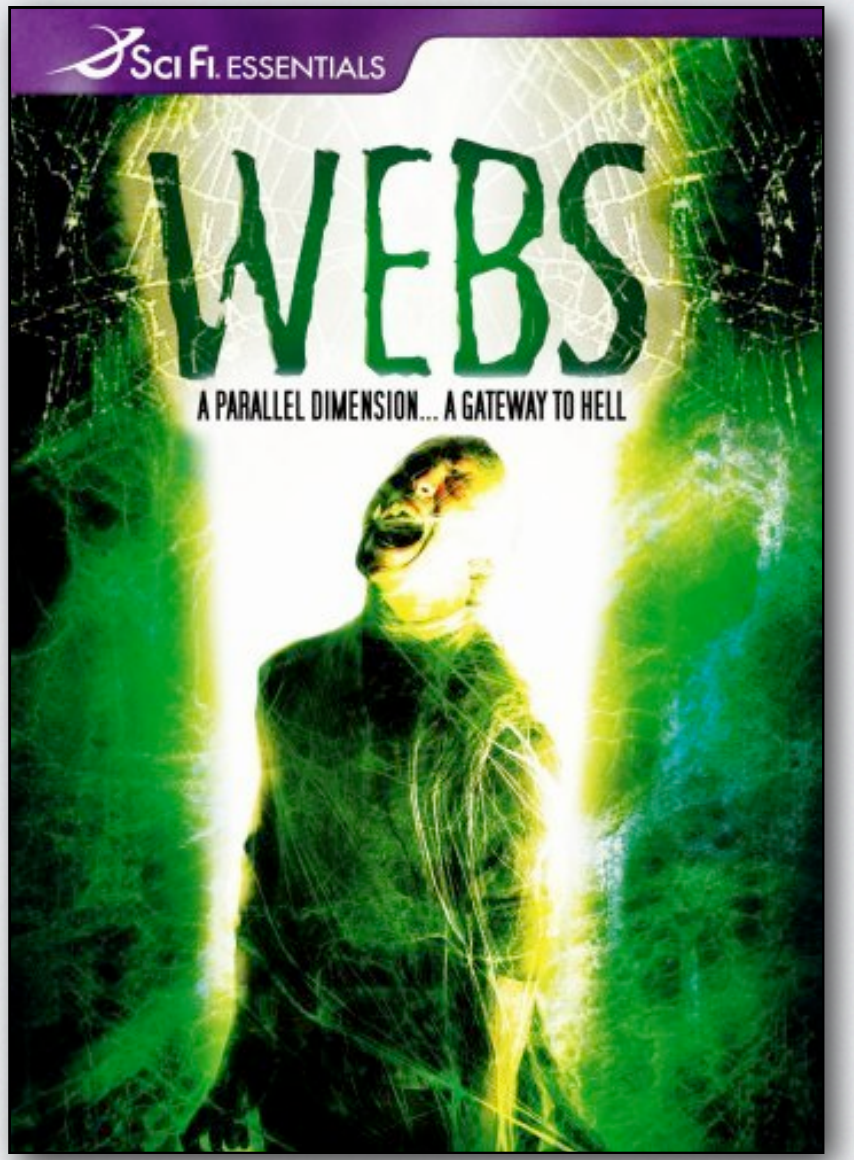


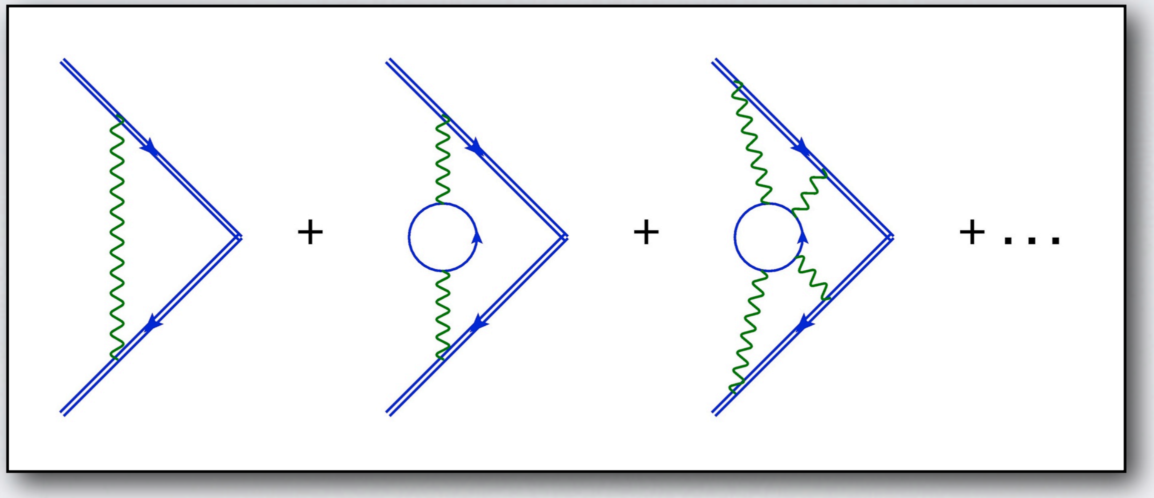
# Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in exponential form.

$$S_n \equiv \langle 0 | \, \Phi_1 \otimes \ldots \otimes \Phi_n \, | 0 \rangle = \exp\left(\omega_n\right)$$

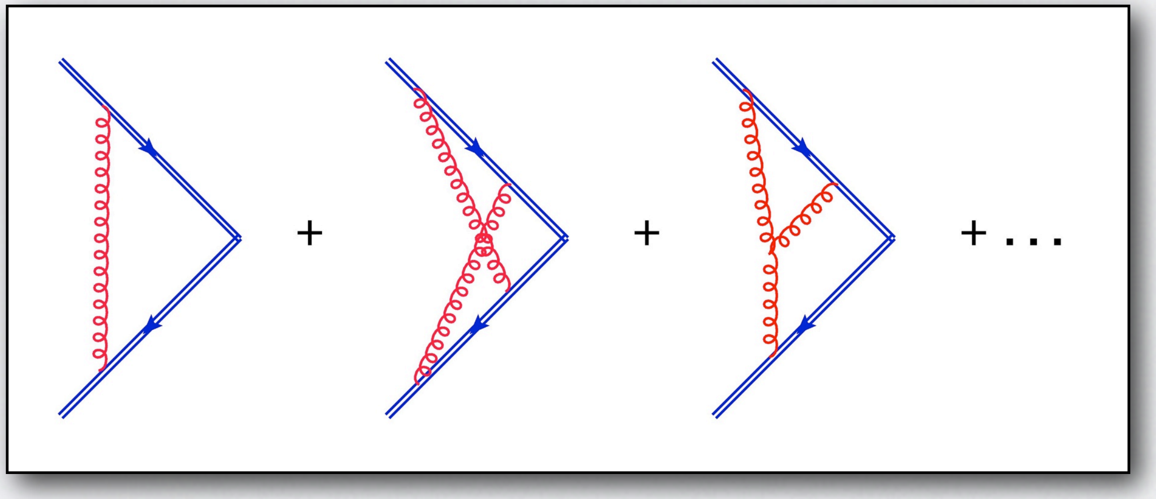
Diagrammatic rules exist to compute directly the logarithm of the correlators.

