String techniques for perturbative calculations in field theory

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Abstract

String theory provides a surprisingly efficient tool for the computation of scattering amplitudes, correlation function and effective actions in a variety of field theories, including QCD. This is illustrated by describing briefly the computation of multiloop bosonic string amplitudes, and then taking different field theory limits. Examples include one–loop gluon amplitudes, two–loop scalar amplitudes, and Euler-Heisenberg effective actions.

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Introduction

- Motivations
  - Difficulty in computing high-order gauge theory amplitude
  - Further understanding of string amplitudes and their generalizations
- Bosonic string perturbation theory
  - The Schottky representation of Riemann surfaces
  - Master formulas for gluons and scalars
  - Advantages and limitations of string amplitudes
- One-loop amplitudes
  - One-loop off-shell gluon master formula
  - The field theory limit: matching conditions, Schwinger parameters, power counting
  - Diagrammatics
- Two-loop amplitudes
  - Two-loop string moduli space
  - Scalar amplitudes: matching conditions, two-loop example
- Euler-Heisenberg effective actions
  - Master formula in a constant gauge field
  - Two-loop Euler-Heisenberg action for scalars
  - The two-loop field theory limit
- Outlook

The Sickness

The computation of perturbative gauge theory amplitudes may appear straightforward ... however: conventional methods become intractable beyond about $\mathcal{O}(g^6)$ ($\mathcal{O}(\alpha^6)$ for cross sections).

- Tree-level symptoms
  For gluon scattering, measuring the size in "number of terms",

$$\sigma\,(2 \to 6)_{\text{tree}} \sim (34300 \cdot 6^6)^2 \sim 2.6 \cdot 10^{18}\ .$$

- One-loop symptoms
  - Each "term" must now be integrated over loop momentum (reduce tensor integrals, evaluate scalar integrals).
  - Spurious kinematic singularities cancel only when results are recombined with a common denominator.
  - One-loop results must be analytically combined with tree-level results to cancel IR singularities.
- Two-loop symptoms
  - Scalar integrals are highly nontrivial.
  - Cancellation of IR singularities is more intricate (must combine with one-loop and tree-level results).
- Harsh consequences if untreated
  Precision physics at colliders requires NNLO calculations.

The Treatments

A variety of techniques have been developed by many authors over the past 15 years to tackle these calculations. They involve

- Decomposing the amplitudes into "basic building blocks" (subamplitudes), by fixing all quantum numbers of external particles (Total Quantum Number Management).
- Exploiting symmetries (Bose, C, P, Gauge, Super, ...) to reduce the number of subamplitudes needed.
- Exploiting special features of perturbation theory (unitarity, factorization in appropriate kinematic limits).

Some specific examples:

- Color Decomposition

  Choosing a basis in color space splits the amplitude into gauge-invariant subamplitudes. At tree-level, for gluons

$$\begin{aligned} \mathcal{A}^{\text{tree}}(1,\ldots,N) &= g^{N-2} \sum_{\sigma \in S_N/Z_N} \text{Tr}\left(\lambda^{\sigma(1)} \ldots \lambda^{\sigma(N)}\right) \\ &\times A_\sigma^{\text{tree}}\left(\sigma(1),\ldots,\sigma(N)\right) , \end{aligned}$$

  Note: This decomposition is string-inspired (Chan-Paton factors) and generalizes to g-loop amplitudes. Subamplitudes can be computed using color-ordered Feynman rules.

- Helicity Amplitudes
  Vast simplifications are achieved by fixing the helicities of external particles, and picking polarization vectors to take maximal advantage of gauge invariance. Typically

$$\epsilon_\mu^+(k;q) = \frac{< q^- | \gamma_\mu | k^- >}{\sqrt{2} < q^- | k^+ >} .$$

  – The reference momentum q is a gauge parameter: it can be picked to maximize the number of vanishing products $\epsilon_i \cdot \epsilon_j$ and $\epsilon_i \cdot k_j$.
  – String theory amplitudes are expressed so that helicity methods can be easily implemented also at loop level.
- Implementation of Symmetries
  – Charge conjugation:

$$A_\sigma^{\rm tree}(1, \ldots, N) = (-1)^N A_\sigma^{\rm tree}(N, \ldots, 1)$$

  – Parity and cyclic symmetry connect partial amplitudes with oppposite helicities and different particle orderings.
  – Supersymmetric Ward identities can be employed directly at tree level and induce useful decompositions into spin multiplets at loop level.
- Unitarity and Factorization
  – Cutkosky rules give the absorptive parts of loop amplitudes in terms of products of tree amplitudes.
  – Factorization in terms of universal splitting functions in the IR/collinear limits provides checks and constraints.

