Relativistic Hydrodynamics / I

Ideal case

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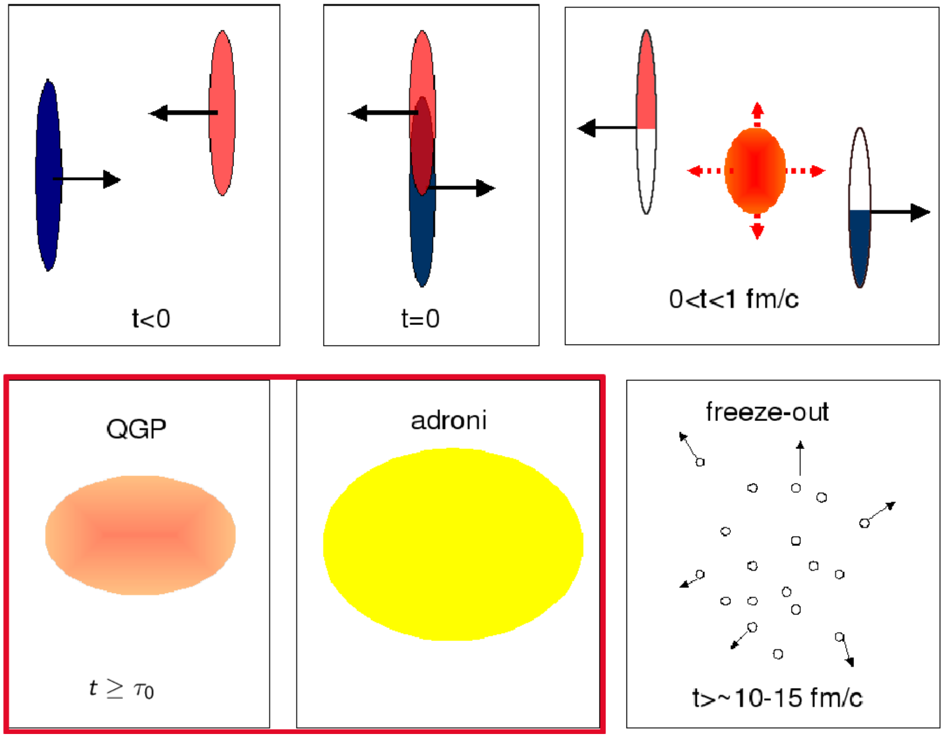


A Heavy-Ion collision (HIC) is a complicate process, undergoing several stages, each of them has to be treated properly.

The expansion and cooling of the QGP and of the subsequent hadron gas is described in terms of relativistic hydrodynamics (from thermalization to freeze-out)

The Hydrodynamical description is appropriate if the fluctuations in the system are small and microscopic dynamics drives such a system rapidly to the state of maximum disorder (i.e. equilibrium). The global behavior of the system can be expressed in terms of a few macroscopic quantities.

Ideal fluid: it reacts istantaneously to any changes of the local macroscopic fields, for instance by readjusting the slope of its particles' momentum distribution (i.e. temperature) locally on an infinitesimally short time scale.

Euler Equations (and their relativistic generalizations) describe how macroscopic pressure gradients generate collective flow of the matter subject to the constraints of local conservation of energy, momentum and conserved charges.

In HIC : typical relaxation times are smaller than 1 fm/c (experimental fit).

Review of thermodynamics

First law of Thermodynamics: $$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N, \qquad (1)$$

In nonrelativistic systems N is the number of particles (conserved). In relativistic systems, particles and anti-particles can be created and N is a conserved quantity (baryon number, electric charge, strangeness,...).
If there are several conserved quantities N_i, $\mu\mathrm{d}N$ is replaced by $\sum_i \mu_i \mathrm{d}N_i$.
In relativistic systems the mass energy mc^2 is included in the internal energy.
The energy is an extensive function of the extensive variables V, S, N:
$U(\lambda V, \lambda S, \lambda N) = \lambda U(V, S, N)$. One can write $U = -PV + TS + \mu N$;
Differentiating and using (1) we get the Gibbs-Duhem relation

$$V\mathrm{d}P = S\mathrm{d}T + N\mathrm{d}\mu \qquad (2)$$

In hydrodynamics the useful variables are the densities: $\epsilon = \frac{U}{V}$, $s = \frac{S}{N}$, $n = \frac{N}{V}$ (intensive quantities). Eqs (1) and (2) become:

$$\epsilon = -P + Ts + \mu n \qquad \mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu \qquad (3)$$

which imply $$\mathrm{d}\epsilon = T\mathrm{d}s + \mu\mathrm{d}n. \qquad (4)$$

Isentropic process

S and N are conserved:

$$\begin{matrix} S = sV = \text{const.} \\ N = nV = \text{const.} \end{matrix} \implies \begin{matrix} \mathrm{d}S = 0 = V\mathrm{d}s + s\mathrm{d}V \\ \mathrm{d}N = 0 = V\mathrm{d}n + n\mathrm{d}V \end{matrix} \implies \frac{\mathrm{d}s}{s} = \frac{\mathrm{d}n}{n} = -\frac{\mathrm{d}V}{V}$$

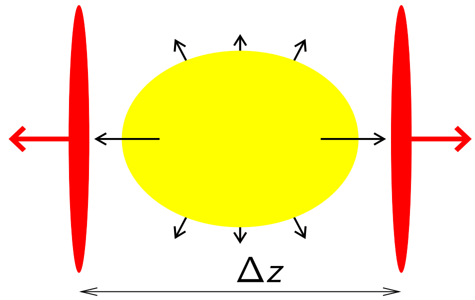
Then Eq. (1) reduces to $\mathrm{d}U = -P\mathrm{d}V$ and

$$\mathrm{d}U = \mathrm{d}(\epsilon V) = \epsilon\mathrm{d}V + V\mathrm{d}\epsilon = -P\mathrm{d}V \implies \mathrm{d}V(P+\epsilon) + V\mathrm{d}\epsilon = 0$$

hence

$$\frac{\mathrm{d}\epsilon}{\epsilon + P} = -\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}s}{s} = \frac{\mathrm{d}n}{n} \qquad (5)$$

