LECTURE 3 : ALL ORDERS

So far : • in lect. 1 : general understanding of IR/C sing's.
• in lect. 2 : one-loop examples, tools.

Now : we move on to study tools that allow to push the analysis to ALL ORDERS in perturbation theory.

3 steps are necessary :

1) Locate potential singularities (necessary conditions)
2) Identify actual singularities (sufficient conditions)
3) Organize singularities into operator matrix elements.

(1) Singularities of (massless...) Feynman diagrams.

Let's begin by studying our master example, the scalar integral contributing to the form factor. Introducing Feynman parameters we get

$$I_0 = 2\int \frac{d^dk}{(2\pi)^d} \int_0^1 \prod_{i=1}^{3} dy_i \, \frac{\delta\left(1-\sum_{i=1}^{3} y_i\right)}{\left[y_1 k^2 + y_2 (p-k)^2 + y_3 (p'+k)^2 + i\eta\right]^3}$$

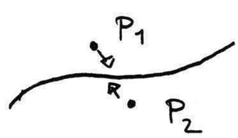
Basic facts :

a) Singularities can ONLY arise where the denominator vanishes (note UV convergence).

b) Let $D_0 \equiv y_1 k^2 + y_2 (p-k)^2 + y_3 (p'+k)^2$. $D_0 = 0$ is NOT a sufficient cond. for a singularity. We are in the complex plane by the $i\eta$ prescriptions. Contours can be DEFORMED.

c) Singularities arise ONLY when DEFORMATION IS IMPOSSIBLE

c1) Contour is trapped between two poles (PINCH singularity)



c2) A pole migrates to the END of a contour

(END-POINT singularity).

Examine I_0.

- dk^μ integrals cannot have endpoint sing's. (UV conv.)
  they can have pinches (D_0 quadratic in k^μ).

Pinch condition $\quad \frac{\partial}{\partial k^\mu} D_0(y_i, k^\mu; p, p') = 0$

- dy_i integrals cannot have pinches (D_0 linear in y_i)
  they can have endpoints (only $y_i = 0$ is relevant).

ALSO: if D_0 vanishes on a surface INDEPENDENT on y_i, then y_i cannot be used to deform the contour.

Landau equations for I_0

A necessary condition for a singularity is that $\boxed{\text{ALL}}$ integration variables be trapped (pinch OR endpoint).
These are the LANDAU EQUATIONS for I_0:

$$\begin{cases} y_1 k^\mu - y_2 (p-k)^\mu + y_3 (p'+k)^\mu = 0 & \to \sum_i y_i \, l_i^\mu \, \sigma_i = 0 \\ y_i = 0 \quad \boxed{\text{OR}} \quad l_i^2 = 0 & (\text{if } D_0 \equiv \sum_i y_i l_i^2) \end{cases}$$

Solutions for I_0

IR: $\quad k^\mu = 0 \quad ; \quad \frac{y_2}{y_1} = \frac{y_3}{y_1} = 0$

C_p: $\quad k^\mu = \alpha p^\mu \quad ; \quad y_3 = 0 \quad ; \quad \alpha y_1 = (1-\alpha) y_2$

$C_{p'}$: $\quad k^\mu = -\beta p'^\mu \quad ; \quad y_2 = 0 \quad ; \quad \beta y_1 = (1-\beta) y_3$

Of course in gen. we need to make sure we have found ALL solutions ... this is relatively EASY, thanks to the

Coleman-Norton physical picture

Observe: $\oplus$ For every off-shell line, i, one must have $y_i = 0$

$\oplus$ To every on-shell line ($y_i \neq 0$) associate

$$\Delta x_i^\mu \equiv y_i \, l_i^\mu = \Delta x_i^0 \cdot v_i^\mu \qquad \left(v_i^\mu = \left(1, \frac{\vec{l}_i}{l_i^0}\right)\right)$$

⊕ The Landau equations become

$$\sum_i \sigma_i \, \Delta x_i^\mu = 0 \quad ; \quad \Delta x_i^\mu = 0$$

on-shell lines $\qquad\qquad$ off-shell lines

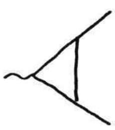
<u>Interpretation</u>

- Δx_i^μ is the displacement in a proper time $\Delta x_i^0 = y_i \, l_i^0$ of a classical particle with 4-velocity v_i^μ (and thus momentum l_i^μ)

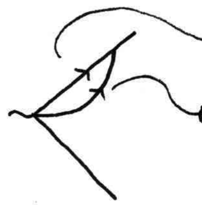
  NOTE: true also for $l^2 = m^2 \neq 0$!.

