Hagedorn's Model

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- R. Hagedorn: "Thermodynamics of strong interactions", CERN-Report 71-12 (1971)
- K. Redlich, H. Satz: "The legacy of Rolf Hagedorn: Statistical Bootstrap and Ultimate Temperature", `arXiv:1501.07523`
- N. Cabibbo, G. Parisi "Exponential Hadronic Spectrum and Quark Liberation", Phys.Lett. B59 (1975) 67-69

Middle '60

In high-energy particle collisions (p,e) many hadrons were created with larger and larger invariant mass.
Theorists were at work to find a consistent description and classification of hadrons. Two possible ways:

- Quark Model (→ QCD, middle '70)
- Hagedorn's Model (1965), based on statistical properties

Hagedorn's approach

Self-similar scheme for the composition and decay of hadrons and their resonances (*"fireballs"*) (Statistical Bootstrap Model)

A heavy particle is a resonant state formed by lighter particles, in a *self similarity pattern*:

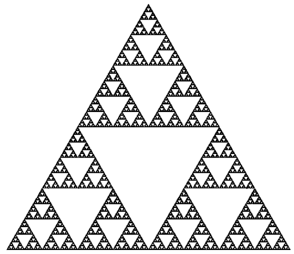
> *a fireball consists of fireballs, which in turn consist of fireballs, and so on...*[1]

System of non-interacting particles, in which the formation and decay of resonances simulates the interaction.

[1] R. Hagedorn: "Statistical thermodynamics of strong interactions at high energies", Nuovo Cim. Suppl. 3, 147 (1965)

Self-similarity in mathematics

- Sierpinski Triangle (1915)



- Benoît Mandelbrot: "Les Objects Fractals: Forme, Hazard et Dimension" (1975)
- Number theory: how many ways are there of decomposing an integer into a sum of integers?

1=1 $p(1) = 1 = 2^{n-1}$
2=2 ,1+1 $p(2) = 2 = 2^{n-1}$
3= 3, 2+1, 1+2, 1+1+1 $p(3) = 4 = 2^{n-1}$
4= 4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1 $p(4) = 8 = 2^{n-1}$
...

There are $p(n) = 2^{n-1} = \frac{1}{2}\, e^{n \ln 2}$ ways of partitioning an integer n into ordered partitions: $p(n)$ grows exponentially in n.

The Statistical Bootstrap Model

Let us consider a system of non-interacting particles with momentum $\vec{p}_\alpha$, mass m_γ, energy $\varepsilon_{\alpha\gamma} = \sqrt{\vec{p}_\alpha^2 + m_\gamma^2}$; let $\nu_{\alpha\gamma}$ be their multiplicity.
The total energy of the system is:

$$E = \sum_{\alpha\gamma}^{\infty} \nu_{\alpha\gamma}\varepsilon_{\alpha\gamma}$$

The Grand-Partition Function (Grand-canonical description with $\mu = 0$) is:

$$\mathcal{Z}(V, T) = \sum_{\{\nu\}} \exp\left\{-\frac{1}{T}\sum_{\alpha\gamma}^{\infty} \nu_{\alpha\gamma}\varepsilon_{\alpha\gamma}\right\}$$

or, in the continuum limit,

$$\mathcal{Z}(V, T) = \int_0^\infty \sigma(E, V)\, \exp\left\{-\frac{E}{T}\right\} \mathrm{d}E$$

Short-hand notation: $x_{\alpha\gamma} \equiv \exp\left\{-\frac{\varepsilon_{\alpha\gamma}}{T}\right\}$:

$$\mathcal{Z}(V,T) = \sum_{\{\nu\}} \exp\left\{-\frac{1}{T}\sum_{\alpha\gamma}^{\infty} \nu_{\alpha\gamma}\varepsilon_{\alpha\gamma}\right\} = \sum_{\{\nu\}}\prod_{\alpha\gamma} x_{\alpha\gamma}{}^{\nu_{\alpha\gamma}} = \prod_{\alpha\gamma}\left[\sum_{\{\nu\}} x_{\alpha\gamma}{}^{\nu_{\alpha\gamma}}\right]$$

with the occupation numbers: $\nu_{\alpha\gamma} \Longrightarrow \begin{cases} \nu_{\alpha\beta} = 0, 1, 2, \ldots & \textbf{Bosons} \\ \nu_{\alpha\phi} = 0, 1 & \textbf{Fermions} \end{cases}$