Nucleus-Nucleus collisions at high energy



The two nuclei are, in the c.m. frame, contracted ($\gamma_{SPS} \simeq 10$, $\gamma_{RHIC} \simeq 100$, $\gamma_{LHC} \simeq 2500 - 7000$)
They collide at $z = t = 0$, thousands of particles are produced in the interactions between the nucleons of the nuclei.
The expansion of the matter produced in HIC is, initially, mostly longitudinal. At later times the transverse expansion is no longer negligible.
The velocity of a fluid element at point (z, t) is z/t (Hubble's law)
In the midrapidity region: approximate invariance for Lorentz boosts parallel to the collision axis (z).

Assume entropy conservation and local equilibrium: $s(t)t =$const., $T^3 t =$const. $\Longrightarrow$

$$\epsilon \sim P \sim T^4 \sim t^{-4/3}$$

Let us give an estimate of the initial energy density:

$Pb - Pb$ at SPS: 600 pions in $-0.5 < y < 0.5$ (y= rapidity).

Assume $\tau \sim 1$ fm/c: particles in the rapidity interval $|y| < 0.5$ have the maximal longitudinal velocity: $\beta_z = \frac{v_z}{c} = \tanh y = \tanh 0.5 \sim 0.5$.

Lenght of the system at $t = \tau_0$: $\quad \Delta z = 2 v_z \tau_0 \sim 1$ fm.

Average energy per pion $E_\pi \sim 0.5$ GeV: $\Longrightarrow$

$$\epsilon_0 \simeq \frac{E_\pi}{\pi R_A^2 \Delta z} \frac{\mathrm{d}N}{\mathrm{d}y} \sim 3 \text{ GeV/fm}^3$$

Fundaments of relativistic hydrodynamics

Let us define the 4-velocity of a fluid element $u^\mu = \gamma(u)(1, \vec{u})$. The 4 components are not independent: $u^\mu u_\mu = 1$ (natural units: $c = 1$).
In the local rest frame: $u^\mu_{(0)} = (1, \vec{0})$.
Particles of the fluid element have different velocities with $\langle v_x \rangle = \frac{1}{N}\sum_i v_{xi} = 0$, $\langle v_y \rangle = \langle v_z \rangle = 0$. Local thermodynamic equilibrium. Thermodynamic quantities are defined in the rest frame.
The Energy-momentum Tensor is a 2-rank contravariant tensor $T^{\mu\nu}(x)$ whose components have the physical meaning of:

$$\begin{array}{ll} T^{00} = \text{energy density} & T^{i0} = \text{energy flux} \\ T^{0i} = \text{momentum density} & T^{ij} = \text{momentum flux} \end{array}$$

For a fluid element in its rest frame spatial isotropy implies $T^{0i} = T^{i0} = 0$, $T^{ij} = P\delta^{ij}$:

$$T^{\mu\nu}_{(0)}(x) = \begin{pmatrix} \epsilon(x) & 0 & 0 & 0 \\ 0 & P(x) & 0 & 0 \\ 0 & 0 & P(x) & 0 \\ 0 & 0 & 0 & P(x) \end{pmatrix}$$

$T^{\mu\nu}(x)$ for a moving fluid element is obtained from $T^{\mu\nu}_{(0)}(x)$ with a Lorentz boost $\Lambda(\vec{v})$:

$$T^{\mu\nu} = \Lambda(\vec{v})^\mu_\alpha \Lambda(\vec{v})^\nu_\beta T^{\alpha\beta}_{(0)} \qquad \text{or} \qquad T = \Lambda T_{(0)} \Lambda^T$$

To first order in $\vec{v}$: $\Lambda(\vec{v}) \simeq \begin{pmatrix} 1 & v_x & v_y & v_z \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ v_z & 0 & 0 & 1 \end{pmatrix}$

$$T^{\mu\nu} \simeq \begin{pmatrix} \epsilon & (\epsilon+P)v_x & (\epsilon+P)v_y & (\epsilon+P)v_z \\ (\epsilon+P)v_x & P & 0 & 0 \\ (\epsilon+P)v_y & 0 & P & 0 \\ (\epsilon+P)v_z & 0 & 0 & P \end{pmatrix} \tag{6}$$

$T^{\mu\nu}$ is symmetric in the relativistic form: $T^{i0} = T^{0i}$, while in NR fluid dynamics they differ.
Momentum density: $(\epsilon + P)\vec{v}$, in the NR limit $P \ll \epsilon$ and $\epsilon \simeq \rho$ therefore $(\epsilon + P)\vec{v} \overset{NR}{\to} \rho\vec{v}$. Pressure contributes to the inertia of a relativistic fluid.

The exact form of $T^{\mu\nu}$ for moving fluid element can be obtained with a finite boost, or by the following argument:

to respect covariance $T^{\mu\nu}$ must have the form: $T^{\mu\nu} = ag^{\mu\nu} + bu^\mu u^\nu$, going to the local rest frame ($u^\mu = (1, \vec{0})$), $T^{\mu\nu}$ must reduce to $T^{\mu\nu}_{(0)}$:

$$T^{00} = a + b = \epsilon \qquad T^{11} = -a = p \quad \Longrightarrow \quad b = \epsilon + P$$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - pg^{\mu\nu}$$

ϵ and P are defined in the local rest frame.