$\omega_{2,\mathrm{QED}} =$




Only connected photon subdiagrams contribute to the logarithm.

$\omega_{2,\mathrm{QCD}} =$




Only gluon subdiagrams which are two-eikonal irreducible contribute to the logarithm. They have modified color factors.

For eikonal form factors, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

# Multiparticle webs

The concept of web generalizes non-trivially to the case of multiple Wilson lines. (Gardi, Smillie, White, *et al*).

A **web** is a set of diagrams which differ only by the order of the gluon attachments on each Wilson line. They are weighted by modified color factors.

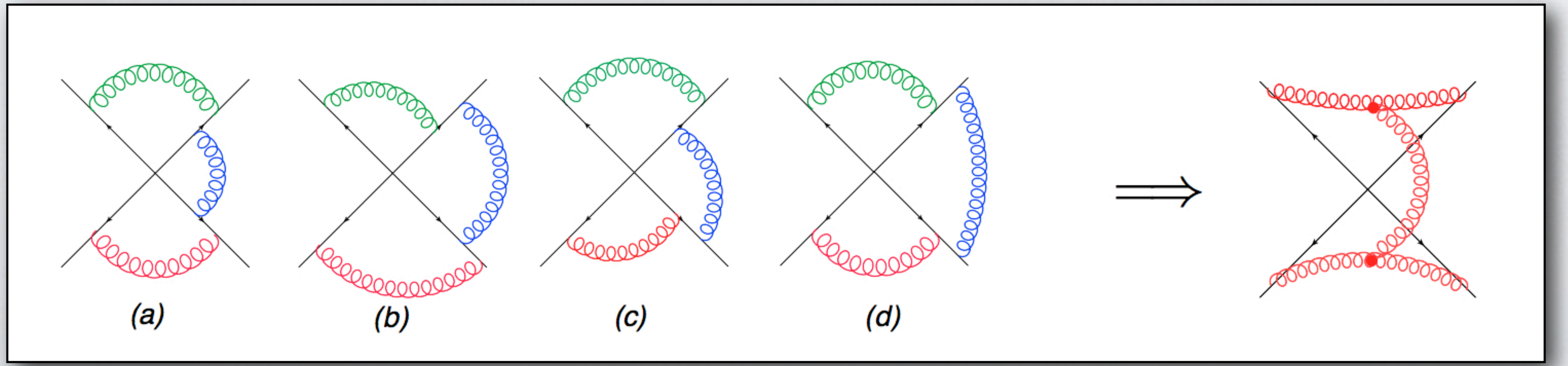
Writing each diagram as the product of its natural color factor and a kinematic factor

$$D = C(D)\mathcal{F}(D)$$

a web W can be expressed as a sum of diagrams in terms of a web mixing matrix R

$$W = \sum_{D} \widetilde{C}(D)\mathcal{F}(D) = \sum_{D,D'} C(D')R(D',D)\mathcal{F}(D)$$

The non-abelian exponentiation theorem holds: each web has the color factor of a fully connected gluon subdiagram (Gardi, Smillie, White).



Gardi, Smillie, White 2010-2012;
Mitov, Sterman, Sung 2010

# Computing webs

Bare Wilson-line correlators vanish beyond tree level in dimensional regularization: they are given by scale-less integrals. We require renormalized correlators, which depend on the Minkowsky angles between the Wilson lines.

$$S_{\rm ren}\left(\gamma_{ij},\alpha_s,\epsilon\right) \, = \, S_{\rm bare}\left(\gamma_{ij},\alpha_s,\epsilon\right) Z\left(\gamma_{ij},\alpha_s,\epsilon\right) \, = \, Z\left(\gamma_{ij},\alpha_s,\epsilon\right) \, , \qquad \gamma_{ij} \, = \, \frac{2\beta_i\cdot\beta_j}{\sqrt{\beta_i^2\beta_j^2}}$$

To compute the counterterm Z we make use of an auxiliary, IR-regularized correlator

$$\begin{aligned} \widehat{S}_{\rm ren}\left(\gamma_{ij},\alpha_s,\epsilon,m\right) \quad &= \quad \widehat{S}_{\rm bare}\left(\gamma_{ij},\alpha_s,\epsilon,m\right) Z\left(\gamma_{ij},\alpha_s,\epsilon\right) \\ &\equiv \quad \exp\left(\omega\right)\, \exp\left(\zeta\right) \, = \, \exp\left\{\omega+\zeta+\frac{1}{2}\left[\omega,\zeta\right]+\ldots\right\} \end{aligned}$$

The expression of Z in terms of the anomalous dimension Γ follows from RG arguments

$$Z \, = \, \exp\left[\frac{\alpha_s}{\pi}\frac{1}{2\epsilon}\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{1}{4\epsilon}\Gamma^{(2)} - \frac{b_0}{4\epsilon^2}\Gamma^{(1)}\right) + \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{1}{6\epsilon}\Gamma^{(3)} + \frac{1}{48\epsilon^2}\left[\Gamma^{(1)},\Gamma^{(2)}\right] + \ldots\right)\right]$$

