Analytic Resummation and Power Corrections in QCD

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The Landau pole in $d > 4$

QCD resummations (renormalons, threshold,...) typically yield expressions of the form

$$f_a(Q^2) \;=\; \int_0^{Q^2} \frac{dk^2}{k^2} (k^2)^a \alpha_s(k^2) \;,$$

which are ill–defined because of the Landau pole in the running coupling. Expansion in powers of $\alpha_s(Q^2)$ yields

$$f_a(Q^2) \;=\; \sum_{n=0}^{\infty} c_n^{(a)} \left(\alpha_s(Q^2)\right)^n$$
$$\rightarrow \;\; c_n^{(a)} \propto n! \rightarrow \delta f_a(Q^2) \propto \left(\frac{\Lambda^2}{Q^2}\right)^a \;.$$

The ambiguity in the resummed perturbative result signals a nonperturbative power correction.

Consider however using consistently dimensional regularization, with $d = 4 - 2\epsilon > 4 \rightarrow \epsilon < 0$. Then

$$\beta(\epsilon, \alpha_s) \;=\; \mu \frac{\partial \alpha_s}{\partial \mu} = -2\epsilon\alpha_s + \hat{\beta}(\alpha_s) \;,$$
$$\hat{\beta}(\alpha_s) \;=\; -\frac{\alpha_s^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\alpha_s}{\pi}\right)^n \;.$$

The one–loop β–function in $d > 4$ has *two* fixed points: an asymptotically free one at $\alpha_s = -4\pi\epsilon/b_0$, and a Wilson-Fisher fixed point at $\alpha_s = 0$. Thus, at one loop, the running coupling

$$\overline{\alpha}\left(\mu^2\right) = \frac{\alpha_s(\mu_0^2)}{\left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon} - \frac{1}{\epsilon}\left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon}\right)\frac{b_0}{4\pi}\alpha_s(\mu_0^2)} \ ,$$

- *vanishes* as $\mu^2 \to 0$ for $\epsilon < 0$

- has a Landau pole in the complex μ^2–plane at

$$\mu^2 = \Lambda^2 \equiv Q^2\left(1 + \frac{4\pi\epsilon}{b_0\alpha_s(Q^2)}\right)^{-1/\epsilon} \ ,$$

  The pole is *not* on the real axis in the μ^2 plane, *i.e. not* on the integration contour of resummed formulas, provided

$$\epsilon < -b_0\alpha_s(Q^2)/(4\pi) \ .$$

We then expect resummed expressions to be *integrable* for general ϵ: scale integrals will yield *analytic functions* of ϵ and α_s, with a "Landau cut".

A neat example: the quark form factor

The on-shell, timelike, dimensionally regularized electromagnetic quark form factor,

$$\begin{aligned}\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) &= \langle 0|J_\mu(0)|p_1,p_2\rangle \\ &= -\mathrm{i}ee_q\, \overline{v}(p_2)\gamma_\mu u(p_1)\ \Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\ ,\end{aligned}$$

satisfies

$$Q^\mu \Gamma_\mu(p_1,p_2;\mu^2,\epsilon) = 0\ ,$$
$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\epsilon,\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s,\epsilon\right) \ = \ 0\ .$$

Double logarithms of the energy are known to exponentiate (Sudakov, 1956)

$$\Gamma(Q^2)\Big|_{LLA} = \exp\left[-\frac{\alpha_s C_F}{4\pi}\log^2(-Q^2)\right]\ .$$

The ratio of the timelike to the spacelike form factor

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2_{LLA} = \exp\left[\frac{\alpha_s C_F}{2\pi}\pi^2\right]\ ,$$

is phenomenologically relevant, *e.g.* for the Drell–Yan cross section (Parisi, 1980).

