Solutions to Collider Phenomenology Course problem sheets

Marco Musich[1]

(1) University of Torino and INFN, Via Pietro Giuria 1, Torino 10125, Italy

1. Problem Sheet 1

1.1. Problem 1

In the lectures the quark mass threshold has been approximated by a step-function. What actually happens at the threshold?

Solution

The production cross-section for the process $e^+(q_2)e^-(q_1) \to hadrons(\{p_i\})$ can be derived from first principle starting with the matrix element to produce n hadrons in the final state:

$$\mathcal{M} = \{\overline{v}(q_2)e\gamma_\mu u(q_1)\}\frac{-ig^{\mu\nu}}{q^2}T_\nu(n,q,\{p_1,...p_n\})$$

with $q = q_1 + q_2$ and T is a guess (parametrization) of the the unknow matrix element of an of-shell photon to decay to n hadrons.

This matrix element gives the total cross-section:

$$\begin{aligned}
\sigma &= \frac{1}{2s}\sum_n \int dPs_n \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 \\
&= \frac{1}{2s}\frac{1}{4}\frac{e^2}{s}\left(\sum_{spins}\{\overline{u}(q_2)\gamma^\mu u(q_1)u(q_1)\gamma^\nu v(q_2)\}\right)\sum_n \int dPs_n T^*_\mu(n,q,\{p_X\})T_\nu(n,q,\{p_X\}) \\
&= \frac{1}{2s}\frac{1}{4}\frac{e^2}{s}\mathrm{Tr}\left[\not{q}_2\gamma^\mu\not{q}_1\gamma^\nu\right]\sum_n \int dPs_n T^*_\mu(n,q,\{p_X\})T_\nu(n,q,\{p_X\}) \\
= &= \frac{1}{2s}\frac{1}{4}\frac{e^2}{s}L^{\mu\nu}(q_1,q_2)H_{\mu\nu}(q)
\end{aligned}$$

Defining the hadronic tensor as:

$$H_{\mu\nu}(q) = \sum_n \int dPs_n T^*_\mu(n,q,\{p_X\})T_\nu(n,q,\{p_X\})$$

Lorentz covariance implies that $H_{\mu\nu}(q)$ is a rank two symmetric tensor,being the only two symmetric tensor that can be be built from q^μ:

$$H_{\mu\nu}(q) = A(q^2)g_{\mu\nu} + B(q^2)q_\mu q_\nu$$

with $A(q^2)$ and $B(q^2)$ functions of the only Lorentz scalar available q^2. Gauge invariance implies:

$$q^\mu H_{\mu\nu}(q) = q^\nu H_{\mu\nu}(q) = 0 \Rightarrow A(q^2)q_\nu + B(q^2)q^2 q_\nu = 0$$

So that: $A(q^2) = -q^2 B(q^2)$. The cross-section is therefore:

$$\begin{aligned}
\sigma &= \frac{1}{2s}\frac{1}{4}\frac{e^2}{s}L^{\mu\nu}(q_1,q_2)[B(q^2)(q_{\mu\nu} - q^2 g_{\mu\nu})] \\
&= \frac{e^2}{2s}B(q^2) \Rightarrow \boxed{\sigma(e^+e^- \to hadrons) = \frac{e^2}{2s}B(s)}
\end{aligned}$$

This implies that, since $B(s)$ is required from dimensional reasoning to be adimensional:

$$\mathcal{R}_{e^+e^-}(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = constant$$

To predict the exact value of $\mathcal{R}_{e^+e^-}(s)$, one can use a space-time argument. Since the photon exchanged in the process is highly virtual, it is produced and decays to quark in a small space-time volume, $t_{hard} \simeq 1/\sqrt{s}$. On the other hand, the wave function of a hadron with mass m_{had} has a spatial extent $\simeq 1/m_{had}$; hence confinement of a quark into the hadron takes $t_{had} \simeq 1/m_{had}$. Thus since: $t_{hard} \gg t_{had}$ we can expect:

$$\sigma(e^+e^- \to hadrons) = \sum_q \sigma(e^+e^- \to q\bar{q}) \times \left(1 + \mathcal{O}\left(\frac{m_{had}}{\sqrt{s}}\right)^n\right)$$

the cross-section $\sigma(e^+e^- \to q\bar{q})$ is simple to calculate and this gives us the clue to obtain $\mathcal{R}_{e^+e^-}(s)$

$$\mathcal{R}_{e^+e^-}(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} \simeq \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_q e_q^2 N_C$$

This simple results has been obtained at LO neglecting all QCD corrections. It is a step function of s, since in the sum one should consider only the quarks with mass lower the than the s of the process.

$$\mathcal{R}_{e^+e^-}(s) = \sum_{q, m_q < \sqrt{s}/2} e_q^2 N_C$$

The width of the quark would model the threshold behaviour more smoothly. More importantly, one has the formation of hadronic resonances. This cannot be modeled in perturbative QCD. The following plot shows some resonances explicitely.

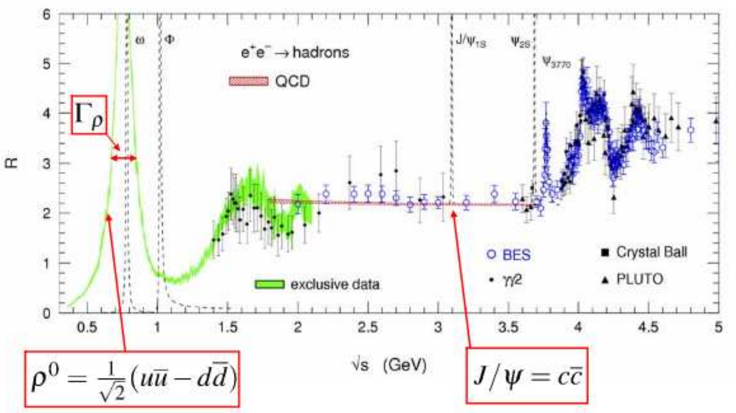


Fig. 1. Data on $\mathcal{R}_{e^+e^-}(s)$ as a function of centre-of-mass energy

1.2. Problem 2

Since $R(e^+e^-)$ is the only quantity for which we have NNNLO results, it is our only chance to calibrate how good different schemes for fixing μ are. Looking at the figure for μ scale dependence, $K^{(n)}$ vs μ, discuss the relative merits of the three schemes defined there.

Solution

Altough a physical quantity do not depend on μ the renormalization scale, a calculation truncated at a finite order of perturbation theory does. To illustrate this, let's write the perturbative expansion of the adimensional $R(e^+e^-)$ quantity. Since it is an adimensional quantity it should depend on μ via the dimensionful variables s and μ^2 only through their ratio $t = s/\mu^2$

$$\mathcal{R}[s/\mu^2, \alpha_S(\mu^2)] = \mathcal{R}_0[1 + \alpha_s R_1(t) + \alpha_s R_2(t) + ...] = \mathcal{R}_0 \sum_{n=0}^{\infty} \alpha_s^n R_n(t) \quad \mathcal{R}_0 = \sum_q e_q^2 N_C$$

We can use the fact that R, as a physical quantity, must be independent of the value of μ, and the chain rule for partial derivatives, to write:

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2}\mathcal{R}(s/\mu^2, \alpha_S) = 0 &= \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}\frac{\partial}{\partial \alpha_s}\right]\mathcal{R} \\ &\equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right]\mathcal{R} \end{aligned}$$

a physical solution of this renormalization group equation is given by:

$$\mathcal{R} = \mathcal{R}(1, \alpha_S(s)) = \mathcal{R}_0(1 + R_1\alpha_S(s) + R_2\alpha_S^2(s) + ...)$$

i.e. by setting the renormalization scale equal to the physical scale in the problem (in our case the centre-of-mass energy of the annihilation).

Let's now suppose to know $\mathcal{R}$ at leading order in QCD.

$$\mathcal{R} = \mathcal{R}(1, \alpha_S(s)) = \mathcal{R}_0(1 + R_1\alpha_S(s) + ...)$$

now the renormalization scale dependence becomes evident trough:

$$\alpha_S(s) = \frac{\alpha_s(\mu^2)}{1 + \alpha_S(\mu^2)\beta_0 \ln \frac{s}{\mu^2}}$$

where the running of α_S in known at one loop too. We have:

$$\begin{aligned}\mathcal{R} &= \mathcal{R}(1, \alpha_S(s)) = \mathcal{R}_0(1 + R_1\alpha_S(s) + ...) \\ &= \mathcal{R}_0\left(1 + R_1\alpha_S(\mu^2)\left[1 - \beta_0\alpha_S(\mu^2) ln\frac{s}{\mu^2} + \beta_0^2\alpha_S^2(\mu^2) ln^2\frac{s}{\mu^2} + ...\right]\right)\end{aligned}$$

The leading-order result in renormalized perturbation theory is the first term of this series, i.e., $\mathcal{R}_0(1+R_1\alpha(\mu^2))$. It is therefore clear that although μ^2 is completely arbitrary, choosing it far from s guarantees a large truncation error (note that the converse is not true). One should therefore choose μ^2 "close" to s , but how close is close?

There are three popular choices:

- Put the physical scale $\mu = \sqrt{s}$ in our case. This has the merit to be simplest educated guess to be used.
- Principal of Minimal Sensitivity: where $d\sigma/d\mu = 0$. This method has the merit to take the renormalization scale to with one is the least sensible to it, i.e. by varying μ of a small quantity the cross-section does not change much.
- F.C.A. (Fastest Apparent Convergence): where NLO = LO, i.e. where the effects of truncating at some order the perturbative series has the smallest consequence.

1.3. Problem 3

Convince yourself that the thrust of a three-parton configuration is given by $max(x_1, x_2, x_3)$ where $x_1 = 2E_i/\sqrt{s}$, with E_i being the parton energy and $\sqrt{s}$ the collider energy.

Solution

We know that the thrust for a three-parton configuration is defined as:

$$T_3(p_i) = \frac{max_{\widehat{\mathbf{u}}} \sum_{i=1}^{3} |\mathbf{p_i} \cdot \widehat{\mathbf{u}}|}{\sum_{i=1}^{3} |\mathbf{p_i}|} \tag{1}$$

now in the centre-of-mass rest frame $\sum_{i=1}^{3} \mathbf{p_i} = 0$ and the decay is therefore planar. Projecting along $\widehat{\mathbf{p_q}}$:

$$\begin{aligned}\left(\sum_{i=1}^{3} \mathbf{p_i}\right) \cdot \widehat{\mathbf{p_q}} &= \mathbf{p_q} \cdot \widehat{\mathbf{p_q}} + \mathbf{p_{\overline{q}}} \cdot \widehat{\mathbf{p_q}} + \mathbf{p_{\overline{g}}} \cdot \widehat{\mathbf{p_q}} \\ &= |\mathbf{p_q}| \cos\theta_{qq} + |\mathbf{p_{\overline{q}}}| \cos\theta_{q\overline{q}} + |\mathbf{p_g}| \cos\theta_{qg} \\ &= E_q + E_{\overline{q}} \cos\theta_{q\overline{q}} + E_g \cos\theta_{qg} = \frac{\sqrt{s}}{2}\left(\frac{2E_q}{\sqrt{s}} + \frac{2E_{\overline{q}}}{\sqrt{s}} \cos\theta_{q\overline{q}} + \frac{E_g}{\sqrt{s}} \cos\theta_{qg}\right) \\ &= \frac{\sqrt{s}}{2}(x_q + x_{\overline{q}} \cos\theta_{q\overline{q}} + x_g \cos\theta_{qg}) = 0\end{aligned}$$

in the case of a leading quark ($x_q > x_q, x_{\overline{q}}$):

$$x_q = -x_{\overline{q}} \cos\theta_{q\overline{q}} - x_g \cos\theta_{qg} = x_{\overline{q}} |\cos\theta_{q\overline{q}}| - x_g |\cos\theta_{qg}|$$

since in both cases: $\cos\theta_{q\overline{q}}; \cos\theta_{qg} < 0$. It is clear that this configuaration maximizes all the projections and:

$$T_3(x_q > x_q, x_{\overline{q}}) = \frac{|\frac{\sqrt{s}}{2}(x_q + x_{\overline{q}}\cos\theta_{q\overline{q}} + x_g\cos\theta_{qg})|}{\frac{\sqrt{s}}{2}(x_q + x_{\overline{q}} + x_g)} = \frac{\frac{1}{2}|(x_q + x_{\overline{q}}\cos\theta_{q\overline{q}} + x_g\cos\theta_{qg})|}{\frac{1}{2}\sum_i x_i}$$

$$= \frac{1}{2}\left(x_q + \underbrace{x_{\overline{q}}|\cos\theta_{q\overline{q}}| + x_g|\cos\theta_{qg}|}_{x_q}\right) = \frac{1}{2}\left(x_q + x_q\right) = x_q$$

The result can be generalized to:

$$T_3(x_1 > x_2, x_3) = x_1 \Longrightarrow \quad \boxed{T_3(x_1 > x_2, x_3) = max\{x_1, x_2, x_3\}}$$

2. Problem Sheet 2

2.1. Problem 1

Solve the differential equation:

$$\mu^2 \frac{d\alpha_S(\mu^2)}{d\mu^2} = \beta(\alpha_S(\mu^2)) = -\beta_0\alpha_s^2(\mu^2) + \mathcal{O}(\alpha_s^3)$$

(also called α_s evolution equation) through first order: i.e. ignore the higher order terms $\mathcal{O}(\alpha_s^3)$. Recall that in QCD:

$$\beta_0 \equiv \frac{11N_C - 2n_f}{12\pi} = \frac{33 - 2n_f}{12\pi}$$

Determine the integration constant by setting $\alpha_s(\mu^2 = M_Z^2) \equiv \alpha_s(M_Z^2) = 0.12$ and calculate the value of α_s at μ = 10 GeV. (Use the fact that there are n_f five active flavours and $M_Z = 91.1$ GeV)

Solution

Let's start by usign the fact that the logarithmic derivative can be written as:

$$\frac{d}{dln\mu^2} = \frac{\partial}{\partial\mu^2}\frac{\partial\mu^2}{\partial ln\mu^2} = \left[\frac{\partial ln\mu^2}{\partial\mu^2}\right]^{-1}\frac{\partial}{\partial\mu^2} = \left[\frac{1}{\mu^2}\right]^{-1}\frac{\partial}{\partial\mu^2} = \mu^2\frac{\partial}{\partial\mu^2}$$

so we end up with:

$$\mu^2\frac{d\alpha_s(\mu^2)}{d\mu^2} = \frac{\partial\alpha_s(\mu^2)}{\partial ln\mu^2} = -\beta(\alpha_s) = \beta_0\alpha_s^2 + \mathcal{O}(\alpha_s^3)$$

hence ignoring higher order in the perturbation series $\mathcal{O}(\alpha_s^3)$ we get a simple first order separable differential equation:

$$\frac{\partial\alpha_s(\mu^2)}{\partial ln\mu^2} = -\beta_0\alpha_s^2(\mu^2) \Rightarrow \frac{d\alpha_s}{\alpha_s} = -\beta_0 ln\mu^2$$

so integrating between the integral bounds $\alpha_s(\mu^2)$ and $\alpha_s(\mu_0^2)$ we have:

$$\int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)}\frac{d\alpha_s}{\alpha_s} = -\beta_0\int_{ln\mu_0^2}^{ln\mu^2} dln\mu^2 \Rightarrow \left[-\frac{1}{\alpha_s}\right]_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} = -\beta_0\left[ln\mu^2\right]_{ln\mu_0^2}^{ln\mu^2}$$

so we have solving the algebra:

$$\frac{1}{\alpha_S(\mu^2)} - \frac{1}{\alpha_S(\mu_0^2)} = \beta_0\left(ln\mu^2 - ln\mu_0^2\right) \Rightarrow \frac{1}{\alpha_s(\mu^2)} = \frac{1}{\alpha_s(\mu_0^2)} + \beta_0 ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

and so finally:

$$\frac{1}{\alpha_s(\mu^2)} = \frac{1 + \alpha_S(\mu_0^2)\beta_0 ln\left(\frac{\mu^2}{\mu_0^2}\right)}{\alpha_s(\mu^2)} \Rightarrow \boxed{\alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + \alpha_s(\mu_0^2)\beta_0 ln\left(\frac{\mu^2}{\mu_0^2}\right)}} \quad (2)$$

The equation is solved so we can now put $\mu_0^2 \equiv M_Z^2$ and use the mesaured value for $\alpha_s(M_z^2) = 0.12$ and use the evolution equation to determine the value of the strong coupling constant at the $\mu = 10$ GeV scale.

$$\begin{aligned}\alpha_s(Q^2) &= \frac{\alpha_S(M_Z^2)}{1 + \alpha_S(M_z^2)\left(\frac{11N_c - 2n_f}{12\pi}\right) ln\left(\frac{Q^2}{M_Z^2}\right)} \\ &= \frac{0.12}{1 + 0.12\left(\frac{11\cdot 3 - 2\cdot 10}{12\pi}\right) ln\left(\frac{100}{(91.18)^2}\right)} = \frac{0.12}{1 + 0.12\left(\frac{33-20}{12\pi}\right) ln\left(0.012\right)} \\ &= \frac{0.12}{1 - 0.12(0.34)4.42} = \frac{0.12}{0.81} = 0.148\end{aligned}$$

the final result is $\alpha_s(Q^2) = 0.148$ which is higher than the value at $Q^2 = M_Z^2$ as expected from the structure of the RGE solution for α_S as can be seen in Fig. 2.1. .

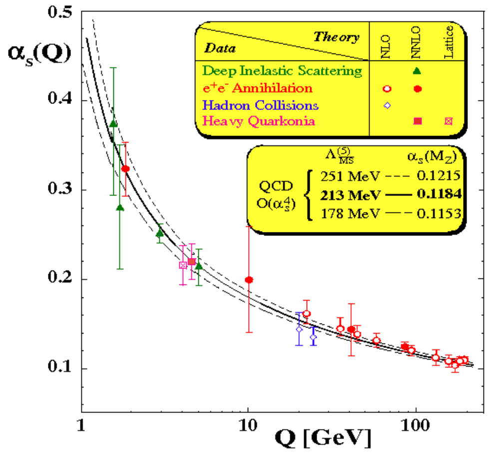


Fig. 2. Plot of α_s vs. Q^2 compared with experimental data

2.2. *Problem 2*

In the lecture it has been said that F_2 and F_L depend only on x and Q^2, and that the DIS cross-section (in the one photon exchange approximation) is given by the linear combination:

$$\frac{d^2\sigma^{ep}}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^2}\left\{[1+(1-y^2)]F_2(x,Q^2) - y^2 F_L(x,Q^2)\right\} \tag{3}$$

How would you measure F_2 and F_L separately?

Solution

Looking to Eq. 3 it is convenient to define a function of the scattering inelestaticity y defined as follows:

$$Y_+(y) = [1+(1-y^2)]$$

using this definition one can rewrite Eq. 3 as:

$$\frac{d^2\sigma^{ep}}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^2}[Y_+F_2(x,Q^2) - y^2F_L(x,Q^2)] = \frac{2\pi\alpha^2 Y_+}{xQ^2}[F_2(x,Q^2) - \frac{y^2}{Y_+}F_L(x,Q^2)]$$

and one can define a reduced cross-section as:

$$\sigma_r(x,Q^2,y) \equiv \frac{d^2\sigma^{ep}}{dxQ^2}\cdot\frac{Q^2x}{2\pi\alpha^2Y_+} = F_2(x,Q^2) - \frac{y^2}{Y_+}F_L(x,Q^2)$$

This implies that the two structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$ can be written as:

$$F_2(x,Q^2) = \sigma_r(x,Q^2,y=0) \qquad F_L(x,Q^2) = -\frac{\partial\sigma_r(x,Q^2,y)}{\partial\,(y^2/Y_+)}$$

since $\sigma_r(x,Q^2,y)$ is expected to be a linear function of y^2/Y_+ one can take the finite difference instead of the derivative:

$$F_L(x,Q^2) = -\frac{\Delta\sigma_r}{\Delta\left(\frac{y^2}{Y_+}\right)}$$

So in order to measure the structure function $F_L(x,Q^2)$ we need to measure the "reduced cross-section" at different values of the variable y^2/Y_+ which depends only on y but keeping the same values of Q^2 and x. Since Q^2, x, y are related to the center of mass energy of the system via the relation: $Q^2 = sxy$, one can experimentally determine the value of $F_L(x,Q^2)$ measuring the reduced cross-section at different center-of-mass energies $\sqrt{s} = \sqrt{Q^2/xy}$.

$$F_L(x,Q^2) = -\frac{\sigma_r(s_1)-\sigma_r(s_2)}{\left(\frac{y^2}{Y_+}\right)_{s_1} - \left(\frac{y^2}{Y_+}\right)_{s_2}} = \frac{[\sigma_r(s_1)-\sigma_r(s_2)]Y_{+,s_1}Y_{+,s_2}}{y^2_{s_1}Y_{+,s_2} - y^2_{s_2}Y_{+,s_1}}$$

$F_2(x,Q^2)$ can be easily extracted from the the formula $F_2(x,Q^2) = \sigma_r(x,Q^2,y=0)$ by looking for the intercept of the reduced cross-section as a function of y, with the $y=0$ axis.

2.3. Problem 3

The most general form of the DIS cross-section has three non-zero structure functions, F_1,F_2 and F_3 while the e^+e^- cross-section has only one (called B). Why the difference?

Solution

The most general cross-section for a deep inelastic scattering can be written as:

$$d\sigma = \frac{1}{2s}\sum_X \int d\Phi \frac{1}{4}\sum_{spins} |\mathcal{M}_{e(\nu)P\rightarrow eX}|^2$$

the phase space factor can be written as the product of two contributions, one coming from the outcoming lepton and the other from the recoiling hadronic mass:

$$d\Phi = \frac{1}{(2\pi)^3}\frac{d^3k'}{2E'}d\Phi_X = \frac{1}{(2\pi)^3}\frac{d|k'|^2dk'd\Omega}{2E'}d\Phi_X$$

Concentrating first on the electromagnetic scattering process:

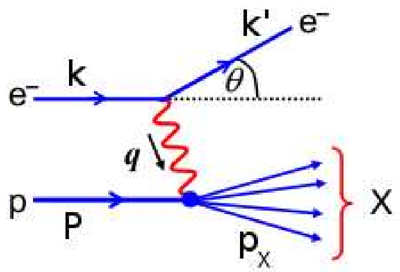


Fig. 3. Lowest order QED diagram for $e(k') + P(P) \longrightarrow e(k') + X(p_X)$

$$e(k') + P(P) \longrightarrow e(k') + X(\{p_i\}) \qquad q = k - k'$$

Let's define some convenient kinematic variables. For an electron of momentum k to scatter to one of momentum k' by exchanging a photon of momentum q with a proton of momentum p we have for fixed centre-of-mass energy s that:

$$s = (k+p)^2 \qquad t = q^2 = -Q^2 \qquad x = \frac{Q^2}{2P\cdot q}$$

where x is the inelasticity of the process. In terms of those wariables we can define another two commonly used variables:

$$W^2 = (P+q)^2 = P^2 + q^2 + 2P\cdot q \simeq -Q^2 + \frac{Q^2}{x} = Q^2\frac{1-x}{x} \tag{4}$$

$$y = \frac{p\cdot q}{P\cdot k} = \frac{Q^2}{x2P\cdot k} \simeq \frac{Q^2}{xs} \tag{5}$$

The lowest order QED amplitude for the process is:

$$\mathcal{M} = \overline{u}(k')(ie\gamma^\mu)u(k)\frac{-ig_{\mu\nu}}{q^2}(ie)\langle X|J^\nu_{p,EM}|P,P\rangle$$

where $\langle X|J^\nu_{p,EM}|P,P\rangle$ is the electromagnetic current associated to the proton inelastic scattering

The amplitude can be re-written:

$$\mathcal{M} = (ie)^2\overline{u}(k')\gamma^\mu u(k)\frac{1}{q^2}\langle X|J^{p,EM}_\mu|P,P\rangle$$

squaring the amplitude we get:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{e^4}{q^4}\overline{u}(k')\gamma^\mu u(k)\Big(\overline{u}(k')\gamma^\nu u(k)\Big)^*\langle X|J^{p,EM}_\mu|P,P\rangle\Big(\langle X|J^{p,EM}_\mu|P,P\rangle\Big)^*$$

and using the γ matrices properties we get to:

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\overline{u}(k')\gamma^\mu u(k)\overline{u}(k)\gamma^\nu u(k')\langle P,P|J^{*p,EM}_\nu|X\rangle\langle X|J^{p,EM}_\mu|P,P\rangle$$

the total cross-section can be written as:

$$d\sigma = \frac{1}{F}\frac{d^3k'}{(2\pi)^3 2E'}\sum_X (2\pi)^4 \int d\Phi_X \delta^4(k+P-k'-p_X)\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

where we have assumed that the sum of all momenta of the recoiling hadron mass is $p_X = \sum_i p_i$ and the phase space associated to that subsystem is:

$$d\Phi_X = \prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2p_0^i}$$

now the squared and averaged (summed) on initial (final) states matrix element is:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}\sum_{spins}\frac{e^4}{q^4}\overline{u}(k')\gamma^\mu u(k)\overline{u}(k)\gamma^\nu u(k')\langle P,P|J_\nu^{*p,EM}|X\rangle\langle X|J_\mu^{p,EM}|P,P\rangle$$

we can isolate a contribution due to the electron current and a contribution of the hadronic current:

$$\begin{aligned} L^{\mu\nu}(k,k') &= \frac{1}{2}\sum_{spins}\overline{u}(k')\gamma^\mu u(k)\overline{u}(k)\gamma^\nu u(k') = \frac{1}{2}Tr\Big(\sum_{spins}\overline{u}(k')\gamma^\mu u(k)\overline{u}(k)\gamma^\nu u(k')\Big) \\ &= \frac{1}{2}Tr\Big(\sum_{spins}u(k')\overline{u}(k')\gamma^\mu u(k)\overline{u}(k)\gamma^\nu\Big) = \frac{1}{2}Tr\Big(\underbrace{\sum_{s'}u(k',s')\overline{u}(k',s')}_{\not k'+m_e}\gamma^\mu\underbrace{\sum_s u(k,s)\overline{u}(k,s)}_{\not k+m_e}\gamma^\nu\Big)\end{aligned}$$

where we have used the completeness relations for dirac spinors.

$$L^{\mu\nu}(k,k') = \frac{1}{2}[Tr\Big(\not k'\gamma^\mu\not k\gamma^\nu\Big)+m_e^2 Tr\Big(\gamma^\mu\gamma^\nu\Big)] = \frac{1}{2}[k'_\alpha k_\beta\underbrace{Tr\Big(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big)}_{4(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\beta}g^{\mu\nu})}+m_e^2\underbrace{Tr\Big(\gamma^\mu\gamma^\nu\Big)}_{4g^{\mu\nu}}]$$

This gives:

$$L^{\mu\nu}(k,k') = \frac{1}{2}\Big[4\Big(k'^\mu k^\nu + k'^\nu k^\mu - (k\cdot k')g^{\mu\nu}\Big)+4m_e^2 g^{\mu\nu}\Big] = 2\Big(k'^\mu k^\nu + k'^\nu k^\mu - (k\cdot k' - m_e^2)g^{\mu\nu}\Big)$$

neglecting electron mass in the high energy limit we have:

$$L^{\mu\nu}(k,k') = 2\Big(k'^\mu k^\nu + k'^\nu k^\mu - (k\cdot k')g^{\mu\nu}\Big)$$

This implies that the squared and averaged matrix element is:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{q^4}L^{\mu\nu}(k,k')\frac{1}{2}\sum_{s',s_i}\langle P,P,s|J_\nu^{*p,EM}|X,s_i\rangle\langle X,s_i|J_\mu^{p,EM}|P,P,s\rangle$$

We parametrize now the matrix element for a proton of momentum P to absorb a photon of momentum q and Lorentz index μ to produce an arbitrary set of hadrons X with fixed momenta $\{p_i\}$ as:

$$T_\mu(P,q;\{p_i\}) = \langle X|J_\mu^{p,EM}|P,P\rangle$$

This implies for the total matrix element:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = |\overline{\mathcal{M}}|^2 = \frac{e^4}{q^4}L^{\mu\nu}(k,k')\frac{1}{2}\sum_{s,s_i}T_\nu^*(P,q;\{p_i\})T_\mu(P,q;\{p_i\})$$

now it is convenient to evaluate the quantity:

$$\begin{aligned}\sum_X\int d\Phi_X(2\pi)^4\delta(k+P-k'-p_X)|\overline{\mathcal{M}}|^2 &= \sum_X\int d\Phi_X(2\pi)^4\delta(k+P-k'-p_X)\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 \\ &\equiv \frac{e^2}{q^4}L^{\mu\nu}(k,k')H_{\mu\nu}(P,q)\end{aligned}$$

or equivalently:

$$H_{\mu\nu}(P,q) = \frac{1}{2}\sum_{spins}\sum_{X}\int d\Phi_X (2\pi)^4\delta^4(k+P-k'-p_X)T_\mu(P,q;\{p_i\})T^*_\nu(P,q;\{p_i\}) \quad (6)$$

Since we have summed and integrated out all the dependence on X, $H_{\mu\nu}(P,q)$ can only depend on the vectors P^μ and q^μ The most general expression for a generic tensor of rank two containing P^μ and q^μ is:

$$H_{\mu\nu} = -H_1 g_{\mu\nu} + H_2\frac{P_\mu P_\nu}{Q^2} - \frac{i}{Q^2}H_3\varepsilon_{\mu\nu\lambda\sigma}P^\lambda q^\sigma + H_4\frac{q_\mu q_\nu}{Q^2} + H_5\frac{P_\mu q_\nu + q_\mu P_\nu}{Q^2}$$

where the Hs are scalar functions of the only two Lorentz scalars available $q\cdot q = -Q^2$ and $p\cdot q = Q^2/2x$, i.e. of x and Q^2 only (not s). We have neglected $p\cdot p = M_P^2$ since we work in the limit $|q\cdot q|, p\cdot q \gg p\cdot p$

We can now simplify the hadronic tensor $H_{\mu\nu}$ making some considerations. The term proportional to H_3 is antisimmetric in the exchange of the two Lorenzt indexes μ and ν, while the leptonic tensor has a symmetric structure, thus terms proportional to H_3 in the contraction of the two tensors do not contribute to the physical cross-section.