Equations of relativistic Hydrodynamics

Local conservation of energy and momentum, no dissipation (i.e. no viscosity):

$$\partial_\mu T^{\mu\nu} = 0 \qquad \nu = 0, 1, 2, 3$$

Baryon number conservation: $\partial_\mu(nu^\mu) = \partial_\mu j^\mu = 0$. There is a similar equation for any conserved charge: $\partial_\mu(n_i u^\mu) = \partial_\mu j_i^\mu = 0$, $i = 1, \ldots, M$
Equation of state relate the thermodynamic variables: $p(\epsilon, n_i) = 0$.
Altogether there are

- $4 + M + 1 = 5 + M$ equations
- 3 indep. comp. of u^μ + ϵ + P + n_i ($i = 1, \ldots, M$) = $5 + M$ variables

Sound waves

Sound = small disturbance propagating in an uniform fluid at rest. Let us consider an uniform fluid at rest with ϵ_0 and P_0. Now let us add a small perturbation:

$$\begin{aligned} \epsilon(t,\vec{x}) &= \epsilon_0 + \delta\epsilon(t,\vec{x}) \\ P(t,\vec{x}) &= P_0 + \delta P(t,\vec{x}) \end{aligned}$$

The energy and momentum conservation equations give (using (6) for $T^{\mu\nu}$, keeping only 1st order terms in $\vec{v}$ and δ):

$$\partial_\mu T^{\mu 0} = \frac{\partial \epsilon}{\partial t} + \vec{\nabla}\cdot\big((\epsilon+P)\vec{v}\big) = 0 \quad \Longrightarrow \quad \frac{\partial(\delta\epsilon)}{\partial t} + (\epsilon_0+P_0)\vec{\nabla}\cdot\vec{v} = 0 \tag{7}$$

energy conservation (i.e. continuity eq.) in the NR limit

$$\partial_\mu T^{\mu i} = \frac{\partial}{\partial t}\big((\epsilon+P)\vec{v}\big) + \vec{\nabla}P = \vec{0} \quad \Longrightarrow \quad (\epsilon_0+P_0)\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}(\delta P) = \vec{0} \tag{8}$$

Newton's 2nd law

Note: if $\vec{\nabla}\cdot\vec{v} > 0$ the system is expanding;
$-\vec{\nabla}(\delta P)$= force per unit volume (it pushes the fluid towards lower pressure).

Velocity of sound: $c_s = \sqrt{\frac{\partial P}{\partial \epsilon}}$

c_s is inversely proportional to the adiabatic compressibility of the fluid.

$\delta P = c_s^2 \delta\epsilon$ replace in (8) $\Longrightarrow$

$$(\epsilon_0+P_0)\frac{\partial \vec{v}}{\partial t} + c_s^2\vec{\nabla}(\delta\epsilon) = 0 \tag{9}$$

Using (9) and (7):

$$(\epsilon_0+P_0)\vec{\nabla}\cdot\frac{\partial \vec{v}}{\partial t} + c_s^2\Delta(\delta\epsilon) = 0\ , \qquad \frac{\partial^2(\delta\epsilon)}{\partial t^2} + (\epsilon_0+P_0)\vec{\nabla}\cdot\frac{\partial \vec{v}}{\partial t} = 0$$

$$\Delta(\delta\epsilon) - \frac{1}{c_s^2}\frac{\partial^2(\delta\epsilon)}{\partial t^2} = 0$$

Inviscid fluid is non dispersive.

Ideal fluid of massless particles: $P = \frac{\epsilon}{3} \quad \Longrightarrow \quad c_s = \frac{1}{\sqrt{3}}$

Ideal gas

We must define the initial conditions for the hydrodynamical evolution.
If the interaction energies between particles are small compared to their kinetic energy, the ideal gas approximation is applicable. The hydrodynamic quantities are expressed in terms of the individual particle properties.
(We use the notation $v^\mu \equiv (1, \vec{v}) = p^\mu / p_0$)

Baryon current: $$j^\mu = n u^\mu = \frac{1}{V} \sum_{\text{particles}} B \frac{p^\mu}{p^0} = \frac{1}{V} \sum_{\text{particles}} B \; v^\mu$$

Energy-momentum tensor: $$T^{\mu\nu} = \frac{1}{V} \sum_{\text{particles}} \frac{p^\mu p^\nu}{p^0} = \frac{1}{V} \sum_{\text{particles}} v^\mu p^\nu$$

nu^0, T^{00} and T^{0i} correspond to the baryon, energy and momentum densities. The corresponding fluxes are obtained by weighting these quantities with the particle velocity $\vec{v} = \vec{p}/p^0$.

Local thermodynamic equilibrium $\Longrightarrow$ Boltzmann statistics (for simplicity of notation): the mean number of particles with momentum $\vec{p}$ is $\frac{\mathrm{d}^3 N}{\mathrm{d}p^3} \propto \mathrm{e}^{-(E^* - \mu)/T}$, where $E^* = \sqrt{\vec{p}^2 + m^2}$ is the energy of a particle in the fluid element rest frame.

Baryon current: $$j^\mu = n u^\mu = \frac{1}{V} \sum_{\text{particles}} B \frac{p^\mu}{p^0} \to \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{-\frac{E^* - \mu}{T}} B \frac{p^\mu}{p^0}$$

Energy-momentum tensor: $$T^{\mu\nu} = \frac{1}{V} \sum_{\text{particles}} \frac{p^\mu p^\nu}{p^0} \to \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{-\frac{E^* - \mu}{T}} \frac{p^\mu p^\nu}{p^0}$$

Note that j^μ and $T^{\mu\nu}$ are explicitly covariant, since $\dfrac{\mathrm{d}^3 p}{p^0}$ is Lorentz invariant.