Evolution Equation

Summation of *subleading* logarithms requires the techniques of IR/collinear factorization in QCD. One derives the evolution equation (see Collins, 1990)

$$Q^2 \frac{\partial}{\partial Q^2} \log \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] =$$
$$\frac{1}{2} \left[K\left(\epsilon, \alpha_s(\mu^2)\right) + G\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] \ .$$

The functions K and G are characterized by

- $K(\epsilon, \alpha_s)$ is a *pure counterterm*, thus

$$\begin{aligned} K(\epsilon, \alpha_s) &= \sum_{n=1}^{\infty} \frac{K_n(\alpha_s)}{\epsilon^n} \ , \\ K_n(\alpha_s) &= \sum_{m=n}^{\infty} K_n^{(m)} \left(\frac{\alpha_s}{\pi} \right)^m \ . \end{aligned}$$

- $G(x, \alpha_s, \epsilon)$ contains the energy dependence and is *finite* as $\epsilon \to 0$
- K and G renormalize *additively*, to preserve the invariance of Γ,

$$\mu \frac{d}{d\mu} G = -\mu \frac{d}{d\mu} K = \gamma_K(\alpha_s) \ .$$

Solution at $\epsilon < 0$

Dimensional regularization provides the means to construct an explicit solution for the form factor. Recall that

$$\beta(\epsilon, \alpha_s) \;=\; \mu\frac{\partial \alpha_s}{\partial \mu} = -2\epsilon\alpha_s + \widehat{\beta}(\alpha_s) \;,$$

so that, for $\epsilon < 0$

$$\lim_{\mu^2 \to 0} \overline{\alpha}\left(\mu^2, \alpha_s(\mu_0^2), \epsilon\right) \;=\; 0 \;.$$

This establishes the initial condition for the evolution

$$\Gamma\left(0, \alpha_s(\mu^2), \epsilon\right) = 1 \;,$$

In fact note that every perturbative contribution to Γ carries a factor $(\mu^2/(-Q^2))^{m\epsilon}$, $m > 0$. Then (LM, GS 1990)

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\Big[K\left(\epsilon, \alpha_s\right) + \; G\left(-1, \overline{\alpha}\left(\xi^2\right), \epsilon\right) + \frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\overline{\alpha}\left(\lambda^2\right)\right)\Big]\right\}$$

The counterterm function $K(\epsilon, \alpha_s)$

In minimal schemes the counterterm function K has no explicit scale dependence, thus it obeys

$$\beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} K(\epsilon, \alpha_s) = -\gamma_K(\alpha_s) \ .$$

Using the finiteness of the anomalous dimension the RG equation turns into a recursion relation for infrared poles

$$\begin{aligned} \alpha_s \frac{d}{d\alpha_s} K_1(\alpha_s) &= \frac{1}{2} \gamma_K(\alpha_s) \ , \\ \alpha_s \frac{d}{d\alpha_s} K_{n+1}(\alpha_s) &= \frac{1}{2} \widehat{\beta}(\alpha_s) \frac{d}{d\alpha_s} K_n(\alpha_s) \ . \end{aligned}$$

As usual, all counterterms are determined by the residue of the simple pole, which is determined by the anomalous dimension γ_K. It is useful to define the sum of $(\mathrm{next-to})^m$–leading poles as

$$\begin{aligned} K(\epsilon, \alpha_s) &= \sum_{m=0}^{\infty} \mathcal{K}_m(\epsilon, \alpha_s) \ , \\ \mathcal{K}_m(\epsilon, \alpha_s) &= \sum_{n=1}^{\infty} K_n^{(n+m)} \left(\frac{\alpha_s}{\pi} \right)^{n+m} \frac{1}{\epsilon^n} \ . \end{aligned}$$

Solving the recursion relation

One can establish the following facts

- the recursion relation for $K_n^{(m)}$ can be solved including *all* orders in the β function.

- All the resulting series of poles, $\mathcal{K}_m(\epsilon, \alpha_s)$, can be summed.

- Upon summation, $\mathcal{K}_m(\epsilon, \alpha_s)$ is an *analytic* function of α_s and ϵ, *regular* as $\epsilon \to 0$ for $m > 0$. The only singularity at $\epsilon \to 0$ is logarithmic and completely determined by a one loop calculation.

- The finite limits $\mathcal{K}_m(0, \alpha_s), (m > 0)$ can be computed for *all* m in terms of the perturbative coefficients of β and γ_K. They reconstitute a power series in α_s.

At one loop ...