Let's consider now a process in which a photon of momentum q^μ couples with the electromagnatic' "proton- to- X final state" current. The matrix element for such a process is:

$$\mathcal{M}_0 = \varepsilon^\mu(\lambda,q)\langle X|J_\mu^{p,EM}|P,P\rangle = \varepsilon^\mu(\lambda,q)T_\mu(P,q;\{p_i\})$$

charge conservation requires:

$$\partial^\mu J_\mu^{EM}(P\to X) = \partial^\mu T_\mu(P,q;\{p_i\}) = 0:$$

or in an equivalent way:

$$\partial^\mu T_\mu \Rightarrow \frac{d}{dt}\int T_0 d^3x = \int \partial^0 T_0 d^3x =$$
$$\int \nabla\cdot\mathbf{T}d^3x = \int_{S\to\infty}\mathbf{T}\cdot d\mathbf{\Sigma} = 0$$

in momentum space this means: $q^\mu T_\mu = 0$

another way of saying it is that the theory is invariant under a gauge transformation: $\varepsilon^\mu(\lambda,q)\to\varepsilon^\mu(\lambda,q)+q^\mu$ This means the matrix element is invariant:

$$\mathcal{M}_0 \to \mathcal{M}_0' = (\varepsilon^\mu(\lambda,q)+q^\mu)T_\mu = \varepsilon^\mu(\lambda,q)T_\mu = \mathcal{M}_0$$

This translates into a requirement on $H_{\mu\nu}(P,q)$:

$$q^\mu T_\mu = 0 \Rightarrow q^\mu H_{\mu\nu}(P,q) = q^\mu\sum_X\int d\Phi_X T_\mu T_\nu^* = \sum_X\int d\Phi_X q^\mu T_\mu T_\nu^* = 0$$

and in a equivalent way $q^\nu H_{\mu\nu} = 0$. So we have:

$$\begin{aligned} q^\mu H_{\mu\nu} &= q^\mu\left[-H_1 g_{\mu\nu} + H_2\frac{P_\mu P_\nu}{Q^2} + H_4\frac{q_\mu q_\nu}{Q^2} + H_5\frac{P_\mu q_\nu + q_\mu P_\nu}{Q^2}\right] \\ &= -q_\nu H_1 + H_2\frac{P\cdot q}{Q^2}p_\nu + H_4\frac{q^2}{Q^2}q_\nu + H_5\frac{P\cdot q}{Q^2}q_\nu + H_5\frac{q^2}{Q^2}P_\nu \\ &= -q_\nu H_1 + H_2\frac{P\cdot q}{Q^2}p_\nu - H_4 q_\nu + H_5\frac{P\cdot q}{Q^2}q_\nu - H_5 P_\nu \\ &= \left(-H_1 - H_4 + H_5\frac{P\cdot q}{Q^2}\right)q_\nu + \left(H_2\frac{P\cdot q}{Q^2} - H_5\right)P_\nu = 0 \end{aligned}$$

the two terms in parenthesis should be zero. So we have:

$$H_5 = H_2\frac{P\cdot q}{Q^2}$$
$$H_1 + H_4 = H_5\frac{P\cdot q}{Q^2} \Rightarrow H_4 = H_2\left(\frac{P\cdot q}{Q^2}\right)^2 - H_1$$

and the hadronic tensor can be written:

$$
\begin{aligned}
H_{\mu\nu} &= -g_{\mu\nu}H_1 + H_2\frac{P_\mu P_\nu}{Q^2} + H_2\left(\frac{P\cdot q}{Q^2}\right)^2\frac{q_\mu q_\nu}{Q^2} - H_1\frac{q_\mu q_\nu}{Q^2} + H_2\frac{P\cdot q}{Q^2}\frac{P_\mu q_\nu + q_\mu P_\nu}{Q^2} \\
&= \left(-g_{\mu\nu} - \frac{q_\mu q_\nu}{Q^2}\right)H_1 + \left[\frac{P_\mu P_\nu}{Q^2} + \left(\frac{P\cdot q}{Q^2}\right)^2\frac{q_\mu q_\nu}{Q^2} + \frac{P\cdot q}{Q^2}\frac{P_\mu q_\nu + q_\mu P_\nu}{Q^2}\right]H_2 \\
&= \left(-g_{\mu\nu} - \frac{q_\mu q_\nu}{Q^2}\right)H_1 + \left[P_\mu P_\nu + \frac{(P\cdot q)^2}{Q^4}q_\mu q_\nu + \frac{P\cdot q}{Q^2}(P_\mu q_\nu + q_\mu P_\nu)\right]\frac{H_2}{Q^2} \\
&= H_1\left(-g_{\mu\nu} - \frac{q_\mu q_\nu}{Q^2}\right) + \frac{H_2}{Q^2}\left[P_\mu + \frac{P\cdot q}{Q^2}q_\mu\right]\left[P_\nu + \frac{P\cdot q}{Q^2}q_\nu\right]
\end{aligned}
$$

now since $Q^2 = -q^2$ we have finally:

$$
\boxed{H_{\mu\nu}(x,q^2) = H_1(x,q^2)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) - \frac{H_2(x,q^2)}{q^2}\left[P_\mu - \left(\frac{P\cdot q}{q^2}\right)q_\mu\right]\left[P_\nu + \left(\frac{P\cdot q}{Q^2}\right)q_\nu\right]}
$$

which have 2 structure functions $H_i(x,q^2)$! If we want to evaluate the DIS cross-section we have to contract the $H_{\mu\nu}(x,q^2)$ tensor with the leptonic tensor $L^{\mu\nu}(k,k')$ Gauge invariance again implies:

$$
q_\mu j^{e,\mu}(k,k') = 0 \Rightarrow q^\mu L^{\mu\nu}(k,k') = 0
$$

so terms proportional to q^μ or q^μ in the hadronic cross-section do not contribute to the physical cross-section:

$$
\begin{aligned}
L^{\mu\nu}(k,k')H_{\mu\nu}(x,q^2) &= 2\Big(k'^\mu k^\nu + k'^\nu k^\mu - (k\cdot k')g^{\mu\nu}\Big)\Big[-H_1(x,Q^2)g_{\mu\nu} + H_2(x,Q^2)\frac{P_\mu P_\nu}{Q^2}\Big] \\
&= -2H_1(k\cdot k') + 2H_2\frac{(k'\cdot P)(k\cdot P)}{Q^2} - 2H_1(k'\cdot k) \\
&\quad + 2H_2\frac{(k\cdot P)(k'\cdot P)}{Q^2} + 8(k\cdot k')H_1 - \underbrace{2H_2(k\cdot k')\frac{P^2}{Q^2}}_{\propto M_p^2/Q^2\simeq 0}
\end{aligned}
$$

and so neglecting terms proportional to $Mr_P \ll q^2$:

$$
L^{\mu\nu}(k,k')H_{\mu\nu}(x,q^2) = 4H_1(k\cdot k') + 4H_2\frac{(k\cdot P)(k'\cdot P)}{Q^2}
$$

therefore:

$$
\begin{aligned}
\sum_X \int d\Phi_X \delta(k+P-k'-p_X)|\overline{\mathcal{M}}|^2 &\equiv \frac{e^2}{Q^4}L^{\mu\nu}(k,k')H_{\mu\nu}(P,q) = \frac{e^2}{Q^4}\left(4H_1(k\cdot k') + 4H_2\frac{(k\cdot P)(k'\cdot P)}{Q^2}\right) \\
&= \frac{e^2}{Q^4}\left(4H_1(k\cdot k') + 4H_2\frac{(k\cdot P)(k'\cdot P)}{Q^2}\right)
\end{aligned}
$$

Now it is convenient to go to a set of intrinsically lorentz invariant variables:

$$
Q^2 = -q^2 \simeq 2k\cdot k' \qquad x = \frac{Q^2}{2(P\cdot q)} \qquad s = (p+k)^2 \simeq 2(P\cdot k)
$$

and so: $q = k - k' \to k' = k - q$ this implies $P\cdot k' = P\cdot k - p\cdot q$. Using this variables:

$$
\begin{aligned}
L^{\mu\nu}(k,k')H_{\mu\nu}(P,q) &= 4H_1(k\cdot k') + 4H_2\frac{(k\cdot P)(k\cdot P - q\cdot P)}{Q^2} = 4H_1(k\cdot k') + H_2\frac{4(k\cdot P)^2 - 2(q\cdot P)2(k\cdot P)}{Q^2} \\
&= 2H_1Q^2 + H_2\frac{s^2 - 2s(q\cdot P)}{Q^2} = 2H_1Q^2 + H_2\left[\frac{s^2}{Q^2} - s\frac{2(q\cdot P)}{Q^2}\right] = 2H_1Q^2 + H_2\left[\frac{s^2}{Q^2} - \frac{s}{x}\right]
\end{aligned}
$$

we are now ready to proceed and calculate the electromagnetic DIS cross-section at LO:

$$d\sigma = \frac{1}{2s}\frac{d^3k'}{(2\pi)^3 2E'}\sum_X (2\pi)^4 \int d\Phi_X \delta^4(k+P-k'-p_X)\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$
$$= \frac{1}{2s}\frac{d^3k'}{(2\pi)^3 2E'}\frac{e^2}{Q^4}L^{\mu\nu}(k,k')H_{\mu\nu}(P,q)$$

now it can be proved that rhat the electron phase space $d\Phi$ can be rewritten as a function of Q^2 and x (this is done in detail in Problem 4.1).

$$d\Phi = \frac{d^3k'}{(2\pi)^3 2E'} = \frac{1}{16\pi^2}\frac{Q^2}{sx^2}dxdQ^2$$

and we have the following differential cross-section:

$$d\sigma = \frac{1}{2s}\frac{1}{16\pi^2}\frac{Q^2}{sx^2}dxdQ^2\frac{e^2}{Q^4}L^{\mu\nu}(k,k')H_{\mu\nu}(P,q) = \frac{1}{32\pi^2 x^2 s^2}\frac{e^4}{Q^2}L^{\mu\nu}(k,k')H_{\mu\nu}(P,q)dxdQ^2$$

the differential cross-section is:

$$\frac{d\sigma}{dxdQ^2} = \frac{1}{32\pi^2x^2s^2}\frac{e^4}{Q^2}L^{\mu\nu}(k,k')H_{\mu\nu}(P,q) = \frac{1}{32\pi^2x^2s^2}\frac{e^2}{Q^2}\left[2H_1Q^2 + H_2\left(\frac{s^2}{Q^2}-\frac{s}{x}\right)\right]$$
$$= \frac{1}{32\pi^2x^2s^2}\frac{16\pi^2\alpha^2}{Q^2}\left[2H_1Q^2 + H_2\left(\frac{s^2}{Q^2}-\frac{s}{x}\right)\right] = \frac{\alpha^2}{2x^2s^2}\frac{1}{Q^2}\left[2H_1Q^2 + H_2\left(\frac{s^2}{Q^2}-\frac{s}{x}\right)\right]$$

now redefining (just a matter of convention) $H_1 = 4\pi F_1$ and $H_2 = 8\pi x F_2$ we have:

$$\frac{d\sigma}{dxdQ^2} = \frac{\alpha^2}{2x^2s^2}\frac{1}{Q^2}\left[8\pi F_1Q^2 + 8\pi xF_2\left(\frac{s^2}{Q^2}-\frac{s}{x}\right)\right] = \frac{8\pi\alpha^2}{2x^2s^2}\frac{1}{Q^2}\left[F_1Q^2 + F_2\left(\frac{s^2x}{Q^2}-s\right)\right]$$
$$= \frac{4\pi\alpha^2}{x^2s^2}\frac{1}{Q^2}\left[F_1Q^2 + F_2\left(\frac{sx}{Q^2}s - s\right)\right] = \frac{4\pi\alpha^2}{x^2s^2}\frac{1}{Q^2}\left[F_1Q^2 + F_2\left(\frac{s}{y}-s\right)\right]$$
$$= \frac{4\pi\alpha^2}{xQ^2}\left[F_1\frac{Q^2}{xs^2} + \frac{1}{xs^2}F_2\left(\frac{s}{y}-s\right)\right] = \frac{4\pi\alpha^2}{xQ^2}\left[F_1x\frac{Q^2}{x^2s^2} + \frac{1}{xs}F_2\left(\frac{1}{y}-1\right)\right] =$$
$$= \frac{4\pi\alpha^2}{xQ^2}\left[F_1\frac{x}{Q^2}\frac{Q^4}{x^2s^2} + \frac{1}{Q^2}\frac{Q^2}{xs}F_2\left(\frac{1}{y}-1\right)\right] = \frac{4\pi\alpha^2}{xQ^4}\left[F_1xy^2 + F_2y\left(\frac{1}{y}-1\right)\right]$$

and finally we have the cross-section:

$$\boxed{\frac{d\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^4}\left[xy^2F_1(x,Q^2) + (1-y)F_2(x,Q^2)\right]} \tag{7}$$

The Fs are called the structure functions of the proton. It is common to see other linear combinations of the structure functions:

$$F_T(x,Q^2) = 2xF_1(x,Q^2)$$
$$F_L(x,Q^2) = F_2(x,Q^2) - 2xF_1(x,Q^2)$$

which correspond to scattering of transverse and longitudinally polarized photons respectively. We therefore have:

$$F_1(x,Q^2) = \frac{1}{2x}F_T(x,Q^2)$$
$$F_2(x,Q^2) = F_L(x,Q^2) + 2xF_1(x,Q^2) = F_L(x,Q^2) + F_T(x,Q^2)$$

the cross-section becomes:

$$\frac{d\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{xQ^4}\left\{xy^2\frac{1}{2x}F_T(x,Q^2) + (1-y)(F_L(x,Q^2)+F_T(x,Q^2))\right\}$$
$$= \frac{4\pi\alpha^2}{xQ^4}\left\{\left[\frac{y^2}{2}+(1-y)\right]F_T(x,Q^2)+(1-y)F_L(x,Q^2)\right\}$$
$$= \frac{2\pi\alpha^2}{xQ^4}\left\{2\frac{y^2+2-2y}{2}F_T(x,Q^2)+2(1-y)F_L(x,Q^2)\right\}$$
$$= \frac{2\pi\alpha^2}{xQ^4}\left\{[1+(1-y)^2]F_T(x,Q^2)+2(1-y)F_L(x,Q^2)\right\}$$

now if we re-express $F_T = F_2 - F_L$:

$$\frac{d\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}\left\{[1+(1-y)^2]F_2(x,Q^2)+[2(1-y)-1-(1-y^2)]F_L(x,Q^2)\right\}$$
$$=\frac{2\pi\alpha^2}{xQ^4}\left\{[1+(1-y)^2]F_2(x,Q^2)+[2-2y-1-1-y^2+2y)]F_L(x,Q^2)\right\}$$

we have finally the DIS cross-section as it is written nowadays:

$$\boxed{\left.\frac{d\sigma}{dxdQ^2}\right|_{eP\to eX} = \frac{2\pi\alpha^2}{xQ^4}\left\{[1+(1-y)^2]F_2(x,Q^2)-y^2F_L(x,Q^2)\right\}} \quad (8)$$

which has 2 structure functions!

Let's consider now the charged current neutrino DIS process:

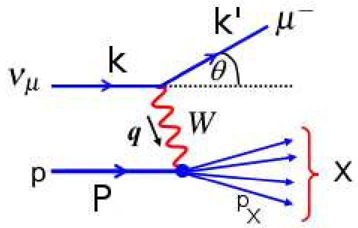


Fig. 4. Lowest order SM diagram for $\nu_\mu(k')P(P) \longrightarrow \mu^-(k)X(\{p_X\})$

The matrix element can be written, in the same notation of the electromagnetic scattering:

$$\mathcal{M} = \overline{u}(k')\frac{-ig_w}{2\sqrt{2}}\gamma_\mu(1-\gamma_5)u(k)\frac{-ig^{\mu\nu}}{q^2-M_W^2}\frac{-ig_W}{2\sqrt{2}}\langle X|J_\nu^{W,P}|P,P\rangle$$
$$= \frac{ig_W^2}{8(q^2-M_W^2)}\overline{u}(k')\gamma_\mu(1-\gamma_5)u(k)\langle X|J^\mu_{W,P}|P,P\rangle$$

squaring the matrix element we get:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \frac{g_W^4}{64(q^2-M_W^2)^2}\overline{u}(k')\gamma_\mu(1-\gamma_5)u(k)\overline{u}(k)\gamma_\nu(1-\gamma_5)u(k')\langle X|J^\mu_{W,P}|P,P\rangle\langle P,P|J^{\nu*}_{W,P}|X\rangle$$

using now the definition of Fermi constant:

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8M_W^2} \Rightarrow= \frac{g_W^4}{64} = \frac{G_F^2M_W^4}{2}$$

this gives that the matrix element can be written:

$$|\mathcal{M}|^2 = \frac{G_F^2}{2}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2\overline{u}(k')\gamma_\mu(1-\gamma_5)u(k)\overline{u}(k)\gamma_\nu(1-\gamma_5)u(k')\langle X|J^\mu_{W,P}|P,P\rangle\langle P,P|J^{\nu*}_{W,P}|X\rangle$$
$$= \frac{G_F^2}{2}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2\overline{u}(k')\gamma_\mu(1-\gamma_5)u(k)\overline{u}(k)\gamma_\nu(1-\gamma_5)u(k')T_W^\mu(\{p_X\})T_W^{\nu*}(\{p_X\})$$

now the cross section is given by:

$$d\sigma = \frac{1}{F}\frac{d^3k'}{(2\pi)^3 2E'}\sum_X (2\pi)^4 \int d\Phi_X \delta^4(k+P-k'-p_X)\frac{1}{2}\sum_{spins}|\mathcal{M}|^2$$

summing on final state and averaging on initial states (neutrinos have only one state of polarizations if we assume them massless) we get:

$$\begin{aligned}
\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 &= \frac{G_F^2}{4}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 \sum_{spins}[\overline{u}(k')\gamma_\mu(1-\gamma_5)u(k)\overline{u}(k)\gamma_\nu(1-\gamma_5)u(k')]\sum_{spins}T_W^\mu(\{p_X\})T_W^{\nu*}(\{p_X\})\\
&= \frac{G_F^2}{4}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 \mathrm{Tr}[\sum_{s'}u(k',s')\overline{u}(k',s')\gamma_\mu(1-\gamma_5)\sum_s u(k,s)\overline{u}(k,s)\gamma_\nu(1-\gamma_5)]\times\\
&\times \sum_{spins}T_W^\mu(\{p_X\})T_W^{\nu*}(\{p_X\})\\
&= \frac{G_F^2}{4}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 \mathrm{Tr}[\not k'\gamma_\mu(1-\gamma_5)\not k\gamma_\nu(1-\gamma_5)]\sum_{spins}T_W^\mu(\{p_X\})T_W^{\nu*}(\{p_X\})\\
&= G_F^2\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 L_{\mu\nu}(k,k')\sum_{spins}T_W^\mu(\{p_X\})T_W^{\nu*}(\{p_X\})
\end{aligned}$$

Let's now define:

$$\sum_X (2\pi)^4\int d\Phi_X\delta^4(k+P-k'-p_X)\frac{1}{2}\sum_{spins}|\mathcal{M}|^2 = G_F^2\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 L_{\mu\nu}(k,k')H^{\mu\nu}(Q^2,x)$$

where the hadronic tensor is defined as:

$$H^{\mu\nu}(Q^2,x) = \sum_X(2\pi)^4\int d\Phi_X\delta^4(k+P-k'-p_X)\sum_{spins}T_W^\mu(\{p_X\})T_W^{\nu*}(\{p_X\})$$

Going ahead in the same way as for the electromagnetic scattering we have:

$$L_{\mu\nu}(k,k') = \mathrm{Tr}[\not k'\gamma_\mu(1-\gamma_5)\not k\gamma_\nu(1-\gamma_5)] = L^e_{\mu\nu}(k,k') + 2i\varepsilon_{\mu\nu\rho\sigma}k^\rho k^\sigma$$

while the hadronic tensor has the same form as before, but for the coefficient H_3 which is allowed since weak interactions violating parity allow for an additional Lorentz structure:

$$H^{\mu\nu}(Q^2,x) = -H_1 g^{\mu\nu} + H_2\frac{P^\mu P^\nu}{Q^2} - \frac{i}{Q^2}\varepsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma H_3$$

The contraction of the two tensor gives:

$$L_{\mu\nu}(k,k')H^{\mu\nu}(Q^2,x) = 2Q^2H_1 + Q^2\frac{1-y}{x^2y^2}H_2 + \frac{Q^2}{xy}H_3\left(1-\frac{y}{2}\right)$$

the cross-section is therefore, using the results for the electron phase space derived explicitely in the Problem Sheet 4:

$$\begin{aligned}
d\sigma &= \frac{1}{2s}\frac{Q^2}{16\pi^2 sx^2}dQ^2dx\; G_F^2\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 L_{\mu\nu}(k,k')H^{\mu\nu}(Q^2,x)\\
&= \frac{Q^2}{32\pi^2s^2x^2}G_F^2\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 L_{\mu\nu}(k,k')H^{\mu\nu}(Q^2,x)dQ^2dx
\end{aligned}$$

performing the algebra to get the cross-section we get:

$$\frac{d\sigma}{dxdQ^2} = \frac{Q^2}{32\pi^2 s^2 x^2} G_F^2 \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[2Q^2 H_1 + Q^2 \frac{1-y}{x^2 y^2} H_2 + \frac{Q^2}{xy} H_3 \left(1 - \frac{y}{2}\right) \right]$$
$$= \frac{Q^4}{32\pi^2 s^2 x^2} G_F^2 \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[2H_1 + \frac{1-y}{x^2 y^2} H_2 + \frac{\left(1 - \frac{y}{2}\right)}{xy} H_3 \right]$$
$$= \frac{y^2}{32\pi^2} G_F^2 \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[2H_1 + \frac{1-y}{x^2 y^2} H_2 + \frac{\left(1 - \frac{y}{2}\right)}{xy} H_3 \right]$$

Now we can re-define the structure functions as:

$$H_1 \equiv 4\pi F_1 \quad H_2 \equiv 8\pi x F_2 \quad H_3 \equiv 8\pi x F_3$$

so that the cross-section is:

$$\frac{d\sigma}{dxdQ^2} = \frac{y^2}{32\pi^2} G_F^2 \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[8\pi F_1 + 8\pi \frac{1-y}{xy^2} F_2 + 8\pi \frac{\left(1 - \frac{y}{2}\right)}{y} F_3 \right]$$
$$= \frac{1}{4\pi^2} G_F^2 \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[y^2 F_1 + \frac{1-y}{x} F_2 + \left(y - \frac{y^2}{2} \right) F_3 \right]$$
$$= \frac{G_F^2}{4\pi^2 x} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[xy^2 F_1 + (1-y) F_2 + \left(y - \frac{y^2}{2} \right) x F_3 \right]$$

Let's now define an alternative set of structure functions:

$$F_T = 2xF_1 \to F_1 = F_T/2x$$
$$F_L = F_2 - 2xF_1 \to F_2 = F_L + F_T$$

and finally the cross-section becomes:

$$\frac{d\sigma}{dxdQ^2} = \frac{G_F^2}{4\pi^2 x} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[xy^2 \frac{F_T}{2x} + (1-y) F_L + (1-y) F_L + \left(y - \frac{y^2}{2} \right) x F_3 \right]$$
$$= \frac{G_F^2}{4\pi^2 x} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[\frac{1}{2} (y^2 - 2 - 2y) F_T + (1-y) F_L + \left(y - \frac{y^2}{2} \right) x F_3 \right]$$
$$= \frac{G_F^2}{4\pi^2 x} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[(1 + (1-y^2)) F_T + 2(1-y) F_L + \left(2y - y^2 \right) x F_3 \right]$$
$$= \frac{G_F^2}{4\pi^2 x} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left[(1 + (1-y^2)) F_T + 2(1-y) F_T + (1 - (1-y^2)) x F_3 \right]$$

and finally returning to the structure function F_2:

$$\boxed{\frac{d\sigma}{dxdQ^2} = \frac{G_F^2}{4\pi^2 x} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 \left\{ (1 + (1-y^2)) F_2(x, Q^2) - y^2 F_L(x, Q^2) + (1 - (1-y^2)) x F_3(x, q^2) \right\}}$$