- ALL solutions of the Landau eqn. can be represented as <u>REDUCED DIAGRAMS</u> where
  - all off-shell lines have been contracted to a point
  - all on-shell lines form loops that correspond to ALLOWED CLASSICAL TRAJECTORIES of on-shell particles with the given momenta.

<u>Examples</u> :



⊳ reduce



$\Delta x_2^\mu = y_2 (p-k)^\mu = y_2 (1-\alpha) p^\mu$

$\Delta x_1^\mu = y_1 k^\mu = \alpha y_1 p^\mu$

⊳ The solution $\Delta x_1^\mu - \Delta x_2^\mu = 0$, $\Delta x_3^\mu = 0$, given by $\alpha y_1 = (1-\alpha) y_2$ describes two particles (light-like here) starting at $t=0$ in the origin in direction p^μ and MEETING AGAIN at $t = \Delta x_1^0 = \Delta x_2^0$.

Note: the IR solution has reduced diagram  with ALL propagators on-shell. All $\Delta x_i^0 = 0$, corresponding to an instantaneous circuit in the CN picture (reminiscent of the $\lambda_{DB} \to \infty$ property of $k^\mu \to 0$ particles).

<u>Generalization</u> : The CN picture for the general solution of the Landau eqn's. survives to ALL ORDERS. Indeed:

⊕ Any Feynman diagram can be parametrized down to

$$G(p_i) = \int_0^1 \prod_{\text{lines } i} dy_i \; \delta\Big(1 - \sum_{\text{lines } i} y_i\Big) \cdot \int \prod_{\text{loops } l} \frac{d^d k_l}{(2\pi)^d} \, \frac{\mathcal{N}(y_i, k_l, p_r)}{\left[\mathcal{D}(y_i, k_l, p_r)\right]^N}$$

⊕ The denominator is a sum of propagators

$$\mathcal{D}(y_i, k_\ell, p_r) = \sum_{\text{lines } i} y_i \left(\ell_i^2 (k_\ell, p_r) - m_i^2 \right) + i\eta$$

with ℓ_i^μ linear in k_ℓ^μ and p_r^μ.

⊕ Pinches only arise in k_ℓ^μ integrals, endpoints only at $y=0$

⊕ Landau eqn's become

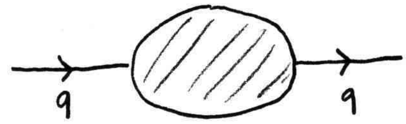
$$\begin{cases} \sum_i \varepsilon_{ji} \, \Delta x_i^\mu = 0 & \forall j \,/\, i \in j \quad \text{(loop)} \qquad \text{[on-shell lines]} \\ \Delta x_i^\mu = 0 & \text{[off-shell lines]} \end{cases}$$

⊕ The CN interpretation still holds!

This is enough to give intuitive arguments for important all-order results ...

$\boxed{\text{EXAMPLE}}$ The two-point function.

Consider the two-point function $G(q^2, m^2)$ in a theory with a single particle species of mass m^2.

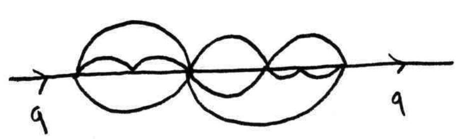
$G(q^2, m^2) \equiv$ 

THEOREM : the only possible singularities of the (renormalized) $G(q^2, m^2)$ are NORMAL THRESHOLDS : $q^2 = n^2 m^2$, $n = 2, 3, \ldots$

PROOF : a) Normal thresholds are solutions of the Landau eqs.

⊕ Choose frame $q^\mu = (\sqrt{q^2}, \vec{0})$ for $q^2 > 0$

⊕ CN process : create n particles at rest. They don't move until reabsorbed.