... using

$$K_1^{(m)} = \frac{1}{2m}\gamma_K^{(m)} \ ,$$

and truncating the β function at $\mathcal{O}(\alpha_s^2)$, one readily finds

$$K_n^{(m)} = \frac{1}{2m}\left(-\frac{b_0}{4}\right)^{n-1} \gamma_K^{(m-n+1)} \ ,$$

which is exact for $n = m$. Then

$$\begin{aligned}\mathcal{K}_0(\epsilon, \alpha_s) &= \sum_{n=1}^{\infty} K_n^{(n)} \left(\frac{\alpha_s}{\pi\epsilon}\right)^n \\ &= \frac{2\gamma_K^{(1)}}{b_0} \ln\left(1 + \frac{b_0\alpha_s}{4\pi\epsilon}\right) \ .\end{aligned}$$

At two loops ...

... one can sum next-to-leading poles, obtaining

$$\begin{aligned}\mathcal{K}_1(\epsilon, \alpha_s) &= \sum_{n=1}^{\infty} K_n^{(n+1)} \left(\frac{\alpha_s}{\pi}\right)^{n+1} \frac{1}{\epsilon^n} \\ &= \frac{2\alpha_s}{\pi b_0}\left[\gamma_K^{(2)} - \frac{\gamma_K^{(1)} b_1}{b_0}\right] + \mathcal{O}(\epsilon) \ .\end{aligned}$$

Including *all* loops

Formally, one can perform the resummation including *all* orders in the perturbative expansion of the functions β and γ_K. All the series of poles, $\mathcal{K}_m(\epsilon, \alpha_s)$ sum up to polylogarithms, and the resulting functions have finite, calculable limits as $\epsilon \to 0$. Explicitly

$$K(\epsilon, \alpha_s) = K_{DIV}(\epsilon, \alpha_s) + K_{FIN}(\alpha_s) + \mathcal{O}(\epsilon)$$

where

$$K_{DIV}(\epsilon, \alpha_s) \;=\; \frac{2\gamma_K^{(1)}}{b_0} \ln\left(1 + \frac{b_0 \alpha_s}{4\pi\epsilon}\right) \; .$$

The finite terms are given by

$$K_{FIN}(\alpha_s) \;=\; \frac{2}{b_0} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\alpha_s}{\pi}\right)^m \mathcal{A}_m \; .$$

with the coefficients $\mathcal{A}_m$ given by

$$\mathcal{A}_m = \sum_{p=0}^{m} B_p^{(m)} \gamma_K^{(m+1-p)} \; .$$

and $B_p^{(m)}$ constructed out of the coefficients b_i of the β function.

Analytic resummation of one loop effects

Retaining only one-loop effects in the exponent of Γ, and in the β function, direct integration yields

$$\ln \Gamma\left(\frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = -\frac{2C_F}{b_0}\left\{\frac{1}{\epsilon}\mathrm{Li}_2\left[\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\frac{a(\mu^2)}{a(\mu^2)+\epsilon}\right] - C(\epsilon)\ \ln\left[1-\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\frac{a(\mu^2)}{a(\mu^2)+\epsilon}\right]\right\},$$

where $a(\mu^2) = b_0\alpha_s(\mu^2)/(4\pi)$ and

$$C(\epsilon) = G^{(1)}(-1,\epsilon)/C_F = 3/2 + \mathcal{O}(\epsilon)\ .$$

Since we used the solution of the one-loop RG equation to perform the integral, and Γ is RG-invariant, the above should be independent of μ^2. *It is ...* in fact

$$\log\Gamma\left(\frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \log\Gamma\left(-1, \alpha_s(Q^2), \epsilon\right)$$
$$= -\frac{2C_F}{b_0}\left\{\frac{1}{\epsilon}\mathrm{Li}_2\left[\frac{a(Q^2)}{a(Q^2)+\epsilon}\right] + C(\epsilon)\log\left[1+\frac{a(Q^2)}{\epsilon}\right]\right\}.$$

The quark form factor is expressed in terms of a simple analytic function of the coupling and of ϵ, which is manifestly independent of the renormalization scale, and resums the first two towers of IR/collinear poles.

Features of the resummation

- RG invariance is natural if one *uses the coupling* α_s *as integration variable*

$$\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta(\alpha_s)} = -\frac{d\alpha_s}{\alpha_s}\frac{1}{\epsilon + \frac{b_0 \alpha_s}{4\pi}} \ ,$$

  One observes

  - the Landau singularity as the *new fixed point* of the β function at $\epsilon < 0$;
  - a natural generalization to higher orders in β.