This cross-section contains 3 structure functions!
Let's finally consider the simple case of the electron-positron annihilation to hadrons. The cross-section for this process can be written:

$$\sigma = \frac{1}{2s} \sum_X \int d\Phi_X (2\pi)^4 \delta^4 (k + k' - p_X) \frac{1}{4} \sum_{spins} |\mathcal{M}_{e^+ e^- \to X}|^2$$

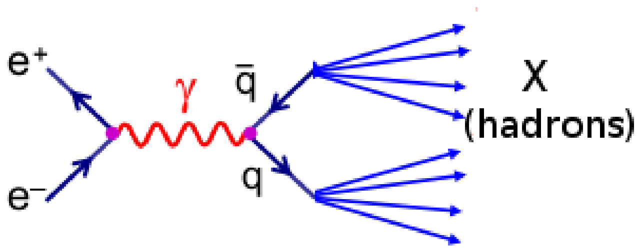


Fig. 5. Lowest order QED diagram for $e^+(k') + e^-(k) \to \gamma^*(q = k + k') \longrightarrow X(\{p_X\})$

The lowest order QED diagram for the process is:
The matrix element is the following:

$$\mathcal{M}_X = e\langle X|J_h^\nu|0\rangle \frac{-ig_{\mu\nu}}{q^2}\overline{v}(k')(ie\gamma^\mu)u(k) = \frac{e^2}{q^2}\langle X|J_h^\mu|0\rangle\overline{v}(k')\gamma_\mu u(k)$$

squaring the matrix element we get:

$$|\mathcal{M}_X|^2 = \mathcal{M}_X\mathcal{M}_X^* = \frac{e^4}{q^4}\overline{v}(k')\gamma_\mu u(k)\overline{u}(k)\gamma_\nu v(k')\langle 0|J_h^{\nu *}|X\rangle\langle X|J_h^\mu|0\rangle$$

and the averaged matrix element is:

$$|\overline{\mathcal{M}_X}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}_{e^+e^-\to X}|^2 = \frac{1}{4}\sum_{spins}\sum_{X spins}\frac{e^4}{q^4}\overline{v}(k')\gamma_\mu u(k)\overline{u}(k)\gamma_\nu v(k')\langle 0|J_h^{\nu *}|X\rangle\langle X|J_h^\mu|0\rangle$$

we now postulate that the matrix element for the sum of all diagrams in which a virtual photon with Lorentz index ν and momentum q produces a particular set of n hadrons with momenta $\{p_1, ..., p_n\} = \{p_X\}$ is known, and we parametrize it by a function:

$$\langle X|J_h^\mu|0\rangle \equiv T_\mu(n, q, \{p_1, ..., p_n\})$$

returning now to the cross-section expression:

$$\sigma = \frac{1}{2s}\sum_X\int d\Phi_X\frac{1}{4}\sum_{spins}\sum_{X spins}(2\pi)^4\delta^4(k+k'-p_X)\frac{e^4}{q^4}\overline{v}(k')\gamma_\mu u(k)\overline{u}(k)\gamma_\nu v(k')T_\nu^*(n,q,\{p_X\})T_\mu(n,q,\{p_X\})$$

and it is convenient to rewrite this expression as:

$$\begin{aligned}\sigma &= \frac{1}{2s}\frac{1}{4}\frac{e^4}{q^4}\underbrace{\sum_{spins}\overline{v}(k')\gamma_\mu u(k)\overline{u}(k)\gamma_\nu v(k')}_{L_{\mu\nu}(k,k')}\sum_X\sum_{X spins}(2\pi)^4\delta^4(q-p_X)\int d\Phi_X T^{*\nu}(n,q,\{p_X\})T^\mu(n,q,\{p_X\}) \\ &= \frac{1}{2s}\frac{1}{4}\frac{e^4}{q^4}L_{\mu\nu}(k,k')H^{\mu\nu}(q)\end{aligned}$$

where we have defined the hadronic tensor $H^{\mu\nu}(q)$ as:

$$\boxed{H^{\mu\nu}(q) = \sum_X\sum_{X spins}\int d\Phi_X(2\pi)^4\delta^4(k+k'-p_X)T^{*\nu}(n,q,\{p_X\})T^\mu(n,q,\{p_X\})}$$

this tensor after the integration over all of possible final states and summation over all possible spin configurations, can only be a function of the four-vector available at the photon vertex, i.e. q^μ. Now the only two possible Lorentz covariant two-index tensor functions of one four vector are $g^{\mu\nu}$ and $q^\mu q^\nu$. We therefore parametrize $H_{\mu\nu}$ as a linear combination of these, with coefficients that are functions of the only available Lorentz scalar q^2,

$$H^{\mu\nu} = A(q^2)g^{\mu\nu} + B(q^2)q^\mu q^\nu$$

finally since the theory is gauge invariant, charge conservation implies that the matrix element for the absorbtion process of a photon by the hadronic current is invariant under a gauge transformation, i.e.:

$$\begin{aligned}\mathcal{M}_0 &= \varepsilon_\mu(q,\lambda)T^\mu(n,q,\{p_X\}) \to (\varepsilon_\mu(q,\lambda) + q_\mu)T^\mu(n,q,\{p_X\}) = \mathcal{M}_0 \\ &\Rightarrow q^\mu T_\mu(n,q,\{p_X\}) = 0\end{aligned}$$

this implies $q_\mu H^{\mu\nu}(q) = q_\nu H^{\mu\nu}(q) = 0$:

$$q_\mu H^{\mu\nu}(q) = \sum_X \sum_{X spins} \int d\Phi_X (2\pi)^4 \delta^4(k+k'-p_X) T^{*\nu}(n,q,\{p_X\}) \Big[q_\mu T^\mu(n,q,\{p_X\}) \Big] = 0$$

therefore:

$$\begin{aligned} q_\mu H^{\mu\nu}(q) &= q_\mu [A(q^2) g^{\mu\nu} + B(q^2) q^\mu q^\nu] = A(q^2) q^\nu + B(q^2) q^2 q^\nu = [A(q^2) + q^2 B(q^2)] q^\nu = 0 \\ &\Rightarrow \quad A(q^2) = -q^2 B(q^2) \end{aligned}$$

the hadronic tensor is therefore written:

$$H^{\mu\nu} = -q^2 B(q^2) g^{\mu\nu} + B(q^2) q^\mu q^\nu = B(q^2)[q^\mu q^\nu - q^2 g^{\mu\nu}]$$

on the other hand the leptonic tensor is:

$$\begin{aligned} L_{\mu\nu}(k,k') &= \sum_{spins} \overline{v}(k')\gamma_\mu u(k) \overline{u}(k) \gamma_\nu v(k') = Tr\left[\sum_{s'} v(k',s')\overline{v}(k',s')\gamma_\mu \sum_s u(k,s)\overline{u}(k,s)\gamma_\nu \right] \\ &= Tr\left[(\not{k}' - m_e)\gamma_\mu (\not{k} + m_e)\gamma_\nu \right] = k'^\rho k^\sigma Tr\left[\gamma_\rho \gamma_\mu \gamma_\sigma \gamma_\nu \right] - m^2 Tr[\gamma_\mu \gamma_\nu] \\ &= k'^\rho k^\sigma 4(g_{\rho\mu} g_{\sigma\nu} + g_{\rho\nu} g_{\sigma\mu} - g_{\rho\sigma} g_{\mu\nu}) - 4m_e^2 g_{\mu\nu} = 4(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k' + m_e^2) g_{\mu\nu}) \end{aligned}$$

in the high energy limit the electron mass is negligible and the leptonic tensor is given by:

$$L_{\mu\nu}(k,k') = 4(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k') g_{\mu\nu})$$

this gives:

$$\begin{aligned} \sum_X \int d\Phi_X (2\pi)^4 \delta^4(k+k'-p_X) \frac{1}{4} |\overline{\mathcal{M}}|^2 &= \frac{1}{4}\frac{e^4}{q^4} L_{\mu\nu}(k,k') H^{\mu\nu} \\ &= \frac{1}{4}\frac{e^4}{q^4} 4(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k') g_{\mu\nu}) B(q^2)[q^\mu q^\nu - q^2 g^{\mu\nu}] \\ &= -q^2 B(q^2) \frac{e^4}{q^4} (k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k') g_{\mu\nu}) g^{\mu\nu} \\ &= -B(q^2)\frac{e^4}{q^2}[(k\cdot k') + (k\cdot k') - 4(k\cdot k')] = B(q^2)\frac{e^4}{q^2} \underbrace{2(k\cdot k')}_{=q^2} = e^4 B(q^2) \end{aligned}$$

where we have used: $q^2 = (k+k')^2 \simeq 2k\cdot k'$.
In the calculation we have neglected terms proportional to $q^\mu L_{\mu\nu}(k,k')$ since this are vanishing because of gauge invariance, since the electron current is conserved. Let's derive this directly:

$$\begin{aligned} q^\mu L_{\mu\nu}(k,k') &= q^\mu 4(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k') g_{\mu\nu}) = 4((k\cdot q)k'_\nu + k_\nu (k\cdot q) - (k\cdot k') q_\nu) \\ &= 4(\underbrace{(k\cdot k)k'_\nu}_{\propto m_e^2 \simeq 0} + (k\cdot k')k'_\nu + \underbrace{k_\nu (k\cdot k)}_{m_e^2 \simeq 0} + k_\nu (k\cdot k') - (k\cdot k') q_\nu) \\ &= 4(k\cdot k')(k'_\nu + k_\nu - q_\nu) = 0 \end{aligned}$$

the cross-section is:

$$\sigma = \frac{1}{2s}\frac{1}{4}\frac{e^4}{q^4} L_{\mu\nu}(k,k') H^{\mu\nu} = \frac{e^4 B(q^2)}{2s} = \frac{(4\pi\alpha)^2 B(q^2)}{2s} = \frac{8\pi^2\alpha^2}{s} B(s)$$

The final step is to realize that $B(s)$ has to be dimensionless because of the physical dimensions of $[\sigma] = [E^{-2}]$.
Since it is a function of only one dimensionful parameter, it must therefore be constant.
It is convenient now to rewrite the cross-section as:

$$\sigma(e^+e^- \to X) = \frac{8\pi^2\alpha^2}{s}B(s) = \frac{4\pi\alpha^2}{3s}\underbrace{\frac{6}{\pi}B(s)}_{\mathcal{R}(s)}$$

this gives us finally:

$$\boxed{\sigma(e^+e^- \to X(hadrons)) = \frac{4\pi\alpha^2}{3s}\mathcal{R}(s)}$$

We therefore have the fundamental prediction that (for energies above all hadron masses) the cross-section to produce any number of hadrons is proportional to that to produce a muon-antimuon pair:

$$\mathcal{R}(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = constant$$

So summarizing:

- The process $\sigma(e^+e^- \to hadrons)$ has only one structure function since after requirements on Lorentz and gauge invariance there is only one out of two structure functions which are non-zero, (this is due to the fact that there is only one Lorentz vector available at the vertex of annihilation)
- The process $\sigma(e^-P \to e^-X)$ has two structure functions since with two Lorentz vectors available at the scattering vertex is possible to construct six independent symmetric rank two Lorentz tensor. Requiring gauge invariance and Lorentz invariance only 2 are independent.
- The process $\sigma(\nu_\mu P \to \mu^-X)$ has three structure functions, one more w.r.t the electromagnetic process. This happens since weak interactions violate parity allowing for an extra Lorentz structure which is anti-symmetric in the Lorentz indices.

2.4. Problem 4

Given the +-distribution defined by:

$$\int_0^1 \frac{f(x)}{(1-x)_+}dx = \int \frac{f(x)-f(1)}{1-x}dx$$

and:

$$\frac{1}{(1-x)_+} = \frac{1}{1-x} \qquad for \;\; 0 \le x < 1,$$

show that:

$$\int_0^1 P_{qq}^{(0)}(x)dx = 0$$

where:

$$P_{qq}^{(0)}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

What is the significance of this result?

Solution

The calculation is straigthforward:

$$\int_0^1 P_{qq}^{(0)}(x)dx = \int_0^1 C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]dx = C_F\left[\frac{3}{2} + \int_0^1 \frac{1+x^2}{(1-x)_+}dx\right]$$

$$= C_F\left[\frac{3}{2} + \int_0^1 \frac{(1+x^2)|_x - (1+x^2)|_1}{(1-x)}\right]dx = C_F\left[\frac{3}{2} + \int_0^1 \frac{1+x^2-(1+1^2)}{(1-x)}dx\right]$$

$$= C_F\left[\frac{3}{2} + \int_0^1 \frac{x^2-1}{(1-x)}dx\right] = C_F\left[\frac{3}{2} + \int_0^1 \frac{(x-1)(x+1)}{(1-x)}dx\right] = C_F\left[\frac{3}{2} + \int_0^1 -(1+x)dx\right]$$

and finally:

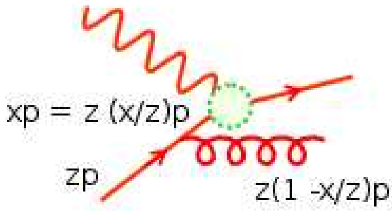
$$\int_0^1 P_{qq}^{(0)}(x)dx = C_F\left[\frac{3}{2} - \left[\frac{x^2}{2}+x\right]_0^1 dx\right] = C_F\left[\frac{3}{2}-\left(1+\frac{1}{2}\right)\right] = 0$$

When studying the P.D.F. evolution, at first order in α_S one has to apply the following correction:

$$f_q^{(0)}(x) \longrightarrow f_q^{(1)}(x,Q^2) = \int_x^1 \frac{dz}{z} f_q^{(0)}(z)\mathcal{P}_{qq}(\frac{x}{z},Q^2)$$

with the kernel splitting function:

$$\mathcal{P}_{qq}(\xi,Q^2) = \delta(1-\xi) + \frac{\alpha_S}{\pi}P_{qq}(\xi)\ln\frac{Q^2}{\lambda^2}$$



$\mathcal{P}_{qq}(\xi,Q^2)$ can be regarded as the probability for a quark to split into a quark carrying a fraction ξ of momentum and a gluon with a fraction $1-\xi$ (in other words, the probability for the quark to "contain" a quark with a fraction of momentum ξ and a gluon with a fraction $1-\xi$).
Being a probability, its norm must be 1:

$$\int_0^1 d\xi\mathcal{P}_{qq}(\xi,Q^2) = \underbrace{\int_0^1 \delta(1-\xi)d\xi}_{=1} + \frac{\alpha_S}{\pi}\ln\frac{Q^2}{\lambda^2}\int_0^1 P_{qq}(\xi)d\xi = 1$$

Therefore:

$$\int_0^1 P_{qq}(\xi)d\xi \equiv 0$$

Moreover, if $\int_0^1 P_{qq}(\xi)d\xi \neq 0$ the norm of $\mathcal{P}_{qq}(\xi,Q^2)$ would be a function of Q^2.
The physical meaning of this constraing is that the total number of quarks-antiquarks is conserved.

3. Problem Sheet 3

3.1. Problem 1

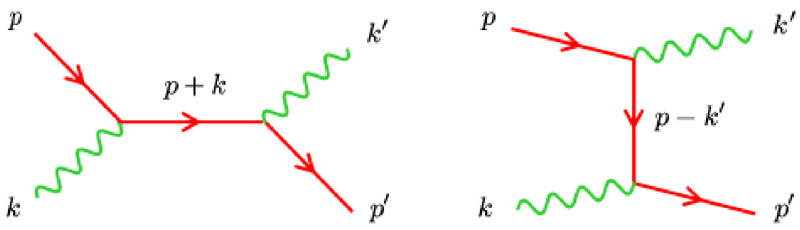
On proving gauge invariance show that in the so-called Landau gauge:

$$\sum_\lambda \varepsilon^\mu(k,\lambda)e^{\nu *}(k,\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2 + i\epsilon}$$

the term $\propto k^\mu k^\nu$ added to the photon spin sum $-g^{\mu\nu}$ of the Feynman gauge does not contribute to QED Compton scattering $e^-\gamma \to e^-\gamma$. Neglect the electron mass.

3.2. Solution

Two diagrams contribute at LO to the QED Compton scattering matrix element:



In the Feynman gauge we have the following expressions for the two diagrams :

$$\mathcal{M}_1 = \overline{u}(p')(ie\gamma^\nu)\varepsilon^*_{2,\nu}(\lambda')\frac{i}{\not p + \not k - m}(ie\gamma^\mu)\varepsilon_{1,\mu}(\lambda)u(p) \tag{9}$$

$$\mathcal{M}_2 = \overline{u}(p')(ie\gamma^\nu)\varepsilon_{1,\nu}(\lambda)\frac{i}{\not p - \not k' - m}(ie\gamma^\mu)\varepsilon^*_{2,\mu}(\lambda')u(p) \tag{10}$$

swapping the Lorentz indexes in order to isolate the photon contributions we get:

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \overline{u}(p')(ie\gamma^\nu)\varepsilon^*_{2,\nu}(\lambda')\frac{i}{\not p + \not k - m}(ie\gamma^\mu)\varepsilon_{1,\mu}(\lambda)u(p) + \overline{u}(p')(ie\gamma^\mu)\varepsilon_{1,\mu}(\lambda)\frac{i}{\not p - \not k' - m}(ie\gamma^\nu)\varepsilon^*_{2,\nu}(\lambda')u(p)$$

and after some algebra:

$$\mathcal{M} = -ie^2\overline{u}(p')\left[\gamma^\nu\frac{1}{\not p + \not k - m}\gamma^\mu + \gamma^\mu\frac{1}{\not p - \not k' - m}\gamma^\nu\right]u(p)\varepsilon_{1,\mu}(\lambda)\varepsilon^*_{2,\nu}(\lambda')$$

This means that the amplitude can be factorized:

$$\mathcal{M} = \mathcal{T}^{\mu\nu}\varepsilon_{1,\mu}(\lambda)\varepsilon^*_{2,\nu}(\lambda') \tag{11}$$

now neglecting electron masses $m_e \equiv m = 0$ we can write:

$$\mathcal{M} = -ie^2\overline{u}(p')\left[\gamma^\nu\frac{1}{\not p + \not k}\gamma^\mu + \gamma^\mu\frac{1}{\not p - \not k'}\gamma^\nu\right]u(p)\varepsilon_{1,\mu}(\lambda)\varepsilon^*_{2,\nu}(\lambda')$$

now we can use the following fact:

$$\not p^2 = \not p\not p = \gamma^\mu p_\mu\gamma^\nu p_\nu = (\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu)\,p_\mu p_\nu = (2g^{\mu\nu} - \gamma^\mu\gamma^\nu)\,p_\mu p_\nu = 2p^2 - \not p^2 \Rightarrow \boxed{\not p^2 = p^2}$$

so we get:

$$\frac{1}{\not p} = \frac{1}{\not p}\cdot\frac{\not p}{\not p} = \frac{\not p}{p^2}$$

and the amplitude for Compton scattering can be written as:

$$\mathcal{M} = -ie^2\overline{u}(p')\left[\gamma^\nu\frac{\not p + \not k}{(p+k)^2}\gamma^\mu + \gamma^\mu\varepsilon_{1,\mu}\frac{\not p - \not k'}{(p-k')^2}\gamma^\nu\right]u(p)\varepsilon_{1,\mu}(\lambda)\varepsilon^*_{2,\nu}(\lambda')$$

we can then make use of Mandelstamm invariants for a $2 \to 2$ reaction: $s = (p+k)^2$, $t = (p-k)^2$, $u = (p-k')^2$ to further simplify the expression for the invariant amplitude:

$$\mathcal{M} = -ie^2 \overline{u}(p') \left[\gamma^\nu \frac{\not{p} + \not{k}}{s} \gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k}'}{u} \gamma^\nu \right] u(p) \varepsilon_{1,\mu}(\lambda) \varepsilon^*_{2,\nu}(\lambda')$$

We are now ready to square the invariant amplitude in order to get the matrix element for the considered reaction:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \mathcal{T}^{\mu\nu}\mathcal{T}^{*\rho\sigma} \varepsilon_{1,\mu}(\lambda) \varepsilon^*_{2,\nu}(\lambda') \varepsilon^*_{1,\rho}(\lambda) \varepsilon_{2,\sigma}(\lambda')$$

now summing on final states and averaging on initial states we get:

$$|\mathcal{M}|^2 = \frac{1}{4} \sum_{s,s'} \left[\mathcal{T}^{\mu\nu}\mathcal{T}^{*\rho\sigma} \right] \left(\sum_\lambda \varepsilon_{1,\mu}(\lambda) \varepsilon^*_{1,\rho}(\lambda) \right) \left(\sum_{\lambda'} \varepsilon^*_{2,\nu}(\lambda') \varepsilon_{2,\sigma}(\lambda') \right)$$

and this imply that to calculate the matrix element squared we nee to evaluate the spin sum of the photon. As seen in the text of the problem this spin sum in the Landau Gauge contain terms proportional to the photon momenta.
This terms does not yield any physical content to the matrix element since they are canceled because of gauge invariance. This can be easily proved: Let's consider the Euler-Lagrange equations of motion for the photon field (Maxwell Equations) in absence of external currents:

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu \left(\partial^\mu A_\mu \right) = 0$$

These motion equations are left unchanged if we perform a gauge transformation on the photon field:

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi$$

We can use this freedom to *choose* the A^μ field to satisfy the condition:

$$\boxed{\partial^\mu A_\mu = 0} \tag{12}$$

which is colled *Lorentz Condition*. The condition in Eq. 12 is very useful since it decouples the different components of the photon field and leaves us with a very simple equation of motion:

$$\Box A^\mu = 0$$

This equation has plane wave solutions of the form:

$$A^\mu = \mathcal{N} \varepsilon^\mu e^{-ik\cdot x}$$

provided that $k^2 = 0$, i.e. $\mathbf{k^2}= k_0^2$ since:

$$\Box A^\mu = \Box \left[\mathcal{N} \varepsilon^\mu e^{-ik\cdot x} \right] = -k^2 \mathcal{N} \varepsilon^\mu e^{-ik\cdot x} = 0$$

this implies that (from Eq. 12):

$$\partial^\mu A_\mu = \partial^\mu \mathcal{N} \varepsilon_\mu e^{-ik\cdot x} = -\mathcal{N} k^\mu \varepsilon_\mu e^{-ik\cdot x} \Rightarrow \boxed{\varepsilon \cdot k = 0} \tag{13}$$

However this has not exhausted our gauge freedom, we are still free to make another shift in the potential:

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \tilde{\chi}$$

provided that the $\tilde{\chi}$ field obeys the massless Klein-Gordon equation $\Box\tilde{\chi} = 0$, and the resulting potential still satisfies the Lorentz condition in Eq.12. This $\tilde{\chi}$ field has plane wave solutions of the type:

$$\tilde{\chi} = Ae^{ik\cdot x} \to \Box\tilde{\chi} = -k^2 Ae^{-ik\cdot x}$$

this means that the photon field in the new gauge has the form:

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \tilde{\chi} = \mathcal{N} \varepsilon^\mu e^{-ik\cdot x} + ik^\mu A e^{-ik\cdot x} = \mathcal{N} \left(\varepsilon^\mu + \beta k^\mu \right) e^{-ik\cdot x}$$

This means that producing a gauge transformation implies a shift of the polarization vector:

$$\varepsilon^\mu \to \varepsilon'^\mu = \varepsilon^\mu + \beta k^\mu \tag{14}$$

this field still satisfy the transversity condition in Eq.13 ($k \cdot \varepsilon = 0$), since $k^2 = 0$ for free field solutions of the equation of motion. Let's now consider a process with one external photon leg, with momentum k^μ, whose polarization state is described by the polarization vector $\varepsilon^\mu(k, \lambda)$:
the amplitude associated to this process is in general separable between a contribution due to the external photon and one associated to the current to which the photon is coupled:

$$\mathcal{M}_\gamma = \varepsilon^\mu(k,\lambda) \mathcal{T}_\mu$$

With the Lorentz choice of trasversality ($k \cdot \varepsilon$) we have that the amplitude must be gauge invariant, as should be any physical quantity, under a tranformation of the type of Eq. 14, which means:

$$\mathcal{M}_\gamma \to M'_\gamma = \left(\varepsilon^\mu(k,\lambda) + \beta k^\mu \right) \mathcal{T}_\mu = \mathcal{M}_\gamma \Rightarrow k^\mu \mathcal{T}_\mu = 0$$

$$\boxed{k^\mu \mathcal{T}_\mu = 0} \tag{15}$$

This important and general result, stated in Eq. 15 is known as *Ward Identity*. Let's check in detail that this identity holds for the Compton scattering amplitude. The amplitude can be written separating the contribution from the external photons as in Eq. 11. Making the substitution $\varepsilon_{1,\mu}(k,\lambda) \to k^\mu$ we get:

$$k_\mu \mathcal{T}^\mu = -ie^2 \overline{u}(p') \left[\gamma^\nu \frac{\not{p} + \not{k}}{s} \gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k}'}{u} \gamma^\nu \right] u(p) k_\mu \varepsilon^*_{2,\nu}(\lambda')$$

let's work separately now on each term coming from one of the two diagrams contributing to the process at L0. we have:

$$k_\mu \mathcal{D}_1^\mu = -i\frac{e^2}{s} \overline{u}(p')\gamma^\nu(\not{p} + \not{k})\not{k} u(p) \varepsilon^*_{2,\nu}(\lambda') \tag{16}$$

$$k_\mu \mathcal{D}_2^\mu = -i\frac{e^2}{u} \overline{u}(p')\not{k}(\not{p} - \not{k}')\gamma^\nu u(p) \varepsilon^*_{2,\nu}(\lambda') \tag{17}$$

working on Eq. 16 one can notice that:

$$\not{k} u(p) = (\not{k}' + \not{p}' - \not{p})u(p) = (\not{k} + \not{p} - m)u(p) = (\not{k} + \not{p})u(p)$$

having used 4-momentum conservation ($k + p = k' + p'$) and Dirac Equation ($(\not{p})u(p) = mu(p)$), and neglecting electron mass in the last step. Eq. 16 becomes:

$$k_\mu \mathcal{D}_1^\mu = -i\frac{e^2}{s} \overline{u}(p')\gamma^\nu(\not{p} + \not{k})^2 u(p) \varepsilon^*_{2,\nu}(\lambda') = -i\frac{e^2}{s} \overline{u}(p')\gamma^\nu(p + k)^2 u(p) \varepsilon^*_{2,\nu}(\lambda') = -ie^2 \overline{u}(p')\not{\varepsilon}_2(k,\lambda')u(p)$$

which is manifestly non-vanishing. working on Eq. 17 one can notice that:

$$\overline{u}(p')\not{k} = \overline{u}(p')(\not{p}' + \not{k}' - \not{p}) = \overline{u}(p')(m + \not{k}' - \not{p}) = -\overline{u}(p')(\not{p} - \not{k}')$$

having used again momentum conservation and the Dirac Equation. Eq. 17 becomes:

$$k_\mu \mathcal{D}_2^\mu = i\frac{e^2}{u} \overline{u}(p')(\not{p} - \not{k}')^2\gamma^\nu u(p) \varepsilon^*_{2,\nu}(\lambda') = i\frac{e^2}{u} \overline{u}(p')\gamma^\nu(p - k')^2 u(p) \varepsilon^*_{2,\nu}(\lambda') = ie^2 \overline{u}(p')\not{\varepsilon}_2(k,\lambda')u(p)$$

which is again manifestly non-vanishing. Combining the two results instead we obtain:

$$k_\mu \mathcal{T}^\mu = k_\mu(\mathcal{D}_1^\mu + \mathcal{D}_2^\mu) = -ie^2 \overline{u}(p')\not{\varepsilon}_2(k,\lambda')u(p) - ie^2 \overline{u}(p')\not{\varepsilon}_2(k,\lambda')u(p) = 0$$

