LECTURE 2 : TOOLS

① Infrared safety : the total cross section.

Goal : outline a well-known calculation to extract general informations and useful tools.

DEFINITIONS

- $$\sigma_{TOT}(q^2) = \frac{1}{2q^2} \sum_X \int d\Gamma_X \, \frac{1}{4} \sum_{spin} |\mathcal{M}(k_1 + k_2 \to X)|^2$$

$$\to \quad R \equiv \frac{\sigma_{TOT}(e^+e^- \to \text{hadrons})}{\sigma_{TOT}(e^+e^- \to \mu^+\mu^-)}$$

- Hadronic tensor : $\sigma_{TOT}(q^2) \equiv \frac{1}{2q^2} L_{\mu\nu} H^{\mu\nu}$

$$L^{\mu\nu} = \frac{e^2}{q^4} \left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1 \cdot k_2 \, g^{\mu\nu} \right)$$

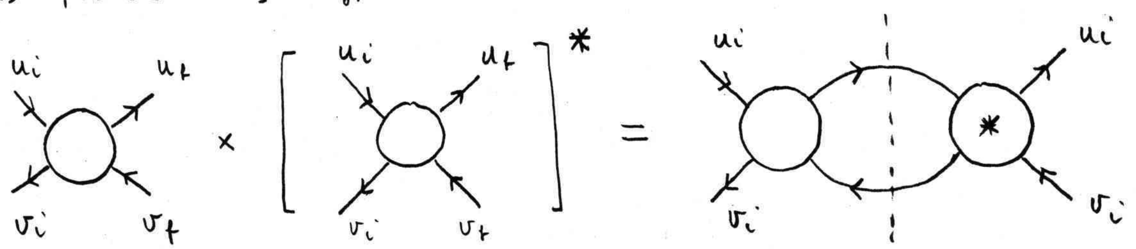
$$H_{\mu\nu} = e^2 q_f^2 \sum_X \langle 0 | J_\mu(0) | X \rangle \langle X | J_\nu(0) | 0 \rangle \cdot (2\pi)^4 \delta^4(q - p_X)$$

$$\to H_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu}) H(q^2) \Rightarrow -g^{\mu\nu} H_{\mu\nu}(q) = \overset{\to 3-2\varepsilon\ldots}{3}\, q^2 H(q^2)$$

Then $$\sigma_{TOT}(q^2) = \frac{e^2}{2q^4} \cdot \frac{1}{3} \left(-g^{\mu\nu} H_{\mu\nu}(q) \right)$$

TOOL 1 Cut diagrams.

It is pictorially efficient and intuitive to use



where $\underset{p \;\vdots\; p}{\longrightarrow\!\!\vdots\!\!\longrightarrow} = 2\pi \, \delta(p^2) \, \theta(p_0) \, \not{p} = 2\pi \, \delta_+(p^2) \sum \text{pol}.$

and right of the cut fermion arrows are reversed and $i \to -i$.

<u>Note</u> :

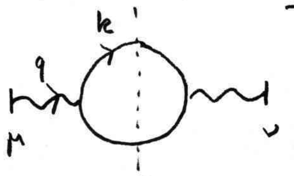
- True thanks to the identities

$$\left(\bar{w}_1 \left(\gamma_{\mu_1} \cdots \gamma_{\mu_n}\right) w_2\right)^* = \bar{w}_2 \left(\gamma_{\mu_n} \cdots \gamma_{\mu_1}\right) w_1$$

and $\left[(T_a)_{ij}\right]^* = (T_a)_{ji}$

- True for FIXED final state momenta, one may (or may not) integrate them with any measure.

- Nice : virtual and real diagrams "look the same" useful to map cancellations.

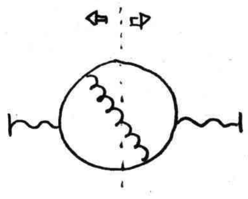
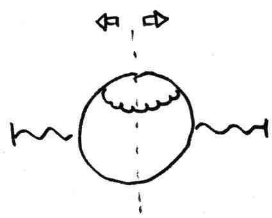
$\boxed{\text{EXAMPLE}}$ $\quad \sigma^{(0)}_{TOT}(q^2) = \dfrac{e^2}{2q^4} \cdot \dfrac{1}{3} \left(-g^{\mu\nu}\right) \sum_{c,f} \Big[$  $\Big]$

$$[\ \ ] = -e^2 q_f^2 \int \frac{d^4k}{(2\pi)^4} (2\pi)\,\delta_+(k^2)\,(2\pi)\,\delta_+((k-q)^2)\ \mathrm{Tr}\left[\not{k}\,\gamma_\mu\,(\not{k}-\not{q})\,\gamma_\nu\right]$$