- It is possible to study the "physical" limit $\epsilon \to 0$:

$$\log \Gamma\left(-1, \alpha_s(Q^2), \epsilon\right) = \frac{2C_F}{b_0}\Big[-\frac{\zeta(2)}{\epsilon} + \frac{1}{a(Q^2)} + \left(\frac{1}{a(Q^2)} - \frac{3}{2}\right) \log\left(\frac{a(Q^2)}{\epsilon}\right) + \mathcal{O}(\epsilon, \epsilon \log \epsilon)\Big] \ .$$

  Note that

  - all leading and next-to-leading poles resum to a *single* pole, whose residue receives *no* corrections at two loops;
  - a term generating power behavior as $\epsilon \to 0$ arises in a gauge invariant manner.

Partonic cross sections: threshold resummation in DIS and Drell–Yan

The simplest finite cross section to which this idea can be applied is DIS. Consider the following expression for the Mellin transform of $F_2(x, Q^2/\mu^2, \alpha_s(\mu^2), \epsilon)$, where one resums leading logarithms of N

$$F_2\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = F_2(1) \ \ \exp\left[\frac{C_F}{\pi}\int_0^1 dz \frac{z^{N-1}-1}{1-z} \right.$$
$$\left. \times \int_0^{(1-z)Q^2} \frac{d\xi^2}{\xi^2}\, \bar{\alpha}\left(\frac{\xi^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] .$$

Integration of the running coupling around $\xi^2 = 0$ generates the collinear divergence. It can be factorized by subtracting the resummed parton distribution (CLS)

$$\psi_{MS}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left[\frac{C_F}{\pi}\int_0^1 dz \frac{z^{N-1}-1}{1-z} \right.$$
$$\left. \times \int_0^{Q^2} \frac{d\xi^2}{\xi^2}\, \bar{\alpha}\left(\frac{\xi^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] .$$

The IR and collinear finite, resummed partonic DIS cross section is then defined by

$$\widehat{F}_2\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \frac{F_2\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\psi_{MS}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)} .$$

Using again $d\mu/\mu = d\alpha_s/\beta(\epsilon,\alpha_s)$ one easily performs the inner integrals obtaining the RG invariant expression

$$\widehat{F}_2\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \widehat{F}_2\,(1) \exp\left[-\frac{4\pi C_F}{b_0}\int_0^1 \frac{z^{N-1}-1}{1-z} \times \log\left(\frac{4\pi\epsilon + b_0\ \bar{\alpha}((1-z)Q^2)}{4\pi\epsilon + b_0\ \bar{\alpha}(Q^2)}\right)\right] ,$$

manifestly finite as $\epsilon \to 0$.

As was done for the form factor, the expected power correction can be evaluated by taking the limit $\epsilon \to 0$ with $\alpha_s(Q^2)$ fixed. One is lead to

$$\begin{aligned} L_2\left(N, \alpha_s(Q^2)\right) &\equiv \log\left[\frac{\widehat{F}_2\left(N, 1, \alpha_s(Q^2), 0\right)}{\widehat{F}_2\,(1)}\right] \\ &= -\frac{4\pi C_F}{b_0}\sum_{k=0}^{N-2} I_k\left(\alpha_s(Q^2)\right) \ , \end{aligned}$$

where

$$I_k\left(\alpha_s(Q^2)\right) = \int_0^1 dz z^k \log\left[1 + \frac{b_0\alpha_s(Q^2)}{4\pi}\log(1-z)\right] \ .$$

Each of these integrals carries an imaginary part due to the cut, proportional to integer powers of $\exp(-1/a(Q^2))$, with $a(Q^2) = b_0\alpha_s(Q^2)/(4\pi)$, as above. Thus we find, collecting the leading power corrections

$$\delta L_2\left(N, \alpha_s(Q^2)\right) \propto N\frac{\Lambda^2}{Q^2}\left(1 + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)\right) \ ,$$

as expected in DIS.