This result is expected because of gauge invariance of QED. On the other hand this implies that the terms $\propto k^\mu k^\nu$ in the Landau gauge form of the foton spin sum yield no physics content, since they are "gauged away".
As a side point this line of reasoning allows to derive in the Feynman gauge the photon spin sum in a covariant way. In effect considering the general "'one photon" process with amplitude $A_\gamma = \varepsilon^\mu(k, \lambda\mathcal{T}_\mu)$. The photon polarizations physically allowed are $\varepsilon^\mu(k,1) = (0,1,0,0)$ and $\varepsilon^\mu(k,2) = (0,0,1,0)$ and the physical 4-momentum is $k^\mu = (k,0,0,k)$. Then the required polarization sum would be:

$$|\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon^\mu(k,\lambda)\mathcal{T}_\mu \varepsilon^{*\nu}(k,\lambda)\mathcal{T}^*_\nu = |\mathcal{T}_1|^2 + |\mathcal{T}_2|^2$$

However, we have also the Ward Identity in Eq. 15 $k^\mu \mathcal{T}_\mu = 0$. This tells us:

$$k^0\mathcal{T}_0 + k^3\mathcal{T}_3 = k(\mathcal{T}_0 - \mathcal{T}_3)$$

hence the squared amplitude is:

$$|\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda)\mathcal{T}_\mu\mathcal{T}^*_\nu = |\mathcal{T}_1|^2 + |\mathcal{T}_2|^2 + |\mathcal{T}_3|^2 - |\mathcal{T}_0|^2 = -g^{\mu\nu}\mathcal{T}_\mu\mathcal{T}^*_\nu$$

and we get the identity for the photon spin sum:

$$\boxed{\sum_{\lambda=1,2} \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu}}$$

4. Problem Sheet 4

4.1. Problem 1

Consider the process (photon exchange only)

$$e^- + Q(\xi p) \rightarrow e^{'}(k') + Q(\xi p + q)$$

where Q represents a generic massless quark of EM charge e_q and e^- a massless electron. Schematically:

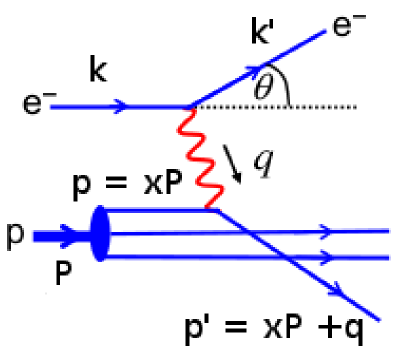


Fig. 6. Elastic scattering process $e^- + Q(\xi p) \rightarrow e^{'}(k') + Q(\xi p + q)$

show that the double-differential cross-section may be written:

$$\frac{d^2\widehat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^2}\left[1 + (1 - y^2)\right]\frac{1}{2}e_q^2\delta(x - \xi)$$

One may use the expression:

$$|\overline{\mathcal{M}}|^2 = 2e_q^2 e^4 \frac{\widehat{s}^2 + \widehat{u}^2}{\widehat{t}^2}$$

for the matrix element squared summed/averaged over final/initial colours and spins.(Hint: express $\widehat{s}$, $\widehat{t}$, $\widehat{u}$ in terms od the standard DIS variables)

$$Q^2 = -q^2, \qquad x = \frac{-q^2}{2q \cdot P}, \qquad \text{and} \quad y = \frac{q \cdot P}{k \cdot P}$$

Solution

The matrix element for the elementary electron-parton scattering in the single photon approximation is:

$$\mathcal{M} = \overline{u}(p')(iee_q\gamma^\mu)u(p)\frac{-ig_{\mu\nu}}{(k - k')^2}\overline{u}(k')(ie\gamma^\nu)u(k)$$

where the incoming(outgoing) electron has momentum $k(k')$ while the incoming(outgoing) parton has momentum $p = \xi P(p')$. Momentum conservation implies: $k - k' = q = p' - p$. So the amplitude can be written:

$$\mathcal{M} = -i\frac{e_q e^2}{q^2}\Big(\overline{u}(p')\gamma^\mu u(p)\Big)\Big(\overline{u}(k')\gamma_\mu u(k)\Big)$$

Averaging on initial and summing over final spins (assuming the scattering happens on a charged parton of spin 1/2) we get:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}\sum_{spins}\mathcal{M}\mathcal{M}^* = \frac{1}{4}\sum_{spins}N_C\frac{e_q^2e^4}{q^4}\Big(\overline{u}(p')\gamma^\mu u(p)\overline{u}(k')\gamma_\mu u(k)\Big)\Big(\overline{u}(p')\gamma^\mu u(p)\overline{u}(k')\gamma_\mu u(k)\Big)^*$$

using the fact that:

$$[\overline{u}(p')\gamma^\mu u(p)]^* = u^\dagger(p)(\gamma^\mu)^\dagger(\gamma^0)^\dagger u(p') = u^\dagger(p)(\gamma^\mu)^\dagger(\gamma^0)u(p') = u^\dagger(p)\gamma^0\gamma^\mu u(p') = \overline{u}(p)(\gamma^\mu)u(p')$$

the squared and averaged ME can be written:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}\sum_{spins}N_C\frac{e_q^2e^4}{q^4}\Big(\overline{u}(p')\gamma^\mu u(p)\overline{u}(k')\gamma_\mu u(k)\Big)\Big(\overline{u}(k)\gamma_\nu u(k')\overline{u}(p)\gamma^\nu u(p')\Big)$$

or re-shuffling the elements $(\overline{u}\gamma^\lambda u)$:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{4}\sum_{spins} N_C \frac{e_q^2 e^4}{q^4}\Big(\overline{u}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu u(p')\Big)\Big(\overline{u}(k)\gamma_\nu u(k')\overline{u}(k')\gamma_\mu u(k)\Big)$$

so arranging the spin sums for each term:

$$\frac{1}{4}|\overline{\mathcal{M}}|^2 = \frac{1}{4}N_C\frac{e_q^2 e^4}{q^4}\Big(\sum_{rr'}\overline{u}(p',r')\gamma^\mu u(p,r)\overline{u}(p,r)\gamma^\nu u(p',r')\Big)\Big(\sum_{ss'}\overline{u}(k,s)\gamma_\nu u(k',s')\overline{u}(k',s')\gamma_\mu u(k,s)\Big)$$

One can take trace since the terms in parenthesis are scalars and then use the cyclic property of traces:

$$\frac{1}{4}|\overline{\mathcal{M}}|^2 = \frac{1}{4}N_C\frac{e_q^2 e^4}{q^4}Tr\Big(\sum_{r'}u(p',r')\overline{u}(p',r')\gamma^\mu\sum_{r}u(p,r)\overline{u}(p,r)\gamma^\nu\Big)Tr\Big(\gamma_\mu\sum_{s}u(k,s)\overline{u}(k,s)\gamma_\nu\sum_{s'}u(k',s')\overline{u}(k',s')\Big)$$

then one can use the identity:

$$\sum_s u(k,s)\overline{u}(k,s) = \not{p} + m$$

to get:

$$\frac{1}{4}|\overline{\mathcal{M}}|^2 = \frac{1}{4}N_C\frac{e_q^2 e^4}{q^4}Tr\Big((\not{p}' + m_q)\gamma^\mu(\not{p} + m_q)\gamma^\nu\Big)Tr\Big(\gamma_\mu(\not{k} + m_e)\gamma_\nu(\not{k}' + m_e)\Big)$$

Now we can identify the same tensorial structure in the two traces and introduce the so called leptonic tensor $L^{\mu\nu}(k,k')$ and the partonic one $M^{\mu\nu}(p,p')$ as:

$$L_{\mu\nu}(k,k') = \frac{1}{2}Tr\Big((\not{k}' + m_e)\gamma_\mu(\not{k} + m_e)\gamma_\nu\Big) \qquad M^{\mu\nu} = \frac{1}{2}Tr\Big((\not{p}' + m_q)\gamma^\mu(\not{p} + m_q)\gamma^\nu\Big)$$

Let's concentrate on the leptonic tensor:

$$L_{\mu\nu}(k,k') = \frac{1}{2}Tr\Big((\not{k}'+m_e)\gamma_\mu(\not{k}+m_e)\gamma_\nu\Big) = \frac{1}{2}\Big[Tr\Big(\not{k}'\gamma_\mu\not{k}\gamma_\nu\Big) + m_e^2 Tr\Big(\gamma_\mu\gamma_\nu\Big)\Big] = \frac{1}{2}\Big[k'^\rho k^\sigma Tr\Big(\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\Big) + 4m_e^2 g_{\mu\nu}\Big]$$

where one has used $Tr\Big(\text{odd } \# \text{ of } \gamma' s\Big) = 0$, and that $Tr\Big(\gamma_\mu\gamma_\nu\Big) = 4g_{\mu\nu}$. Now using the trace theorem with 4 γ matrices:

$$Tr\Big(\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\Big) = 4\left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu}\right)$$

we have:

$$L_{\mu\nu}(k,k') = \frac{1}{2}\left[k'^\rho k^\sigma 4\left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu}\right) + 4m_e^2 g_{\mu\nu}\right] = 2\left[k'_\mu k_\nu + k_\mu k'_\nu - \left(k\cdot k' - m_e^2\right)g_{\mu\nu}\right]$$

and so we get:

$$\boxed{L_{\mu\nu}(k,k') = 2\Big[k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}\left(k\cdot k' - m_e^2\right)\Big]} \tag{18}$$

in the same way we get for the partonic tensor the expression:

$$M^{\mu\nu}(p,p') = 2\Big[p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu}\left(p\cdot p' - m_q^2\right)\Big]$$

contracting the two tensors we have:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = N_C\frac{e_q^2 e^4}{q^4}L_{\mu\nu}(k,k')M^{\mu\nu}(p,p') = 4N_C\frac{e_q^2 e^4}{q^4}\Big[k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}\left(k\cdot k' - m_e^2\right)\Big]\Big[p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu}\left(p\cdot p' - m_q^2\right)\Big]$$

In the high energy regime we can neglect the incoming electron and parton masses ($m_e = m_q = 0$):

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 4N_C\frac{e_q^2 e^4}{q^4}\Big[k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}\left(k\cdot k'\right)\Big]\Big[p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu}\left(p\cdot p'\right)\Big]$$

dotting the two expressions:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 4N_C\frac{e_q^2e^4}{q^4}\Big[(k'\cdot p')(k\cdot p)+(k'\cdot p)(k\cdot p')-(k\cdot k')(p\cdot p')+$$
$$(k\cdot p')(k'\cdot p)+(k\cdot p)(k'\cdot p)-(k\cdot k')(p'\cdot p)-(p\cdot p')(k\cdot k')-(p'\cdot p)(k\cdot k')+4(k\cdot k')(p\cdot p')\Big]$$

simplifying:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 4N_C\frac{e_q^2e^4}{q^4}\Big[2(k'\cdot p')(k\cdot p)+2(k'\cdot p)(k\cdot p')\Big] = 8N_C\frac{e_q^2e^4}{q^4}\Big[(k'\cdot p')(k\cdot p)+(k'\cdot p)(k\cdot p')\Big]$$

Now is convenient to introduce the Mandelstam variables in the partonic scattering reference frame $\widehat{s}$,$\widehat{t}$,$\widehat{u}$.

$$\widehat{s} = (p+k)^2 = p^2+k^2+2k\cdot p = m_e^2+m_q^2+2k\cdot p \simeq 2k\cdot p = 2k'\cdot p'$$
$$\widehat{t} = q^2 = (k-k')^2 = k^2+k'^2-2k\cdot k' = m_e^2+m_e^2-2k\cdot k' \simeq -2k\cdot k' = -2p\cdot p'$$
$$\widehat{u} = (k-p')^2 = k^2+p'^2-2k\cdot p' = m_e^2+m_q^2-2k\cdot p' \simeq -2k\cdot p' = -2k'\cdot p$$

were we have neglected the particle's masses and used momentum conservation ($k+p=k'+p'$) in the last steps. So:

$$(k\cdot p) = (k'\cdot p') = \frac{\widehat{s}}{2}, \qquad (k'\cdot p) = (k\cdot p') = -\frac{\widehat{u}}{2}, \qquad q^2 \simeq -2(k\cdot k') = -2(p\cdot p') = \widehat{t}$$

so we have the invariant squared amplitude defined in terms of Mandelstam variables:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 8N_C\frac{e_q^2e^4}{\widehat{t}^2}\Big[\frac{\widehat{s}^2}{4}+\frac{\widehat{u}^2}{4}\Big]$$

and finally we obtain the expression for the invariant squared matrix element:

$$\boxed{\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 2N_Ce_q^2e^4\Big[\frac{\widehat{s}^2+\widehat{u}^2}{\widehat{t}^2}\Big]} \tag{19}$$

We are now ready to calculate the cross-section:

$$\sigma = \frac{1}{2s}\int\frac{1}{S}\sum_{spins}|\mathcal{M}|^2dPs \tag{20}$$

an element of n-body space phase is given by:

$$dPs = \prod_{i=1}^{n}\left(\frac{d^4p_i}{(2\pi)^4}(2\pi)\delta(p_i^2-m_i^2)\right)(2\pi)^4\delta^4\left(p_{tot}-\sum_{i=1}^{n}p_i\right)$$
$$= \prod_{i=1}^{n}\left(\frac{d^3\mathbf{p_i}}{(2\pi)^32E_i}\right)(2\pi)^4\delta^4\left(p_{tot}-\sum_{i=1}^{n}p_i\right)$$

and in our case:

$$\sigma = \frac{1}{2\widehat{s}}\frac{1}{4N_c}\int\sum_{spins}|\mathcal{M}|^2dPs_2 = \frac{1}{2\widehat{s}}\frac{1}{4N_c}\int\sum_{spins}|\mathcal{M}|^2\frac{d^3\mathbf{k'}}{(2\pi)^32k'_0}\frac{d^3\mathbf{p'}}{(2\pi)^32p'_0}(2\pi)^4\delta^4\left(p+k-p'-k'\right)$$

Now it's convenient to re-express the two body phase-space in terms of variables used in the DIS phenomenology (i.e. isolating the electron contribution):

$$dPs_2 = \frac{d^3k'}{(2\pi)^32k'_0}\frac{d^3p'}{(2\pi)^32p'_0}(2\pi)^4\delta^4\left(p+k-p'-k'\right) = \frac{d^3\mathbf{k'}}{(2\pi)^32k'_0}dPs_X = \frac{d|\mathbf{k'}||\mathbf{k'}|^2d\Omega}{(2\pi)^32k'_0}dPs_X$$

where the dPs_X contribution denotes the phase space due to the hadronic mass recoiling off the scattered electron (in this case just a on-shell parton). Now assuming high energy particles (therefore neglecting the masses): $m_e^2 \simeq 0 = k_0'^2 - |\mathbf{k'}|^2 \rightarrow |\mathbf{k'}|^2 = k_0'^2$

$$dPs_2 = \frac{dk_0' k_0'^2 d\cos\theta d\varphi}{(2\pi)^3 2k_0'} dPs_X = \frac{dk_0' k_0' d\cos\theta}{8\pi^2} dPs_X$$

where we have integrated out the φ variable, since does not contribute to relevent physics. Now switching for simplicity from k_0' to E' (energy of the outgoing electron) we can go from the variables $(E', \cos\theta)$ to the Lorentz-invariant set (x, Q^2).

$$dE' d\cos\theta = \left|\frac{\partial(E', \cos\theta)}{\partial(x, Q^2)}\right| dx dQ^2 = \left|\frac{\partial(x, Q^2)}{\partial(E', \cos\theta)}\right|^{-1} dx dQ^2$$

now one has to calculate the determinant of the Jacobian matrix for the change of variables:

$$J(x, Q^2; E', \cos\theta) = \left|\frac{\partial(x, Q^2)}{\partial(E', \cos\theta)}\right| = \left|\begin{pmatrix} \frac{\partial x}{\partial E'} & \frac{\partial Q^2}{\partial E'} \\ \frac{\partial x}{\partial \cos\theta} & \frac{\partial Q^2}{\partial \cos\theta} \end{pmatrix}\right|$$

to solve the determinant one should express x and Q^2 in terms of the variables E' and $\cos\theta$ We have:

$$Q^2 = -q^2 = -(k - k')^2 = -(k^2 + k'^2 - 2k \cdot k') = -2m_e^2 + 2(EE' - \mathbf{k} \cdot \mathbf{k}') \simeq 2EE'(1 - \cos\theta)$$

$$x = \frac{-q^2}{2P \cdot q} = \frac{-q^2}{2(M, 0) \cdot (E - E', \mathbf{k} \cdot \mathbf{k}')} = \frac{EE'(1 - \cos\theta}{M(E - E')}$$

were we have used $m_e^2 \simeq 0 = E'^2 - |\mathbf{k}'|^2$. In the last step we have evaluated the invariant $P \cdot q$ in the rest reference frame of the proton (from which comes the parton in the elementary hard scattering). So one has:

$$\left|\frac{\partial(x, Q^2)}{\partial(E', \cos\theta)}\right| = \left|\begin{pmatrix} \frac{\partial\left(\frac{EE'(1-\cos\theta}{M(E-E')}\right)}{\partial E'} & \frac{\partial(2EE'(1-\cos\theta))}{\partial E'} \\ \frac{\partial\left(\frac{EE'(1-\cos\theta}{M(E-E')}\right)}{\partial \cos\theta} & \frac{\partial(2EE'(1-\cos\theta))}{\partial \cos\theta} \end{pmatrix}\right|$$

taking the derivatives:

$$\frac{\partial x}{\partial E'} = \frac{\partial\left(\frac{EE'(1-\cos\theta}{M(E-E')}\right)}{\partial E'} = \frac{E(1-\cos\theta)M(E-E') + MEE'(1-\cos\theta)}{M^2(E-E')^2} = \frac{E^2(1-\cos\theta)}{M(E-E')^2}$$

$$\frac{\partial Q^2}{\partial E'} = \frac{\partial(2EE'(1-\cos\theta))}{\partial E'} = 2E(1-\cos\theta)$$

$$\frac{\partial x}{\partial \cos\theta} = \frac{\partial\left(\frac{EE'(1-\cos\theta}{M(E-E')}\right)}{\partial \cos\theta} = \frac{-EE'}{M(E-E')}$$

$$\frac{\partial Q^2}{\partial \cos\theta} = \frac{\partial(2EE'(1-\cos\theta))}{\partial \cos\theta} = -2EE'$$

Then Jacobian is:

$$\left|\frac{\partial(x, Q^2)}{\partial(E', \cos\theta)}\right| = \left|\begin{pmatrix} \frac{E^2(1-\cos\theta)}{M(E-E')^2} & 2E(1-\cos\theta) \\ \frac{-EE'}{M(E-E')} & -2EE' \end{pmatrix}\right| = \frac{-2E'E^3(1-\cos\theta)}{M(E-E')^2} + \frac{2E'E^2(1-\cos\theta)}{M(E-E')}$$

doing all the algebra:

$$= \frac{-2E'E^3(1-\cos\theta) + (E-E')2E'E^2(1-\cos\theta)}{M(E-E')^2} = \frac{-2E'E^3(1-\cos\theta) + (E-E')2E'E^2(1-\cos\theta)}{M(E-E')^2} =$$

$$\frac{-2E'E^3(1-\cos\theta) + 2E'E^3(1-\cos\theta) - 2E'^2E^2(1-\cos\theta)}{M(E-E')^2} = \frac{-2E'^2E^2(1-\cos\theta)}{M(E-E')^2} = \frac{-2E'2E(1-\cos\theta)}{M(E-E')} \frac{E'E}{(E-E')}$$

and so finally:

$$\left|\frac{\partial(x, Q^2)}{\partial(E', \cos\theta)}\right| = 2x\frac{E'E}{(E-E')} \Rightarrow \left|\frac{\partial(E', \cos\theta)}{\partial(x, Q^2)}\right| = \frac{1}{2x}\frac{E-E'}{EE'}$$

now returning to the two body space phase dPs_2

$$dPs_2 = \frac{dE'E'd\cos\theta}{8\pi^2} dPs_X = \frac{E'}{8\pi^2}\left|\frac{\partial(E', \cos\theta)}{\partial(x, Q^2)}\right| dx dQ^2 dPs_X = \frac{E'}{8\pi^2}\frac{1}{2x}\frac{E-E'}{EE'} dx dQ^2 dPs_X = \frac{1}{8\pi^2}\frac{1}{2x}\left(1 - \frac{E'}{E}\right) dx dQ^2 dPs_X$$

now it is convenient to introduce the inelestacity y variable evaluated in the "proton at rest" frame:

$$y = \frac{P \cdot q}{P \cdot k} = \frac{(M,0) \cdot (E - E', \mathbf{k} - \mathbf{k}')}{(M,0) \cdot (E, \mathbf{k})} = \frac{M(E - E')}{ME} = \frac{E - E'}{E} = 1 - \frac{E'}{E}$$

on the other hand the variables $s = (P+k)^2$, x and y are not independent, since:

$$s = (P+k)^2 = P^2 + k^2 + 2P \cdot k = m_e^2 + M_P^2 + 2 \cdot k \simeq 2P \cdot k$$

so that:

$$P \cdot k = \frac{s}{2} \qquad P \cdot q = \frac{Q^2}{2x} \Rightarrow \boxed{y = \frac{Q^2}{2x}\frac{2}{s} = \frac{Q^2}{sx}}$$

i.e. the inelasticiy is:

$$y = 1 - \frac{E'}{E} = \frac{Q^2}{sx} \tag{21}$$

and finally:

$$\boxed{dPs_2 = \frac{1}{8\pi^2}\frac{1}{2x}\frac{Q^2}{sx} dx dQ^2 dPs_X = \frac{1}{16\pi^2}\frac{Q^2}{sx^2} dx dQ^2 dPs_X}$$

since X consists just of one massless parton, we have:

$$dPs_X = \frac{d^3\mathbf{p}'}{(2\pi)^3 2p_0'}(2\pi)^4 \delta^4(p + k - k' - p') = \frac{d^4 p_X}{(2\pi)^3}\delta(p_X^2)\delta(\xi P + q - p_X) = (2\pi)\delta\left((\xi P + q)^2\right)$$

where we have used momentum conservation: $k + \xi P = k' + p_X \Rightarrow p_X = \xi P + k - k' = \xi P + q$
so now:

$$(\xi P + q)^2 = (\xi P)^2 + q^2 + 2\xi P \cdot q = m_q^2 + q^2 + 2\xi P \cdot q \simeq q^2 + 2\xi P \cdot q$$

and so the Dirac delta becomes:

$$\delta\left(q^2 + 2\xi P \cdot q\right) = \delta\left(\left(\frac{q^2}{2p \cdot q} + \xi\right) 2p \cdot q\right) = \delta\left((\xi - x)\frac{-q^2}{x}\right) = \delta\left((\xi - x)\frac{Q^2}{x}\right) = \frac{x}{Q^2}\delta(\xi - x)$$

where we have used $\delta(ax) = 1/a\delta(x)$.
the phase space of the parton is:

$$dPs_X = \frac{(2\pi x)}{Q^2}\delta(\xi - x)$$

we are now ready to put together the pieces and get the full cross-section:

$$d^2\sigma = \frac{1}{2\widehat{s}}\frac{1}{4N_c}\sum_{spins}|\mathcal{M}|^2 dPs_2 = \frac{1}{2\widehat{s}}\frac{1}{4N_c}\sum_{spins}|\mathcal{M}|^2 \frac{1}{16\pi^2}\frac{Q^2}{sx^2} dx dQ^2 dPs_X$$

and therefore:

$$d^2\sigma = \frac{1}{4N_c}\frac{1}{2\widehat{s}}\frac{Q^2}{16\pi^2 sx^2}\frac{2\pi x}{Q^2}\delta(\xi - x) dx dQ^2 \sum_{spins}|\mathcal{M}|^2 = \frac{1}{N_c}\frac{1}{2\widehat{s}}\frac{1}{8\pi sx}\delta(\xi - x) dx dQ^2 \frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

now since: $(q + \xi P)^2 = 0 = 2\xi P \cdot q + q^2 \Rightarrow \xi \equiv x$:

$$\frac{d^2\sigma}{dx dQ^2} = \frac{1}{N_c}\frac{1}{2xs}\frac{1}{8\pi sx}\delta(\xi - x)\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{1}{N_c}\frac{1}{16\pi s^2 x^2}\delta(\xi - x)\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

but from Eq. 21 $Q^2 = xys \rightarrow 1/s^2x^2 = y^2/Q^4$ so that:

$$\frac{d^2\sigma}{dx dQ^2} = \frac{1}{N_c}\frac{y^2}{16\pi Q^4}\delta(\xi - x)\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

the last step is to evaluate the matrix element squared in Eq. 19 in the DIS variables:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 2N_C e_q^2 e^4\left[\frac{\widehat{s}^2 + \widehat{u}^2}{\widehat{t}^2}\right]$$

Now one can use the Mandelstam invariants properties:

$$\widehat{s}+\widehat{t}+\widehat{u}=\sum_i m_i^2=0\Rightarrow \widehat{u}^2=\left(-\widehat{s}-\widehat{t}\right)^2$$

so that:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2=2N_Ce_q^2e^4\Big[\frac{\widehat{s}^2+\left(\widehat{s}+\widehat{t}\right)^2}{\widehat{t}^2}\Big]=2N_Ce_q^2e^4\Big[\frac{\widehat{s}^2+\widehat{s}^2+\widehat{t}^2+2\widehat{s}\widehat{t}}{\widehat{t}^2}\Big]=\frac{2N_Ce_q^2e^4}{\widehat{t}^2}\left(2\widehat{s}^2+\widehat{t}^2+2\widehat{s}\widehat{t}\right)$$

$$=2N_Ce_q^2e^4\frac{\widehat{s}^2}{\widehat{t}^2}\left(2+\frac{\widehat{t}^2}{\widehat{s}^2}+2\frac{\widehat{t}}{\widehat{s}}\right)=2N_Ce_q^2e^4\frac{\widehat{s}^2}{\widehat{t}^2}\left[1+\left(1+\frac{\widehat{t}^2}{\widehat{s}^2}+2\frac{\widehat{t}}{\widehat{s}}\right)\right]=2N_Ce_q^2e^4\frac{\widehat{s}^2}{\widehat{t}^2}\left[1+\left(1+\frac{\widehat{t}}{\widehat{s}}\right)^2\right]$$

now using: $\widehat{t}=-Q^2\quad \widehat{s}=\xi s=xs\quad Q^2=xys$ we have:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2=2N_Ce_q^2e^4\frac{x^2s^2}{Q^4}\left[1+\left(1-\frac{Q^2}{xs}\right)^2\right]=2N_Ce_q^2e^4\frac{\left(1+(1-y)^2\right)}{y^2}$$

now since $e^2=4\pi\alpha$ in terms of the fine structure constant (electromagnetic coupling strength):