Recent Developments

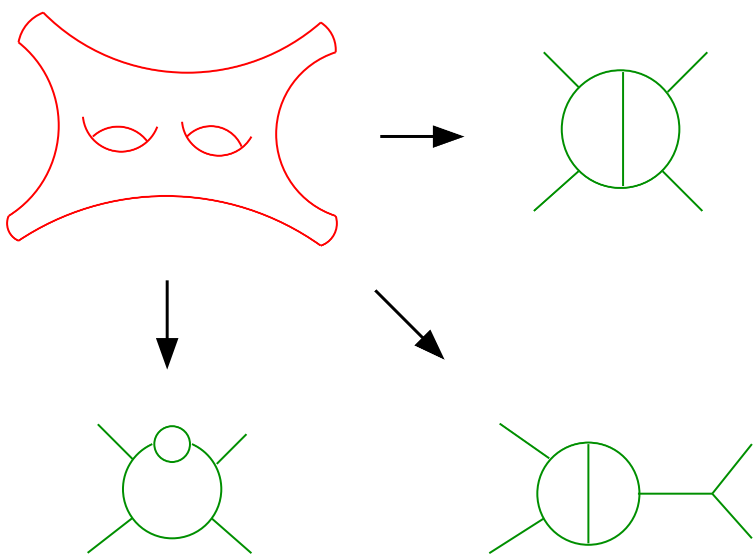
The connection between gauge theories and string theory is now a focus of great interest both in view of formal developments and of phenomenological applications.

- Gauge-Gravity correspondence
  - After the Maldacena conjecture ($\mathcal{N} = 4$ super Yang-Mills $\Leftrightarrow$ strings on $\mathrm{AdS}_5 \otimes S_5$), surprising links between gravity and $\mathcal{N} = 2, 1$ SUSY gauge theories emerged (Di Vecchia).
  - The field theory limit of string amplitudes including D-branes yields nonperturbative information on gauge theories (instanton effects, moduli space) (Billò).
  - A subtle pattern of exponentiation has emerged in $\mathcal{N} = 4$ super Yang-Mills amplitudes (Bern).
- Twistor techniques
  - String theory inspired powerful techniques based on analiticity properties to compute gluon amplitudes (Witten).
  - "Twistor" techniques have lead to recursion relations at tree level and one loop (Britto).
  - Generalizations to $\mathcal{N} = 0$, scalars, fermions are of immediate relevance to collider phenomenology (Bern).

$\Rightarrow$ Controlling precisely the field theory limit is important. New surprises may be forthcoming ...

The uses of string theory

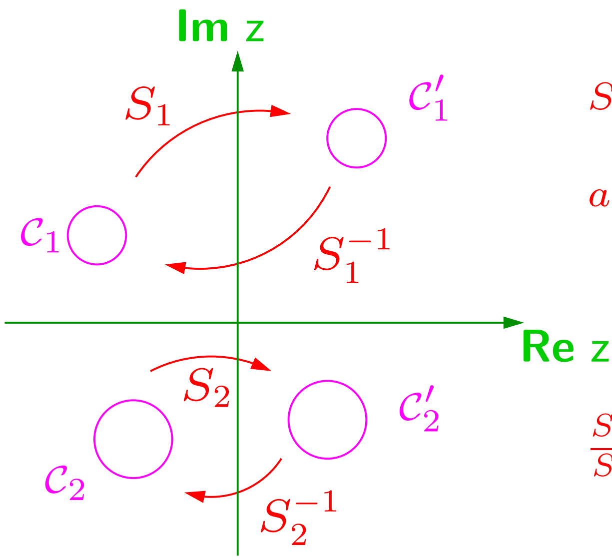
- String theory expresses on-shell scattering amplitudes of a d-dimensional field theory in terms of correlation functions of operators of a two-dimensional free field theory.
- String theory is first-quantised: the number of string loops is fixed at the outset ($2d$ field theory on a Riemann surface of genus g).
- String theory has an infinite number of massive states with masses $M_n^2 \propto n/\alpha' \propto nT$. Tuning the limit $\alpha' \to 0$ for different strings one may get different effective field theories, including scalar, gravity, gauge and SUSY gauge theories.
- In the field theory limit ($\alpha' \to 0$) the Riemann surface degenerates into a set of Feynman-like graphs.



- Is it practical?

Schottky representation of Riemann surfaces

- A Riemann surface of genus g can be represented by cutting and identifying g pairs of circles on the Riemann sphere, via projective transformations.



$$S_i(z) = \frac{a_i z + b_i}{c_i z + d_i}$$

$$a_i d_i - b_i c_i = 1$$

$$\frac{S_i(z) - \eta_i}{S_i(z) - \xi_i} = k_i \frac{z - \eta_i}{z - \xi_i}$$

$$S_i(\mathcal{C}_i) = \mathcal{C}'_i$$

The Riemann surface is then $\Sigma_g = (\mathbf{C} \cup \infty) / \mathcal{S}_g$, where $\mathcal{S}_g$ is the genus g Schottky group generated by the projective transformations S_i.

- $\{\eta_i, \xi_i\}$ are fixed points of the trasformation S_i.
- The multipliers k_i are proportional to the radii of the circles $\mathcal{C}_i$, and they drive the field theory limit, $k_i \to 0$.
- The shape of the genus g Riemann surface is determined by $3\mathrm{g} - 3$ (complex) moduli (subtracting one overall projective trasformation on the sphere).