$H_{\mu\nu}$

$$\Rightarrow 2e^2 q_f^2 \cdot \frac{q^2}{4} \frac{4\pi}{(2\pi)^2} \quad \Rightarrow \ \cdots \ \Rightarrow \quad R^{(0)} = N_c \sum_f q_f^2 .$$

$\boxed{\text{AT ONE LOOP}}$

$H_{\mu\nu} = \sum_{\text{cut positions}} \Big[$  $+$  $+ \ c.t. \Big]$

It would be nice to be able to compute directly the (weighted) sum, which is FINITE .... however ....

a) Real emission.

Phase space integrals will diverge, go to $d = 4 - 2\varepsilon$ $(\varepsilon < 0\,!)$

$$H^{(1,R)}_{\mu\nu} = \int \frac{d^dp\ d^dk}{(2\pi)^{2d-3}}\ \delta_+(p^2)\ \delta_+(k^2)\ \delta_+((q-p-k)^2)\ \mathcal{H}_{\mu\nu}$$

integrate p_0, k_0 $\qquad$ depend only on $\cos\theta_{pk}$

All angular integrals can be done, except for θ_{pk}.

Note : $\varepsilon < 0$ regularizes both IR and collinear div's.

$$d^d k \ \delta_+(k^2) \to \frac{d^{d-1} k}{k} \to dk \ k^{1-2\varepsilon} \ d\Omega_{d-2}$$

$$d\Omega_{d-2} = \underbrace{d\theta_{d-2} (\sin\theta_{d-2})^{d-3}}_{d\cos\theta \ (1-\cos^2\theta)^{-\varepsilon}} \cdot \ldots \cdot d\theta_2 \ \sin\theta_2 \cdot d\theta_1$$

Choosing $z = 2k/\sqrt{q^2}$ and $y = (1-\cos\theta_{pk})/2$ (somewhat unconventional ...) as integration variables one gets

$$H_{\mu\nu}^{(1,R)} = \# \cdot (q^2)^{1-2\varepsilon} \int_0^1 dz \ z^{1-2\varepsilon} \int_0^1 dy \ [y(1-y)]^{-\varepsilon} \ldots \mathcal{H}_{\mu\nu}$$

The leading IR/C behavior of $\mathcal{H}_{\mu\nu}$ is, as expected

$$\mathcal{H}_{\mu\nu} \sim \frac{1}{z^2 \ y \ (1-y)} \Rightarrow (\text{in } H_{\mu\nu}) \begin{cases} z^{-1-2\varepsilon} & (\text{IR}) \\ [y(1-y)]^{-1-\varepsilon} & (\text{C}) \end{cases}$$

yielding

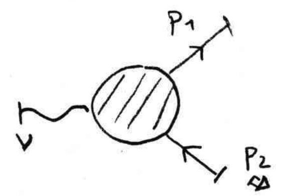
$$\left(H^\mu{}_\mu\right)^{1,R} = \# \ \frac{\alpha_s}{\pi} C_F \cdot q^2 \cdot \left(\frac{4\pi\mu^2}{q^2}\right)^{2\varepsilon} \cdot \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - \pi^2 + \ldots\right]$$

the characteristic structure of double (IR/C) and single (IR+C) poles.

b) Virtual correction.

We need the one-loop contribution to the QUARK FORM FACTOR

$$\Gamma_\nu(p_1, p_2; \mu^2, \varepsilon) =$$



which we'll meet again later, so we pause to note some formal properties.

Note: another long history ... Sudakov 1956 ...

⊕ Γ_μ is a matrix element of a conserved current.
In a MASSLESS theory, it is expressed in terms of just ONE "structure function" (or "scalar" form factor)

$$\Gamma_\mu(p_1, p_2; \mu^2, \varepsilon) \equiv \langle p_1, p_2 | J_\mu(0) | 0 \rangle =$$
$$= \bar{u}(p_1)\gamma_\mu v(p_2) \; \Gamma\left(\frac{q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right)$$

⊕ Being the matrix el. of a conserved current, Γ_μ is RG invariant (its anomalous dimension vanishes...
it does not violate the QED Ward identity $Z_1 = Z_\psi$...
strong interactions do not renormalize electric charges...)