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2=2N_Ce_q^2(4\pi\alpha)^2\frac{\left(1+(1-y)^2\right)}{y^2}$$

finally putting everything together:

$$\frac{d^2\sigma}{dxdQ^2}=\frac{1}{N_c}\frac{y^2}{16\pi Q^4}\delta(\xi-x)2N_Ce_q^2(4\pi\alpha)^2\frac{\left(1+(1-y)^2\right)}{y^2}=\frac{32\pi^2\alpha^2}{16\pi Q^4}e_q^2\left(1+(1-y)^2\right)\delta(\xi-x)$$

and the final expression is:

$$\boxed{\left.\frac{d^2\widehat{\sigma}}{dxdQ^2}\right|_{e^-q\to e^-q}=\frac{4\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\frac{1}{2}e_q^2\delta(\xi-x)}\tag{22}$$

Let's now suppose that the proton consists of a bundle of comoving partons, which carry a range of the proton's momentum. We posit probability distribution functions (called parton distribution functions, *pdf*s), such that the partons od the type q carry a fraction of the proton's momemtum between η and $\eta+d\eta$ a fraction $f_q(\eta)d\eta$ of the time. Provided that these partons are pointlike $r^2\ll 1/Q^2$ and dilute $f_q(\eta)\ll Q^2R^2$, the photons will scatter incoherently off individual partons. Now we can calculate what we expect for the $e^-P\to e^-X$ cross-section given this naive parton model: the cross section can be factorized as the convolution of the *pdf*s with the cross-section for parton scattering:

$$\left.\frac{d^2\sigma}{dxdQ^2}\right|_{e^-P\to e^-X}=\sum_q\int_0^1 d\eta f_{q/P}(\eta)\frac{d^2\widehat{\sigma}(e+q(\eta P))}{dxdQ^2}\tag{23}$$

If we assume that the scattering is elastic, then the outgoing parton must be on mass-shell, and assuming also that partons are massless, then on can obtain the relation:

$$(q+\eta P)=2\eta p\cdot q-Q^2=0\longrightarrow\eta=x$$

Combining now Eq. 22 and 23 we have:

$$\begin{aligned}\left.\frac{d^2\sigma}{dxdQ^2}\right|_{e^-P\to e^-X}&=\sum_q\int_0^1 d\eta f_{q/P}(\eta)\frac{4\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\frac{1}{2}e_q^2\delta(\eta-x)\\&=\frac{2\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\sum_q\int_0^1 d\eta f_{q/P}(\eta)e_q^2\delta(\eta-x)\\&=\frac{2\pi\alpha^2}{xQ^4}\left[1+(1-y)^2\right]\sum_q e_q^2xf_{q/P}(x)\end{aligned}$$

the cross section of DIS in the lowest order of QED in the naive parton model frame can be wirtten:

$$\left.\frac{d^2\sigma}{dxdQ^2}\right|_{e^-P\to e^-X} = \frac{2\pi\alpha^2}{xQ^4}\left[1+(1-y)^2\right]\underbrace{\sum_q e_q^2 x f_{q/P}(x)}_{structure\quad function} \tag{24}$$

from direct comparison of Eq. 24 with Eq. 8 one gets:

$$F_2(x) = \sum_q e_q^2 x f_{q/P}(x)$$
$$F_L(x) = 0$$

Note that $F_2(x)$ is Q^2-independent showing Bjorken scaling. Simply from helicity conservation, one can show that, if we assume that the struck partons are the quark of the quark model (which are fermions), $F_L(x) = 0$. This is known as Callan-Gross relation. (If the partons were instead scalars we would have $F_T = 0$ and hence a completely different y-dependence of the cross-section).

4.2. Problem 2

In the calculation of the Z^0 lineshape in the lectures the interference between the γ and Z contributions has been neglected (i.e. the correct result should have been obtained by computing $|\mathcal{M}_\gamma + \mathcal{M}_{Z^0}|^2$ instead of $|\mathcal{M}_\gamma|^2$ and $|\mathcal{M}_{Z^0}|^2$ separately. Estimate the interference contribution. What is its value on the peak ($\sqrt{s} = M_Z^0$)?)

Solution

At LO two diagrams are contributing to the cross-section of electron positron annihilation in to fermion-antifermion (other than electron!) pairs:

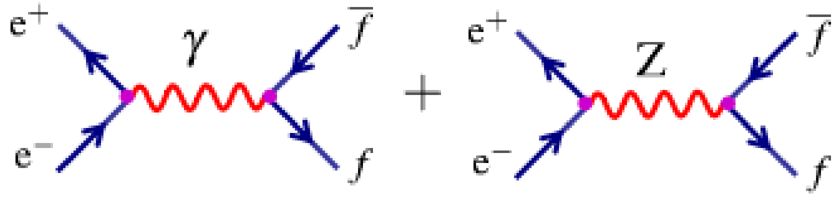


Fig. 7. Electron positron annihilation $e^-(p_1)e^+(p_2) \to f(k_1)\overline{f}(k_2)$

In the lectures it was stated that the Z lineshape at tree level can be written:

$$\sigma_{e^+e^-\to Z\to f\overline{f}}(s) = \sigma_\gamma(s) + \sigma_Z(s) = \underbrace{\frac{4\pi\alpha^2}{3s}Q_f^2 N_f^C}_{\gamma\ \text{contribution}} + \underbrace{\frac{4\pi\alpha^2}{3s}N_f^C \frac{s^2}{(s-M_Z^2)^2 + M_Z^2\Gamma_Z^2}\mathcal{C}_e\mathcal{C}_f}_{Z\ \text{contribution}} \underbrace{\left[1+\Delta_Z\right]}_{\gamma-Z\ \text{interference}} \tag{25}$$

let's derive this expression in detail to check how the interference term comes out. The cross section is proportional to the square of the matrix element:

$$\sigma \propto |\mathcal{M}_\gamma + \mathcal{M}_{Z^0}|^2 = \left(\mathcal{M}_\gamma + \mathcal{M}_{Z^0}\right)\left(\mathcal{M}_\gamma + \mathcal{M}_{Z^0}\right)^* = |\mathcal{M}_\gamma|^2 + |\mathcal{M}_{Z^0}|^2 + \underbrace{(\mathcal{M}_\gamma\mathcal{M}_{Z^0}^* + \mathcal{M}_\gamma^*\mathcal{M}_{Z^0})}_{interference}$$

and so we can express the interference term as:

$$\mathcal{M}_\gamma\mathcal{M}_{Z^0}^* + \mathcal{M}_\gamma^*\mathcal{M}_{Z^0} = \mathcal{M}_{\gamma,Z} + \mathcal{M}_{\gamma,Z}^* = \mathfrak{Re}\mathcal{M}_{\gamma,Z} + i\mathfrak{Im}\mathcal{M}_{\gamma,Z} + \mathfrak{Re}\mathcal{M}_{\gamma,Z} - i\mathfrak{Im}\mathcal{M}_{\gamma,Z} = 2\mathfrak{Re}\mathcal{M}_{\gamma,Z}$$

The cross-section is then:

$$\boxed{d\sigma_{TOT} = \frac{1}{2s}\left(|\mathcal{M}_\gamma|^2 + |\mathcal{M}_{Z^0}|^2 + 2\mathfrak{Re}\mathcal{M}_{\gamma,Z}\right) dPs_2} \tag{26}$$

The matrix element for the γ exchange contribution is a simple QED matrix element:

$$\mathcal{M}_\gamma = \overline{v}(p_2)(ie\gamma^\mu)u(p_1)\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\overline{u}(k_1)(-ieQ_f\gamma^\nu)v(k_2) \tag{27}$$

where the incoming electron (positron) has momentum $p_1(p_2)$ while the outgoing fermion (antifermion) has momentum k_1 (k_2). Momentum conservation implies: $p_1 + p_2 = q = k_1 + k_2$. So the amplitude can be written:

$$\mathcal{M}_\gamma = \frac{-ie^2Q_f}{q^2}\left(\overline{v}(p_2)\gamma^\mu u(p_1)\right)\left(\overline{u}(k_1)\gamma_\mu v(k_2)\right)$$

squaring and averaging(summing) on initial(final) degrees of freedom, whe have:

$$|\overline{\mathcal{M}_\gamma}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2 = \frac{1}{4}\sum_{spins}\frac{e^4Q_f^2}{q^4}\left(\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(k_1)\gamma_\mu v(k_2)\right)\left(\overline{v}(p_2)\gamma^\nu u(p_1)\overline{u}(k_1)\gamma_\nu v(k_2)\right)^*$$

using the fact:

$$[\overline{u}(k_1)\gamma^\mu v(k_2)]^* = v^\dagger(k_2)(\gamma^\mu)^\dagger(\gamma^0)^\dagger u(k_1) = v^\dagger(k_2)(\gamma^\mu)^\dagger(\gamma^0)u(p_1) = v^\dagger(k_2)\gamma^0\gamma^\mu u(k_1) = \overline{v}(k_2)(\gamma^\mu)u(k_1)$$

the squared amplitude can be written:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2 = \frac{1}{4}\sum_{spins}\frac{e^4Q_f^2}{q^4}\left(\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(k_1)\gamma_\mu v(k_2)\right)\left(\overline{v}(k_2)\gamma_\nu u(k_1)\overline{u}(p_1)\gamma^\nu v(p_2)\right)$$

or:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2=\frac{1}{4}\sum_{spins}\frac{e^4Q_f^2}{q^4}\Big(\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(p_1)\gamma^\nu v(p_2)\Big)\Big(\overline{u}(k_1)\gamma_\mu v(k_2)\overline{v}(k_2)\gamma_\nu u(k_1)\Big)$$

arranging the spin sums for each particle:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2=\frac{1}{4}\frac{e^4Q_f^2}{q^4}\Big(\sum_{r,r'}\overline{v}(p_2,r')\gamma^\mu u(p_1,r)\overline{u}(p_1,r)\gamma^\nu v(p_2,r')\Big)\Big(\sum_{s,s'}\overline{u}(k_1,s)\gamma_\mu v(k_2,s')\overline{v}(k_2,s')\gamma_\nu u(k_1,s)\Big)$$

i.e. taking the trace and using the cyclic property of the traces:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2=\frac{1}{4}\frac{e^4Q_f^2}{q^4}Tr\Big(\sum_{r'}\overline{v}(p_2,r')v(p_2,r')\gamma^\mu\sum_{r}u(p_1,r)\overline{u}(p_1,r)\gamma^\nu\Big)Tr\Big(\sum_{s}u(k_1,s)\overline{u}(k_1,s)\gamma_\mu\sum_{s'}v(k_2,s')\overline{v}(k_2,s')\gamma_\nu\Big)$$

now using the identities:

$$\sum_s u(k_1,s)\overline{u}(k_1,s)=\not{k}_1+m_f \qquad \sum_s v(k_2,s)\overline{v}(k_2,s)=\not{k}_2-m_f$$

we have:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2=\frac{1}{4}\frac{e^4Q_f^2}{q^4}Tr\Big((\not{p}_2-m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\Big)Tr\Big((\not{k}_1+m_f)\gamma_\mu(\not{k}_2-m_f)\gamma_\nu\Big)$$

we can now identify the same tensorial structure in the two traces and introduce the so called leptonic tensor: $L^{\mu\nu}(p_1,p_2)$:

$$L_e^{\mu\nu}(p_1,p_2)=\frac{1}{2}Tr\Big((\not{p}_2-m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\Big) \quad L^f_{\mu\nu}(k_1,k_2)=\frac{1}{2}Tr\Big((\not{k}_1+m_f)\gamma_\mu(\not{k}_2-m_f)\gamma_\nu\Big)$$

Let's consider the first trace. Now since the trace of an odd number of γ matrices yields zero, we have:

$$L_e^{\mu\nu}(p_1,p_2)=\frac{1}{2}Tr\Big((\not{p}_2-m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\Big)=p_{2,\rho}p_{1,\sigma}Tr(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu)-m_e^2Tr(\gamma^\mu\gamma^\nu)$$

so using the usual γ traces theorems:

$$Tr(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu)=4(g^{\rho\mu}g^{\sigma\nu}+g^{\rho\nu}g^{\mu\sigma}-g^{\rho\sigma}g^{\mu\nu})$$
$$Tr(\gamma^\mu\gamma^\nu)=4g^{\mu\nu}$$

and we have therefore:

$$L_e^{\mu\nu}(p_1,p_2)=\frac{1}{2}\Big[4p_{2\rho}p_{1\sigma}(g^{\rho\mu}g^{\sigma\nu}+g^{\rho\nu}g^{\mu\sigma}-g^{\rho\sigma}g^{\mu\nu})-4g^{\mu\nu}m_e^2\Big]=2\Big(p_2^\mu p_1^\nu+p_1^\mu p_2^\nu-(p_1\cdot p_2)g^{\mu\nu}-m_e^2g^{\mu\nu}\Big)$$

and finally the leptonic tensor has the form:

$$L_e^{\mu\nu}(p_1,p_2)=2\Big[p_2^\mu p_1^\nu+p_1^\mu p_2^\nu-(p_1\cdot p_2+m_e^2)g^{\mu\nu}\Big]$$

in the same way the tensor for the outogoing pair of fermions:

$$L^f_{\mu\nu}(k_1,k_2)=2\Big[k_{1,\mu}k_{2,\nu}+k_{2,\mu}k_{1,\nu}-(k_1\cdot k_2+m_f^2)g_{\mu\nu}\Big]$$

the squared matrix element is obtained contracting the two tensors. Neglecting electron and fermion mass, since we want to consider the high energy limit:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}_\gamma|^2=\frac{e^4Q_f^2}{q^4}L_e^{\mu\nu}(p_1,p_2)L^f_{\mu\nu}(k_1,k_2)=4\frac{e^4Q_f^2}{q^4}\Big(p_2^\mu p_1^\nu+p_1^\mu p_2^\nu-(p_1\cdot p_2)g^{\mu\nu}\Big)\Big(k_{1,\mu}k_{2,\nu}+k_{2,\mu}k_{1,\nu}-(k_1\cdot k_2)g_{\mu\nu}\Big)$$
$$=4\frac{e^4Q_f^2}{q^4}\Big[2(p_2\cdot k_1)(p_1\cdot k_2)+2(p_2\cdot k_2)(p_1\cdot k_1)\Big]$$

now using the definition of Mandelstamm invariants:

$$
\begin{aligned}
s &= q^2 = (p_1+p_2)^2 = p_1^2 + p_1^2 + 2p_1 \cdot p_2 = m_e^2 + m_e^2 + 2p_1 \cdot p_2 \simeq 2p_1 \cdot p_2 = 2k_1 \cdot k_2 \\
t &= (p_1 - k_1)^2 = p_1^2 + k_1^2 - 2p_1 \cdot k_2 = m_e^2 + m_f^2 - 2p_1 \cdot k_1 \simeq -2p_1 \cdot k_1 = -2p_2 \cdot k_2 \\
u &= (p_1 - k_2)^2 = p_1^2 + k_2^2 - 2p_1 \cdot k_2 = m_e^2 + m_f^2 - 2p_1 \cdot k_2 \simeq -2p_1 \cdot k_2 = -2p_2 \cdot k_1
\end{aligned}
$$

were we have neglected the particle's masses and used momentum conservation ($p_1+p_2 = k_1+k_2$) in the last steps. So:

$$
q^2 \simeq 2(p_1 \cdot p_2) = 2(k_1 \cdot k_2) = s, \qquad (p_1 \cdot k_1) = (p_2 \cdot k_2) = -\frac{t}{2}, \qquad (p_1 \cdot k_2) = (p_2 \cdot k_1) = -\frac{u}{2}
$$

and the matrix element becomes:

$$
\frac{1}{4}\sum_{spins} |\mathcal{M}_\gamma|^2 = 8\frac{e^4 Q_f^2}{q^4}\Big[(p_2 \cdot k_1)(p_1 \cdot k_2) + (p_2 \cdot k_2)(p_1 \cdot k_1)\Big] = 8\frac{e^4 Q_f^2}{s^2}\left[\frac{t^2}{4} + \frac{u^2}{4}\right]
$$

and finally:

$$
\boxed{\frac{1}{4}\sum_{spins} |\mathcal{M}_\gamma|^2 = 2e^4 Q_f^2 \left[\frac{t^2+u^2}{s^2}\right]} \tag{28}
$$

to complete the calculation of the cross section we need to evaluate the expression:

$$
\sigma = \frac{1}{2s}\int \frac{1}{4}\sum_{spins} |\mathcal{M}_\gamma|^2 dPs_2 = \frac{1}{2s}\int 2e^4 Q_f^2 \left[\frac{t^2+u^2}{s^2}\right] dPs_2
$$

The two body phase space is:

$$
dPs_2 = \frac{d^3\mathbf{k_1}}{(2\pi)^3 2E_1}\frac{d^3\mathbf{k_2}}{(2\pi)^3 2E_2}(2\pi)^4 \delta(p_1 + p_2 - k_1 - k_2)
$$

so going in the center of mass reference frame and integrating on the momenutm k_2:

$$
dPs_2 = \frac{(2\pi)^{-2}}{4E_1E_2} d^3\mathbf{k_1} d^3\mathbf{k_2} \delta(\sqrt{s} - k_1 - k_2)\delta^3(\mathbf{k_1} - \mathbf{k_2}) = \frac{(2\pi)^{-2}}{4E_1E_2}|\mathbf{k_1}|^2 d|\mathbf{k_1}| d\cos\theta d\varphi \delta(\sqrt{s} - k_1 - k_2)
$$

in the high energy limit: $lvert\mathbf{k_1}| = k_1 \simeq E_1$ so:

$$
dPs_2 = \frac{(2\pi)^{-2}}{4k_1k_2} k_1^2 dk_1 d\cos\theta d\varphi \delta(\sqrt{s} - k_1 - k_2)
$$

but the integration on the 3-momenutum yields: $\mathbf{k_1} - \mathbf{k_2} = 0 \Rightarrow k_1 = k_2$ and so:

$$
dPs_2 = \frac{(2\pi)^{-2}}{4{k^*}^2}{k^*}^2 dk^* d\cos\theta d\varphi \delta(\sqrt{s} - 2k^*) = \frac{1}{16\pi^2} dk^* d\cos\theta d\varphi \delta(\sqrt{s} - 2k^*) = \frac{1}{16\pi^2} dk^* d\cos\theta d\varphi \frac{1}{2}\delta\left(\frac{\sqrt{s}}{2} - k^*\right)
$$

with k^* momentum of the outgoing fermions in the Centre-of-mass reference frame. Integrating out the uninteresting azimuthal angle:

$$
dPs_2 = \frac{1}{32\pi^2} dk^* \mathrm{d}\cos\theta 2\pi \delta\left(\frac{\sqrt{s}}{2} - k^*\right) = \frac{\mathrm{d}\cos\theta}{16\pi} dk^* \delta\left(\frac{\sqrt{s}}{2} - k^*\right) = \frac{\mathrm{d}\cos\theta}{16\pi}
$$

and the outgoing momentum is $k^* = \sqrt{s}/2$. Now θ is the scattering angle in the centre-of-mass frame can be re-expressed as:

$$
t \simeq -2p_1 \cdot k_1 = -2{k^*}^2(1 - \cos\theta) = -\frac{s}{2}(1 - \cos\theta) \Rightarrow dt = \frac{s}{2} d\cos\theta
$$

where we have used the fact that neglecting all the masses all momenta (incoming and outgoing) are equal to k^*, and so $s = {k^*}^2$. The very last RHS of equation implies:

$$
dt = \frac{s}{2} d\cos\theta \Rightarrow d\cos\theta = \frac{2}{s} dt
$$

and finally we have:

$$dPs_2 = \frac{d\cos\theta}{16\pi} = \frac{2}{16\pi s}dt = \frac{dt}{8\pi s}$$

the integration bounds are obtained from: $\cos\theta = 1 \rightarrow t = 0 \quad \cos\theta = -1 \rightarrow t = -s$. So the total cross-section is obtained from:

$$\sigma_\gamma = \underbrace{\frac{1}{2s}}_{flux factor} \int_{-s}^{0} |\mathcal{M}_\gamma|^2 dPs_2 = \frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}|\mathcal{M}_\gamma|^2$$

so now using the expression derived here:

$$\sigma_\gamma = \frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}2e^4Q_f^2\left[\frac{t^2+u^2}{s^2}\right] = \frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}2(4\pi\alpha)^2Q_f^2\left[\frac{t^2+u^2}{s^2}\right]$$

now using the property of Mandelstamm invariants:

$$s+t+u = \sum_i m_i^2 = 0 \Rightarrow u^2 = (-s-t)^2 = s^2+t^2+2st$$

we can write:

$$\sigma_\gamma = \frac{1}{2s}\int_{-s}^{0}\frac{32\pi^2\alpha^2Q_f^2}{8\pi}\frac{dt}{s}\left[\frac{t^2+(s^2+t^2+2st)}{s^2}\right] = \frac{4\pi\alpha^2Q_f^2}{2s}\int_{-s}^{0}\frac{dt}{s}\left[\frac{2t^2+s^2+2st}{s^2}\right]$$

and we get:

$$\sigma_\gamma = \frac{2\pi\alpha^2Q_f^2}{s}\int_{-s}^{0}\frac{dt}{s}\left[1+2\frac{t^2}{s^2}+2\frac{t}{s}\right] = \frac{2\pi\alpha^2Q_f^2}{s}\int_{-1}^{0}dx(1+2x^2+2x) = \frac{2\pi\alpha^2Q_f^2}{s}\left(x+x^2+\frac{2x^3}{3}\right)_{-1}^{0}$$
$$= -\frac{2\pi\alpha^2Q_f^2}{s}\left(-1+1-\frac{2}{3}\right) = \frac{4\pi\alpha^2Q_f^2}{3s}$$

Taking into account the number of colours of the fermion N_f^C, the gamma exchange cross-section is:

$$\boxed{\sigma_\gamma = \frac{4\pi\alpha^2}{3s}N_f^CQ_f^2} \tag{29}$$

Let's now concentrate on the pure Z-exchange diagram. The invariant amplitude is ($p_1+p_2 = q_Z = k_1+k_2$):

$$\mathcal{M}_Z = \left\{\overline{v}(p_2)\gamma_\mu\frac{-ig}{2\cos\theta_W}(v_e-a_e\gamma_5)u(p_1)\right\}\frac{-ig^{\mu\nu}}{(q_Z^2-M_Z^2)+iM_Z\Gamma_Z}\left\{\overline{u}(k_1)\gamma_\nu\frac{-ig}{2\cos\theta_W}(v_f-a_f\gamma_5)v(k_2)\right\}$$

and the matrix element is:

$$\mathcal{M}_Z = \frac{ig^2}{4\cos^2\theta_W}\frac{1}{(q_Z^2-M_Z^2)+iM_Z\Gamma_Z}\{\overline{v}(p_2)\gamma^\mu(v_e-a_e\gamma_5)u(p_1)\}\{\overline{u}(k_1)\gamma_\mu(v_f-a_f\gamma_5)v(k_2)\} \tag{30}$$

we have introduced the axial and vector coefficients a_f and v_f defined in the following way:

$$v_f = c_V^f = T_3^f - 2\sin\theta_W^2 Q_f$$
$$a_f = c_a^f = T_3^f$$

where T_f^3 is the third component of the weak isospin, Q_f the fermion electromagnetic charge and $\sin\theta_w$ the weak mixing angle.