Geometric objects

The string operator formalism provides explicit constructions for geometric objects defined on Riemann surfaces, in terms of series defined on the Schottky group.

- Abelian differentials

$$\omega_\mu = \sum_\alpha^{(\mu)} \left(\frac{1}{z - T_\alpha(\eta_\mu)} - \frac{1}{z - T_\alpha(\xi_\mu)} \right) dz$$

- Period matrix

$$\tau_{\mu\nu} = \frac{1}{2\pi \mathrm{i}} \int_{b_\nu} \omega_\mu(z)$$

- Prime form

$$E_g(z,w)\sqrt{dz dw} = (z-w) \hat{\prod}_\alpha \frac{z - T_\alpha(w)}{z - T_\alpha(z)} \frac{w - T_\alpha(z)}{w - T_\alpha(w)}$$

- Bosonic Green function

$$\mathcal{G}_g(z,w) = \log \left[E_g(z,w) \right] - \frac{1}{2} \int_z^w \omega_\mu \left[(2\pi \ \mathbf{Im}\tau)^{-1} \right]^{\mu\nu} \int_z^w \omega_\nu$$

- $T_\alpha = S_i^a \cdot S_j^b \cdot \ldots$ are elements of the Schottky group.
- In the field theory limit $k_i \to 0$ only a handful contribute.
- The relevant terms are easily generated with available software for symbolic manipulations.

Master Formulas

String amplitudes are computed by fixing the quantum numbers of the external states and then evaluating correlation functions of the corresponding vertex operators in the $2d$ theory at the relevant genus. Using open bosonic strings one finds

- For scalar states (tachyons in the adjoint of $U(N)$)

$$A^{(\mathrm{g})}_{M,0}(p_1,\ldots,p_N) = C_{\mathrm{g}}\mathcal{N}^M \int [dm]^M_{\mathrm{g}} \prod_{i<j} \exp\left[2\alpha' p_i \cdot p_j G_{\mathrm{g}}(z_i,z_j)\right]$$

- For vector states (gluons in the adjoint of $U(N)$)

$$A^{(\mathrm{g})}_{M,1}(\epsilon_1,p_1;\ldots;\epsilon_N,p_N) = C_{\mathrm{g}}\mathcal{N}^M \int [dm]^M_{\mathrm{g}} \prod_{i<j} \exp\left[2\alpha' p_i \cdot p_j G_{\mathrm{g}}(z_i,z_j)\right]$$

$$\times \left\{ \exp\left[\sum_{i\neq j} \sqrt{2\alpha'}\epsilon_i \cdot p_j \partial_{z_i} G_{\mathrm{g}}(z_i,z_j) + \frac{1}{2}\sum_{i\neq j} \epsilon_i \cdot \epsilon_j \partial_{z_i}\partial_{z_j} G_{\mathrm{g}}(z_i,z_j)\right]\right\}_{m.l.}$$

- Normalizations:
  $C_{\mathrm{g}} = (2\pi)^{-d\mathrm{g}}(2\alpha')^{-d/2} g_S^{2\mathrm{g}-2}$; $\mathcal{N} = 2g_S(2\alpha')^{d/4-1/2}$.

- The mass-shell condition can be relaxed by introducing

$$G_{\mathrm{g}}(z_i,z_j) = \mathcal{G}_g(z_i,z_j) - \frac{1}{2}\log V_i'(0) - \frac{1}{2}\log V_j'(0)$$

- The projective trasformations $V_i(z)$ are associated with external states and play the role of local coordinates around the punctures of the surface, $V_i(0) = z_i$.

Pluses and Minuses

Remarkably ...

- These formulas exist.
  (no such thing in field theory ...)
- Total quantum number management is (largely) built in.
  - Color decomposition is already performed: color is factorized into Chan-Paton factors.
  - Loop momentum integration is already performed: helicity methods are applicable from the beginning at all loops.
- The field theory limit yields Feynman-like diagrams summing up the contributions of gluons and ghosts.
- Off-shell continuation is possible. The gauge chosen by string theory can then be identified; applications to currents and recursion relations can be envisaged.
- The method is algorithmically implementable.

However ...

- The technique is suited for a limited set of problems: pure gauge theories, gravity, SUSY.
  - Non-supersymmetric fermions are difficult to include
  - Theories with several mass scales (SM, MSSM, ...) cannot be handled
- The problem is reduced to the computation of "scalar integrals with numerators". Techniques to evaluate these must come from elsewhere.