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\varepsilon, \alpha_s)\frac{\partial}{\partial\alpha_s}\right)\Gamma\left(\frac{q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = 0$$

$\boxed{\text{TOOL 2}}$ The d-dimensional running coupling.

In $d = 4 - 2\varepsilon$: $\quad g_0 = Z_g \, g_R \, \mu^\varepsilon \quad \Rightarrow \quad \alpha_s(\mu^2) = Z_\alpha^{-1}\, \alpha_s^{(B)} \mu^{-2\varepsilon}$

then $\mu \dfrac{\partial\alpha_s}{\partial\mu} \equiv \beta(\varepsilon, \alpha_s) = -2\varepsilon\,\alpha_s + \hat{\beta}(\alpha_s)$

where $\hat{\beta}$ is the usual β function in $d=4$: $\hat{\beta}(\alpha_s) = -\dfrac{\alpha_s^2}{2\pi}\displaystyle\sum_{n=0}^{\infty} b_n \left(\frac{\alpha_s}{\pi}\right)^n$

with $b_0 = \dfrac{11 C_A - 2 n_f}{3}$.

This has important consequences!

⊕ At one loop

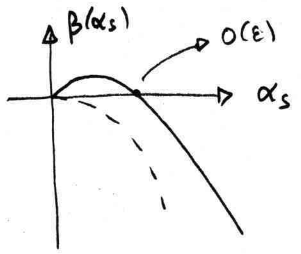
$$\bar{\alpha}\left(\frac{\mu^2}{\mu_0^2}, \alpha_s(\mu_0^2), \varepsilon\right) = \frac{\alpha_s(\mu_0)^2}{\left[\left(\frac{\mu^2}{\mu_0^2}\right)^\varepsilon - \frac{1}{\varepsilon}\left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^\varepsilon\right)\frac{b_0}{4\pi}\alpha_s(\mu_0^2)\right]}$$

is finite and gives the usual result as $\varepsilon \to 0$

⊕ At tree level it scales with its "engineering dimension"

$$\alpha_s(\mu^2) = \left(\frac{\mu^2}{\mu_0^2}\right)^{-\varepsilon}\alpha_s(\mu_0^2) \quad \Rightarrow \quad \alpha_s(0) = 0 \;\text{ for } \varepsilon < 0 \quad !$$

⊕ Why? The β-function starts POSITIVE!



– The theory is IR free in $d>4$.

– The Landau pole is located at

$$\mu^2 \equiv \Lambda^2 = Q^2 \left(1 + \frac{4\pi\varepsilon}{b_0\, \alpha_s(Q^2)}\right)^{-1/\varepsilon}$$

which is COMPLEX for $\varepsilon < - \frac{b_0}{4\pi} \alpha_s(Q^2)$.

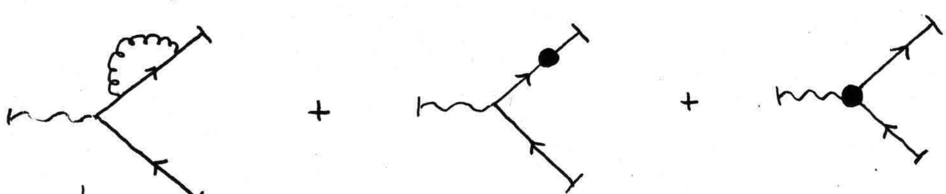
⊕ IR/C divergences will be generated by integration over the scale of the coupling:

$$\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \alpha_s(\lambda^2, \varepsilon) = \alpha_s(\mu^2) \int_0^{\mu^2} \frac{d\lambda^2}{(\lambda^2)^{1+\varepsilon}} \cdot (\mu^2)^{-\varepsilon} =$$

$$= -\frac{1}{\varepsilon} \alpha_s(\mu^2).$$

Now back to the form factor ....