The Z^0 propagator should be written:

$$D_{\mu\nu}(q) = \frac{-i}{q_Z^2-M_Z^2+iM_Z\Gamma_Z}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{M_Z^2}\right)$$

in the expression in Eq. 30 we have neglected terms proportional to $q_\mu q_\nu$ since those cancel, due to gauge invariance of the theory:

$$q_\mu q_\nu \left\{\overline{u}(k_1)\gamma^\nu(v_f - a_f\gamma_5)v(k_2)\right\} = q_\mu q_\nu \left\{\overline{u}(k_1)\gamma^\nu(...)v(k_2)\right\} = q_\mu \left\{\overline{u}(k_1)\not{q}(...)v(k_2)\right\}$$
$$= q_\mu \left\{\overline{u}(k_1)(\not{k}_1 + \not{k}_2)(...)v(k_2)\right\} = q_\mu\{ \underbrace{\overline{u}(k_1)\not{k}_1}_{\overline{u}(k_1)m_f \simeq 0} (...)v(k_2) - \overline{u}(k_1)(...) \underbrace{\not{k}_1 v(k_2)}_{m_f v(k_2)\simeq 0} \}$$

So, squaring the matrix element we get:

$$|\mathcal{M}_Z|^2 = \frac{g^4}{16\cos^4\theta_W}\frac{1}{(q_Z^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}\left\{\overline{v}(p_2)\gamma^\mu(v_e - a_e\gamma_5)u(p_1)\overline{u}(k_1)\gamma_\mu(v_f - a_f\gamma_5)v(k_2)\right\} *$$
$$\left\{\overline{v}(k_2)\gamma_\nu(v_f - a_f\gamma_5)u(k_1)\overline{u}(p_1)\gamma^\nu(v_e - a_e\gamma_5)v(p_2)\right\}$$

now using the Fermi constant G_F, and the relation between the vector boson masses ($M_w = M_Z\cos\theta_W$) we get:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{g^2}{8M_Z^2\cos^2\theta_W}$$

and squaring this expression we have:

$$\frac{G_F^2}{2} = \frac{g^4}{64M_Z^4\cos^4\theta_W} \Rightarrow \frac{g^4}{16\cos^4\theta_W} = \frac{4G_F^2M_Z^4}{2} = 2G_F^2M_Z^4$$

Rearranging the elements in the amplitude expression we have:

$$|\mathcal{M}_Z|^2 = \frac{2G_F^2M_Z^4}{(q_Z^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}\left\{\overline{v}(p_2)\gamma^\mu(v_e - a_e\gamma_5)u(p_1)\overline{u}(p_1)\gamma^\nu(v_e - a_e\gamma_5)v(p_2)\right\} *$$
$$\left\{\overline{v}(k_2)\gamma_\nu(v_f - a_f\gamma_5)u(k_1)\overline{u}(k_1)\gamma_\mu(v_f - a_f\gamma_5)v(k_2)\right\}$$

Averaging on initial spins and summing on final states:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}_Z|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}\left\{\sum_{ss'}\overline{v}(p_2,s')\gamma^\mu(v_e - a_e\gamma_5)u(p_1,s)\overline{u}(p_1,s)\gamma^\nu(v_e - a_e\gamma_5)v(p_2,s')\right\} *$$
$$\left\{\sum_{rr'}\overline{v}(k_2,r')\gamma_\nu(v_f - a_f\gamma_5)u(k_1,r)\overline{u}(k_1,r)\gamma_\mu(v_f - a_f\gamma_5)v(k_2,r')\right\}$$

Now taking the trace of the expressions in brackets we get:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}Tr\left\{\sum_{s'}v(p_2,s')\overline{v}(p_2,s')\gamma^\mu(v_e - a_e\gamma_5)\sum_s u(p_1,s)\overline{u}(p_1,s)\gamma^\nu(v_e - a_e\gamma_5)\right\} *$$
$$Tr\left\{\sum_{r'}v(k_2,r')\overline{v}(k_2,r')\gamma_\nu(v_f - a_f\gamma_5)\sum_r u(k_1,r)\overline{u}(k_1,r)\gamma_\mu(v_f - a_f\gamma_5)\right\}$$

now using the well-known completeness relations: $\sum_{s'} v(p_2,s')\overline{v}(p_2,s') = \not{p}_2 - m_e$ and $\sum_s u(p_1,s)\overline{u}(p_1,s) = \not{p}_1 + m_e$ to get:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}Tr\left\{(\not{p}_2 - m_e)\gamma^\mu(v_e - a_e\gamma_5)(\not{p}_1 + m_e)\gamma^\nu(v_e - a_e\gamma_5)\right\} *$$
$$Tr\left\{(\not{k}_2 - m_f)\gamma_\nu(v_f - a_f\gamma_5)(\not{k}_1 + m_f)\gamma_\mu(v_f - a_f\gamma_5)\right\}$$

neglecting fermion and electron masses $m_e \simeq m_f \simeq 0$:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}Tr\left\{\not{p}_2\gamma^\mu(v_e - a_e\gamma_5)\not{p}_1\gamma^\nu(v_e - a_e\gamma_5)\right\} * Tr\left\{\not{k}_2\gamma_\nu(v_f - a_f\gamma_5)\not{k}_1\gamma_\mu(v_f - a_f\gamma_5)\right\}$$

Let's now consider the first trace:

$$Tr\left\{\not{p}_2\gamma^\mu(v_e-a_e\gamma_5)\not{p}_1\gamma^\nu(v_e-a_e\gamma_5)\right\} = Tr(v_e^2\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu) - Tr(v_e a_e\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\gamma_5)$$
$$- Tr(v_e a_e\not{p}_2\gamma^\mu\gamma_5\not{p}_1\gamma^\nu) + Tr(a_e^2\not{p}_2\gamma^\mu\gamma_5\not{p}_1\gamma^\nu\gamma_5)$$

now using the anticommutation relations $\{\gamma_5,\gamma^\mu\}=0$ we get:

$$Tr\left\{\not{p}_2\gamma^\mu(v_e-a_e\gamma_5)\not{p}_1\gamma^\nu(v_e-a_e\gamma_5)\right\} = (v_e^2+a_e^2)Tr(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu) - 2v_e a_e Tr(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\gamma_5)$$
$$= (v_e^2+a_e^2)p_{\sigma,2}p_{\rho,1}Tr(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu) - 2v_e a_e p_{\sigma,2}p_{\rho,1}Tr(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\gamma_5)$$

now using the traces of gamma matrices:

$$Tr(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu) = 4(g^{\sigma\mu}g^{\rho\nu}+g^{\sigma\nu}g^{\rho\mu}-g^{\sigma\rho}g^{\mu\nu})$$
$$Tr(\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\gamma_5) = 4i\varepsilon^{\rho\mu\sigma\nu}$$

we get for the electron trace:

$$Tr\left\{\not{p}_2\gamma^\mu(v_e-a_e\gamma_5)\not{p}_1\gamma^\nu(v_e-a_e\gamma_5)\right\} = (v_e^2+a_e^2)p_{\sigma,2}p_{\rho,1}4(g^{\sigma\mu}g^{\rho\nu}+g^{\sigma\nu}g^{\rho\mu}-g^{\sigma\rho}g^{\mu\nu}) - 2v_e a_e p_{\sigma,2}p_{\rho,1}(4i\varepsilon^{\rho\mu\sigma\nu})$$
$$= 4(v_e^2+a_e^2)\left[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_1\cdot p_2)g^{\mu\nu}\right] - 8ia_e v_e p_{\sigma,2}p_{\rho,1}\varepsilon^{\rho\mu\sigma\nu}$$

we get the same result for the outcoming fermion tensor:

$$Tr\left\{\not{k}_2\gamma_\nu(v_f-a_f\gamma_5)\not{k}_1\gamma_\mu(v_f-a_f\gamma_5)\right\} = 4(v_f^2+a_f^2)\left[k_{\mu,1}k_{\nu,2}+k_{\nu,1}k_{\mu,2}-(k_1\cdot k_2)g_{\mu\nu}\right] - 8ia_f v_f k_1^\alpha k_2^\beta\varepsilon_{\alpha\nu\beta\mu}$$

this implies that the squared and averaged matrix element is:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2-M_Z^2)^2+M_Z^2\Gamma_Z^2}\left\{4(v_e^2+a_e^2)\left[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_1\cdot p_2)g^{\mu\nu}\right] - 8ia_e v_e p_{\sigma,2}p_{\rho,1}\varepsilon^{\rho\mu\sigma\nu}\right\}*$$
$$\left\{4(v_f^2+a_f^2)\left[k_{\mu,1}k_{\nu,2}+k_{\nu,1}k_{\mu,2}-(k_1\cdot k_2)g_{\mu\nu}\right] - 8ia_f v_f k_1^\alpha k_2^\beta\varepsilon_{\alpha\nu\beta\mu}\right\}$$

now defining the tensor $L^{\mu\nu} = [p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_1\cdot p_2)g^{\mu\nu}]$ this is a clearly symmetric structure in the Lorents indexes μ and ν:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2-M_Z^2)^2+M_Z^2\Gamma_Z^2}\left\{4(v_e^2+a_e^2)L^{\mu\nu}(p_1,p_2) - 8ia_e v_e p_{\sigma,2}p_{\rho,1}\varepsilon^{\rho\mu\sigma\nu}\right\}*$$
$$\left\{4(v_f^2+a_f^2)L_{\mu\nu}(k_1,k_2) - 8ia_f v_f k_1^\alpha k_2^\beta\varepsilon_{\alpha\nu\beta\mu}\right\}$$

so cross-products of the symmetric tensor with the completely antisymmetric Ricci tensor of the type $L^{\mu\nu}(p_1,p_2)\varepsilon_{\alpha\mu\beta\nu}\equiv 0$ are vanishing. What is left is:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{1}{4}\frac{2G_F^2M_Z^4}{(q_Z^2-M_Z^2)^2+M_Z^2\Gamma_Z^2}\left\{16(v_e^2+a_e^2)(v_f^2+a_f^2)L^{\mu\nu}(p_1,p_2)L_{\mu\nu}(k_1,k_2)\right.$$
$$\left. - 64a_e v_e a_f v_f p_{\sigma,2}p_{\rho,1}k_1^\alpha k_2^\beta\varepsilon^{\rho\mu\sigma\nu}\varepsilon_{\alpha\nu\beta\mu}\right\}$$

now the contraction of the $L^{\mu\nu}$ tensors gives:

$$L^{\mu\nu}(p_1,p_2)L_{\mu\nu}(k_1,k_2) = \left[p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - (p_1\cdot p_2)g^{\mu\nu}\right]\left[k_{\mu,1}k_{\nu,2}+k_{\nu,1}k_{\mu,2}-(k_1\cdot k_2)g_{\mu\nu}\right]$$
$$= (p_2\cdot k_1)(p_1\cdot k_2) + (p_2\cdot k_2)(p_1\cdot k_1) - (k_1\cdot k_2)(p_2\cdot p_1) + (p_1\cdot k_2)(p_2\cdot k_1) +$$
$$(p_2\cdot k_1)(p_1\cdot k_2) - (k_1\cdot k_2)(p_1\cdot p_2) - (p_1\cdot p_2)(k_1\cdot k_2) - (p_1\cdot p_2)(k_1\cdot k_2) + 4(p_1\cdot p_2)(k_1\cdot k_2)$$
$$= 2(p_2\cdot k_1)(p_1\cdot k_2) + 2(p_2\cdot k_2)(p_1\cdot k_1)$$

and using the property of the Ricci antisymmetric tensor:

$$\varepsilon^{\rho\mu\sigma\nu}\varepsilon_{\alpha\nu\beta\mu} = \varepsilon^{\rho\mu\sigma\nu}\varepsilon_{\beta\mu\alpha\nu} = -2(\delta^\rho_\beta\delta^\sigma_\alpha - \delta^\rho_\alpha\delta^\sigma_\beta)$$

We have, using the results obtained:

$$\begin{aligned}|\overline{\mathcal{M}_Z}|^2 = &\frac{1}{4}\frac{32G_F^2M_Z^4}{(q_Z^2-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)[2(p_2\cdot k_1)(p_1\cdot k_2)+2(p_2\cdot k_2)(p_1\cdot k_1)] \\ &+ 8a_ev_ea_fv_fp_{\sigma,2}p_{\rho,1}k_1^\alpha k_2^\beta(\delta^\rho_\beta\delta^\sigma_\alpha - \delta^\rho_\alpha\delta^\sigma_\beta)\Big\}\end{aligned}$$

this gives:

$$\begin{aligned}|\overline{\mathcal{M}_Z}|^2 = &\frac{8G_F^2M_Z^4}{(q_Z^2-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)[2(p_2\cdot k_1)(p_1\cdot k_2)+2(p_2\cdot k_2)(p_1\cdot k_1)] \\ &+ 8a_ev_ea_fv_fp_{\sigma,2}p_{\rho,1}(k_2^\rho k_1^\sigma - k_2^\sigma k_1^\rho)\Big\} = \frac{8G_F^2M_Z^4}{(q_Z^2-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)[2(p_2\cdot k_1)(p_1\cdot k_2) \\ &+ 2(p_2\cdot k_2)(p_1\cdot k_1)] + 8a_ev_ea_fv_f[(p_2\cdot k_1)(p_1\cdot k_2)-(p_2\cdot k_2)(p_1\cdot k_1)]\Big\}\end{aligned}$$

using the Mandelstam invariants introduced before:

$$q_Z^2 \simeq 2(p_1\cdot p_2) = 2(k_1\cdot k_2) = s, \qquad (p_1\cdot k_1) = (p_2\cdot k_2) = -\frac{t}{2}, \qquad (p_1\cdot k_2) = (p_2\cdot k_1) = -\frac{u}{2}$$

we have:

$$\begin{aligned}|\overline{\mathcal{M}_Z}|^2 = &\frac{8G_F^2M_Z^4}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)[2\left(-\frac{u}{2}\right)\left(-\frac{u}{2}\right)+2\left(-\frac{t}{2}\right)\left(-\frac{t}{2}\right)] \\ &+ 8a_ev_ea_fv_f[\left(-\frac{u}{2}\right)\left(-\frac{u}{2}\right)-\left(-\frac{t}{2}\right)\left(-\frac{t}{2}\right)]\Big\} \\ =&\frac{8G_F^2M_Z^4}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)\left[\frac{u^2}{2}+\frac{t^2}{2}\right]+8a_ev_ea_fv_f\left[\frac{u^2}{4}-\frac{t^2}{4}\right]\Big\}\end{aligned}$$

and finally we get for the matrix element:

$$\boxed{|\overline{\mathcal{M}_Z}|^2 = \frac{4G_F^2M_Z^4}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)[u^2+t^2]+4a_ev_ea_fv_f[u^2-t^2]\Big\}} \tag{31}$$

it is convenient now to re-express the Mandelstamm invariants using the property:

$$s+t+u=0 \Rightarrow u^2 = (-s-t)^2 = s^2+t^2+2st$$

this implies:

$$\begin{aligned}u^2+t^2 &= s^2+t^2+2st+t^2 = s^2+2t+t^2 = s^2\left(1+2\frac{s}{t}+2\frac{s^2}{t^2}\right) \\ u^2-t^2 &= -t^2+s^2+t^2+2st = -s^2-2st = s^2\left(1+2\frac{s}{t}\right)\end{aligned}$$

this way the matrix element can be written:

$$|\overline{\mathcal{M}_Z}|^2 = \frac{4G_F^2M_Z^4}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\Big\{(v_e^2+a_e^2)(v_f^2+a_f^2)s^2\left(1+2\frac{s}{t}+2\frac{s^2}{t^2}\right)+4a_ev_ea_fv_fs^2\left(1+2\frac{s}{t}\right)\Big\}$$

now using the expression for the cross-section derived before:

$$\sigma_Z = \frac{1}{2s}\int_{-s}^{0}|\mathcal{M}_\mathcal{Z}|^2dPs_2 = \frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}|\mathcal{M}_\mathcal{Z}|^2$$

we can write:

$$\sigma_Z = \frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}\frac{4G_F^2M_Z^4}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)s^2\left(1+2\frac{s}{t}+2\frac{s^2}{t^2}\right)+4a_ev_ea_fv_fs^2\left(1+2\frac{s}{t}\right)\right\}$$

this gives:

$$\begin{aligned}\sigma_Z =&\frac{1}{16\pi s}\frac{4G_F^2M_Z^4s^2}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\int_{-s}^{0}\frac{dt}{s}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\left(1+2\frac{s}{t}+2\frac{s^2}{t^2}\right)+4a_ev_ea_fv_f\left(1+2\frac{s}{t}\right)\right\}\\ =&\frac{G_F^2M_Z^4s}{4\pi[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\int_{-s}^{0}\frac{dt}{s}\left(1+2\frac{s}{t}+2\frac{s^2}{t^2}\right)+4a_ev_ea_fv_f\int_{-s}^{0}\frac{dt}{s}\left(1+2\frac{s}{t}\right)\right\}\end{aligned}$$

defining now $x = t/s$ we have:

$$\sigma_Z = \frac{G_F^2M_Z^4s}{4\pi[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\int_{-1}^{0}dx\left(1+2x+2x^2\right)+4a_ev_ea_fv_f\int_{-1}^{0}dx\left(1+2x\right)\right\}$$

Performing the integrals we have:

$$\begin{aligned}\sigma_Z =& \frac{G_F^2M_Z^4s}{4\pi[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\left(x+x^2+2\frac{x^3}{3}\right)_{-1}^{0}+4a_ev_ea_fv_f\left(x+x^2\right)_{-1}^{0}\right\}\\ =& \frac{G_F^2M_Z^4s}{4\pi[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)(-1)\left(-1+(-1)^2+2\frac{(-1)^3}{3}\right)+4a_ev_ea_fv_f(-1)\left(-1+(-1)^2\right)\right\}\\ =& \frac{2}{3}\frac{G_F^2M_Z^4s}{4\pi[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\right\}\end{aligned}$$

now using the definition of G_F in terms of precision electroweak variables:

$$G_F = \frac{\sqrt{2}(4\pi\alpha)}{8\sin^2\theta_W M_W^2} = \frac{\sqrt{2}(4\pi\alpha)}{8\sin^2\theta_W\cos^2\theta_W M_Z^2}$$

which means, squaring G_F

$$G_F^2 = \frac{2(4\pi\alpha)^2}{64\sin^4\theta_W\cos^4\theta_W M_Z^4}$$

The Z-exchange cross-section can be written:

$$\begin{aligned}\sigma_Z =& \frac{2}{3}\frac{1}{4\pi}\frac{2(4\pi\alpha)^2}{64\sin^4\theta_W\cos^4\theta_W M_Z^4}\frac{M_Z^4 s}{[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\right\}\\ =& \frac{16\pi\alpha^2}{3\cdot 64\sin^2\theta_W\cos^2\theta_W}\frac{s}{[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\left\{(v_e^2+a_e^2)(v_f^2+a_f^2)\right\}\\ =& \frac{4\pi\alpha^2}{3s}\frac{s^2}{[(s-M_Z^2)^2+M_Z^2\Gamma_Z^2]}\frac{(v_e^2+a_e^2)(v_f^2+a_f^2)}{16\sin^4\theta_W\cos^4\theta_W}\end{aligned}$$

now it is convenient to define the coefficients:

$$\mathcal{C}_f = \frac{v_f^2+a_f^2}{4\sin^2\theta_W\cos^2\theta_W} = \frac{(t_{3,f}-2Q_f\sin^2\theta_w)^2+t_{3,f}^2}{4\sin^2\theta_W\cos^2\theta_W}$$

to finally get the expression, taking into account the number of colours of the final state fermions N_f^C:

$$\boxed{\sigma_Z = \frac{4\pi\alpha^2}{3s}N_f^C\frac{s^2}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}\mathcal{C}_e\mathcal{C}_f} \tag{32}$$

Let's now turn to te interference term:

$$2\mathfrak{Re}\mathcal{M}_{\gamma,Z} = 2\mathfrak{Re}(\mathcal{M}_\gamma^*\mathcal{M}_Z) = 2\mathfrak{Re}\bigg(\frac{ie^2Q_f}{q^2}\,(\overline{v}(k_2)\gamma_\nu u(k_1)\overline{u}(p_1)\gamma^\nu v(p_2))*$$
$$*\,\frac{ig^2}{4\cos^2\theta_W}\frac{1}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\,\{\overline{v}(p_2)\gamma^\mu(v_e-a_e\gamma_5)u(p_1)\}\,\{\overline{u}(k_1)\gamma_\mu(v_f-a_f\gamma_5)v(k_2)\}\bigg)$$

so that the interference term is:

$$2\mathfrak{Re}\mathcal{M}_{\gamma,Z} = 2\mathfrak{Re}\bigg(\frac{ie^2Q_f}{q^2}\frac{ig^2}{4\cos^2\theta_W}\frac{1}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\,(\overline{v}(k_2)\gamma_\nu u(k_1)\overline{u}(p_1)\gamma^\nu v(p_2))*$$
$$*\,\{\overline{v}(p_2)\gamma^\mu(v_e-a_e\gamma_5)u(p_1)\}\,\{\overline{u}(k_1)\gamma_\mu(v_f-a_f\gamma_5)v(k_2)\}\bigg)$$

i.e. we have rearranging the terms:

$$2\mathfrak{Re}\mathcal{M}_{\gamma,Z} = 2\mathfrak{Re}\bigg(\frac{-e^2Q_fg^2}{4q^2\cos^2\theta_W}\frac{1}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big(\overline{v}(k_2)\gamma_\nu u(k_1)\overline{u}(k_1)\gamma_\mu(v_f-a_f\gamma_5)v(k_2)\Big)*$$
$$*\,\Big(\overline{u}(p_1)\gamma^\nu v(p_2)\overline{v}(p_2)\gamma^\mu(v_e-a_e\gamma_5)u(p_1)\Big)\bigg)$$

now since:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_Z^2\cos^2\theta_W} \Rightarrow \frac{g^2}{4\cos^2\theta_W} = \frac{2G_FM_Z^2}{\sqrt{2}}$$

we have for the matrix element summed on final states and averaged on the initial:

$$2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} = 2\mathfrak{Re}\bigg(\frac{1}{4}\sum_{spins}\frac{-e^2Q_f2G_FM_Z^2}{q^2\sqrt{2}}\frac{1}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big(\overline{v}(k_2)\gamma_\nu u(k_1)\overline{u}(k_1)\gamma_\mu(v_f-a_f\gamma_5)v(k_2)\Big)*$$
$$*\,\Big(\overline{u}(p_1)\gamma^\nu v(p_2)\overline{v}(p_2)\gamma^\mu(v_e-a_e\gamma_5)u(p_1)\Big)\bigg)$$

which gives:

$$2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} = \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big(\sum_{rr'}\overline{v}(k_2,r')\gamma_\nu u(k_1,r)\overline{u}(k_1,r)\gamma_\mu(v_f-a_f\gamma_5)v(k_2,r')\Big)*$$
$$*\,\Big(\sum_{ss'}\overline{u}(p_1,s)\gamma^\nu v(p_2,s')\overline{v}(p_2,s')\gamma^\mu(v_e-a_e\gamma_5)u(p_1,s)\Big)\bigg)$$

Taking the trace and using the cyclic property of traces:

$$2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} = \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}Tr\Big(\overbrace{\sum_{r'}v(k_2,r')\overline{v}(k_2,r')}^{\not k_2-m_f}\gamma_\nu\overbrace{\sum_{r}u(k_1,r)\overline{u}(k_1,r)}^{\not k_1+m_f}\gamma_\mu(v_f-a_f\gamma_5)\Big)*$$
$$*\,Tr\Big(\underbrace{\sum_{s}u(p_1,s)\overline{u}(p_1,s)}_{\not p_1+m_e}\gamma^\nu\underbrace{\sum_{s}v(p_2,s')\overline{v}(p_2,s')}_{\not p_2+m_e}\gamma^\mu(v_e-a_e\gamma_5)\Big)\bigg)$$

where we have used the completeness relations of spinors to get:

$$2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} = \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}Tr\Big((\not k_2-m_f)\gamma_\nu(\not k_1+m_f)\gamma_\mu(v_f-a_f\gamma_5)\Big)*$$
$$*\,Tr\Big((\not p_1+m_e)\gamma^\nu(\not p_2+m_e)\gamma^\mu(v_e-a_e\gamma_5)\Big)\bigg)$$

Ignoring fermion masses:

$$
\begin{aligned}
2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} &= \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}Tr\Big[v_f(\not{k}_2\gamma_\nu\not{k}_1\gamma_\mu)-a_f(\not{k}_2\gamma_\nu\not{k}_1\gamma_\mu\gamma_5)\Big]* \\
&* Tr\Big[v_e(\not{p}_1\gamma^\nu\not{p}_2\gamma^\mu)-a_e(\not{p}_1\gamma^\nu\not{p}_2\gamma^\mu\gamma_5)\Big]\bigg) \\
&= \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}k_2^\rho k_1^\sigma Tr\Big[v_f(\gamma_\rho\gamma_\nu\gamma_{1\sigma}\gamma_\mu)-a_f(\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_5)\Big]* \\
&* p_{1,\alpha}p_{2,\beta}Tr\Big[v_e(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu)-a_e(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\gamma_5)\Big]\bigg)
\end{aligned}
$$

now using the properties of the γ matrices, we have:

$$
\begin{aligned}
Tr(\gamma_\sigma\gamma_\mu\gamma_\rho\gamma_\nu) &= 4(g_{\sigma\mu}g_{\rho\nu}+g_{\sigma\nu}g_{\rho\mu}-g_{\sigma\rho}g_{\mu\nu}) \\
Tr(\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_5) &= 4i\varepsilon_{\rho\nu\sigma\mu}
\end{aligned}
$$

which means:

$$
\begin{aligned}
2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} &= \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}k_2^\rho k_1^\sigma\Big[4v_f(g_{\sigma\mu}g_{\rho\nu}+g_{\sigma\nu}g_{\rho\mu}-g_{\sigma\rho}g_{\mu\nu})-4a_fi\varepsilon_{\rho\nu\sigma\mu}\Big]* \\
&* p_{1,\alpha}p_{2,\beta}\Big[4v_e(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\rho}g^{\beta\nu})-4ia_e\varepsilon^{\alpha\nu\beta\mu}\Big]\bigg) \\
&= \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big[4v_f\Big(k_{2,\mu}k_{1,\nu}+k_{2,\nu}k_{1,\mu}-(k_1\cdot k_2)g_{\mu\nu}\Big)-4a_fik_2^\rho k_1^\sigma\varepsilon_{\rho\nu\sigma\mu}\Big]* \\
&\Big[[4v_e\Big(p_1^\mu p_2^\nu+p_1^\nu p_2^\mu-(p_1\cdot p_2)g^{\mu\nu}\Big)-4a_fip_{1,\alpha}p_{2,\beta}\varepsilon^{\alpha\nu\beta\mu}\Big]\bigg)
\end{aligned}
$$