Drell–Yan

The resummed expression for the Drell-Yan partonic cross section, at the leading $\log N$ level, is very similar. One finds

$$\widehat{\sigma}_{DY}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \frac{\sigma_{DY}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\psi^2_{MS}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)} ,$$

while σ_{DY} differs from F_2 because of a factor of two in the exponent, and because phase space dictates that the upper limit of the scale integration should be $(1-z)^2 Q^2$. Thus

$$\begin{aligned} L_{DY}\left(N, \alpha_s(Q^2)\right) &\equiv \log\left[\frac{\widehat{\sigma}_{DY}\left(N, 1, \alpha_s(Q^2), 0\right)}{\widehat{\sigma}_{DY}\left(1\right)}\right] \\ &= -\frac{8\pi C_F}{b_0} \sum_{k=0}^{N-2} I_k\left(2\ \alpha_s(Q^2)\right) , \end{aligned}$$

which is *twice* the DIS result with $a(Q^2) \to 2\ a(Q^2)$. Then

$$\delta L_{DY}\left(N, \alpha_s(Q^2)\right) \propto N\frac{\Lambda}{Q}\left(1 + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{\Lambda}{Q}\right)\right) .$$

This result is *wrong*, but it *follows* from the leading log approximation used above (BB, SV). Only including nonlogarithmic terms (suggested by kinematics) one finds that the leading power correction *cancels*.

Dimensionally continued g functions

At NLL accuracy, it is customary (CT) to replace the Mellin integral with the prescription

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z} f(z) \quad \to \quad - \int_0^{1-1/N} dz \frac{1}{1-z} f(z) \ ,$$

which leads to an expansion of the form

$$\begin{aligned} L_{DY}\left(N, \alpha_s(Q^2)\right) &= \log N \; g_1(b_0 \alpha_s \log N) \\ &+ g_2(b_0 \alpha_s \log N) + \mathcal{O}(\alpha_s^k \log^{k-1} N) \ . \end{aligned}$$

This approximation makes it possible to evaluate explicitly the integrals in the exponent, also for $\epsilon < 0$. One finds, in the $\overline{MS}$ scheme,

$$L_{DY}\left(N, \alpha_s(Q^2)\right) = -\frac{8C_F}{b_0}\left[\log N \log\left(-\frac{a(Q^2)}{\epsilon} N^{2\epsilon}\right) \right.$$
$$\left. -\frac{1}{2\epsilon}\left(\mathrm{Li}_2\left(1 + \frac{\epsilon}{a(Q^2)}\right) - \mathrm{Li}_2\left(\frac{a(Q^2)+\epsilon}{a(Q^2) N^{2\epsilon}}\right)\right) - \epsilon \log^2 N\right].$$

One can get a similar expression in the DIS scheme, by dividing σ_{DY} by F_2^2. One gets

$$L_{DY}^{(DIS)}\left(N, \alpha_s(Q^2)\right) = -\frac{8C_F}{b_0}\left[\frac{1}{2\epsilon}\mathrm{Li}_2\left(\frac{a(Q^2)+\epsilon}{a(Q^2) N^{2\epsilon}}\right) \right.$$
$$\left. -\frac{1}{2\epsilon}\mathrm{Li}_2\left(1 + \frac{\epsilon}{a(Q^2)}\right) - \frac{1}{\epsilon}\mathrm{Li}_2\left(\frac{a(Q^2)+\epsilon}{a(Q^2) N^{\epsilon}}\right) - \frac{\epsilon}{2}\log^2 N\right].$$

Similar, lengthier expressions may be obtained at NLL level.

Perspectives

- For $\epsilon < 0$ the QCD β function develops a second fixed point, corresponding to the Landau singularity; for sufficiently large negative ϵ the Landau pole migrates off the real axis; resummed expressions in QCD are then explicitly computable.

- Consistent usage of dimensional regularization leads to explicit expressions for the quark form factor, the DIS and Drell-Yan cross sections, in terms of *RG invariant analytic functions* of ϵ, α_s and N, *to the desired accuracy* in $\beta(\alpha_s)$. These expressions carry reliable information about power corrections in compact form.

- Power corrections exponentiate; scheme dependence is under control; all expressions are gauge invariant.

- Generalization to more complicated QCD amplitudes is hard (tensor structures in color space) but techniques *exist*. Phenomenological applications are possible, as in the case of the exponentiation of π^2 terms for the Drell-Yan cross section. Work is in progress.