⊕ Reduced diagrams  $(n=4)$

b) No other reduced diagrams correspond to a CN process.
(any particle produced with $\vec{p} \neq 0$ must be balanced by another one with $-\vec{p}$. Once emitted they can never meet again)

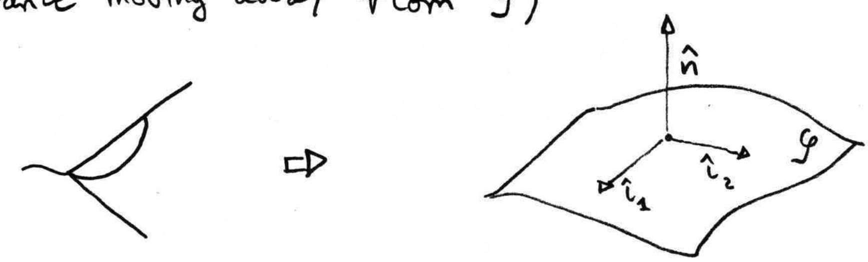
② Infrared and collinear power counting.

Landau eqs. are only NECESSARY conditions for singularities.
Denominator poles can be outweighed by numerators and phase space / loop integrals (e.g.: ϕ^3_6 for IR divergences).

$\boxed{\text{STEPS}}$

⊕ Given a diagram, identify the reduced diagrams corresp. to trapped surfaces $\mathcal{S}$ in the space $\{k^\mu_i, y_i\}$.

⊕ For each RD, identify among the k^μ_i the INTRINSIC COORDINATES (internal to $\mathcal{S}$) and the NORMAL COORDINATES (distance moving away from $\mathcal{S}$)



Ex.: for I_0, $k // p$, k^+ is INTRINSIC, $\{k^-, \vec{k}_\perp\}$ are NORMAL

⊕ Introduce a SCALING VARIABLE λ to weigh integration volume / degree of singularity, for EACH NORMAL VARIABLE : $n_\alpha = \lambda^{a_\alpha} \hat{n}_\alpha$

Ex.: for I_0, $k // p$, $k^- \sim \lambda^2 \sqrt{q^2}$, $|\vec{k}_\perp| \sim \lambda \sqrt{q^2}$

$k \to 0$: $k^\mu = \lambda \sqrt{q^2}$ $\forall \mu$. (ALL NORMAL)

⊕ Construct the HOMOGENEOUS INTEGRAL for $\mathcal{S}$, taking the DOMINANT POWER of λ in EACH FACTOR of the reduced graph.

Ex.: for I_0, $k^\mu \to 0$ it is the EIKONAL integral $I_0^{(EIK)}$.

⊕ Establish the DEGREE OF DIVERGENCE : the power of λ associated with the HOMOGENEOUS INTEGRAL.

If $n_\alpha = \lambda^{a_\alpha} \hat{n}_\alpha$ $(\alpha = 1, .., N_{norm})$, $l_i^2(p,k) - m_i^2 \to \lambda^{A_i} f(\hat{n})$
$(i = 1, .., N_{lines})$

then
$$n_{\mathcal{S}} = \sum_{\alpha=1}^{N_{norm}} a_i - \sum_{i=1}^{N_{lines}} A_i + n_{num}$$

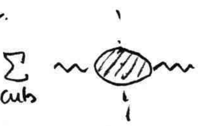
and we have a singularity associated with $\mathcal{S}$ iff $n_{\mathcal{S}} \leq 0$

A QUICK EXAMPLE All-order finiteness of $R_{e^+e^-}$.

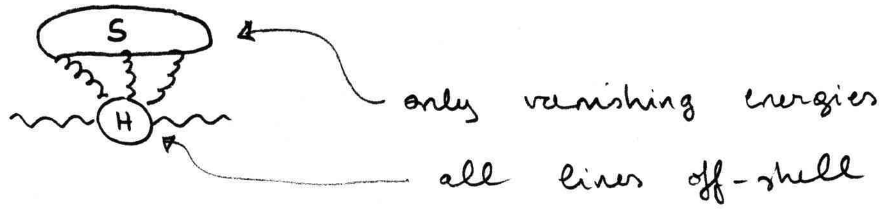
$R_{e^+e^-}$ is related by unitarity to the 2-point function of two EM currents. Define

$$\rho_{\mu\nu}(q) \equiv i e^2 \int d^4x \, e^{iqx} \langle 0 | T [J_\mu(x) J_\nu(0)] | 0 \rangle =$$

$$= (q_\mu q_\nu - q^2 g_{\mu\nu}) \, \Pi(q^2)$$

Then $\sigma_{TOT}(e^+e^- \to \text{hadrons}) = \dfrac{e^2}{q^2} \, \mathrm{Im}\left[\Pi(q^2)\right]$ $\Big(\sum_{\text{cuts}}$  $\Big)$