The contraction of tensors that follow is essentially the same as already calculated for the pure Z exchange amplitude:

$$
\begin{aligned}
2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} &= \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big[16v_fv_e\Big(2(p_2\cdot k_1)(p_1\cdot k_2)+2(p_2\cdot k_2)(p_1\cdot k_1)\Big) \\
&- 16a_ea_fk_2^\rho k_1^\sigma p_{1,\alpha}p_{2,\beta}(-2)(\delta_\rho^\alpha\delta_\sigma^\beta-\delta_\rho^\beta\delta_\sigma^\alpha)\Big]\bigg) \\
&= \mathfrak{Re}\bigg(\frac{1}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big[16v_fv_e\Big(2(p_2\cdot k_1)(p_1\cdot k_2)+2(p_2\cdot k_2)(p_1\cdot k_1)\Big) \\
&+ 32a_ea_fk_2^\rho k_1^\sigma(p_{1,\rho}p_{2,\sigma}-p_{1,\sigma}p_{2,\rho}) \\
&= \mathfrak{Re}\bigg(\frac{8}{q^2\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(q^2-M_Z^2)+iM_Z\Gamma_Z}\Big[2v_fv_e\Big(2(p_2\cdot k_1)(p_1\cdot k_2)+2(p_2\cdot k_2)(p_1\cdot k_1)\Big) \\
&+ 4a_ea_f\Big((p_2\cdot k_1)(p_1\cdot k_2)-(p_1\cdot k_1)(p_2\cdot k_2)\Big)\Big]\bigg)
\end{aligned}
$$

where we have used $\varepsilon_{\rho\nu\sigma\mu}\varepsilon^{\alpha\nu\beta\mu}=(-2)(\delta_\rho^\alpha\delta_\sigma^\beta-\delta_\rho^\beta\delta_\sigma^\alpha)$. Now using the Mandelstam invariants defined before, we get:

$$
2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} = \mathfrak{Re}\bigg(\frac{8}{s\sqrt{2}}\frac{-e^2Q_fG_FM_Z^2}{(s-M_Z^2)+iM_Z\Gamma_Z}\Big[v_fv_e\Big(t^2+u^2\Big)+a_ea_f\Big(u^2-t^2\Big)\Big]\bigg)
$$

so that finally the matrix element can be written:

$$
\boxed{2\mathfrak{Re}\overline{\mathcal{M}_{\gamma,Z}} = -\mathfrak{Re}\bigg(\frac{8e^2Q_fG_FM_Z^2}{s\sqrt{2}[(s-M_Z^2)+iM_Z\Gamma_Z]}\bigg)\Big[v_fv_e\Big(t^2+u^2\Big)+a_ea_f\Big(u^2-t^2\Big)\Big]} \tag{33}
$$

so that the $\gamma - Z$ interference cross-section is:

$$\sigma_{\gamma,Z} = \frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}2\Re\mathfrak{e}\overline{\mathcal{M}_{\gamma,Z}}$$

hence:

$$\begin{aligned}\sigma_{\gamma,Z} &= -\frac{1}{2s}\int_{-s}^{0}\frac{dt}{8\pi s}\Re\mathfrak{e}\left(\frac{8e^2Q_fG_FM_Z^2}{s\sqrt{2}[(s-M_Z^2)+iM_Z\Gamma_Z]}\right)\left[v_fv_2\left(t^2+u^2\right)+a_ea_f\left(u^2-t^2\right)\right]\\ &= -\frac{1}{16\pi s}\Re\mathfrak{e}\left(\frac{8e^2Q_fG_FM_Z^2}{s\sqrt{2}[(s-M_Z^2)+iM_Z\Gamma_Z]}\right)\int_{-s}^{0}\frac{dt}{s}\left[v_fv_2\left(t^2+u^2\right)+a_ea_f\left(u^2-t^2\right)\right]\end{aligned}$$

the integral has already been evaluted before and gives:

$$\sigma_{\gamma,Z} = -\frac{1}{16\pi s}\Re\mathfrak{e}\left(\frac{8e^2Q_fG_FM_Z^2}{s\sqrt{2}[(s-M_Z^2)+iM_Z\Gamma_Z]}\right)\frac{2}{3}s^2v_fv_2 = \frac{-e^2Q_fG_FM_Z^2}{3\sqrt{2}\pi s}v_ev_f\Re\mathfrak{e}\left(\frac{s}{(s-M_Z^2)+iM_Z\Gamma_Z}\right)$$

now using again:

$$G_F = \frac{\sqrt{2}(4\pi\alpha)}{8\sin^2\theta_W M_W^2} = \frac{\sqrt{2}(4\pi\alpha)}{8\sin^2\theta_W\cos^2\theta_W M_Z^2}$$

we get:

$$\begin{aligned}\sigma_{\gamma,Z} &= -\frac{(4\pi\alpha)Q_fM_Z^2}{3\sqrt{2}\pi s}\frac{\sqrt{2}(4\pi\alpha)}{8\sin^2\theta_W\cos^2\theta_W M_Z^2}\Re\mathfrak{e}\left(\frac{s}{(s-M_Z^2)+iM_Z\Gamma_Z}\right)v_ev_f\\ &= -\frac{2\pi\alpha^2Q_f}{3s}\frac{v_ev_f}{\sin^2\theta_W\cos^2\theta_W}\Re\mathfrak{e}\left(\frac{s}{(s-M_Z^2)+iM_Z\Gamma_Z}\right)\end{aligned}$$

defining:

$$\mathcal{V}_f = \frac{v_f}{2\sin\theta_W\cos\theta_W} = \frac{t_f^3-2Q_f\sin^2\theta_W}{2\sin\theta_W\cos\theta_W}$$

and taking into account the number of colours of the final state fermion N_f^C we get the following expression for the cross-section:

$$\boxed{\sigma_{\gamma,Z} = -\frac{4\pi\alpha^2Q_f}{3s}N_f^C 2\Re\mathfrak{e}\left(\frac{s}{(s-M_Z^2)+iM_Z\Gamma_Z}\right)\mathcal{V}_e\mathcal{V}_f} \tag{34}$$

now defining the function:

$$\chi(s) = \frac{s}{(s-M_Z^2)+iM_Z\Gamma_Z}$$

the real part of this function at $\sqrt{s}=M_Z$ is clearly vanishing, so that $\sigma_{\gamma,Z}(\sqrt{s}=M_Z)=0$ i.e. the interference term does not yield any contribution (at tree level) to the cross-section. The total cross-section as a function of the center-of-mass energy (lineshape) is:

$$\sigma_{e^+e^-\to f\overline{f}}(s) = \frac{4\pi\alpha^2}{3s}N_f^C\left(Q_f^2-2Q_f\Re\mathfrak{e}[\chi(s)]\mathcal{V}_e\mathcal{V}_f+|\chi(s)|^2\mathcal{C}_e\mathcal{C}_f\right)$$

We can also write the differential cross-section for the process $e^+e^-\to f\overline{f}$.
The complete matrix element for the process is:

$$\begin{aligned}|\overline{\mathcal{M}}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = &\underbrace{2e^4Q_f^2\left(\frac{t^2+u^2}{s^2}\right)}_{\gamma\text{exchange}}+\\ &+\underbrace{\frac{4G_F^2M_Z^4}{(s-M_Z)^2+\Gamma_Z^2M_Z^2}\left\{(a_e^2+v_e^2)(a_f^2+v_f^2)(u^2+t^2)+4a_ev_ea_fv_f(u^2-t^2)\right\}}_{Z^0\text{exchange}}\\ &-\underbrace{\Re\mathfrak{e}\left(\frac{8e^2Q_fG_FM_Z^2}{s\sqrt{2}[(s-M_Z^2)+i\Gamma_ZM_Z]}\right)\left\{v_ev_f(u^2+t^2)+a_ea_f(u^2-t^2)\right\}}_{Z^0-\gamma\text{interference}}\end{aligned}$$

now using the definition of G_F we have:

$$
\begin{aligned}
|\overline{\mathcal{M}}|^2 = \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= 2(4\pi\alpha^2)Q_f^2\left(\frac{t^2+u^2}{s^2}\right)\\
&+4\frac{8\pi^2\alpha^2}{16\sin^4\theta_W\cos^4\theta_W M_Z^4}\frac{M_Z^4}{(s-M_Z)^2+\Gamma_Z^2M_Z^2}\left\{(a_e^2+v_e^2)(a_f^2+v_f^2)(u^2+t^2)+4a_ev_ea_fv_f(u^2-t^2)\right\}\\
&-\frac{8(4\pi\alpha)^2Q_f\sqrt{2}}{8\cos^2\theta_W\sin^2\theta_W M_Z^2}\mathfrak{Re}\left(\frac{M_Z^2}{s\sqrt{2}[(s-M_Z^2)+i\Gamma_Z M_Z]}\right)\left\{v_ev_f(u^2+t^2)+a_ea_f(u^2-t^2)\right\}
\end{aligned}
$$

Simplifying a bit the expression we get:

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 =&32\pi^2\alpha^2Q_f^2\left(\frac{t^2+u^2}{s^2}\right)\\
&+32\pi^2\alpha^2\frac{s^2}{(s-M_Z)^2+\Gamma_Z^2M_Z^2}\left\{\frac{(a_e^2+v_e^2)(a_f^2+v_f^2)}{16s_W^4c_W^4}\left(\frac{t^2+u^2}{s^2}\right)+\frac{4a_ev_ea_fv_f}{16s_W^4c_W^4}\left(\frac{u^2-t^2}{s^2}\right)\right\}\\
&-64\pi^2\alpha^2Q_f\mathfrak{Re}\left(\frac{s}{(s-M_Z^2)+i\Gamma_ZM_Z}\right)\left\{\frac{v_ev_f}{4s_W^2c_W^2}\left(\frac{t^2+u^2}{s^2}\right)+\frac{a_ea_f}{4s_W^2c_W^2}\left(\frac{u^2-t^2}{s^2}\right)\right\}
\end{aligned}
$$

this gives using the definitions given before:

$$
\begin{aligned}
\mathcal{V}_f &= \frac{v_f}{2s_wc_w} = \frac{t_f^3-2Q_fs_w^2}{2s_wc_w} \qquad \mathcal{A}_f = \frac{a_f}{2s_wc_w} = \frac{t^3}{2s_wc_w}\\
\mathcal{C}_f &= \frac{v_f^2+a_f^2}{4s_W^2c_W^2} = \frac{(t_{3,f}-2Q_fs_w^2)^2+t_{3,f}^2}{4s_W^2s_W^2} \qquad \chi(s) = \frac{s}{(s-M_Z^2)+iM_Z\Gamma_Z}
\end{aligned}
$$

we get the folling squared matrix element:

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 =&32\pi^2\alpha^2\left\{Q_f^2\left(\frac{t^2+u^2}{s^2}\right)+|\chi^2(s)|^2\left[\mathcal{C}_e\mathcal{C}_f\left(\frac{t^2+u^2}{s^2}\right)+4\mathcal{A}_e\mathcal{A}_f\mathcal{V}_e\mathcal{V}_f\left(\frac{u^2-t^2}{s^2}\right)\right]\right.\\
&\left.-2Q_f\mathfrak{Re}\chi(s)\left[\mathcal{V}_e\mathcal{V}_f\left(\frac{t^2+u^2}{s^2}\right)+\mathcal{A}_e\mathcal{A}_f\left(\frac{u^2-t^2}{s^2}\right)\right]\right\}
\end{aligned}
$$

now using the Mandelstamm invariants expressed in terms of the polar angle in the centre-of-mass frame, we have:

$$
\begin{aligned}
s &= (p_1+p_2)^2 \simeq 2p_1\cdot p_2 = 4E_1E_2 = 4k^{*2}\\
t &= (p_1-k_1)^2 \simeq -2p_1\cdot k_1 = -2E_1^2(1-\cos\theta) = -2k^{*2}(1-\cos\theta) = -\frac{s}{2}(1-\cos\theta)\\
u &= (p_1-k_2)^2 \simeq -2p_1\cdot k_2 = -2E_1E_2(1-\cos(\theta-\pi)) = -2k^{*2}(1+\cos\theta) = -\frac{s}{2}(1+\cos\theta)
\end{aligned}
$$

so that:

$$
\begin{aligned}
\left(\frac{t^2+u^2}{s^2}\right) &= \frac{\frac{s^2}{4}(1-\cos\theta)^2+\frac{s^2}{4}(1+\cos\theta)^2}{s^2} = \frac{1}{4}\left(1+\cos^2\theta-2\cos\theta+1+\cos^2\theta+2\cos\theta\right) = \frac{1}{2}(1+\cos^2\theta)\\
\left(\frac{u^2-t^2}{s^2}\right) &= \frac{\frac{s^2}{4}(1+\cos\theta)^2-\frac{s^2}{4}(1-\cos\theta)^2}{s^2} = \frac{1}{4}\left(1+\cos^2\theta+2\cos\theta-1-\cos^2\theta+2\cos\theta\right) = \cos\theta
\end{aligned}
$$

and so the squared matrix element is:

$$
\begin{aligned}
\frac{1}{4}\sum|\mathcal{M}|^2 =&32\pi^2\alpha^2\left\{Q_f^2\frac{1}{2}(1+\cos^2\theta)+|\chi^2(s)|^2\left[\mathcal{C}_e\mathcal{C}_f\frac{1}{2}(1+\cos^2\theta)+4\mathcal{A}_e\mathcal{A}_f\mathcal{V}_e\mathcal{V}_f\cos\theta\right]\right.\\
&\left.-2Q_f\mathfrak{Re}\chi(s)\left[\mathcal{V}_e\mathcal{V}_f\frac{1}{2}(1+\cos^2\theta)+\mathcal{A}_e\mathcal{A}_f\cos\theta\right]\right\}
\end{aligned}
$$

Simplyfying:

$$
\begin{aligned}
\frac{1}{4}\sum|\mathcal{M}|^2 =&16\pi^2\alpha^2\Big\{Q_f^2(1+\cos^2\theta)+|\chi^2(s)|^2\left[\mathcal{C}_e\mathcal{C}_f(1+\cos^2\theta)+8\mathcal{A}_e\mathcal{A}_f\mathcal{V}_e\mathcal{V}_f\cos\theta\right]\\
&-2Q_f\mathfrak{Re}\chi(s)\left[\mathcal{V}_e\mathcal{V}_f(1+\cos^2\theta)+2\mathcal{A}_e\mathcal{A}_f\cos\theta\right]\Big\}
\end{aligned}
$$

Now we can finally evaluate the differential cross-section:

$$
d\sigma_{e^+e^-\to f\overline{f}}=\frac{1}{2s}|\overline{\mathcal{M}}|^2 dPs_2=\frac{1}{2s}\frac{1}{4}\sum|\mathcal{M}|^2 dPs_2
$$

The phase space has already been evaluated before:

$$
\begin{aligned}
dPs_2 &= \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)=\frac{1}{(2\pi)^2}\frac{d^3k_1}{4E_1E_2}\delta(\sqrt{s}-E_1-E_2)\\
&=\frac{1}{16\pi^2}\frac{E_1^2 dE_1}{E_1E_2}d\cos\vartheta d\cos\varphi\,\delta(\sqrt{s}-E_1-E_2)=\frac{1}{8\pi}dE_1 d\cos\vartheta\frac{1}{2}\delta(\frac{\sqrt{s}}{2}-E_1)\\
&\Longrightarrow dPs_2=\frac{d\cos\theta}{16\pi}
\end{aligned}
$$

so one gets:

$$
d\sigma=\frac{1}{2s}|\overline{\mathcal{M}}|^2\frac{d\cos\theta}{16\pi}\Longrightarrow\frac{d\sigma}{d\cos\theta}=\frac{1}{32\pi s}\left(\frac{1}{4}\sum|\mathcal{M}|^2\right)
$$

and the differential cross-section for the scattering of electron positron at high energies becomes (taking into account the number of colours for each fermion):

$$
\begin{aligned}
\frac{d\sigma}{d\cos\theta}(e^+e^-\to f\overline{f})=\frac{\pi\alpha^2}{2s}N_C^f\Big\{&Q_f^2(1+\cos^2\theta) && (\gamma\ \text{ exchange})\\
&-2Q_f\mathfrak{Re}\chi(s)\Big[\mathcal{V}_e\mathcal{V}_f(1+\cos^2\theta)+2\mathcal{A}_e\mathcal{A}_f\cos\theta\Big] && (\gamma-Z\ \text{ interference})\\
&+|\chi^2(s)|^2\Big[\mathcal{C}_e\mathcal{C}_f(1+\cos^2\theta)+8\mathcal{A}_e\mathcal{A}_f\mathcal{V}_e\mathcal{V}_f\cos\theta\Big]\Big\} && (Z\ \text{ exchange})
\end{aligned}
$$

4.3. Problem 3

Given that neutrinos cannot be detected, how is Γ_ν, the partial decay width of a Z^0 into neutrinos, measured?

Solution

First let's calculate the decay width of a Z^0 boson into a pair of fermions. The lowest order contribuion in the Standard Model to the Z decay to fermions is given by the diagram:

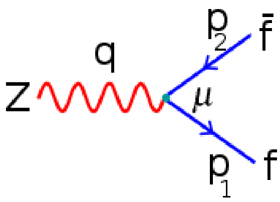


Fig. 8. Feynman diagram for a Z vector boson decaying into a pair of fermions ($Z^0(q) \to \overline{f}(p_1) f(p_2)$)

the matrix element is given by:

$$\mathcal{M} = \overline{u}(p_1)\left(\frac{-ie}{2\sin\theta_W\cos\theta_w}\right)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\varepsilon_\mu(q,\lambda) \tag{35}$$

the decay rate for the process $Z^0 \to f\overline{f}$ is given by the expression:

$$\Gamma_{Z^0\to f\overline{f}} = \frac{1}{2M_Z}\int dPs_2|\overline{M}|^2 = \frac{1}{2M_Z}\frac{1}{3}\sum_{pol}|\overline{M}|^2 2\pi^4\delta^4(q_Z - p_1 - p_2)\frac{d^3p_1}{(2\pi)^3 2p_1^0}\frac{d^3p_2}{(2\pi)^3 2p_2^0}$$

To obtain the decay rate in two fermions we need to evaluate the square of the matrix element:

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \varepsilon_\nu^*(q,\lambda)v^\dagger(p_2)(v_f - a_f\gamma_5)\gamma^{\nu\dagger}\left(\frac{+ie}{2\sin\theta_w\cos\theta_w}\right)\gamma_0 u(p_1)*$$
$$\overline{u}(p_1)\left(\frac{-ie}{2\sin\theta_W\cos\theta_w}\right)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\varepsilon_\nu(q,\lambda)$$

so that:

$$\begin{aligned}
|\mathcal{M}|^2 &= \left(\frac{e^2}{4\sin^2\theta_w\cos^2\theta_w}\right)v^\dagger(p_2)(v_f - a_f\gamma_5)\gamma^{\nu\dagger}\gamma_0 u(p_1)\overline{u}(p_1)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\varepsilon_\nu^*(q,\lambda)\varepsilon_\mu(q,\lambda)\\
&= \left(\frac{e^2}{4s_w^2c_w^2}\right)v^\dagger(p_2)(v_f - a_f\gamma_5)\gamma_0\gamma^\nu u(p_1)\overline{u}(p_1)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\varepsilon_\nu^*(q,\lambda)\varepsilon_\mu(q,\lambda)\\
&= \left(\frac{e^2}{4s_w^2c_w^2}\right)v^\dagger(p_2)\gamma_0(v_f + a_f\gamma_5)\gamma^\nu u(p_1)\overline{u}(p_1)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\varepsilon_\nu^*(q,\lambda)\varepsilon_\mu(q,\lambda)
\end{aligned}$$

where we have used $\gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu \to \gamma^{\nu\dagger}\gamma^0 = \gamma^0\gamma^\nu$ and the anticommutation properties of the γ matrices $\{\gamma^\mu, \gamma^5\} = 0$

and finally we have:

$$|\mathcal{M}|^2 = \left(\frac{e^2}{4s_w^2c_w^2}\right)\overline{v}(p_2)\gamma^\nu(v_f - a_f\gamma_5)u(p_1)\overline{u}(p_1)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\varepsilon_\nu^*(q,\lambda)\varepsilon_\mu(q,\lambda)$$

averaging (summing) on initial (final) states we have:

$$|\overline{\mathcal{M}}|^2 = \frac{1}{3}\sum_{s,s'}\sum_\lambda|\mathcal{M}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right)\sum_{s,s'}\overline{v}(p_2,s')\gamma^\nu(v_f - a_f\gamma_5)u(p_1,s)\overline{u}(p_1,s)\gamma^\mu(v_f - a_f\gamma_5)v(p_2,s')\sum_\lambda\varepsilon_\nu^*(q,\lambda)\varepsilon_\mu(q,\lambda)$$

now taking the trace of the first term and using the invariance property of traces under cyclic permutations:

$$|\overline{\mathcal{M}}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right)Tr[\underbrace{\sum_{s'}v(p_2,s')\overline{v}(p_2,s')}_{\not{p}_2 - m_f}\gamma^\nu(v_f - a_f\gamma_5)\underbrace{\sum_s u(p_1,s)\overline{u}(p_1,s)}_{\not{p}_1 + m_f}\gamma^\mu(v_f - a_f\gamma_5)]\sum_\lambda\varepsilon_\nu^*(q,\lambda)\varepsilon_\mu(q,\lambda)$$

where we have used the completeness relationships. So the squared matrix element is:

$$|\overline{\mathcal{M}}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) Tr[(\not{p}_2 - m_f)\gamma^\nu(v_f - a_f\gamma_5)(\not{p}_1 + m_f)\gamma^\mu(v_f - a_f\gamma_5)] \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$
$$= \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) Tr[\not{p}_2\gamma^\nu(v_f - a_f\gamma_5)\not{p}_1\gamma^\mu(v_f - a_f\gamma_5)] \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$

where we have neglected fermion masses in the approximation $m_f \ll p_1,\ p_2,\ q_Z\ , M_Z$. Expanding the products we get:

$$|\overline{\mathcal{M}}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) Tr[\not{p}_2\gamma^\nu v_f^2\not{p}_1\gamma^\mu - \not{p}_2\gamma^\nu a_f\gamma_5\not{p}_1\gamma^\mu c_f - \not{p}_2\gamma^\nu c_f\not{p}_1\gamma^\mu a_f\gamma_5 + \not{p}_2\gamma^\nu a_f^2\gamma_5\not{p}_1\gamma^\mu\gamma_5] \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$
$$= \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) Tr[v_f^2(\not{p}_2\gamma^\nu\not{p}_1\gamma^\mu) - c_f a_f(\not{p}_2\gamma^\nu\gamma_5\not{p}_1\gamma^\mu) - c_f a_f(\not{p}_2\gamma^\nu\not{p}_1\gamma^\mu\gamma_5) + a_f^2(\not{p}_2\gamma^\nu\gamma_5\not{p}_1\gamma^\mu\gamma_5)] \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$
$$= \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) \left\{(v_f^2 + a_f^2)Tr\left[\not{p}_2\gamma^\nu\not{p}_1\gamma^\mu\right] - 2c_f a_f Tr\left[\not{p}_2\gamma^\nu\not{p}_1\gamma^\mu\gamma_5\right]\right\} \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$

where in the last step we have used the anticommutation of γ_5 with the other γ matrices. We get finally:

$$\frac{1}{3}\sum_{pol}|\mathcal{M}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) \left\{(v_f^2 + a_f^2)p_{2,\rho}p_{1,\sigma}Tr\left[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu\right] - 2c_f a_f p_{2,\rho}p_{1,\sigma}Tr\left[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu\gamma_5\right]\right\} \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$

the sum over polarizations for a massive vector boson gives:

$$\sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}$$

and taking the traces:

$$\frac{1}{3}\sum_{pol}|\mathcal{M}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) p_{2,\rho}p_{1,\sigma}\left\{(v_f^2 + a_f^2)4\left(g^{\rho\nu}g^{\sigma\mu} + g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu}\right) - 8ic_f a_f\varepsilon^{\rho\nu\sigma\mu}\gamma^\nu\right\} \sum_\lambda \varepsilon^*_\nu(q,\lambda)\varepsilon_\mu(q,\lambda)$$
$$= \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) p_{2,\rho}p_{1,\sigma}\left\{\underbrace{(v_f^2 + a_f^2)4\left(g^{\rho\nu}g^{\sigma\mu} + g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu}\right)}_{\text{symmetric}\quad \mu\leftrightarrow\nu}\ \underbrace{-8ic_f a_f\varepsilon^{\rho\nu\sigma\mu}\gamma^\nu}_{\text{antisymmetric}\quad \mu\leftrightarrow\nu}\right\} \underbrace{\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)}_{\text{symmetric}\quad \mu\leftrightarrow\nu}$$

contracting the two tensors only the symmetric part contributes to the decay rate, since the contraction of a symmetric with an antisymmetric tensor yields zero.

$$\frac{1}{3}\sum_{pol}|\mathcal{M}|^2 = \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) p_{2,\rho}p_{1,\sigma}\left\{(v_f^2 + a_f^2)4\left(g^{\rho\nu}g^{\sigma\mu} + g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu}\right)\right\}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)$$
$$= \frac{1}{3}\left(\frac{e^2}{4s_w^2c_w^2}\right) 4(v_f^2 + a_f^2)[p_2^\nu p_1^\mu + p_1^\mu p_2^\nu - (p_1\cdot p_2)g^{\mu\nu}]\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)$$
$$= \frac{1}{3}\left(\frac{e^2}{s_w^2c_w^2}\right)(v_f^2 + a_f^2)\left[-(p_2\cdot p_1) - (p_2\cdot p_1) + 4(p_2\cdot p_1) + \frac{(p_1\cdot q)(p_2\cdot q)}{M_Z^2} + \frac{(p_1\cdot q)(p_2\cdot q)}{M_Z^2} - \frac{(p_1\cdot p_2)q^2}{M_Z^2}\right]$$
$$= \frac{1}{3}\left(\frac{e^2}{s_w^2c_w^2}\right)(v_f^2 + a_f^2)\left[2(p_2\cdot p_1) - \frac{(p_1\cdot p_1)q^2}{M_Z^2} + 2\frac{(p_1\cdot q)(p_2\cdot q)}{M_Z^2}\right]$$

but now since $q^2 = M_Z^2$ we have:

$$\frac{1}{3}\sum_{pol}|\mathcal{M}|^2 = \frac{1}{3}\left(\frac{e^2}{s_w^2c_w^2}\right)(v_f^2 + a_f^2)\left[(p_2\cdot p_1) + 2\frac{(p_1\cdot q)(p_2\cdot q)}{M_Z^2}\right]$$

In the reference frame where the Z boson is at rest:

$$M_Z = E_1 + E_2$$
$$\mathbf{0} = \mathbf{p_1} + \mathbf{p_2} \Rightarrow |\mathbf{p_1}| = |\mathbf{p_2}|$$

This implies:

$$E_1 = (|\mathbf{p_1}|^2 + m_f^2)^{1/2} = (|\mathbf{p_1}|^2 + m_f^2)^{1/2} = E_2$$

so since $E_1 = E_2$ and $E_1 + E_2 = M_Z$ we have $E_1 = E_2 = M_Z/2$ Therefore if $m_f \ll M_Z$ $|\mathbf{p_1}| = |\mathbf{p_2}| = E_1 = E_2 = M_Z/2$ the scalar products are:

$$\begin{aligned}
(p_1 \cdot p_2) &= (\frac{M_z}{2}, \frac{M_Z}{2}\mathbf{e})(\frac{M_Z}{2}, -\frac{M_Z}{2}\mathbf{e}) = \frac{M_Z^2}{4} + \frac{M_Z^2}{4} = \frac{M_Z^2}{2} \\
(p_1 \cdot q) &= (\frac{M_Z}{2}, \frac{M_Z}{2}\mathbf{e})(M_Z, \mathbf{0}) = \frac{M_Z^2}{4} \\
(p_2 \cdot q) &= (\frac{M_Z}{2}, -\frac{M_Z}{2}\mathbf{e})(M_Z, \mathbf{0}) = \frac{M_Z^2}{4}
\end{aligned}$$

and the matrix element becomes:

$$\begin{aligned}
|\overline{\mathcal{M}}|^2 = \frac{1}{3}\sum_{pol}|\mathcal{M}|^2 &= \frac{1}{3}\left(\frac{e^2}{s_w^2 c_w^2}\right)(v_f^2 + a_f^2)\left[\frac{M_Z^2}{2} + 2\frac{\frac{M_Z^2}{2}\cdot\frac{M_Z^2}{2}}{M_Z^2}\right] \\
&= \frac{1}{3}\left(\frac{e^2}{s_w^2 c_w^2}\right)(v_f^2 + a_f^2)\left[\frac{M_Z^2}{2} + \frac{M_Z^2}{2}\right]
\end{aligned}$$

and finally the matrix element is:

$$\boxed{|\overline{\mathcal{M}}|^2 = \frac{1}{3}\left(\frac{e^2}{s_w^2 c_w^2}\right)(v_f^2 + a_f^2)M_Z^2} \tag{36}$$