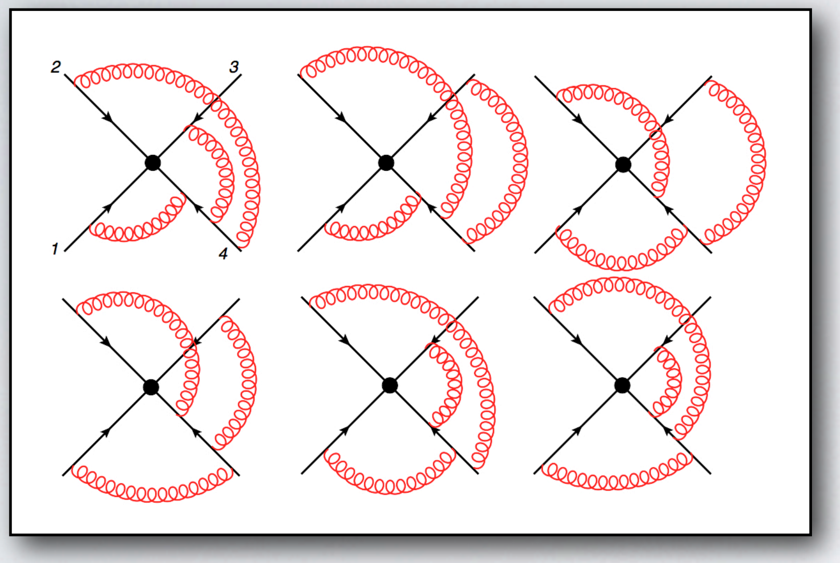
Combining informations one can get Γ directly from the logarithm of the regularized S

$$\begin{aligned} \Gamma^{(1)} \quad &= \quad -2\,\omega^{(1,-1)} \\ \Gamma^{(2)} \quad &= \quad -4\,\omega^{(2,-1)} - 2\left[\omega^{(1,-1)},\omega^{(1,0)}\right] \end{aligned} \qquad \omega = \sum_{n=1}^{\infty}\sum_{k=-n}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \epsilon^k\,\omega^{(n,k)}$$

Computing regularized webs is a game of combinatorics and renormalization theory.

# Three-loop progress

The computation of the three-loop multi-particle soft anomalous dimension is under way.




(1113) web

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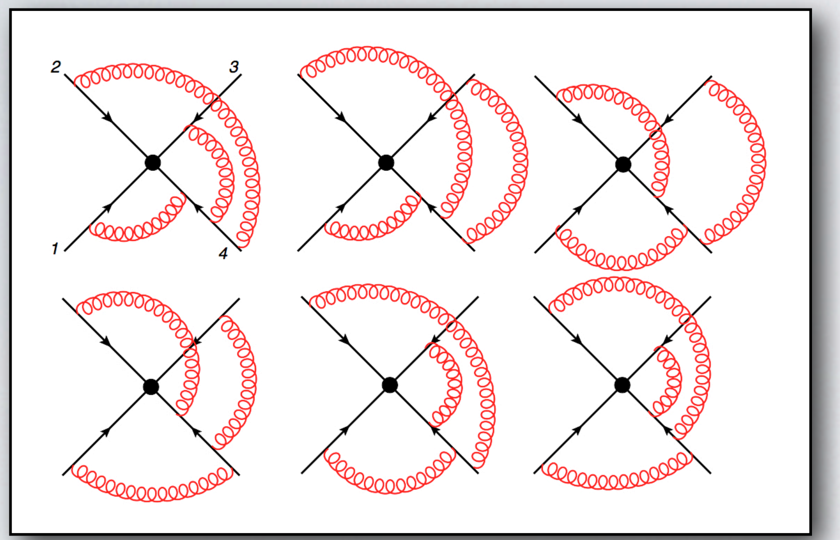


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✓ (Gardi)

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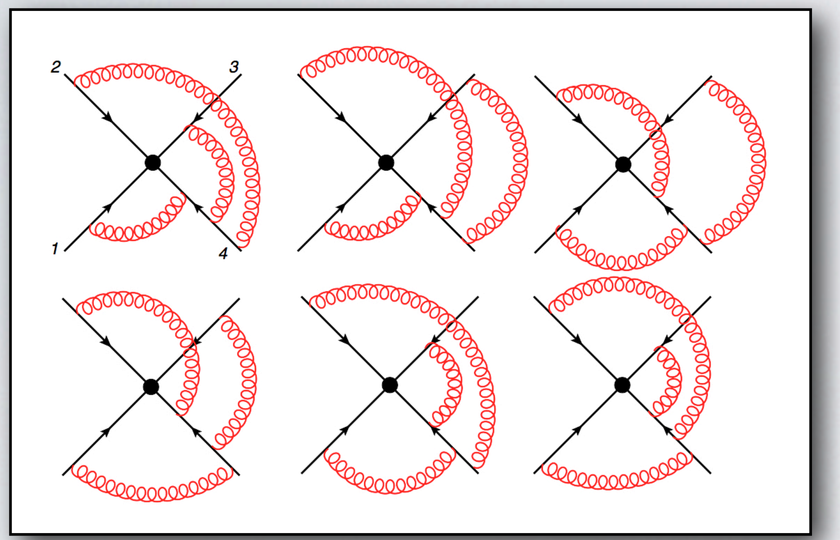


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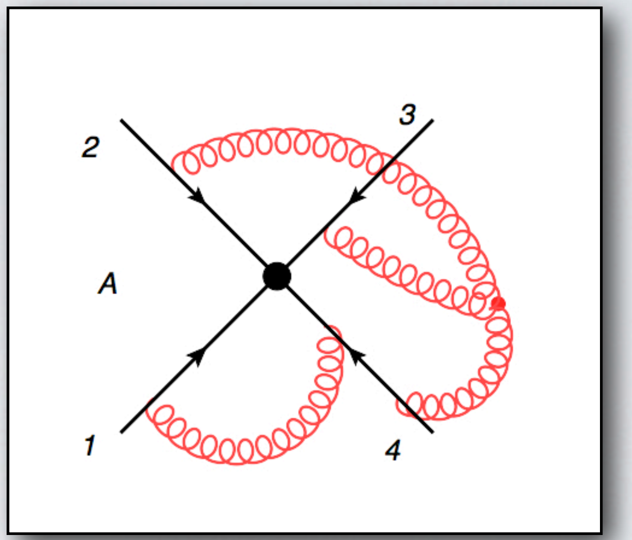