⊕ One-particle reducible graphs on external lines should be "included only once" (the reduction formulas multiply each external leg times $R^{-1/2}$...), and they "vanish" in dim. reg. for massless quarks. More precisely



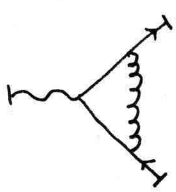
$= 0$

$$0 = \left(\frac{c}{\varepsilon}\right)_{UV} - \left(\frac{c}{\varepsilon}\right)_{IR} \quad ; \quad -\left(\frac{c}{\varepsilon}\right)_{UV} \quad ; \quad +\left(\frac{c}{\varepsilon}\right)_{UV}$$

must cancel since $Z_1 = Z_4$

Note: not true in axial gauges, can build $\frac{(n\cdot p)^2}{n^2}$ dependence.

⊕ At one loop, only one graph needs to be computed

$\Gamma_\mu^{(1)} =$  $= C_\mu I_0 + C_{\mu\alpha} I^\alpha + C_{\mu\alpha\beta} I^{\alpha\beta}$

where

$$I_0, I^\alpha, I^{\alpha\beta} \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1,\ k^\alpha,\ k^\alpha k^\beta}{(k^2+i\eta)\left[(p_1-k)^2+i\eta\right]\left[(p_2+k)^2+i\eta\right]}$$

Note: I_0 $(I^{\alpha\beta})$ clearly diverges in the IR (UV) by obvious power counting. I^α does NOT, but it has COLLINEAR poles $\left(I^+ \propto \int \frac{dk^-}{k^-} \dots\right)$: we need POWER COUNTING TOOLS!

And the result is . . .

$$\Gamma^{(1)} = -\frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{-q^2}\right)^\varepsilon \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \dots\right]$$

NOTE : $(-q^2+i\eta)^{-\varepsilon} = (q^2)^{-\varepsilon}\, e^{-i\pi\varepsilon} = (q^2)^{-\varepsilon}\left(1 - i\pi\varepsilon - \frac{\pi^2}{2}\varepsilon^2 + \dots\right)$

We observe the "KLN" cancellation and get

$$\boxed{R = N_c \sum_f q_f^2 \left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)}$$

A LESSON

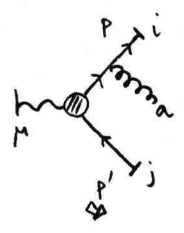
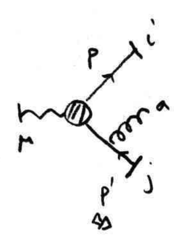
- The virtual correction is proportional to the Born result
- Therefore : ANY MODIFICATION of the real emission contribution BEHAVING LIKE THE BORN σ in the singular limits ($z \to 0$, $y \to 0, 1$) WILL PRESERVE the cancellation of divergences!
- Can we make this systematic ?

$\to$ . . . IR/C safe obs.; jets; event shapes; ⋮

(2) The soft approximation.

a) Real emission. (straightforward)

Consider

$M^{a\mu}_{ij} \equiv$  + 

$$= g\, T^a_{ij}\, \bar{u}(p) \left[\frac{\not\varepsilon(k)(\not p + \not k)\Gamma_\mu}{2p\cdot k} - \frac{\Gamma_\mu(\not p' + \not k)\not\varepsilon(k)}{2p'\cdot k}\right] v(p')$$

NOTE! $-p'-k$ along $\leftarrow$

Now:
- Neglect k in the num. AND in the def. of p'.
- Commute $\not{p}$ and $\not{p}'$ to hit the spinors

  where $\not{p}' v(p') = \bar{u}(p) \not{p} = 0$.

Then

$$\mathcal{M}^{a\mu}_{ij}\Big|_{SOFT} = g\, T^a_{ij} \left[\frac{p\cdot\varepsilon}{p\cdot k} - \frac{p'\cdot\varepsilon}{p'\cdot k} \right] \mathcal{M}^{\mu}_{BORN}$$

with $\mathcal{M}^{\mu}_{BORN} = \bar{u}(p)\, \Gamma_\mu\, v(p')$, <u>whatever</u> Γ_μ.

$\boxed{\text{NOTES}}$

- $\mathcal{M}_{SOFT}$ is <u>gauge-invariant</u> (vanishes for $\varepsilon \propto k$)

  [subnote: crucial − sign is absorbed in the def. of color operators T^a for antiquarks in Catani-Seymour et al.]

- $\mathcal{M}_{SOFT}$ is <u>universal</u> : information on SPIN, ENERGY and SHORT DISTANCE structure DECOUPLES. Only information on COLOR CHARGE and DIRECTION survives

  Indeed : can replace $p^\mu, p'^\mu \to \beta^\mu, \beta'^\mu$, $p^\mu = \beta^\mu Q$ ....