- Replicate analysis for $G(q^2, m^2)$: in the frame $q^\mu = (\sqrt{q^2}, \vec{0})$ and with ONLY MASSLESS PARTICLES, there are NO CN processes <u>except</u> ones with vanishing momenta.
- The only reduced diagrams are of the form



- Zero momentum fermions are less singular than gluons. Consider then GLUON ONLY in S, in d dimensions

$$n_g = d \cdot L_s - 2 g_s = 2(1-\varepsilon) g_s > 0 \quad \text{for } d > 2 \,!$$

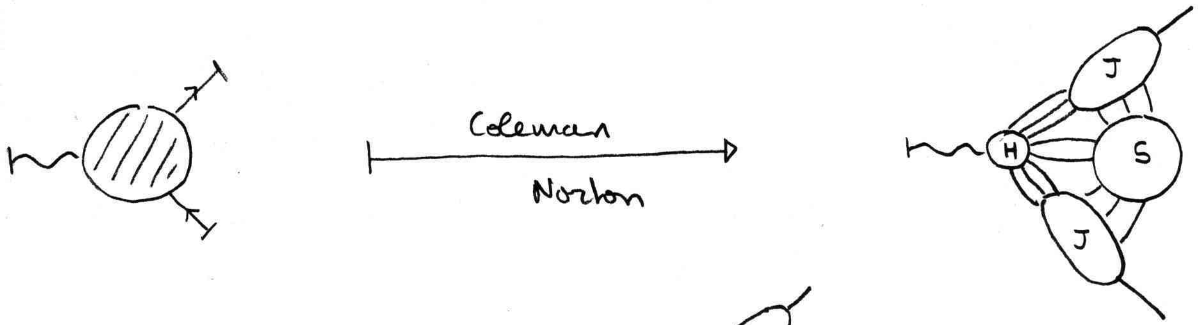
$\# \text{ Loops in } S = \# \text{ gluon lines in } S$

③ Organizing singularities into operator matrix elements.

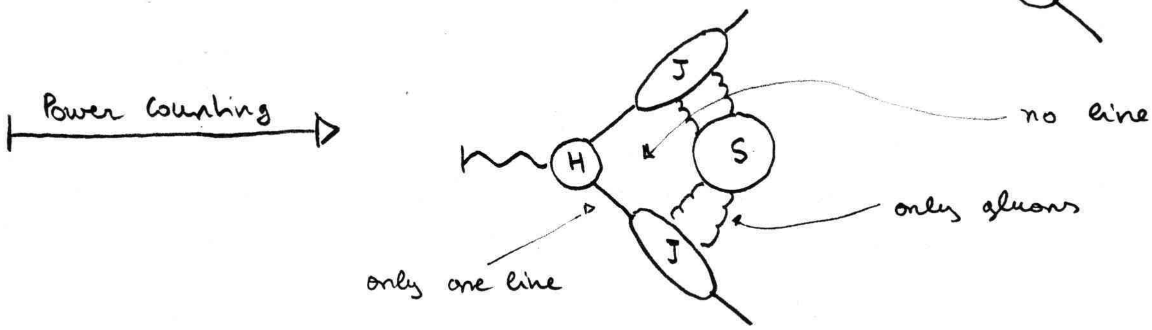
Having located singularities, the process of FACTORIZING amplitudes or cross sections is still long and involved. One needs to

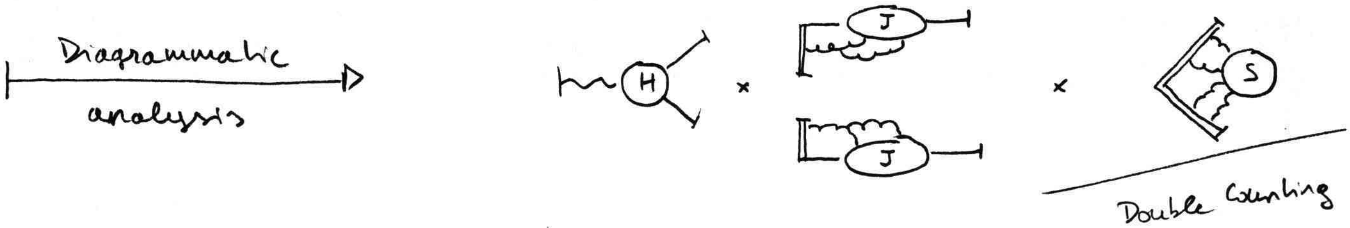
- Use systematic approximations to simplify the structure of Feynman diagrams
- Identify operator matrix elements mimicking the structure of singularities
- Be careful about double countings of singular regions.