Initial conditions

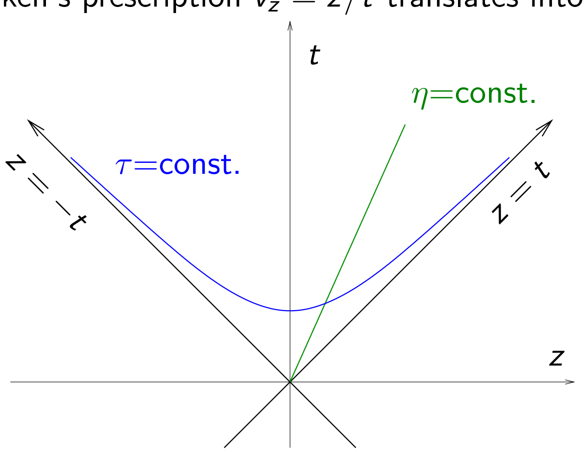
The collision starts at $z = t = 0$, the two nuclei pass through each other in a time $t_{coll} \sim 0.1$ fm/c (RHIC) or $t_{coll} \sim 0.01$ (LHC).
Initial conditions are fixed at some time t_0 or, more precisely, at some proper time τ_0 over a space-like surface.
Isotropy implies that, if the collision is central (i.e. impact parameter $b = 0$) there is no preferred direction in the transverse plane (x, y), and $\langle p_x \rangle = \langle p_y \rangle = 0$ in a fluid element. There is no transverse collective flow in the initial system: if it is present and the end of the evolution, it has been created during the hydrodynamical expansion.
All particles are produced in a very short interval around $z = t = 0$, with longitudinal velocity $v_z = z/t$ (Bjorken's prescription), the same as the fluid element. This prescription is "boost invariant", because for a Lorentz boost in the z direction: $z' = \gamma(z - vt) = \gamma t(v_z - v)$ $\quad t' = \gamma(t - vz) = \gamma t(1 - vv_z)$

$$v'_z = \frac{v_z - v}{1 - vv_z} = \frac{z'}{t'}$$

New coordinates: proper time τ, space-time rapidity η_s, fluid rapidity Y

$$\begin{array}{ccc} t = \tau \cosh \eta_s & z = \tau \sinh \eta_s & v_z = \tanh Y \\ \tau = \sqrt{t^2 - z^2} & \eta_s = \frac{1}{2} \log \frac{t+z}{t-z} & Y = \frac{1}{2} \log \frac{1+v_z}{1-v_z} \end{array} \tag{10}$$

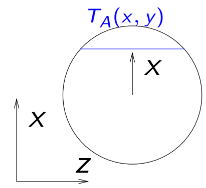
Under a Lorents boost in the z direction τ is unchanged, η_s and Y are shifted by a constant.
Initial conditions are specified at a proper time τ_0.
Bjorken's prescription $v_z = z/t$ translates into $Y = \eta_s$.



Nucleus-nucleus collision in the (z, t) plane.
$z = \pm t$ are the trajectories of the two nuclei.

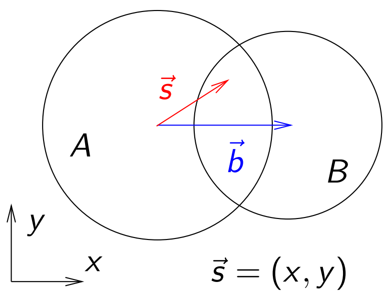
Initial density profile

Initial energy (or entropy) density is specified as a function of x, y, η_s.
Locality: in a given point (x, y) in the trasverse plane, the initial density can only depend on the thickness functions $T_A(x, y)$ and $T_B(x, y)$ of the two colliding nuclei at this point, defined as:



$$T_A(x, y) = \int_{-\infty}^{\infty} \rho_A(x, y, z)\mathrm{d}z$$

where $\rho_A(x, y, z)$ is the density of nucleons per unit volume in the nucleus A. A similar definition holds for T_B.



The initial density $\epsilon(x, y, \eta_s)$ is a function of $T_A(x, y)$ and $T_B(x, y)$, for instance:

- proportional to the density of binary collisions $T_A(x, y)T_B(x, y)$;
- proportional to the density of participants $T_A(x, y) + T_B(x, y)$;
- Color Glass Condensate gives a prescription for the η_s dependence.

To write down readable formulas, we will assume, in the following, a gaussian entropy density profile (which is in rough agreement with the most common prescriptions):

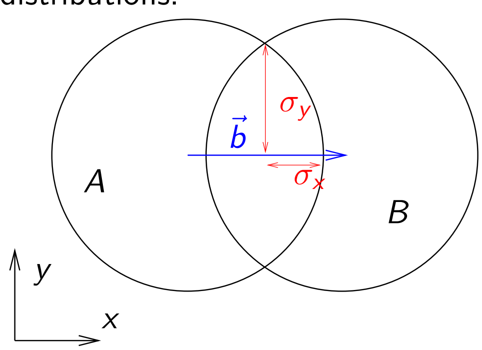
$$s(x, y, \eta_s) \propto \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{\eta_s^2}{2\sigma_\eta^2}\right) \tag{11}$$

where $\sigma_{x,y}$ are the rms of the transverse distributions.

For a central collision: $\sigma_x = \sigma_y \simeq 3$ fm.