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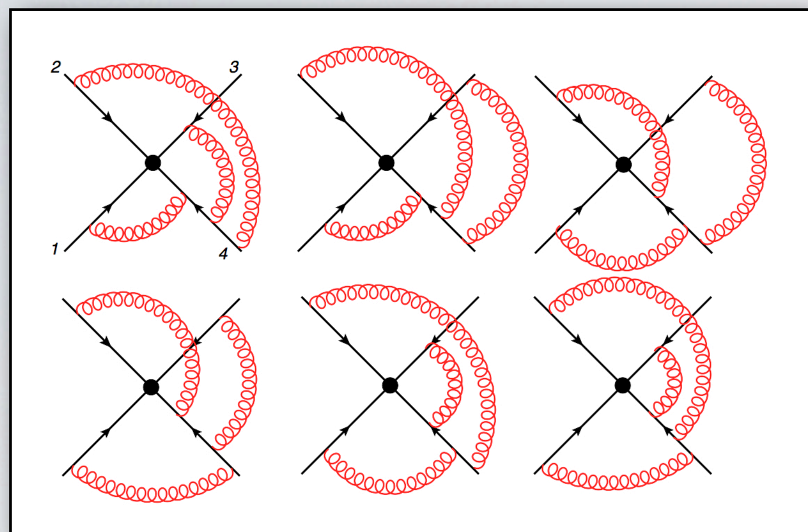


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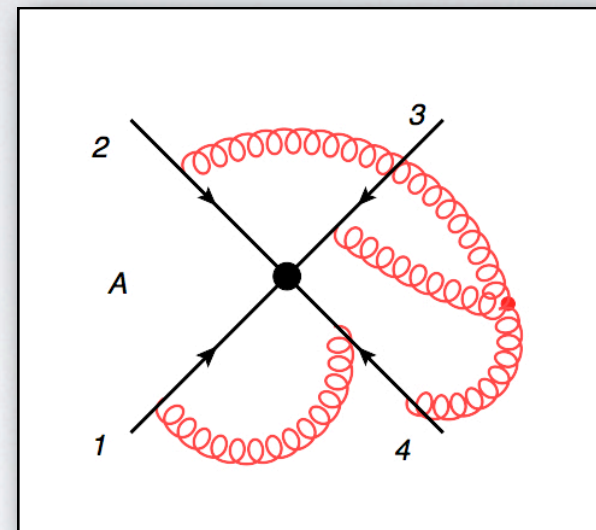


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In progress

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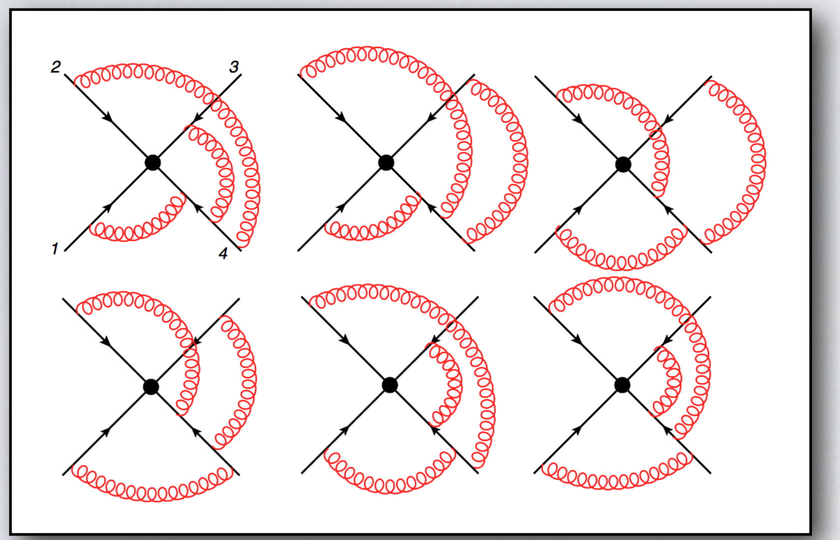


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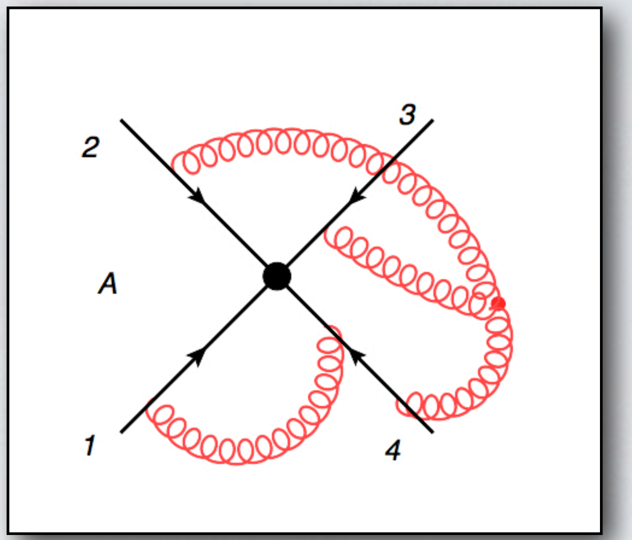


In progress

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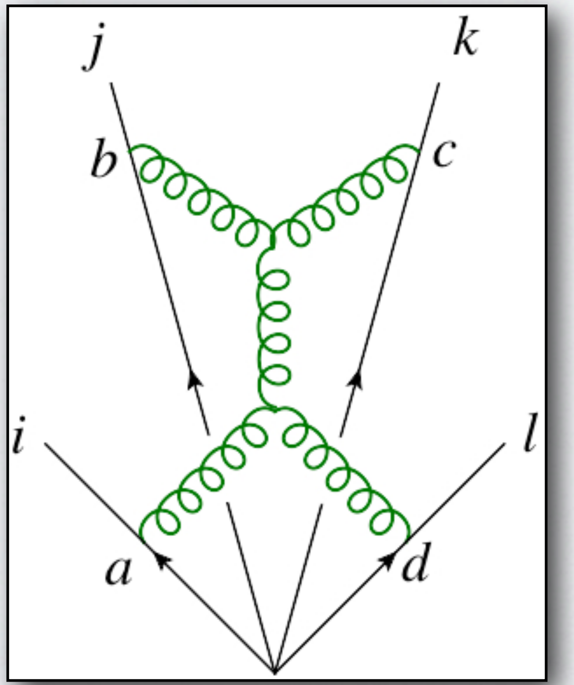