Let us illustrate the process using the form factor as an example. Graphically



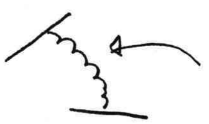
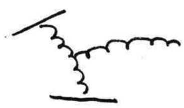
only IR/C regions contribute!





a) Notes on power counting for Γ.

⊕ Obvious that no lines join H and S directly

 $\frac{1}{k^2}$ in H, $k^2 \sim q^2$ vs.  $\frac{k^\mu}{k^4} \sim \frac{1}{q^3}$

⊕ Non-trivial that only one line joins H and J

In axial gauge $\quad G_{\mu\nu}(k) = \frac{1}{k^2+i\eta}\left(-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} - n^2\frac{k_\mu k_\nu}{(n\cdot k)^2}\right)$

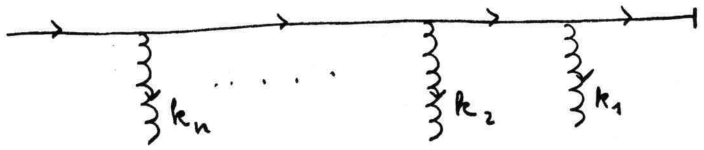
$\Rightarrow \quad k^\mu G_{\mu\nu}(k) = \frac{n_\nu}{n\cdot k} - n^2 \frac{k_\nu}{(n\cdot k)^2} \qquad$ NO POLE at $k^2=0$.

(useful also in complicated diagrams when all lines are parallel ...)

In Feynman gauge the effect persists: many gluons can attach J to H, but they are forced to be LONGITUDINALLY POLARIZED ($\varepsilon_\mu \propto k_\mu$), so they can be decoupled using Ward identities

b) Soft diagrammatics.

The one-loop soft approximation generalizes to all orders ...
Consider multiple emission from a hard fermion line



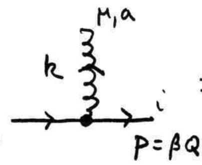
$\longrightarrow$ by the 1-gluon tricks spin decouples ...

$$\rightarrow p^{\mu_1} \cdots p^{\mu_n} \; \frac{1}{p\cdot k_1} \, \frac{1}{p\cdot(k_1+k_2)} \cdots\cdots \frac{1}{p\cdot\left(\sum_{i=1}^{n} k_i\right)}$$

Summing over all gluon orderings, a major simplification using the eikonal identity

$$\sum_{\pi} \left[\frac{1}{p\cdot k_{\pi(1)}} \cdot \frac{1}{p\cdot(k_{\pi(1)}+k_{\pi(2)})} \cdots \frac{1}{p\cdot\left(\sum_{i=1}^{n} k_{\pi(i)}\right)} \right] = \prod_{i=1}^{n} \frac{1}{p\cdot k_i}$$

This (Bose symmetric) result can be generated by the Feynman rules



$$= ig \frac{\beta^\mu}{\beta\cdot k} T^a_{ij}$$

which correspond to CORRELATORS OF WILSON LINES (as expected, a hard colored particle in a cloud of soft gluons DOES NOT RECOIL, sticks to the CLASSICAL TRAJECTORY, and picks up a (nonabelian) PHASE).

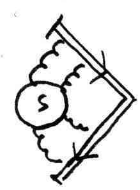
Indeed, define

$$\Phi_n(\lambda_2,\lambda_1) \equiv P\exp\left[ig\int_{\lambda_1}^{\lambda_2} d\lambda \; n\cdot A(\lambda n)\right]$$

and consider

$$S(\beta_1\cdot\beta_2, \alpha_s(\mu^2), \varepsilon) \equiv \langle 0|\, \Phi_{\beta_2}(\infty,0)\, \Phi_{\beta_1}(0,-\infty)\,|0\rangle$$

corresponding to



This is easily seen to reproduce the 1-loop eikonal result of lect. 2 :

$$S^{(1)} = -g^2 \int_0^\infty d\lambda_2 \int_{-\infty}^{0} d\lambda_1 \; \beta_2^\mu \beta_1^\nu \; \langle 0|\, A_\mu(\lambda_2\beta_2)\, A_\nu(\lambda_1\beta_1)\,|0\rangle \sim I_0^{(EIK)} \; !$$

$$\hookrightarrow \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot(\lambda_2\beta_2-\lambda_1\beta_1)}}{k^2+i\eta}$$

We have shown that soft gluons factorize from a hard LINE. In fact we need them to factorize from a JET of parallel moving hard lines ...