One Loop: Gluons

- One-loop gluon master formula

$$A^{(1)}_{M,1}(\epsilon_1, p_1; \ldots ; \epsilon_M, p_M) = \frac{g_d^M}{(4\pi)^{d/2}} (2\alpha')^{\frac{M-d}{2}} \int_0^1 \frac{dk}{k^2} \prod_{n=1}^{\infty} (1-k^n)^{2-d}$$

$$\times \int_k^1 dz_M \int_{z_M}^1 dz_{M-1} \cdots \int_{z_3}^1 dz_2 \left(-\frac{\log k}{2}\right)^{-\frac{d}{2}} \prod_{i<j} \left(\exp\left[G(z_i, z_j)\right]\right)^{2\alpha' p_i \cdot p_j}$$

$$\times \left\{ \exp\left[\sum_{i \neq j} \left(\sqrt{2\alpha'}\, p_j \cdot \epsilon_i \, \partial_i G(z_i, z_j) + \frac{1}{2} \epsilon_i \cdot \epsilon_j \, \partial_i \partial_j G(z_i, z_j)\right)\right]\right\}_{\text{m.l.}}$$

- Matching condition (from tree-level): $g_S = g_d \, (2\alpha')^{1-d/4}/2$
- One-loop Green function:

$$\begin{aligned} G(z_i, z_j) &= \log\left(\left|\sqrt{\frac{z_i}{z_j}} - \sqrt{\frac{z_j}{z_i}}\right|\right) + \frac{1}{2\log k}\left(\log\frac{z_i}{z_j}\right)^2 \\ &+ \log\left[\prod_{n=1}^{\infty} \frac{\left(1 - k^n \frac{z_j}{z_i}\right)\left(1 - k^n \frac{z_i}{z_j}\right)}{(1-k^n)^2}\right] \end{aligned}$$

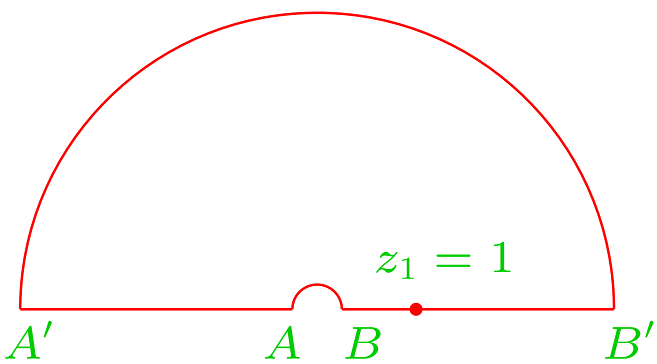
- $V_i'(0) = (\omega(z_i))^{-1} = z_i$, from geometry
- Modular invariance:
  $G(z_i/z_j; k) = G(z_j/z_i; k) = G(k z_i/z_j; k)$

The annulus

At one loop, one can choose

$$\eta = 0 \quad ; \quad \xi = \infty \quad ; \quad z_1 = 1$$

The resulting Schottky representation of the annulus is



where

$$B = -A = \sqrt{k} \quad ; \quad B' = -A' = 1/\sqrt{k}$$

The region of integration is determined by

- Symmetry considerations: for example the transformation $k \longrightarrow 1/k$ does not change the geometry.
- Cyclic ordering of the punctures z_i in accordance with the chosen color ordering.
- Mapping the region $1 < z_i < 1/\sqrt{k}$ onto $k < z_i < \sqrt{k}$ by modular invariance.

The field theory limit

- From the string operator formalism we know that Laurent expansion of the integrand in powers of k counts the mass level of the state propagating in the loop.
  $k^{-2} \longrightarrow$ tachyon ; $k^{-1} \longrightarrow$ gluon ...
- The master formula has an overall power of α'. String moduli defining the shape of the surface must be expressed in units of α' in order to take the limit $\alpha' \to 0$
  Hint: measure of integration is $d \log k$.
- Pedestrian field theory limit (exact for scalars):

$$\log k = -\frac{t}{\alpha'} \quad ; \quad \log z_i = -\frac{t_i}{\alpha'}$$

  Note: t and t_i will be identified with with sums of Schwinger parameters associated with propagators around the loop.
- α' power counting.
  For gluons, the overall power p of α' after the change of variables is not uniform: $-M/2 < p < 0$. One must locate all further sources of positive powers of α'.
  - Four-point vertices: $(t_i - t_{i-1})/\alpha' = \mathcal{O}(1)$.
  - Expansion of the exponential:

$$\exp(2\alpha' p_i \cdot p_j G_{ij}) \to \exp\Big(c_0(t_i) + \alpha' c_1(t_i)\Big)$$

Diagrammatics

- Diagrams with cubic vertices
  Change variables to $\{t, t_i\}$, expand the exponential to the required power, expand the resulting integrand in powers of $\{k, z_i\}$, isolate terms independent of α'.
  Result: Schwinger parameter integrand of the corresponding Yang-Mills diagram (gluons + ghosts).

- Diagrams with quartic vertices
  - For each deleted propagator, between punctures z_i and z_{i+1}, set $z_i = \exp(-t_i/\alpha')$ and $z_{i+1} = y_i z_i$, integrate over y_i in a neighbourhood of $y_i = 1$.
  - Note: regularization is required for singularities of the form $\int_0 dx/x^2$. It is provided by analytic continuation, retaining subleading α' dependence.