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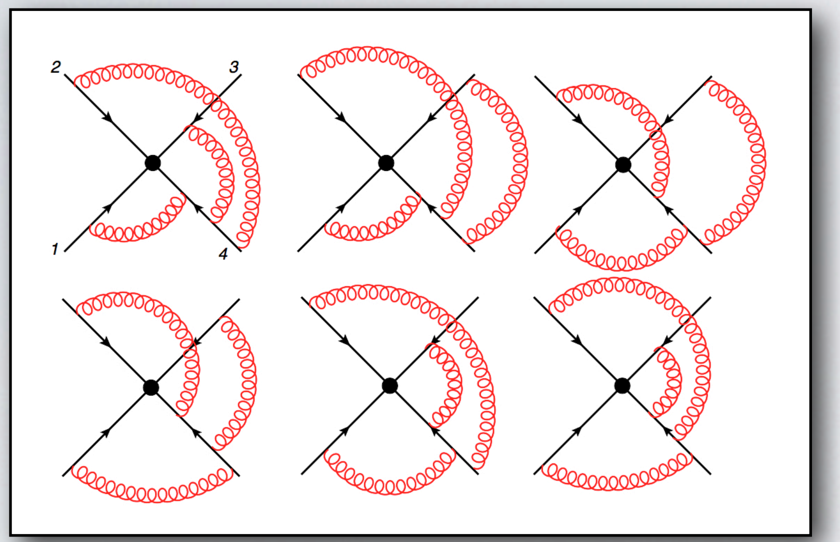


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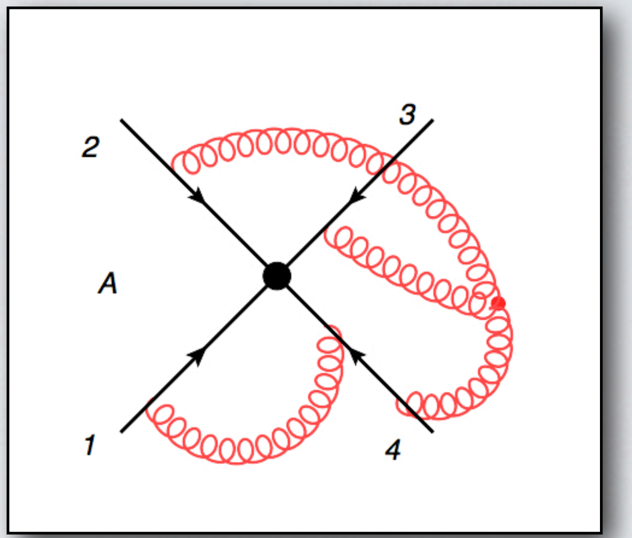


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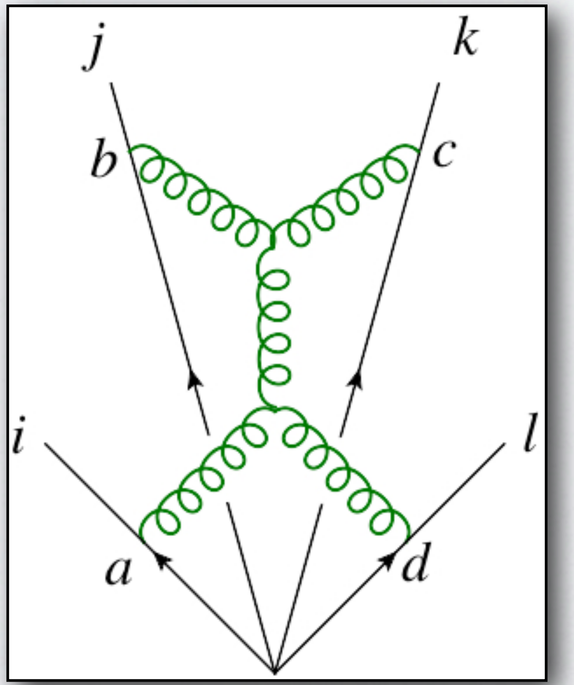


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Gardi 2013; Henn, Huber 2013;
Gardi, Falcioni, Harley, LM, White 2014

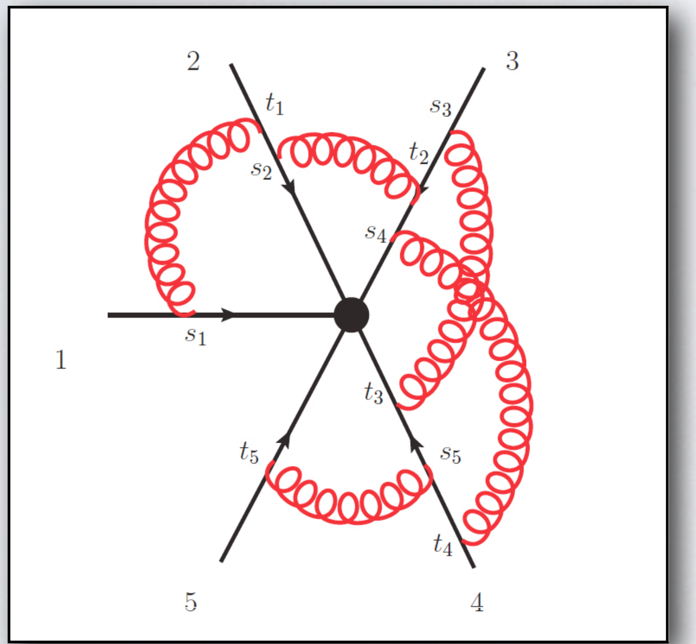
# Multiple Gluon Exchange Webs

Multiple Gluon Exchange Webs (MGEWs) arise from a path integral weighted with the free part of the quantum YM action

$$\mathcal{S}\left(\gamma_{ij}, \alpha_s(\mu), \epsilon, \frac{m}{\mu}\right)\Big|_{\rm MGEW} \equiv \int [DA]\, \Phi^{(m)}_{\beta_1} \otimes \Phi^{(m)}_{\beta_2} \otimes \ldots \otimes \Phi^{(m)}_{\beta_L} \exp\left\{ {\rm i} S_0[A] \right\},$$

A general integral representation for diagrams $D$ contributing to MGEWs can be written down to all orders, starting from a coordinate space representation of the Wilson lines

$$\mathcal{F}^{(n)}(D) = \kappa^n\, \Gamma(2n\epsilon) \int_0^1 \prod_{k=1}^{n} \Big[ dx_k\, \gamma_k\, P_\epsilon(x_k, \gamma_k) \Big]\, \phi_D^{(n)}(x_i;\epsilon)$$



A five-loop MGEW diagram

The variables $x_k$ measure collinearity to Wilson lines, the overall UV pole is extracted, the coordinate-space gluon propagators give

$$P_\epsilon(x, \gamma) \equiv \left[ x^2 + (1-x)^2 - \gamma\, x(1-x) \right]^{-1+\epsilon}$$

Non-abelian information is encoded in the order of gluon attachments on each Wilson line, through the kernel

$$\phi_D^{(n)}(x_i;\epsilon) = \int_0^1 \prod_{k=1}^{n-1} dy_k \left[ (1-y_k)^{-1+2\epsilon}\, y_k^{-1+2k\epsilon} \right] \Theta_D\big[ \{x_k, y_k\} \big]$$

Replacing the set of $\vartheta$ functions by unity one recovers abelian exponentiation.