One can finally build a SOFT CROSS SECTION, using

$\sum_{pol} \varepsilon_\mu \varepsilon^*_\nu = - g_{\mu\nu}$,

$$\sum_{pol} |\mathcal{M}_{SOFT}|^2 = g^2 C_F\, |\mathcal{M}_{BORN}|^2\, \frac{2\, p\cdot p'}{p\cdot k\;\; p'\cdot k}$$

PHASE SPACE factorizes in the soft limit (less trivial in the collinear limit) so

$$\sigma^{(q\bar{q}g)}_{SOFT} = g^2\, C_F\, \sigma^{(q\bar{q})} \int \frac{d^3 k}{(2\pi)^3\, 2|\vec{k}|}\, \frac{2\, p\cdot p'}{p\cdot k\;\; p'\cdot k} =$$

$$= \frac{\alpha_s}{4\pi}\, C_F\, \sigma_{q\bar{q}} \int_{-1}^{1} d\cos\theta_{pk} \int_0^Q \frac{dk}{k}\, \frac{1}{(1-\cos\theta_{pk})(1+\cos\theta_{pk})}$$

⇨ The structure of IR/C singularities is recovered

⇨ The pure collinear pole is wrong (hard collinear is missing)

⇨ Important applications (angular ordering!).

b) Virtual corrections (care required)

The main difference is that here $k^2 \neq 0$, although of course we know that singularities come from $k^2 \sim 0$ ...

Pick light cone coordinates and the frame in which p and p' are back to back

$$p^\mu = (p^+, 0, \vec{0}_\perp) \quad ; \quad p'^\mu = (0, p^-, \vec{0}_\perp)$$

(recall : $v^\mu = (v^+, v^-, \vec{v}_\perp) \Rightarrow v^2 = 2v^+v^- - |\vec{v}_\perp|^2$; $v^\pm = \dfrac{v^0 \pm v^3}{\sqrt{2}}$)

⊕ Straightforward scaling : all components soft

$$k^\mu \ll q \equiv \sqrt{q^2} \quad , \quad \text{or} \quad k^\mu = \lambda q \quad , \quad \lambda \to 0$$

Then $p \cdot k \ll k^2$, match with real emission case.

→ EIKONAL approximation , captures ALL PURELY SOFT singularities.

$\boxed{\text{EXAMPLE}}$

$$I_0^{(EIK)} = \frac{1}{(2\pi)^{4-2\varepsilon}} \int \frac{dk^+ dk^- d^{2-2\varepsilon} k_\perp}{(2k^+k^- - |k_\perp|^2 + i\eta)(-k^- + i\eta)(k^+ + i\eta)}$$

Highly singular! We'll see how to handle it ... NOTE : in $d=4$ it is UV divergent (besides IR).

⊕ Less straightforward scalings (require different approx.)

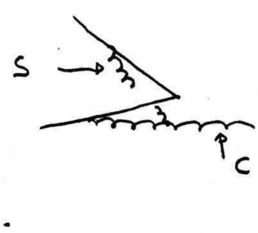
Collinear : $k^\pm \sim q \quad ; \quad k^\mp \sim \lambda^2 q \quad ; \quad |\vec{k}_\perp| \sim \lambda q$

(the $k^2 + i\eta$ denom. still vanishes homogeneously as λ^2)

Are there others ?

Glauber : $k^\pm \sim \lambda^2 q \quad , \quad |\vec{k}_\perp| \sim \lambda q \quad \Rightarrow \quad k^2 \sim -|\vec{k}_\perp|^2$

- Irrelevant at one loop
- Dangerous at $g \geq 2$ loop for INITIAL STATE soft/collinear configs.



- Cancel upon summing over diagrams.

TOOL 3 Eikonal integrals.

Let us examine more closely the integral that gives the one-loop eikonal approximation to the form factor.

With all proper factors it is

$$I_0^{(EIK)} = i\, g^2 \mu^{2\varepsilon}\, C_F\, \beta_1 \cdot \beta_2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\eta)(-\beta_1 \cdot k + i\eta)(\beta_2 \cdot k + i\eta)}$$

where we reintroduced the velocity vectors β_i / $p_i^\mu = Q \beta_i^\mu$.