- Reducible diagrams
  Take the limit $z_i \to z_{i+1}$ before the limit $k \to 0$. Then

$$G(z_i, z_{i+1}) \to \log(z_i - z_{i+1})$$

  Integrate over z_{i+1}, isolate poles of the form $(n - \alpha' s_{i,i+1})^{-1}$. They correspond to the propagation of the $(n+1) - \text{th}$ string state in the "pinched channel".

- Nonplanar diagrams
  Arise when punctures z_i are inserted on both boundaries of the annulus. They reduce to combinations of planar amplitudes with specified coefficients.

Results

- Two algorithms exist to compute on-shell scattering amplitudes. Four- and five-gluon amplitudes have been computed directly from string theory.
- Off-shell continuation, with identification of individual diagrams, identifies the field-theory gauge automatically selected by string theory.
  - Irreducible diagrams: Background Field method, with Feynman gauge chosen for the quantum fields in the loop.
  - Reducible diagrams: Gervais-Neveu gauge for the classical field on tree subdiagram.

$$S_{GN} = \int d^d x \left\{ -\frac{1}{4} \mathrm{Tr} \left(F_{\mu\nu}^2 \right) - \frac{1}{2} \mathrm{Tr} \left[\left(\partial \cdot A - \mathrm{i} g_d A^2 \right)^2 \right] \right\}$$

- Bosonic string theory is well-defined only in the critical dimension $d = 26$. No consequence in the field theory limit: amplitudes have the correct d dependence (dimensional regularization *à la* 't Hooft-Veltman).
- Bosonic string theory has a tachyon. It can be decoupled by hand. Tachyons in loops have IR divergences *not* regulated dimensionally. Tachyon effects remain as contact interactions. Tachyon amplitudes can be used to compute scalar amplitudes in field theory by the replacement $dx/x^2 \to [\exp(\alpha' m^2 \log x)]\, dx/x$.

Two-loop: scalars

- Projective gauge choice

$$\eta_1 = 0 \quad ; \quad \xi_1 = \infty \quad ; \quad \xi_2 = 1$$

- Matching condition (from tree-level)

$$g_S = g_3 \, (2\alpha')^{(6-d)/4}/4$$

- Two-loop Green function (in the limit $k_i \to 0$)

$$\begin{aligned} \mathcal{G}_2(z_i, z_j) &= \log\left(\left|z_i - z_j\right|\right) \\ &+ \frac{1}{2}\frac{\log k_1 \log k_2 - \log^2 S}{\log^2 T \log k_2 + \log^2 U \log k_1 - 2\log T \log U \log S} \end{aligned}$$

In this projective gauge

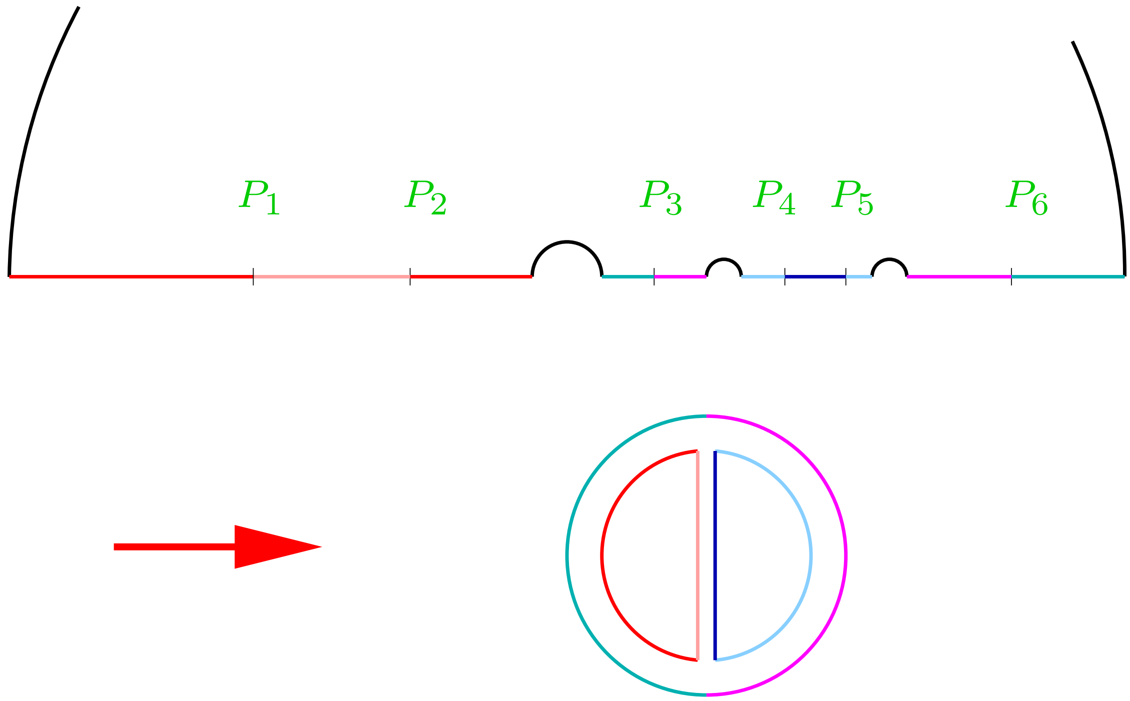
$$S = \eta_2 \quad ; \quad T = \frac{z_i}{z_j} \quad ; \quad U = \frac{(z_j - \eta_2)(z_i - 1)}{(z_i - \eta_2)(z_j - 1)}$$

- $V_i'(0) = \left(a_1\omega_1(z_i) + a_2\omega_2(z_i)\right)^{-1}$, with coefficients picked according to the boundary where the puncture is inserted.