To get to the decay width we evaluate the phase space factor:

$$\Gamma_Z = \frac{1}{2M_Z}\int|\overline{\mathcal{M}}|^2 dPs_2 = \frac{1}{2M_Z}\int\frac{1}{3}\sum_{pol}|\mathcal{M}|^2(2\pi)^4\delta^4(q - p_1 - p_2)\frac{d^3p_1}{(2\pi)^3 2p_1^0}\frac{d^3p_2}{(2\pi)^3 2p_2^0}$$

and so:

$$\begin{aligned}
dPs_2 &= (2\pi)^4\delta^4(q - p_1 - p_2)\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2} \\
&= (2\pi)^4\delta(q^0 - p_1^0 - p_2^0)\delta^3(\mathbf{q} - \mathbf{p_1} - \mathbf{p_2})\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2} \\
&= (2\pi)^4\delta(M_Z - E_1 - E_2)\delta^3(-\mathbf{p_1} - \mathbf{p_2})\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2} \\
&= \frac{1}{(2\pi)^2}\frac{1}{4E_1E_2}\delta(M_Z - E_1 - E_2)\delta^3(\mathbf{p_1} + \mathbf{p_2})d^3p_1 d^3p_2
\end{aligned}$$

integrating over p_2 we get:

$$\begin{aligned}
dPs_2 &= \int d^3p_2\frac{d^3p_1}{(2\pi)^2 4E_1E_2}\delta(M_Z - E_1 - E_2)\delta^3(\mathbf{p_1} + \mathbf{p_2}) \\
&= \frac{d^3p_1}{(2\pi)^2 4E_1E_2}\delta(M_Z - E_1 - E_2)\Big|_{(\mathbf{p_1} = -\mathbf{p_2})}
\end{aligned}$$

but expressing the differential in polar coordinates: $d^3p_1 = |\mathbf{p_1}|^2 d|\mathbf{p_1}|d\Omega$ and putting $p_1 = |\mathbf{p_1}|$:

$$dPs_2 = \frac{p_1^2 dp_1 d\Omega}{(2\pi)^2 4E_1E_2}\delta(M_Z - E_1 - E_2)\Big|_{(\mathbf{p_1} = -\mathbf{p_2})}$$

and the energies can be written:

$$\delta(M_z - E_1 - E_2) = \delta(M_Z - \sqrt{m_1^2 + \mathbf{p_1}^2} - \sqrt{m_2^2 + \mathbf{p_1^2}}) = \delta(M_Z - 2\sqrt{m_1^2 + \mathbf{p_1}^2})$$

where in the last step we have used $m_1 \equiv m_2$ (which is guaranteed by CPT theorem) and $\mathbf{p_1} = -\mathbf{p_2}$.
Let's now use the Dirac δ-distribution property:

$$\delta(f(p_1)) = \sum_i \frac{\delta(p_1 - p_i)}{|\frac{\partial f}{\partial p_1}|_{p=p_i}} \quad f(p_i) = 0$$

in our case $f(p_1) = M_Z - 2\sqrt{m_1^2 + p_1^2}$ and

$$\left|\frac{\partial f}{\partial p_1}\right| = \left|\frac{1}{2\sqrt{m_f^2 + p_1^2}} \cdot (-2)(2p_1)\right| = \frac{2p_1}{E_1}$$

using this expression in the Lorentz invariant phase-space:

$$\begin{aligned} dPs_2 &= \frac{p_1^2 dp_1 d\Omega}{(2\pi)^2 4E_1 E_2} \frac{E_1}{2p_1} \delta(p_1 - p') \\ &= \frac{p_1 dp_1 d\Omega}{(2\pi)^2 8E_2} \delta(p_1 - p') \end{aligned}$$

since $E_2 = \sqrt{m_f^2 + p_1^2}$ neglecting fermion mass, we have: $E_2 \simeq p_1$, and this gives:

$$\begin{aligned} dPs_2 &= \frac{dp_1 d\Omega}{32\pi^2} \delta(p_1 - p') \\ &= \int \frac{dp_1 d\Omega}{32\pi^2} \delta(p_1 - p') = \frac{d\Omega}{32\pi^2} \end{aligned}$$

where $p_1 = p'$ such as:

$$f(p') = M_Z - 2\sqrt{m_f^2 + p'^2} = 0 \Rightarrow m_f^2 + p'^2 = \frac{M_Z^2}{4} \Rightarrow \boxed{p' = \frac{\sqrt{M_Z^2 - 4m_f^2}}{2}}$$

This implies:

$$\begin{aligned} \Gamma_Z &= \frac{1}{2M_Z} \int |\overline{\mathcal{M}}|^2 dPs_2 = \frac{1}{2M_Z} \int \frac{1}{3} \sum_{pol} |\mathcal{M}|^2 \frac{d\Omega}{32\pi^2} \\ &= \frac{1}{2M_Z} \int \frac{1}{3} \left(\frac{e^2}{s_w^2 c_w^2} \right) (v_f^2 + a_f^2) M_Z^2 \frac{d\Omega}{32\pi^2} \\ &= \int \frac{d\Omega}{4\pi} \frac{1}{48\pi} \left(\frac{e^2}{s_w^2 c_w^2} \right) (v_f^2 + a_f^2) M_Z \end{aligned}$$

and thus we come finally to the deacy width expressions:

$$\frac{d\Gamma_Z}{d\Omega^*} = \frac{1}{192\pi^2} \left(\frac{e^2}{s_w^2 c_w^2} \right) (v_f^2 + a_f^2) M_Z$$

taking into account the quantum number of colour by a factor N_f^C:

$$\boxed{\Gamma_{Z \to f\bar{f}} = \frac{N_f^C}{48\pi} \left(\frac{e^2}{\sin^2\theta_w \cos^2\theta_w} \right) (v_f^2 + a_f^2) M_Z} \qquad (37)$$

using the conventions one have used for the calculation of the Z^0 lineshape we can write:

$$\begin{aligned}
\Gamma(Z \to f\overline{f}) &= \frac{N_f^C}{48\pi} \frac{e^2}{\sin^2\theta_w \cos^2\theta_w}(v_f^2 + a_f^2) M_Z \\
&= \frac{N_f^C}{48\pi} \frac{(4\pi\alpha)}{\sin^2\theta_w \cos^2\theta_w}(v_f^2 + a_f^2) M_Z \\
&= \frac{\alpha}{12} \frac{N_C^f}{\sin^2\theta_w \cos^2\theta_w}(v_f^2 + a_f^2) M_Z \\
&= N_f^C \frac{\alpha}{3} \underbrace{\frac{(v_f^2 + a_f^2)}{4\sin^2\theta_w \cos^2\theta_w}}_{\mathcal{C}_f} M_Z \qquad \Rightarrow \boxed{\Gamma(Z \to f\overline{f}) = \frac{\alpha}{3} N_f^C \mathcal{C}_f M_Z}
\end{aligned}$$

This expression allows to re-write the cross-section of $e^+e^- \to f\overline{f}$ near the Z pole. We have written the cross-section as:

$$\sigma_{e^+e^-\to f\overline{f}}(s) = \frac{4\pi\alpha^2}{3s} N_f^C \left(Q_f^2 - 2Q_f \mathfrak{Re}[\chi(s)]\mathcal{V}_e\mathcal{V}_f + |\chi(s)|^2 \mathcal{C}_e\mathcal{C}_f \right)$$

near the Z pole, the Z-exchange term is dominating the cross-section:

$$\begin{aligned}
\sigma_{e^+e^-\to f\overline{f}}(s \simeq M_Z) &= \frac{4\pi\alpha^2}{3s} N_f^C \frac{s^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \overbrace{\left(\frac{v_e^2 + a_e^2}{4\sin\theta_W^2 \cos^2\theta_W} \right)}^{\mathcal{C}_e} \overbrace{\left(\frac{v_f^2 + a_f^2}{4\sin^2\theta_c \cos^2\theta_W} \right)}^{\mathcal{C}_f} \\
&= \frac{4\pi\alpha^2}{3s} N_f^C \frac{s^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \mathcal{C}_e\mathcal{C}_f = \frac{12\pi}{s} \frac{\alpha}{3}\mathcal{C}_e \frac{\alpha}{3} N_f^C \mathcal{C}_f \frac{s^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \\
&= \frac{12\pi}{s} \frac{\Gamma(Z \to e^+e^-)}{M_Z} \frac{\Gamma(Z \to f\overline{f})}{M_Z} \frac{s^2}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \\
&= \frac{12\pi s}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to f\overline{f})}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}
\end{aligned}$$

so that:

$$\boxed{\sigma_{e^+e^-\to f\overline{f}}(s \simeq M_Z) \simeq \frac{12\pi s}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to f\overline{f})}{(s-M_Z^2)^2 + \Gamma_Z^2 M_Z^2}}$$

and exactly at the peak we have:

$$\sigma_{e^+e^-\to f\overline{f}}(s = M_Z) \simeq \frac{12\pi}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to f\overline{f})}{\Gamma_Z^2} = \frac{12\pi}{M_Z^2} BR(Z \to e^+e^-) BR(Z \to f\overline{f})$$

We want to measure the partial decay width of Z^0 into neutrinos. The total width of the Z is the sum of the partial widths:

$$\begin{aligned}
\Gamma_Z &= \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{hadrons} + \underbrace{\Gamma_{\nu_1\nu_1} + \Gamma_{\nu_2\nu_2} + \Gamma_{\nu_3\nu_3} + ...}_{\Gamma(Z\to\text{invisible})\equiv\Gamma_{inv}} \\
&= \sum_l \Gamma_l + \Gamma_{had} + N_\nu \Gamma_\nu
\end{aligned}$$

this implies:

$$\Gamma_{inv} = \Gamma_Z - \Gamma_{had} - \sum_l \Gamma_l = \Gamma_Z - \Gamma_{had} - 3\Gamma_{ee}$$

where in the last step we have used lepton universality ($\Gamma_{ee} = \Gamma_{\mu\mu} = \Gamma_{\tau\tau}$).

now the hadronic decay width is not an independent parameter, because if we are condidering the process $e^+e^- \to Z \to hadrons$ at the Z peak the cross section is:

$$\sigma_{e^+e^-\to hadrons}(s = M_Z) \equiv \sigma_h^0 \simeq \frac{12\pi}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to hadrons)}{\Gamma_Z^2} = \frac{12\pi}{M_Z^2} \frac{\Gamma_{ee}\Gamma_{had}}{\Gamma_Z^2}$$

and this means:

$$\Gamma_{had} = \frac{M_Z^2 \sigma_h^0}{12\pi} \frac{\Gamma_Z^2}{\Gamma_{ee}}$$

so that:

$$\Gamma_{inv} = N_\nu \Gamma_\nu = \Gamma_Z - \Gamma_{had} - 3\Gamma_{ee} = \Gamma_Z - \frac{M_Z^2 \sigma_h^0}{12\pi} \frac{\Gamma_Z^2}{\Gamma_{ee}} - 3\Gamma_{ee}$$

and if we exclude the possibility of the decay of $Z \to \nu_4 \nu_4$ with ν_4 a new fourth generation neutrino species we are left with 3 neutrino species:

$$\boxed{\Gamma_\nu = \frac{1}{3}\left(\Gamma_Z - \frac{M_Z^2 \sigma_h^0}{12\pi} \frac{\Gamma_Z^2}{\Gamma_{ee}} - 3\Gamma_{ee}\right)}$$

One can for example extract from the fit to the lineshape of the $e^+e^- \to Z \to hadrons$ cross section, the mass of the Z (M_Z) the total width Γ_Z and the cross-section at the peak (σ_h^0)

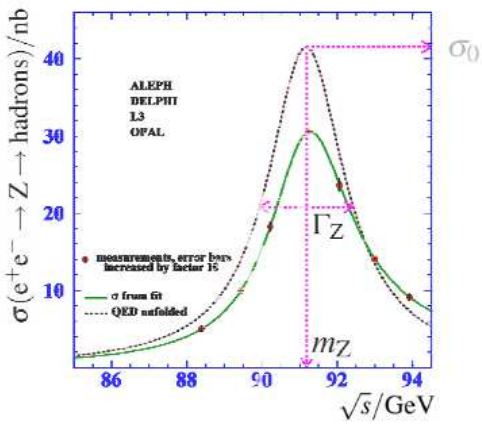


Fig. 9. Fit to the $e^+e^- \to Z \to hadrons$ cross section by the four LEP experiments

and the after measuring the branching ratio to leptons $BR(Z \to \ell\ell)$ compute the neutrino width.
Conversely if one does not fix the number of neutrinos, one can compute the $Z \to \nu\overline{\nu}$ decay width from Eq. 37 and extract an estimate of the number of active neutrinos in the process. This is a stringent and non-trivial test of the Standard Model.

5. Problem sheet 5

5.1. *Problem 1*

Show that in the Standard Model the decay width Γ_{WW} of the Higgs boson into W bosons is:

$$\Gamma_{WW} = \frac{\sqrt{2}G_F M_W^2 M_H}{8\pi}\frac{\sqrt{1-x_W}}{2x_w}(3x_W^2 - 4x_w + 4),$$

where:

$$G_F = \sqrt{2}\frac{4\pi\alpha}{8sin^2\theta_W M_w^2}$$

and:

$$x_w = \frac{4M_W^2}{M_H^2}$$

It is useful the following expression for the calculation on the decay width:

$$\Gamma_{WW} = \int d\Omega \frac{|\overline{\mathcal{M}}|^2}{32\pi^2}\frac{|\mathbf{q}_{CM}|}{M_H^2}$$

where $|\mathbf{q}_{CM}|$ is the modulus of the three-momentum of either W boson in the Centre-of-Mass (CM) frame (i.e., where the Higgs boson is at rest).

Solution

The Feynman diagram of Higgs decay into two gauge bosons is:

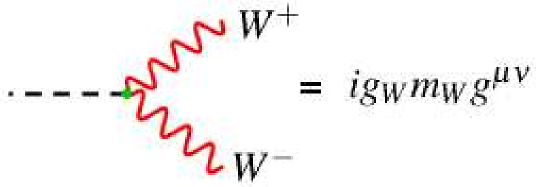


Fig. 10. Feynman diagram for scalar SM Higgs decaying into W's ($h^0(q) \to W^+(k_1)W^-(k_2)$)

The invariant amplitude for the process in the Leading Order of Standard model is:

$$\boxed{\mathcal{M}_{H\to W^+W^-} = ig_W M_W g_{\mu\nu}\varepsilon^{\mu}(k_1,\lambda_1)\varepsilon^{*\nu}(k_2,\lambda_2)} \quad (38)$$

The rate is:

$$\Gamma(H \to W^+W^-) = \frac{1}{2M_H}\int|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 dPs = \frac{1}{2M_H}g_W^2 M_W^2 \int dPs \sum_{\lambda_1,\lambda_2}|\varepsilon^{\mu}(k_1,\lambda_1)\varepsilon^*_{\mu}(k_2,\lambda_2)|^2$$

Let's concentrate on the matrix element calculation, squaring and summing on final state spins Eq. 38:

$$|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 = g_W^2 M_W^2 \sum_{\lambda_1,\lambda_2}|\varepsilon^{\mu}(k_1,\lambda_1)\varepsilon^*_{\mu}(k_2,\lambda_2)|^2 = g_W^2 M_W^2 \sum_{\lambda_1,\lambda_2}\varepsilon^{\mu}(k_1,\lambda_1)\varepsilon^*_{\mu}(k_2,\lambda_2)\varepsilon^{*\nu}(k_1,\lambda_1)\varepsilon_{\nu}(k_2,\lambda_2)$$

arranging the spin sums:

$$|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 = g_W^2 M_W^2 \left(\sum_{\lambda_1}\varepsilon^{\mu}(k_1,\lambda_1)\varepsilon^{*\nu}(k_1,\lambda_1)\right)\left(\sum_{\lambda_2}\varepsilon^*_{\mu}(k_2,\lambda_2)\varepsilon_{\nu}(k_2,\lambda_2)\right)$$

The polarization sum for a massive vector yields:

$$\sum_{\lambda}\varepsilon^{\mu}(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2} \quad (39)$$

so using Eq 39 we have:

$$|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 = g_W^2 M_W^2 \underbrace{\sum_{\lambda_1}\varepsilon^{\mu}(k_1,\lambda_1)\varepsilon^{*\nu}(k_1,\lambda_1)}_{-g^{\mu\nu}+\frac{k_1^{\mu}k_1^{\nu}}{M_W^2}}\underbrace{\sum_{\lambda_2}\varepsilon^*_{\mu}(k_2,\lambda_2)\varepsilon_{\nu}(k_2,\lambda_2)}_{-g_{\mu\nu}+\frac{k_{2\mu}k_{2\nu}}{M_W^2}}$$

i.e.:

$$|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 = g_W^2 M_W^2\Big(-g^{\mu\nu} + \frac{k_1^\mu k_1^\nu}{M_W^2}\Big)\Big(-g_{\mu\nu} + \frac{k_{2\mu}k_{2\nu}}{M_W^2}\Big)$$

Performing the tensorial contraction:

$$|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 = g_W^2 M_W^2\Big(4 - \frac{k_2\cdot k_2}{M_W^2} - \frac{k_1\cdot k_1}{M_W^2} + \frac{(k_1\cdot k_2)^2}{M_W^4}\Big)$$

now since we are assuming the are producing two on-shell gauge bosons we have $k_1^2 = k_1\cdot k_1 = M_W^2$ and $k_2^2 = k_2\cdot k_2 = M_W^2$. The fourth term in the sum is proportional to: $(k_1\cdot k_2)^2$. We can use momentum conservation to write:

$$q = k_1 + k_2 \Rightarrow q^2 = (k_1+k_2)^2 = k_1^2 + k_2^2 + 2k_1\cdot k_2 = 2M_w^2 + 2(k_1\cdot k_2)$$

now using the on-shell mass condition on the Higgs we have $q^2 = M_H^2$ and so: $M_H^2 = 2M_w^2 + 2(k_1\cdot k_2)$ and so:

$$M_H^2 = 2M_w^2 + 2(k_1\cdot k_2) \Rightarrow \boxed{(k_1\cdot k_2) = \frac{M_H^2 - 2M_W^2}{2}}$$

so the squared matrix element is:

$$|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 = g_W^2 M_W^2\Big(4 - \frac{M_W^2}{M_W^2} - \frac{M_W^2}{M_W^2} + \frac{(M_H^2-2M_W^2)^2}{4}\cdot\frac{1}{M_W^4}\Big) = g_W^2 M_W^2\Big(2 + \frac{(M_H^2-2M_W^2)^2}{4M_W^4}\Big)$$

Performing the algebra:

$$\begin{aligned}|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 &= g_W^2 M_W^2\Big(2 + \frac{(M_H^2-2M_W^2)^2}{4M_W^4}\Big) = g_W^2 M_W^2\Big(\frac{M_H^4 + 4M_W^4 - 4M_H^2M_W^2 + 8M_W^4}{4M_W^4}\Big)\\ &= g_W^2 M_W^2\Big(\frac{M_H^4 - 4M_H^2M_W^2 + 12M_W^4}{4M_W^4}\Big) = \frac{g_W^2 M_H^4}{4M_W^2}\Big(1 - 4\frac{M_W^2}{M_H^2} + 12\frac{M_W^4}{M_H^4}\Big)\end{aligned}$$

let's now focus on the phase space factor:

$$d\Gamma(H\to W^+W^-) = \frac{1}{2M_H}|\overline{\mathcal{M}}_{H\to W^+W^-}|^2 dPs = \frac{1}{2M_H}|\overline{\mathcal{M}}_{H\to W^+W^-}|^2\frac{d^3\mathbf{k_1}}{(2\pi)^3 2E_1}\frac{d^3\mathbf{k_2}}{(2\pi)^3 2E_2}(2\pi)^4\delta(q-k_1-k_2)$$

so we have

$$\Gamma = \frac{1}{2M_H}|\overline{\mathcal{M}}|^2 dPs = \frac{(2\pi)^{-2}}{2M_H}\int|\overline{\mathcal{M}}|^2\frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}\delta(q-k_1-k_2)$$

so that:

$$\Gamma = \frac{(2\pi)^{-2}}{2M_H}\int|\overline{\mathcal{M}}|^2\delta(M_H-E_1-E_2)\delta^3(\mathbf{q}-\mathbf{k_1}-\mathbf{k_2})\frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}$$

where it is understood that $E_1 = \sqrt{k_1^2+M_W^2}$ and $E_2 = \sqrt{k_2^2+M_W^2}$ with $k_1 \equiv |\mathbf{k_1}|$ and $k_2 \equiv |\mathbf{k_2}|$. In the rest frame of the Higgs (the centre of mass frame), we have $E_H = M_H$, $p_H = q^* = 0$ and the decay rate becomes:

$$\Gamma = \frac{(2\pi)^{-2}}{2M_H}\int|\overline{\mathcal{M}}|^2\delta(M_H-E_1-E_2)\delta^3(\mathbf{k_1}+\mathbf{k_2})\frac{d^3k_1}{2E_1}\frac{d^3k_2}{2E_2}$$

carrying out the integral $\int d^3k_2\delta^3(\mathbf{k_1}+\mathbf{k_2})$ (and the δ function imposes $\mathbf{k_1} = -\mathbf{k_2}$ gives:

$$\Gamma = \frac{1}{8\pi^2 M_H}\int|\overline{\mathcal{M}}|^2\delta(M_H-E_1-E_2)\frac{d^3k_1}{4E_1E_2}\Big|_{\mathbf{k_2}=-\mathbf{k_1}}$$

where E_2 is now to be understood as $E_2 = \sqrt{k_2^2+M_W^2} = \sqrt{k_1^2+M_W^2}$.
Let's now introduce the function:

$$f(k_1) = M_H - \sqrt{k_1^2+M_W^2} - \sqrt{k_2^2+M_W^2} = -\sqrt{k_1^2+M_W^2} - \sqrt{k_1^2+M_W^2} \quad (40)$$

and setting: $d^3k_1 = k_1^2 dk_1 d\cos\theta d\phi$ in spherical polar coordinates we have:

$$\Gamma = \frac{1}{8\pi^2 M_H} \int |\overline{\mathcal{M}}|^2 \delta(f(k_1)) \frac{k_1^2 dk_1}{4E_1E_2}\bigg|_{\mathbf{k_1}=-\mathbf{k_2}} d\cos\theta d\phi \tag{41}$$

The function $f(k_1)$ has a single zero at a fixed value $k_1 = k^*$ given by the solution of the equation:

$$f(k^*) = M_H - \sqrt{(k^*)^2 + M_W^2} - \sqrt{(k^*)^2 + M_W^2} = 0 \tag{42}$$

This equation expresses the requirement that energy is conserved in the decay, $M_H = E_1 + E_2$. The constant k^* is simply the value of p_1 which results from the constraint of overall energy-momentum conservation; it is the physically allowed momentum of either of the final state decay particles in the centre of mass frame.
We can now use the general result:

$$\int g(k_1)\delta(f(k_1))dk_1 = \left[\frac{g(k_1)}{|df/dk_1|}\right]_{f(k_1)=0}$$

to carry out the integral over k_1 in Eq. 41. Differentiating Equation 40 gives:

$$\frac{df}{dk_1} = \frac{-2k_1}{2\sqrt{k_1^2 + M_W^2}} - \frac{-2k_1}{2\sqrt{k_1^2 + M_W^2}} = -\frac{k_1}{E_1} - \frac{k_1}{E_2} = -k_1\frac{E_1 + E_2}{E_1E_2}$$

hence the rate becomes:

$$\Gamma = \frac{1}{8\pi^2 M_H} \int |\overline{\mathcal{M}}|^2 \left|\frac{1}{k_1}\frac{E_1E_2}{E_1+E_2}\frac{k_1^2}{4E_1E_2}\right|_{k=k^*} d\cos\theta d\phi = \frac{1}{32\pi^2 M_H} \int |\overline{\mathcal{M}}|^2 \left|\frac{k_1}{E_1+E_2}\right|_{k=k^*} d\cos\theta d\phi$$

when$k_1 = k^*$ the sum $E_1 + E_2$ is given simply by the momentum conservation constraint of Equation 42: $E_1 + E_2 = M_H$. Finally therefore, the rate for the two body decay $H \to WW$ in the rest frame of the decaying Higgs particle is obtained as:

$$\Gamma = \frac{k^*}{32\pi^2 M_H^2} \int |\overline{\mathcal{M}}|^2 d\cos\theta d\phi$$

where $k^* = |\mathbf{k_1}| = |\mathbf{k_2}|$ is the momemtum of either of the final state particles. For an isotropic decay of a spin 0 particle (such as the Higgs boson), there is no preferred spatial direction in the system and $\overline{\mathcal{M}}|$ must be independent of θ and ϕ. Using $\int d\cos\theta d\phi = 4\pi$, we obtain:

$$\boxed{\Gamma = \frac{k^*}{8\pi M_H^2}|\overline{\mathcal{M}}|^2} \tag{43}$$

we have now to determine k^* by solving Eq. 42:

$$M_H - 2\sqrt{(k^*)^2 + M_W^2} = 0 \Rightarrow M_H^2 = 4((k^*)^2 + M_W^2)$$

so that:

$$k^{*2} = \frac{M_H^2 - 4M_W^2}{4} \Rightarrow \boxed{k^{*2} = \frac{\sqrt{M_H^2 - 4M_W^2}}{2}}$$

so we are ready to write the final expression for the Higgs width into W bosons:

$$\begin{aligned}
\Gamma(H \to W^+W^-) &= \frac{\sqrt{M_H^2 - 4M_W^2}}{2 \cdot 8\pi M_H^2}|\overline{\mathcal{M}}|^2 = \frac{\sqrt{M_H^2 - 4M_W^2}}{16\pi M_H^2}\frac{g_W^2 M_H^4}{4M_W^2}\Big(1 - 4\frac{M_W^2}{M_H^2} + 12\frac{M_W^4}{M_H^4}\Big) \\
&= \frac{g_W^2 M_H^2}{64\pi M_W^2}\sqrt{M_H^2\Big(1 - 4\frac{M_W^2}{M_H^2}\Big)}\Big(1 - 4\frac{M_W^2}{M_H^2} + 12\frac{M_W^4}{M_H^4}\Big) \\
&= \frac{g_W^2 M_H \cdot M_H^2}{16\pi \cdot 4M_W^2}\sqrt{1 - 4\frac{M_W^2}{M_H^2}}\Big(1 - 4\frac{M_W^2}{M_H^2} + \frac{3}{4}\frac{16M_W^4}{M_H^4}\Big)
\end{aligned}$$

now using the definition of the adimensional parameter $x_W = 4M_W^2/M_H^2$ we have:

$$\Gamma(H \to W^+W^-) = \frac{g_W^2 M_H}{16\pi x_W}\sqrt{1 - x_W}\Big(1 - x_W + \frac{3}{4}x_W^2\Big) = \frac{g_W^2 M_H}{32\pi}\frac{\sqrt{1 - x_W}}{2x_W}\Big(4 - 4x_W + 3x_W^2\Big)$$

now using the definition of G_F the Fermi decay constant:

$$G_F = \frac{\sqrt{2}g_W^2}{8M_W^2} = \frac{\sqrt{2}g_W^2}{8M_W^2} = \frac{\sqrt{2}4\pi\alpha}{8sin^2\theta_W M_W^2}$$

since $g_W = e/sin\theta_W$. We have finally:

$$\Gamma(H \to W^+W^-) = \frac{G_F M_H M_W^2}{4\sqrt{2}\pi}\frac{\sqrt{1-x_W}}{2x_W}\Big(4 - 4x_W + 3x_W^2\Big)$$

then multiplying by a factor $\sqrt{2}/\sqrt{2}$ we obtain:

$$\boxed{\Gamma(H \to W^+W^-) = \frac{\sqrt{2}G_F M_H M_W^2}{8\pi}\frac{\sqrt{1-x_W}}{2x_W}\Big(4 - 4x_W + 3x_W^2\Big); \qquad x_W = \frac{4M_W^2}{M_H^2}}$$