$$\mathcal{Z}(V,T) = \prod_{\alpha\phi}(1 + x_{\alpha\phi})\prod_{\alpha\beta}\frac{1}{1 - x_{\alpha\beta}}$$

$$\log\mathcal{Z}(V,T) = \sum_{\alpha\phi}\log(1 + x_{\alpha\phi}) - \sum_{\alpha\beta}\log(1 - x_{\alpha\beta})$$

$$\begin{aligned}\log\mathcal{Z}(V,T) &= \frac{V}{2\pi^2}\int_0^{\infty} p^2 \mathrm{d}p \left[\int_0^{\infty} \rho_F(m)\log(1 + x_{p,m})\mathrm{d}m \right. \\ &\qquad \left. - \int_0^{\infty} \rho_B(m)\log(1 - x_{p,m})\mathrm{d}m\right]\end{aligned}$$

$\log(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \ldots \qquad (x \equiv \exp\{-\frac{\varepsilon}{T}\} \Rightarrow 0 < x < 1)$

$$\log\mathcal{Z}(V,T) = \frac{V}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty} p^2\rho(m;n)\, x_{p,m}^n\, \mathrm{d}p\,\mathrm{d}m$$

where

$$x_{p,m}^n \equiv \exp\left\{-\frac{n}{T}\sqrt{p^2 + m^2}\right\} \qquad \rho(m;n) = \rho_B(m) - (-1)^n\rho_F(m)$$

By integrating over momenta (✍):

$$\mathcal{Z}(V,T) = \exp\left\{\frac{VT}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{m_0}^{\infty}\rho(m;n)\, m^2\, K_2\left(\frac{nm}{T}\right)\mathrm{d}m\right\}$$

We have obtained two expressions for $\mathcal{Z}$:

$$\mathcal{Z}(V,T) = \int_0^\infty \sigma(E,V)\,\exp\left\{-\frac{E}{T}\right\}\mathrm{d}E$$
$$= \exp\left\{\frac{VT}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{m_0}^{\infty}\rho(m;n)\,m^2\,K_2\left(\frac{nm}{T}\right)\mathrm{d}m\right\}$$

Hagedorn imposed the *logaritm bootstrap condition* ("weak condition")

$$\frac{\log\rho(m;n)}{\log\sigma(m,V_0)} \underset{m\to\infty}{\longrightarrow} 1$$

i.e. asymptotic equality of entropies.
σ and ρ differ by som algebraic factor in m: $\sigma(E)$ counts all the states of the system in V_0, including, for instance, those whit very large angular momentum (=collective motion) which are not "fireballs" and therefore are not counted in ρ.

Hagedorn solved this equation by iterations:

$$\rho^{(0)} \longrightarrow \mathcal{Z}^{(0)} \longrightarrow \sigma^{(0)}$$
$$\log\sigma^{(0)} \equiv \log\rho^{(1)}$$
$$\rho^{(1)} \longrightarrow \mathcal{Z}^{(1)} \longrightarrow \sigma^{(1)}$$
$$\log\sigma^{(1)} \equiv \log\rho^{(2)}$$
$$\ldots$$

Starting from a simple $\rho^{(0)}(m)$, in a few iterations one gets an exponential behaviour:

$$\rho(m) = Am^a e^{m/T_H} \qquad\qquad \sigma(m) = Bm^b e^{m/T_H}$$

The logaritm bootstrap condition is satisfied:

$$\frac{\log\rho(m;n)}{\log\sigma(m,V_0)} = \frac{m/T_H + a\log m + \log A}{m/T_H + b\log m + \log B} \underset{m\to\infty}{\longrightarrow} 1$$

Hagedorn's solution: $a = -\frac{5}{2}$

$$\rho_H(m) = A\, m^{-5/2} e^{m/T_H}$$

with $T_H \sim 150 - 180$ MeV.
A few years later, Nahm[2] solved the equation analytically (with a "strong" condition, conservation laws), and found $a = -3$:

$$\rho(m) = A\, m^{-3} e^{m/T_H}$$

with:

$$\frac{V_0 T_H^3}{2\pi^2} \left(\frac{m_0}{T_H}\right)^2 K_2\left(\frac{m_0}{T_H}\right) = 2\log 2 - 1$$

For $m_0 = m_\pi$: $T_H \simeq 150$ MeV.