# Subtracted Webs

Individual diagrams contain multiple UV poles and give uniform weight results. One must: combine them into webs (where leading poles cancel); subtract subdivergences via commutator counterterms; organize the result in a color basis. In the "right" variables

$$\gamma \to -\alpha - \frac{1}{\alpha}\,, \qquad\qquad P_\epsilon\left(x,\gamma\right) \to p_\epsilon\left(x,\alpha\right)$$

$$p_\epsilon\left(x,\alpha\right) \, = \, -\left(\alpha + \frac{1}{\alpha}\right)\left[q\left(x,\alpha\right)\right]^{-1+\epsilon}\,, \qquad q\left(x,\alpha\right) \, = \, x^2 + (1-x)^2 + \left(\alpha + \frac{1}{\alpha}\right) x(1-x)$$

Expansion in powers of $\epsilon$ will generate logarithms of $q(x,\alpha)$. Assembling the results

$$\overline{\omega}^{(n,-1)}_{a_1,\ldots,a_L} \, = \sum_A C^{(A)}_{a_1,\ldots,a_L}\, F^{(n)}_A\left(\alpha_1,\ldots\alpha_n\right)$$

$$F^{(n)}_A\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right) \, = \, \int_0^1 \left[\prod_{k=1}^n dx_k\, p_0(x_k,\alpha_k)\right] \mathcal{G}_n\Big(x_1,\ldots,x_n; q(x_1,\alpha_1),\ldots q(x_n,\alpha_n)\Big)$$

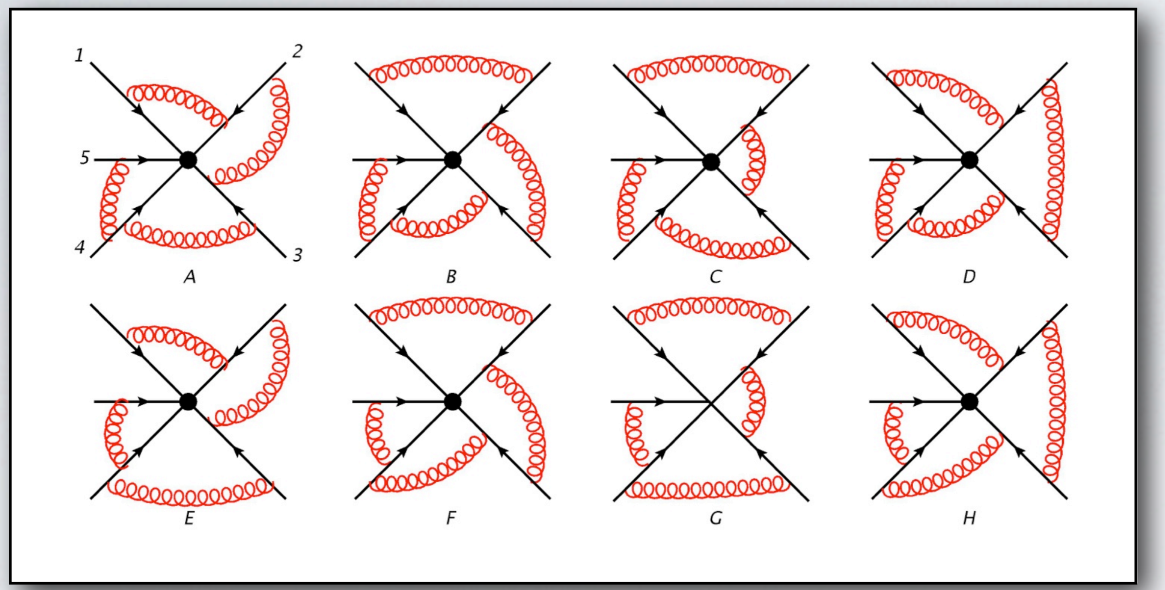
A lot of experimental and conceptual evidence is accumulating favoring the following

- Factorization conjecture: the subtracted MGEW kernel $\mathcal{G}$ is a sum of products of logarithmic functions of individual cusp variables of uniform weight n-1.
- Alphabet conjecture: the Symbol of all subtracted MGEW kinematic coefficients $F_A$ is restricted to the letters

$$\left\{\alpha_k,\ \eta_k \equiv \alpha_k/\left(1-\alpha_k^2\right)\right\}$$

# Four loops, five legs

The explicit evaluation of all three-loop MGEWs is nearing completion. We can however push further into the L&L space ....




The four-loop five-leg MGEW with attachments 1-2-2-2-1

The four-loop, five-line MGEW 1-2-2-2-1

- Contributes to a single color structure of the four-loop anomalous dimension.
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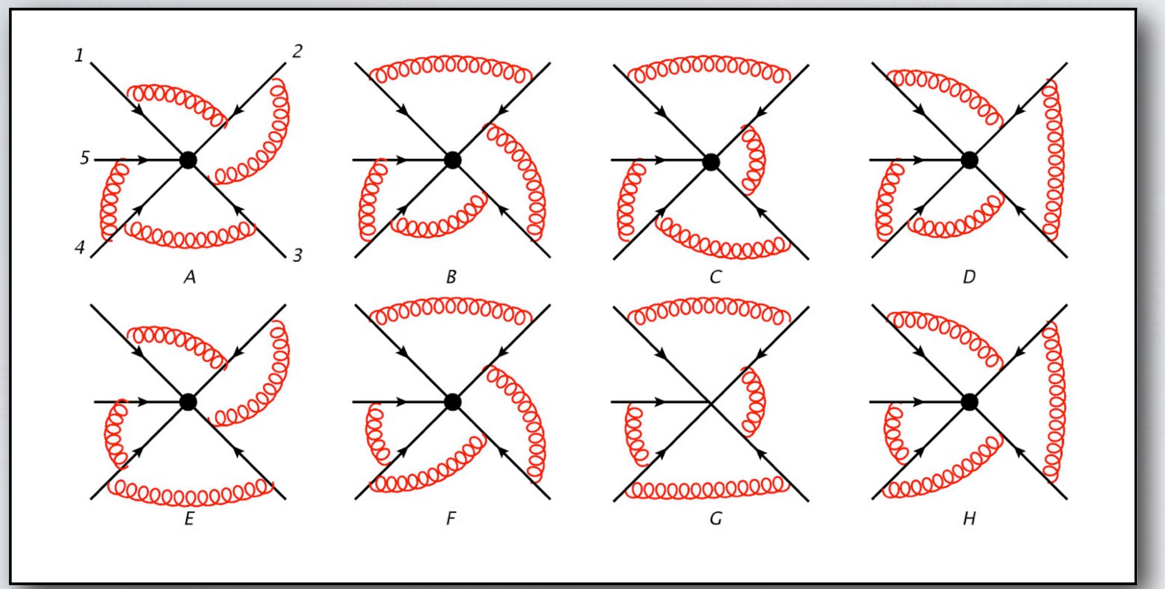