For a non-central collision: the $x-$axis is chosen in the direction of the impact parameter, and $\sigma_x < \sigma_y$

Rapidity distribution: $\sigma_\eta \sim 2 \div 3$



Longitudinal acceleration

Initial condition on longitudinal fluid velocity: $v_z = z/t$ or $Y = \eta_s$.
Consider the midrapidity fluid element: $z = 0$, $v_z = 0$.
From (8) (valid to first order in v_z): $\frac{\partial}{\partial t}((\epsilon + P)v_z) + \frac{\partial}{\partial z}P = 0$
Initially $v_z = 0$, therefore we get the initial acceleration (using also (5)):

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\epsilon + P}\frac{\partial P}{\partial z} = -\frac{\partial \epsilon/\partial z}{\epsilon + P}\frac{\partial P}{\partial \epsilon} = -c_s^2 \frac{1}{s}\frac{\partial s}{\partial z} = -c_s^2 \frac{\partial \log s}{\partial z} \tag{12}$$

If the initial density s depends on z the fluid is accelerated and the initial condition $v_z = 0$ is not preserved by the hydrodynamical evolution.
With the coordinates defined in (10): $\frac{\partial Y}{\partial \tau} = -\frac{c_s^2}{\tau}\frac{\partial \log s}{\partial \eta_s}$ valid for all η_s.
The Bjorken's condition $Y = \eta_s$ corresponds to a flat rapidity spectrum, we see that it is not true at all times: $Y(\tau) = \left(1 + \frac{c_s^2}{\sigma_\eta^2}\log\frac{\tau}{\tau_0}\right)\eta_s > \eta_s$: the fluid element is accelerated! Still, $Y \simeq \eta_s$ is a good approximation around midrapidity.
But there is also the transverse expansion which acts as a cut off for the longitudinal expansion at a time $\tau \sim 3 \div 4$ fm/c (central Au-Au). The increase of the rapidity width is, at the end, quite modest $\sim 10\%$ at RHIC energies.

Longitudinal cooling

We study the evolution of baryon density, energy density and entropy density in a uniform longitudinal expansion (Bjorken's scenario), neglecting transverse components of the velocity (i.e. transverse expansion).
Baryon density: $\partial_\mu(nu^\mu) = 0$, at $z = 0$ (with $v_z = 0$, $\frac{\partial v_z}{\partial z} = \frac{1}{t}$) it becomes:

$$\frac{\partial n}{\partial t} + \frac{n}{t} = 0 \qquad \Longrightarrow \qquad nt = \text{const.} \tag{13}$$

which expresses the baryon number conservation in a comoving fluid element, whose volume increases as t.
Similarly, for the energy density, from (7):

$$\frac{\partial \epsilon}{\partial t} + \frac{\epsilon + P}{t} = 0 \qquad \Longrightarrow \qquad t\mathrm{d}\epsilon + \epsilon\mathrm{d}t = -P\mathrm{d}t \qquad \mathrm{d}(\epsilon t) = -P\mathrm{d}t \tag{14}$$

and we see that the comoving energy is not constant: it decreases, because of the work done by pressure force. A non-zero longitudinal pressure can only appear as a result of a thermalization process.
Longitudinal cooling implies a higher initial energy, for a given (observed) final energy. The most promising observables are electromagnetic probes, i.e. "thermal" dileptons and photons emitted at the early stages, sensitive to the temperature, but affected by huge backgrounds.

Finally, for the entropy density we get, by combining (13) and (14), using (3) and (4):

$$0 = \mu\left(\frac{\partial n}{\partial t} + \frac{n}{t}\right) - \left(\frac{\partial \epsilon}{\partial t} + \frac{\epsilon + P}{t}\right) = \frac{\mu \mathrm{d}n - \mathrm{d}\epsilon}{\mathrm{d}t} + \frac{\mu n - (\epsilon + P)}{t} = -\frac{T\mathrm{d}s}{\mathrm{d}t} - \frac{Ts}{t}$$

or

$$\frac{\partial s}{\partial t} + \frac{s}{t} = 0 \qquad \Longrightarrow \qquad st = \text{const.}$$

which shows that the entropy is conserved.

This is a general result for non-viscous hydrodynamics; physically it means that there is no heat diffusion between fluid cells.

The onset of transverse expansion

The initial transverse velocity of the fluid is zero, but the acceleration in general is not zero. It is given by an equation similar to (12):

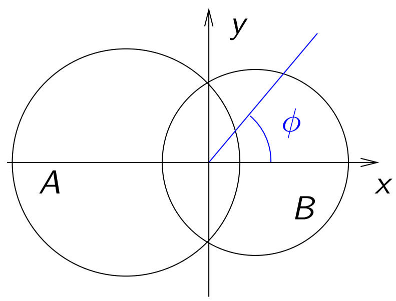
$$\frac{\partial v_x}{\partial t} = -c_s^2 \frac{\partial \log s}{\partial x} \; .$$

A similar equation holds for the y component. Using (11), assuming constant c_s, we integrate over t to obtain, for small t:

$$v_x = \frac{c_s^2 x}{\sigma_x^2} t \qquad\qquad v_y = \frac{c_s^2 y}{\sigma_y^2} t$$

Note that we have integrated from $t = 0$. Thermalization requires some time and hydrodynamics can not apply at very early times. On the other hand, the system is freely expanding in the vacuum, and the transverse expansion starts immediately, before the thermalization.

The typical time scale for the transverse expansion is σ_x/c_s, which means that it is negligible for $t \ll \sigma_x/c_s$: for such small times, the longitudinal expansion dominates.