[2] W. Nahm: "Analytical solution of the statistical bootstrap model", Nucl. Phys. B 45, 525 (1972)

Experimental estimate of T_H

J.Orear "Universality of Transverse Momentum distribution in High Energy Physics", Phys. Rev. Lett. 13, 190, (1964)

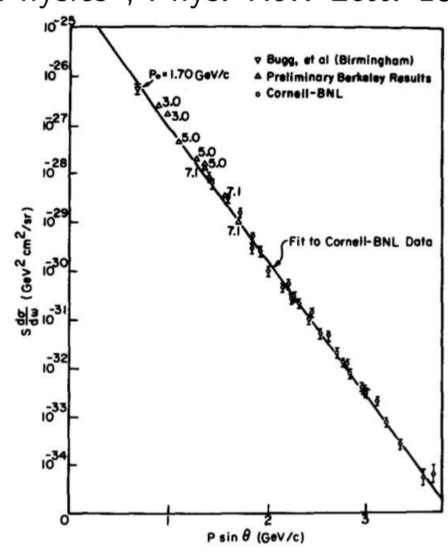


Fig. 1. Plot of large angle p-p elastic scattering data vs. transverse momentum. The line is the least squares fit of eq. (2) to the 29 points of the Cornell-Brookhaven group.

[. . .]

Finally we discuss the transverse momentum distribution of secondaries produced in P-P collisions of fixed energies. In 1961, Cocconi, Koester and Perkins pointed out that the distribution function

$$dN/dp_\perp \propto p_\perp \exp(-ap_\perp), \text{ where } 1/a = 165 \text{ MeV}/c \quad (4)$$

fits the 10 to 30 GeV pion production data from CERN and Brookhaven as well as cosmic ray data up to 10^5 GeV [15]. We note that eqs. (1) or (2) yields the same transverse momentum distribution function for elastically scattered protons as long as we keep away from angles near 90°. The

[. . .]

the plots are consistent with eq. (4) and a value of $1/a \approx 160$ MeV/c.
No firm theoretical explanation has yet been given of why a simple exponential, $\exp(-ap_\perp)$ should appear to dominate high energy physics. Recent

Hagedorn calculated the transverse momentum distribution in its model and found a natural explanation!

A closer look...

Let us study the high-temperature limit of the Hagedorn's partition function (only $n = 1$):

$$\log \mathcal{Z}(T, V) \simeq \frac{VT}{2\pi^2} \int_{m_0}^{\infty} m^2 \rho(m, 1) K_2\left(\frac{m}{T}\right) \mathrm{d}m$$
$$\propto \frac{VT}{2\pi^2} \int_{m_0}^{\infty} m^2 m^a \mathrm{e}^{m/T_H} K_2\left(\frac{m}{T}\right) \mathrm{d}m$$

for $z \to \infty$: $K_2(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} + \mathcal{O}(z^{-2})$

$$\mathrm{e}^{m/T_H} K_2\left(\frac{m}{T}\right) \sim \mathrm{e}^{-m\left(\frac{1}{T} - \frac{1}{T_H}\right)}$$

$\mathcal{Z}$ diverges exponentially if $T > T_H$!

Hagedorn: T_H is the limiting temperature for hadronic matter

A new phase

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EXPONENTIAL HADRONIC SPECTRUM AND QUARK LIBERATION

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The exponentially increasing spectrum proposed by Hagedorn is not necessarily connected with a limiting temperature, but it is present in any system which undergoes a second order phase transition. We suggest that the "observed" exponential spectrum is connected to the existence of a different phase of the vacuum in which quarks are not confined.

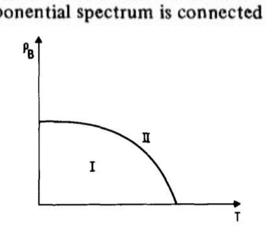

Fig. 1. Schematic phase diagram of hadronic matter. ρ_B is the density of baryonic number. Quarks are confined in phase I and unconfined in phase II.

This is the current interpretation of T_H

Conclusions

- Hagedorn's Model was proposed and studied between 1965 and 1975. It was abandoned in favor of the QCD.
- It was recovered in the middle of '90, when it was observed that the total multiplicity of hadronic particles produced in high energy collisions ($e^+ - e^-$, $p - p$, ...) could be accurately described by thermal models: $N \propto \mathrm{e}^{m/T}$, with $T \sim 150$ MeV.
- At present, its modern version (the Hadron Resonance Gas Model) is widely used to study the hadronic phase in the heavy-ion collisions.