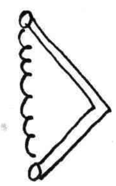
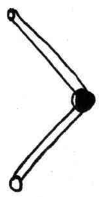
- It is homogeneous of deg. ϕ in β_i (as per eikonal rules)
- It diverges in the IR, C, <u>and</u> UV by power counting
- It is readily evaluated:

$$I_0^{(EIK)} = 0$$

by the rules of dim. reg. (it is a scale-less integral)

<u>NOTE</u> however

- The UV divergence is NOT present in PQCD: I_0 was UV finite
- $I_0^{(EIK)}$ is a good approx in the IR, NOT in the UV.
- We need to RENORMALIZE (subtract) the spurious UV divergence to get the IR singularity we need.

$\Gamma^{(1)}_{EIK}$ =  +  $= \underbrace{\left(\frac{\#}{\varepsilon}\right)_{UV} - \left(\frac{\#}{\varepsilon}\right)_{IR}}_{0} - \left(\frac{\#}{\varepsilon}\right)_{UV}^{CT}$

We find the one-loop coefficient of the anomalous dimension for two light-like Wilson lines (see 3^d lecture) meeting at a cusp: the CUSP ANOMALOUS DIMENSION $\gamma_K(\alpha_s)$

- In the process we observe that the invariance under $\beta_i \to K \beta_i$ is BROKEN by an ANOMALY, due to the presence of a COLLINEAR POLE in the COUNTERTERM.

Evaluation.

⊕ Feynman parametrize linear denominators first.

$$I_0^{(EIK)} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(x\beta_2\cdot k - (1-x)\beta_1\cdot k + i\eta)^2} \frac{1}{k^2 + i\eta} \left(ig^2\mu^{2\varepsilon} C_F \beta_1\cdot\beta_2\right)$$

⊕ Next Feyn. param. these two denominators.

$$I_0^{(EIK)} = ig\mu^{2\varepsilon} \beta_1\cdot\beta_2 C_F \int_0^1 dx\, dy \int \frac{d^d k}{(2\pi)^d} \frac{2(1-y)}{\left[yk^2 + (1-y)(x\beta_2 - (1-x)\beta_1)\cdot k + i\eta\right]^3}$$

⊕ Perform the momentum integral (after extracting y^3).

$$I_0^{(EIK)} = \frac{g^2}{(4\pi)^2}(4\pi\mu^2)^\varepsilon C_F \beta_1\cdot\beta_2 \Gamma(1+\varepsilon) \int_0^1 dx\, dy \cdot \frac{1-y}{y^3} \cdot \left(-\beta_1\cdot\beta_2 \frac{x(1-x)(1-y)^2}{2y^2}\right)^{-1-\varepsilon} =$$

$$= -\frac{g^2}{(4\pi)^2}\left[\frac{4\pi\mu^2}{-\frac{\beta_1\cdot\beta_2}{2}}\right]^\varepsilon \cdot 2C_F \cdot \Gamma(1+\varepsilon) \cdot B(-\varepsilon,-\varepsilon) \cdot \int_0^1 dy\, (1-y)^{-1-2\varepsilon} y^{-1+2\varepsilon}$$

ANOMALY (IF POLE...)

SUM of two single collinear poles $\sim -\frac{2}{\varepsilon} + O(\varepsilon)$

$B(-2\varepsilon, 2\varepsilon) = 0$!

⊕ Identify the UV contribution (comes from $y=0$!)

$$\int_0^1 dy\, y^{-1+2\varepsilon}(1-y)^{-1-2\varepsilon} = \int_0^1 dy\, y^{-1+2\varepsilon}(1-y)^{-1-2\varepsilon}\left(y + (1-y)\right) =$$

$$= B(1+2\varepsilon, -2\varepsilon) + B(2\varepsilon, 1-2\varepsilon) = \underbrace{\Gamma(1+2\varepsilon)\Gamma(-2\varepsilon)}_{IR} + \underbrace{\Gamma(2\varepsilon)\Gamma(1-2\varepsilon)}_{UV}$$

⊕ The required $\overline{MS}$ counterterm is DIVERGENT and ANOMALOUS...

$$\Gamma^{(1)}_{EIK} = -\frac{\alpha_s}{2\pi} C_F \underbrace{\frac{1}{\varepsilon}}_{UV}\left[\underbrace{\frac{1}{\varepsilon}}_{COLL} - \gamma_E + \ln 4\pi + \ln\left(-\frac{\beta_1\cdot\beta_2}{2}\right)\right]$$