In a non-central collision, the overlapping area has an almond shape, which results in $\sigma_x < \sigma_y$, which in turns implies $\langle v_x^2 \rangle > \langle v_y^2 \rangle$: the pressure gradient is larger along the smaller dimension and the expansion is faster.
As a consequence, more particles are emitted near $\phi = 0$ and $\phi = \pi$, i.e. parallel to the $x - axis$, than near $\phi = \pm\pi/2$
This effect corresponds to a $\cos 2\phi$ term in the Fourier decomposition of the azimuthal distribution (at $Y = 0$):

$$\frac{\mathrm{d}N}{\mathrm{d}\phi} \propto 1 + 2v_2 \cos 2\phi \qquad (15)$$

where v_2 is a positive coefficient called "elliptic flow". The observed dependence of v_2 on transverse momentum and particle species is considered the most solid evidence for hydrodynamical behavior in nucleus-nucleus collisions.
Elliptic flow is produced very early: $t_2 = R/c_s$, where $R = \sqrt{\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}}$.
At RHIC energies: $t_2 \sim 2 \div 3.5$ fm/c. Elliptic flow is therefore a signature of early pressure.

Big caveat: the final value of the elliptic flow depends on initial conditions: hydrodynamics predicts that v_2 is proportional to the eccentricity ε of the initial distribution, defined as:

$$\varepsilon = \frac{\sigma_y^2 - \sigma_x^2}{\sigma_y^2 + \sigma_x^2}$$

With ϵ estimated using participant or binary collision scaling, the v_2 calculated with non-viscous hydrodynamics agrees very well with experimental data on azimuthal anisotropy, thus raising the claim that the matter produced in HIC at RHIC is a "perfect fluid".

It was later found that the Color Glass Condensate predicts a much higher initial eccentricity, resulting in a much higher hydrodynamical v_2, thus requiring some viscous effect to be brought in agreement with data.

Another effect may increase the initial eccentricity: one expects that v_2 vanish for central collisions but this is not the case in the experimental data. A possible explanation is due to fluctuations in the position of nucleons within the nuclei. Work is in progress to observe such a fluctuation directly.

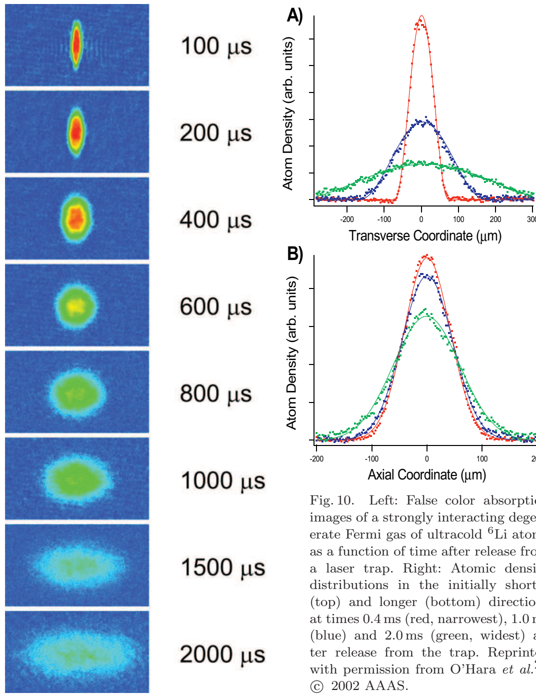


Fig. 10. Left: False color absorption images of a strongly interacting degenerate Fermi gas of ultracold ^{6}Li atoms as a function of time after release from a laser trap. Right: Atomic density distributions in the initially shorter (top) and longer (bottom) directions at times 0.4 ms (red, narrowest), 1.0 ms (blue) and 2.0 ms (green, widest) after release from the trap. Reprinted with permission from O'Hara *et al.*[26] © 2002 AAAS.

Particle spectra and anisotropies

The fluid eventually becomes free particles which reach the detector.
When the fluid density is so small that the average distance between particles is larger than the mean free path, the interactions cease (freeze-out). We can assume that the particle momentum distribution observed in the detector corresponds to the momentum distribution of the particles within the fluid, just before the freeze-out. This assumption forms the basis of the common "Cooper-Frye freeze-out picture".
Let us assume, also, for simplicity, that the fluid is baryonless ($\mu = 0$) and that the momentum distribution in the final stage of the fluid is given by the Boltzmann statistics:

$$\frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p} = \frac{2s+1}{(2\pi)^3}\mathrm{e}^{-E^*/T} \tag{16}$$

where $(2s+1)$ is the spin degeneracy and E^* is the energy of the particle in the fluid rest frame. T is the freeze-out temperature.

Comoving particles and fast particles

The Boltzmann factor (16) is maximum when the energy E^* in the fluid rest frame is minimum.
For a given fluid velocity E^* is minimum when the particle is at rest in the fluid rest frame, in which case $E^* = m$. This means that the particle is comoving with the fluid and has momentum $p^\mu = mu^\mu$.
The fast particles are more interesting: they are faster than the fluid, $E^* > m$.
For a given $\vec{p}$ of the particles, the minimum of E^* occurs if the fluid velocity is parallel to $\vec{p}$ (the fluid and the particle have the same azimuthal angle ϕ and the same rapidity Y).

For simplicity, we study particles emitted at zero rapidity ($p_z = 0$) and we derive the properties of the transverse momentum distributions. Since the transverse momentum is invariant under Lorentz boost along z, our results are valid also at non-zero rapidity.