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The result is a simple function of the logarithms $L_{ij} = \log\left(\frac{q(x_i, \alpha_{ij})}{x_i^2}\right)\,, \quad \Sigma_i = \log\left(\frac{x_i}{1-x_i}\right)$

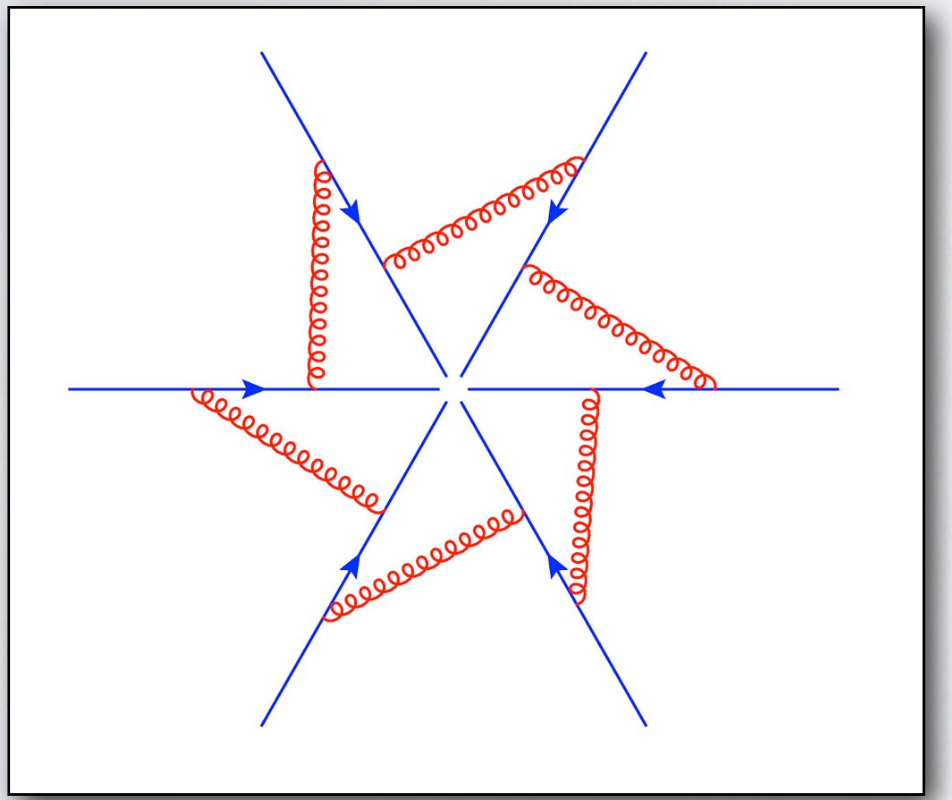
$$\begin{aligned}
\mathcal{G}_{(4)}\left(x_1, x_2, x_3, x_4\right) \quad = \quad & -\frac{1}{144}\Bigg\{L_{12}^3 - 3\,L_{23}^3 + 3\,L_{34}^3 - L_{45}^3 + 3\,L_{12}^2\Big[L_{23} + L_{34} - 3L_{45}\Big] \\
& + 3\,L_{23}^2\Big[L_{12} - 3\,L_{34} + 5\,L_{45}\Big] - 3\,L_{34}^2\Big[5\,L_{12} - 3\,L_{23} + L_{45}\Big] - 3\,L_{45}^2\Big[L_{23} + L_{34} - 3\,L_{12}\Big] \\
& + 6\,\Big[L_{12}L_{23}L_{34} - 3\,L_{12}L_{23}L_{45} + 3\,L_{12}L_{34}L_{45} - L_{23}L_{34}L_{45}\Big] \\
& + 24\Big[\Sigma_2^2\Big(L_{12} + L_{23} + L_{34} - 3\,L_{45}\Big) - \Sigma_3^2\Big(L_{23} + L_{34} + L_{45} - 3\,L_{12}\Big)\Big]\Bigg\}
\end{aligned}$$

# The Escher Staircase

Special diagrams contributing to MGEWs have special features, notably those which do not contain subdivergences. The most symmetric example is the "Escher Staircase", with kernel

$$\phi_{ES}^{(n)}(x_i;\epsilon) = \int_0^\infty \prod_{k=1}^{n-1} d\xi_k \left[\xi_k^{-1+2k\epsilon}(1+\xi_k)^{-2(k+1)\epsilon}\right] \widehat{\Theta}_{ES}\Big[\{x_k,\xi_k\}\Big]$$

$$= \int_{A_1}^{B_1} \frac{d\xi_1}{\xi_1} \int_{A_2(\xi_1)}^{B_2(\xi_1)} \frac{d\xi_2}{\xi_2} \ldots \int_{A_{n-1}(\xi_1,\ldots,\xi_{n-2})}^{B_{n-1}(\xi_1,\ldots,\xi_{n-2})} \frac{d\xi_{n-1}}{\xi_{n-1}} + \mathcal{O}(\epsilon)$$



Escher Staircase with six loops and six legs

where $\xi_k = y_k/(1-y_k)$

- The integral has a *d log* form, with intricate limits.
- The nested theta functions can be made explicit

$$A_k(\xi_1,\ldots,\xi_{k-1}) = \frac{x_{k+1}}{1-x_k}(1+\xi_{k-1})$$

$$B_k(\xi_1,\ldots,\xi_{k-1}) = \frac{\prod_{j=k+1}^{n}(1-x_j)}{\prod_{j=k+2}^{n+1} x_j} \prod_{j=1}^{k-1} \frac{1+\xi_j}{\xi_j}$$