- Modular invariance

The mapping $z \to (z - \eta_2)\,/\,(z - 1)$ maps the two inner boundaries of the two-annulus into each other.

The two-annulus

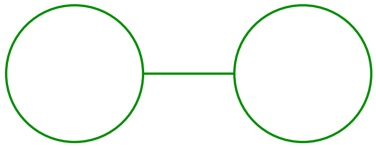
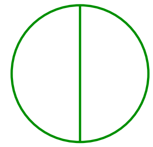


$$P_1 = -1 \quad ; \quad P_2 = -\eta_2 \quad ; \quad P_3 = \eta_2/\beta$$

$$P_4 = \frac{2\eta_2}{1+\eta_2} \quad ; \quad P_5 = \frac{1+\eta_2}{2} \quad ; \quad P_6 = \beta > 1$$

- It is possible to identify precisely on which propagator and on which boundary the punctures are inserted
- "Broken propagators" yield different expressions in different segments, but the results are related by modular transformations, providing highly nontrivial checks on the field theory limit.

Two-loop scalar vacuum bubbles

$D_1 =$  $D_2 =$ 

- Leading regions in the field theory limit: $k_1,\ k_2 \to 0$. Furthermore

$$\eta_1 \to 1 \quad ; \quad \text{or} \quad \eta_1 \to 0$$

suggested by sewing procedure, integration region and measure. η_1 plays the role of 'distance between loops'.

- Schwinger parameters:
  - $\eta_1 \to 1$: $\quad k_i = \mathrm{e}^{-t_i/\alpha'} \quad ; \quad 1 - \eta_1 = \mathrm{e}^{-t_3/\alpha'}$.
  - $\eta_1 \to 0$: $\quad q_i \equiv \frac{k_i}{\eta_1} = \mathrm{e}^{-t_i/\alpha'} \quad ; \quad q_3 \equiv \eta_1 = \mathrm{e}^{-t_3/\alpha'}$.

- Master formula: two-loop Green function does not appear. Change variables to Schwinger parameters, regulate tachyon poles ($dx/x^2 \to [\exp(\alpha' m^2 \log x)]\, dx/x$), get

$$D_1 = \frac{N^3}{(4\pi)^d} \frac{g^2}{32} \int_0^\infty dt_3 \int_0^\infty dt_2 \int_0^{t_2} dt_1 \mathrm{e}^{-m^2(t_1+t_2+t_3)} (t_1 t_2)^{-d/2}$$

$$D_2 = \frac{N^3}{(4\pi)^d} \frac{g^2}{32} \int_0^\infty dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \mathrm{e}^{-m^2(t_1+t_2+t_3)} \times (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-d/2}$$

Agrees with field theory, color and symmetry factors included.

Euler-Heisenberg effective action

Field theory

Consider a scalar field in the adjoint of $U(N)$, coupled to a constant background field strength $\mathcal{F}_{\mu\nu}$.

$$\mathcal{L} = \mathrm{Tr}\left[D_\mu \Phi D^\mu \Phi - m^2 \Phi^2 + \frac{2}{3}\lambda\, \Phi^3\right] .$$

Take $\mathcal{A}^\mu_{AB} = A^\mu\, \delta_{A,N}\delta_{B,N}$, with $A_\mu = B\, x_1\, g_{\mu 2}$, corresponding to a constant color magnetic field. The matrix Φ decomposes as

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\,\Pi(x) & \boldsymbol{\xi}(x) \\ \boldsymbol{\xi}^\dagger(x) & \sigma(x) \end{pmatrix}$$

Only the $U(N-1)$ vector $\boldsymbol{\xi}$ is charged under the background field.

$$\begin{aligned} \mathcal{L} \quad = \quad & \mathrm{Tr}\left[\partial_\mu \Pi\, \partial^\mu \Pi\right] + \frac{1}{2}\partial_\mu \sigma\, \partial^\mu \sigma + D_\mu \boldsymbol{\xi}^\dagger D^\mu \boldsymbol{\xi} \\ & - m^2\, \mathrm{Tr}\left(\Pi^2\right) - \frac{1}{2} m^2 \sigma^2 - m^2 \boldsymbol{\xi}^\dagger \boldsymbol{\xi} \\ & + \frac{2}{3}\lambda\, \mathrm{Tr}\left(\Pi^3\right) + \sqrt{2}\,\frac{\lambda}{6}\sigma^3 + \frac{\lambda}{\sqrt{2}}\sigma \boldsymbol{\xi}^\dagger \boldsymbol{\xi} + \lambda\, \boldsymbol{\xi}^\dagger \Pi\, \boldsymbol{\xi} \; . \end{aligned}$$

The charged propagator can be exactly computed.