First of all, we observe that E^* is Lorentz-invariant and it can be written as $E^* = p^\mu u_\mu$, in fact

$$E^* = p^0 u_0 - p^i u^i = \gamma E - \gamma \vec{p} \cdot \vec{u} = \gamma(E - \vec{p} \cdot \vec{u})$$

where E and $\vec{p}$ are, respectively, the energy and the momentum of the particle in the laboratory frame.
Recall $u^\mu u_\mu = 1 = (u^0)^2 - u^2$ therefore $u^0 = \sqrt{1 + u^2}$. Assuming that the fluid velocity is parallel to the particle velocity, with $p_z = 0$ ($\vec{p} = (p_x, p_y, 0) = \vec{p}_t$), we get

$$E^* = p^\mu u_\mu = \sqrt{m^2 + p_t^2}\, u^0 - p_t u = m_t u^0 - p_t u$$

The definition of fast particle is that its velocity exceeds the maximum fluid velocity, i.e. $p_t > mu$ ($m_t > mu^0$) everywhere. For a fast particle, E^* is minimum if u is maximum: fast particles are emitted from the region where the fluid velocity is largest.

Radial flow

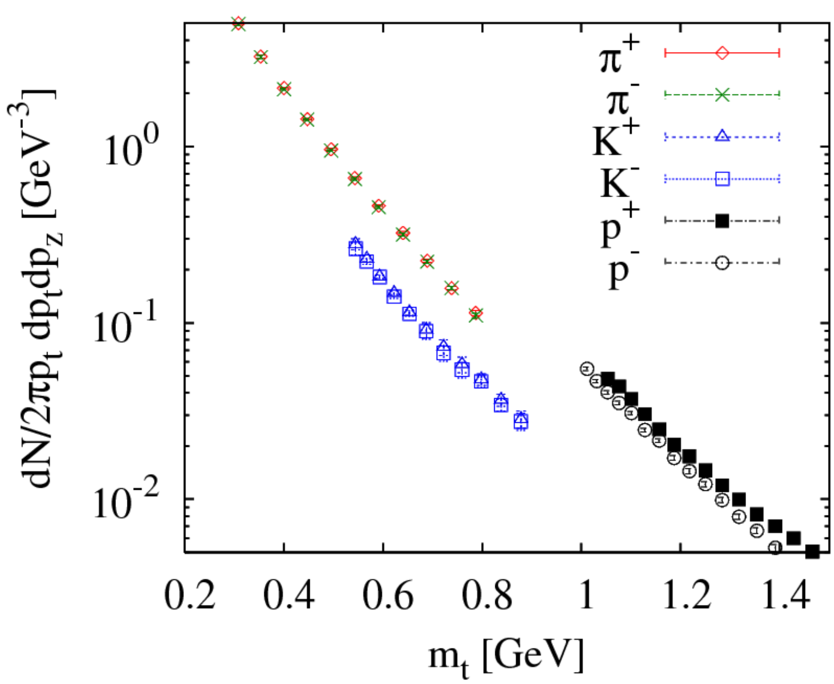
We first consider central collisions. Rotational symmetry in the transverse plane allows us to write $\mathrm{d}p_x\mathrm{d}p_y = 2\pi p_t \mathrm{d}p_t$. The Boltzmann distribution (16) becomes:

$$\frac{\mathrm{d}N}{2\pi p_t \mathrm{d}p_t \mathrm{d}p_z} \propto (2s+1) \exp\left(\frac{-m_t u^0 + p_t u}{T}\right).$$

If the fluid is at rest, i.e. $u = 0$ and $u^0 = 1$, then the spectrum is expected to be exponential in m_t, with the same slope T for all particles. This is precisely what is seen in proton-proton collisions (see figure).
Pions, kaons, protons and antiprotons are on parallel lines.
Protons are slightly above antiprotons: the net baryon number is not exactly zero.
Protons and antiprotons are above the pion line by about a factor 2: spin degeneracy.
Kaon line is well below the other lines: strangeness suppression.

Thermal models give a satisfactory description of particle spectra and abundances in $p - p$ and even in $e^- - e^+$: this does not mean that thermal equilibrium is achieved in these collisions. It could arise from the mechanism of hadronization itself, which is a statistical process.

m_T spectra in p-p collisions at $p_z = 0$



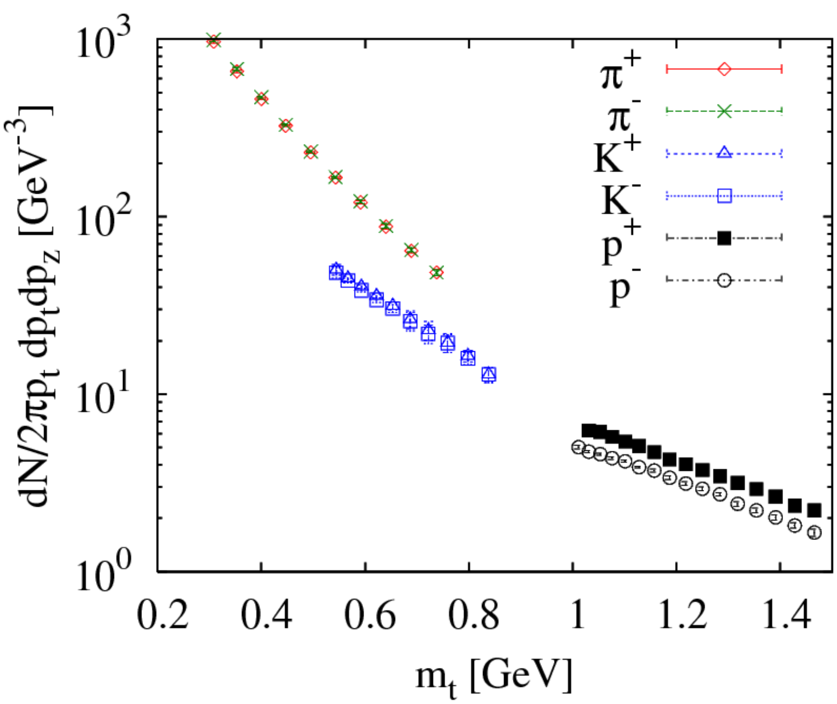
If the fluid is moving, the m_t scaling is broken: on top of thermal motion there is now a collective velocity u, the fluid velocity. The kinetic energy associated with this collective motion increases with the particle mass: heavier particles should have larger kinetic energies.
To see this, let us compute the slope of the m_t spectrum:

$$\frac{\mathrm{d}}{\mathrm{d}m_t}\log\frac{\mathrm{d}N}{2\pi p_t \mathrm{d}p_t \mathrm{d}p_z} = \frac{\mathrm{d}}{\mathrm{d}m_t}\left(\frac{-m_t u^0 + p_t u}{T}\right) = \frac{-u^0 + u m_t/p_t}{T}$$