- physically clear : a long-wavelength gluon only sees the total color charge and direction of the jet.
- diagrammatically difficult : one shows that soft gluons couple to hard lines in the + direction only with the − component of both their momentum AND polarization. Thus they are LONGITUDINAL and decouple by Ward identities.

  NOTE : special analysis required for initial state jets and Glauber gluons!
- The same is elegantly achieved in SCET by a redefinition of collinear fields by a Wilson line (modulo Glauber).

c) Collinear diagrammatics

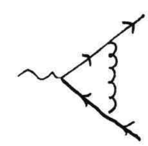
Consider again the form factor, with kinematics

$$p_1^\mu = (p_1^+, 0, \vec{0}_\perp) \;, \qquad p_2^\mu = (0, p_2^-, \vec{0}_\perp) \;, \qquad k^\mu = (k^+, k^-, \vec{k}_\perp)$$

in the collinear region $k^+ \gg (k^-, \vec{k}_\perp)$

The numerator simplifies to

$$\bar{u}(p_1)\, \gamma_\sigma\, (\not{p}_1 - \not{k})\, \gamma_\mu\, (\not{p}_2 + \not{k})\, \gamma^\sigma\, v(p_2) \sim$$

()

$$\sim \bar{u}(p_1)\, \gamma^+\, (\not{p}_1 - \not{k})\, \gamma_\mu\, (\not{p}_2 + \not{k})\, \gamma^-\, v(p_2) =$$

$(\gamma^+ v(p_2) = \bar{u}(p_1)\gamma^- = 0 \ldots)$

$$= \underbrace{\frac{1}{k^+}}_{\hookrightarrow\, k\cdot\beta_2}\, \bar{u}(p_1)\, \underbrace{\gamma^+}_{\hookrightarrow\, \not{\beta}_2}\, (\not{p}_1 - \not{k})\, \gamma_\mu\, (\not{p}_2 + \not{k})\, \not{k}\, v(p_2)$$

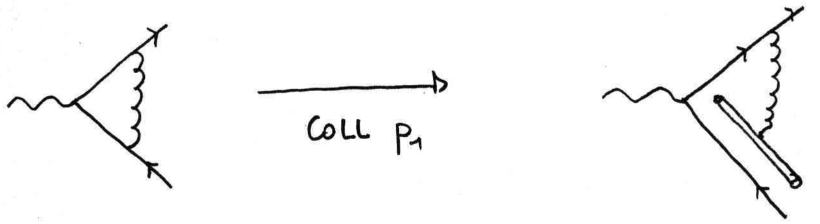
$\rightarrow$ "Ward identity" $\quad \not{k} = (\not{p}_2 + \not{k}) - \not{p}_2$

$\hookrightarrow$ kills denominator; $\not{p}_2$ $\rightarrow$ vanishes on $v(p_2)$

$$\Rightarrow \quad \Gamma_\mu^{(\mathrm{COLL})} \propto \int \frac{d^d k}{(2\pi)^d}\, \frac{\beta_2^\sigma}{k\cdot\beta_2}\, \frac{\bar{u}(p_1)\, \gamma_\sigma\, (\not{p}_1 - \not{k})\, \gamma_\mu\, v(p_2)}{k^2\, (p_1 - k)^2}$$

$\Rightarrow$ The GLUON-ANTIQUARK coupling EIKONALIZES (but NOT the gluon-quark one!)

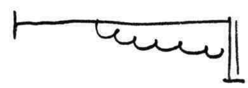
Graphically



Once again, this generalizes to higher orders thanks to eikonal and Ward identities. In order to avoid the collinear div's. associated with the p_2 line (and isolate the ones we need on the p_1 line) it is sufficient to tilt the eikonal line off the light cone, and define

$$J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\ \alpha_s(\mu^2),\ \varepsilon\right) \equiv \langle 0|\, \Phi_n(\infty,0)\, \psi(0)\, |p\rangle$$

and similarly for antiquarks and initial states

(here, say $J^{(1)} =$  $+ \ \dots$).

We now have all the building blocks needed for the full factorization of fixed-angle amplitudes in QCD!