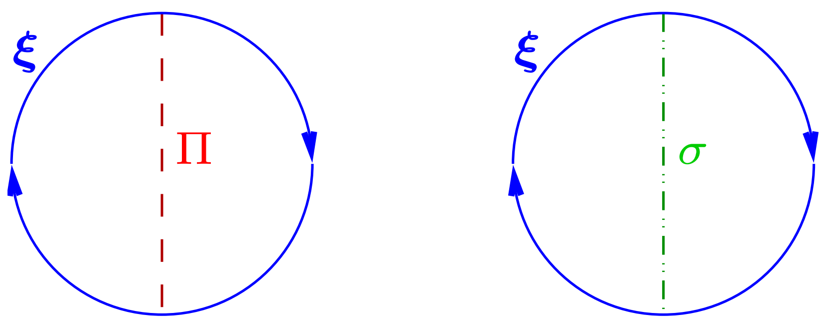
$$G_\xi(x,y) = \frac{\mathrm{e}^{-\mathrm{i}B(x_1+y_1)(x_2-y_2)/2}}{(4\pi)^{d/2}} \int_0^\infty dt\, \mathrm{e}^{-tm^2}\, t^{-d/2+1}\, \frac{B}{\sinh(Bt)}$$

$$\exp\left[\frac{(x_0-y_0)^2 - (\mathbf{x}_\perp - \mathbf{y}_\perp)^2}{4\,t} - \frac{B\,(x_1-y_1)^2 - (x_2-y_2)^2}{4\tanh(Bt)}\right] .$$

Euler-Heisenberg effective action

Two-loop diagrams

The two-loop effective action receives contributions from



Charged diagrams are easily evaluated in coordinate space

$$W^{(2)}_{\xi\Pi}(m,B) = -\lambda^2 \frac{(N-1)^2}{4} \int d^dx\, d^dy\, G_\xi\,(x,y)\, G_\xi\,(y,x)\, G_\Pi\,(x,y)$$

One finds a Schwinger parameter representation

$$\begin{aligned} W^{(2)}_{\xi\Pi}(m,B) &= -\mathrm{i}\, V_d \frac{\lambda^2}{(4\pi)^d} \frac{(N-1)^2}{4} \\ &\times \int_0^\infty dt_1 dt_2 dt_3\; \mathrm{e}^{-m^2(t_1+t_2+t_3)}\; \Delta_0^{-\frac{d}{2}+1}\, \Delta_B^{-1} \end{aligned}$$

with the B dependence encoded in the factor Δ_B,

$$\begin{aligned} \Delta_0 &= t_1 t_2 + t_1 t_3 + t_2 t_3\ , \\ \Delta_B &= \frac{1}{B^2} \sinh(Bt_2) \sinh(Bt_3) + \frac{t_1}{B} \sinh\left[B\,(t_2+t_3)\right]\ . \end{aligned}$$

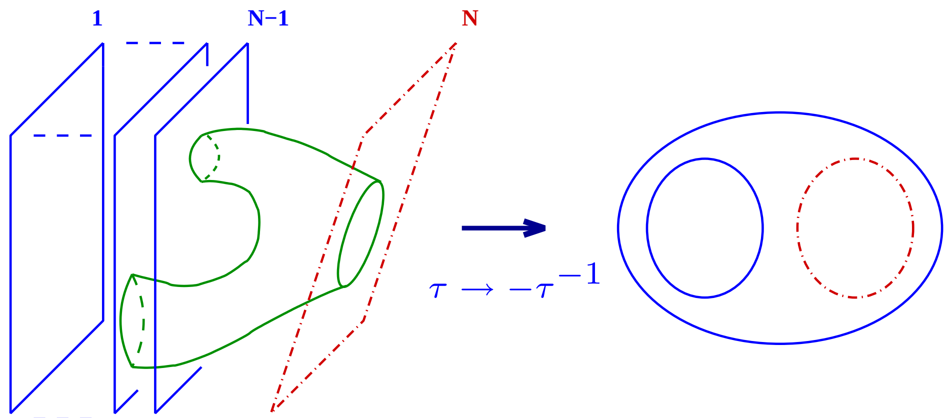
Euler-Heisenberg effective action

String theory

A constant gauge field affects the bosonic string only thorugh boundary conditions.

$$\left[\partial_\sigma X^i + \mathrm{i}\, \partial_\tau X^j F_j{}^{i\,(A)}\right]_{\sigma=0} = 0 \; ,$$

The field-string interaction admits a dual description.