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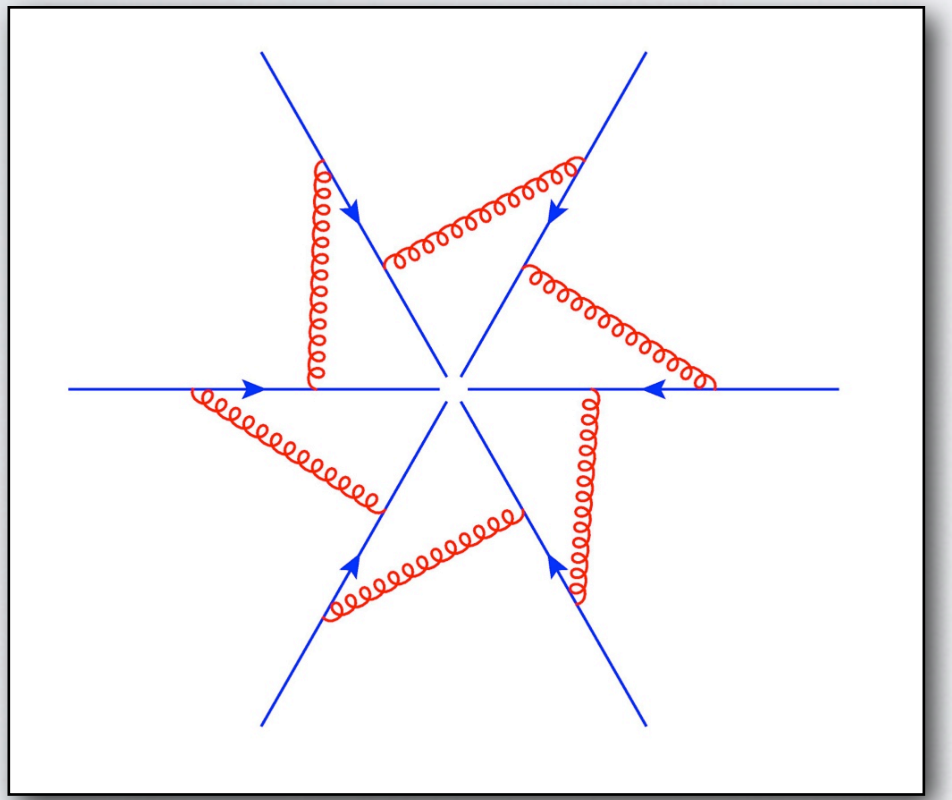

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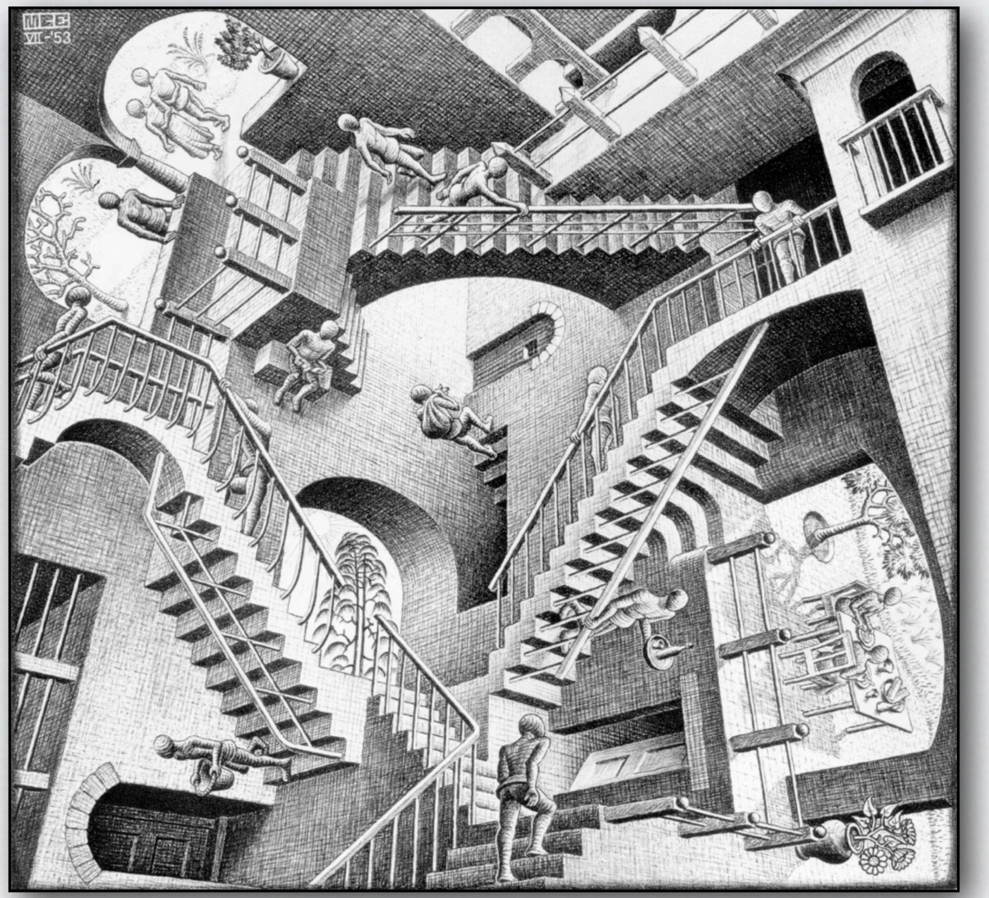
The result is remarkably simple!

$$\phi_{ES}^{(n)}(x_i;0) = \frac{1}{n!}\left[\log\left(\prod_{i=1}^{n}\frac{1-x_i}{x_i}\right)\right]^{n-1} \Theta^{(n)}(x_i)$$

# OUTLOOK



# Summary

- We are developing an ever deeper understanding of the perturbative expansion of gauge field theories to all orders.
- Important tools in the infrared are factorization and evolution equations.
- Conformal gauge theories have interesting special properties.
- Planar N = 4 Super Yang-Mills theory may be exactly solvable.
- A simple dipole formula encodes infrared singularities for any massless gauge theory to a high degree of accuracy.
- Potential corrections to the dipole formula are interesting, highly constrained, and their study is under way.
- We now understand non-abelian infrared exponentiation for multi-particle amplitudes.
- The calculation of the three-loop multi-particle soft anomalous dimension is advancing, using new technologies.
- Controlling IR singularities leads to the resummation of potentially large logarithms in phenomenologically relevant collider cross sections.

# THANK YOU!