A master formula for the g-loop effective action can be derived.

$$Z_\epsilon(g) = \frac{\mathrm{e}^{2\pi \mathrm{i} \epsilon_g} - 1}{\prod_{\mu=1}^{g} \cos \pi \epsilon_\mu} C_g \int [dm]_g^0 \; \mathrm{e}^{-\mathrm{i}\pi \vec{\epsilon} \cdot \tau \cdot \vec{\epsilon}} \; \frac{\det(\tau)}{\det(\tau_{\vec{\epsilon}})} \; \mathcal{R}_g\left(k_\alpha, \vec{\epsilon} \cdot \tau\right)$$

with $F_{12}^{(A)} = -F_{21}^{(A)} = \tan(\pi \epsilon^A)$. All ingredients can be computed explicitly in the Schottky representation. For example

$$\mathcal{R}_g\left(k_\alpha, \vec{\epsilon}\right) = \frac{\prod_\alpha{}' \prod_{n=1}^{\infty} (1 - k_\alpha^n)^2}{\prod_\alpha{}' \prod_{n=1}^{\infty} \left(1 - \mathrm{e}^{-2\pi \mathrm{i} \vec{\epsilon} \cdot \vec{N}_\alpha} k_\alpha^n\right) \left(1 - \mathrm{e}^{2\pi \mathrm{i} \vec{\epsilon} \cdot \vec{N}_\alpha} k_\alpha^n\right)} \; .$$

Euler-Heisenberg effective action

Field theory limit

The field theory limit is defined by fixing the dimensionful physical field B (note $F_{ij}^{(A)}$ is dimensionless).

$$\tan(\pi\epsilon) = 2\pi\alpha' B \ .$$

The matrix $\tau_{\vec{\epsilon}}$ encodes the leading $\vec{\epsilon}$ dependence. It is the period matrix of a set of twisted abelian differentials.

$$\begin{aligned}
(\tau_{\vec{\epsilon}})_{\nu\mu} &= \frac{1}{2\pi \mathrm{i}} \int_{w}^{S_\nu(w)} dz \left[\zeta_\mu^{\vec{\epsilon}\cdot\tau}(z) \mathrm{e}^{\frac{2\pi\mathrm{i}}{g-1}\vec{\epsilon}\cdot\vec{\Delta}_z} \right] , \quad (\nu \neq g) \ , \\
(\tau_{\vec{\epsilon}})_{g\mu} &= \mathrm{e}^{2\pi\mathrm{i}(\vec{\epsilon}\cdot\tau)_\mu} - 1 \ ,
\end{aligned}$$

The twisted differentials $\zeta_\mu^{\vec{\epsilon}}(z)$ obey twisted boundary conditions $\zeta_\mu^{\vec{\epsilon}}\left(S_\nu(z)\right) dS_\nu(z) = \exp\left(2\pi\mathrm{i}\epsilon_\nu\right) \zeta_\mu^{\vec{\epsilon}}(z) dz$. As $\alpha' \to 0$ one finds

$$\det \tau_{\vec{\epsilon}} \to \frac{B}{\mathrm{i}\pi\alpha'} \Delta_B \ ,$$

One can now compute individual diagrams, for example

$$\begin{aligned}
W_{\mathrm{st}}^{(2)}(m, B) &= V_d \frac{\lambda^2}{(4\pi)^d} \frac{(N-1)^2}{2} \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \\
&\quad \times \mathrm{e}^{-m^2(t_1+t_2+t_3)} \Delta_0^{-d/2+1} \Delta_B^{-1} \ .
\end{aligned}$$

Upon symmetrization one recovers the field theory result, with symmetry and color factors acquiring a string interpretation.

Outlook

- String theory provides a powerful and efficient method for the calculation of scattering amplitudes and effectiveactions in a broad class of field theories.
- The field theory limit can be taken in a controlled way: it selects boundaries of string moduli space corresponding to particle graphs of given topology.
- "Master formulas" exist for amplitudes with any number of legs and loops. They yield directly the Feynman parameter integrand of individual diagrams, in dimensional regularization.
- Off-shell continuation is possible, exploiting geometric features of Riemann surfaces.
- Organization of gauge theory amplitudes automatically takes maximal advantage of gauge invariance: color decomposition and helicity methods are implemented, and a subtle field-theory gauge is picked.
- Limitations: difficult to include non-SUSY fermions, not suited for theories with multiple mass scales, generalized scalar integrals still to be computed.
- At one loop, two algorithms exist to compute gluon amplitudes directly from the string master formula. The four- and five-gluon amplitudes have been computed. The major stumbling block to the inclusion of more gluons is the computation ofgeneralized scalar integrals.

- At two loops.
  - The field theory limit of string moduli space is understood, and scalar amplitudes are easily computed.
  - Euler-Heisenberg effective actions for scalar fields can be derived from string theory.
  - Two-loop gluon amplitudes await the understanding of subleading contributions to proper time assignements.

- On the agenda.
  - Further one-loop applications: six gluons, gluon currents, recursion relations?
  - Two-loop gluons?
  - All-order analysis in suitable kinematic limits?