For a given m_t, heavier particles have a smaller p_t and their spectra are flatter

This is clearly seen in Au-Au collisions, where proton and kaon spectra are much flatter than pion spectra. This is considered evidence for transverse flow. Due to its rotational symmetry, it is called "radial flow"

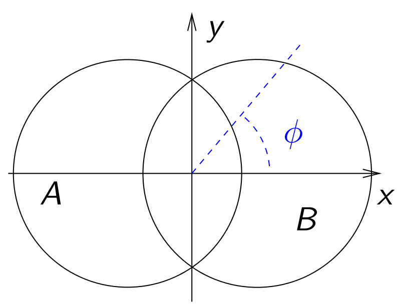
m_T spectra in central Au-Au collisions at $p_z = 0$ Yields normalized per event



Chemical versus kinetic freeze-out

The relative abundancies of pions, and protons (antiprotons) do not change drastically in going from p-p to Au-Au. Only their momentum spectra change. The number of particles of a given species is obtained by integrating the Boltzmann factor over momentum, as a consequence particle ratios depend only on the temperature.
The fact that the particle ratios are the same in p-p and Au-Au means that the "chemical freeze-out temperature" is the same: $T_{ch} \simeq 170$ MeV.
The yield of kaons is enhanced in central Au-Au collision with respect to p-p (the strangeness suppression is removed).
The m_t spectra in Au-Au collision requires a smaller temperature than in p-p: $T_f \simeq 120$ MeV, this is the "kinetic freeze-out". Inelastic collisions, which maintain chemical equilibrium, stop at T_{ch}; particle abundances are frozen, but they interact elastically and maintain kinetic equilibrium until the system cools down to T_f, when the system density is so low that interactions stop completely.

Elliptic flow



We now study non-central collisions. $\mathrm{d}p_x\mathrm{d}p_y = p_t\mathrm{d}p_t\mathrm{d}\phi$ and we rewrite the Boltzmann factor as

$$\frac{\mathrm{d}N}{p_t\mathrm{d}p_t\mathrm{d}p_z\mathrm{d}\phi} \propto \exp\left(\frac{-m_t u^0(\phi) + p_t u(\phi)}{T}\right) .$$

The fluid velocity is larger on the x axis than on the y axis, this effect can be parameterized as

$$u(\phi) = u + 2\alpha\cos 2\phi$$

where u is the average over ϕ and α is positive coefficient characterizing the magnitude of the elliptic flow. Experimental data indicate that α is a very small number, so we can expand $u^0(\phi)$ to first order in α:

$$u^0(\phi) = \sqrt{1 + (u(\phi))^2} = \sqrt{1 + u^2 + 4\alpha u\cos 2\phi + \mathcal{O}(\alpha)} \simeq u_0 + 2\alpha v\cos 2\phi$$

where $v \equiv u/u_0$.

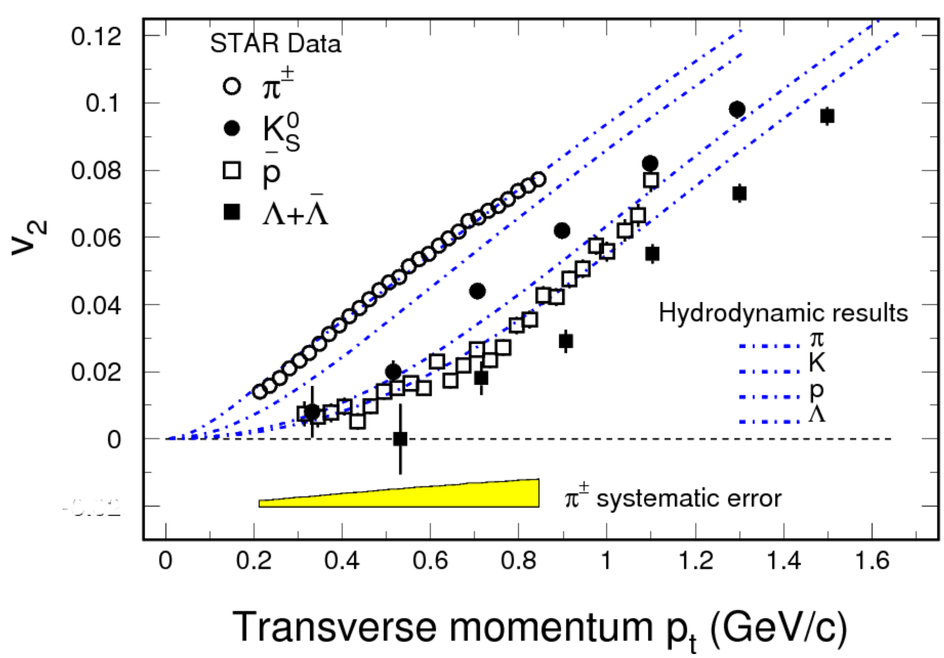
The Boltzmann distribution becomes:

$$
\begin{aligned}
\frac{\mathrm{d}N}{p_t \mathrm{d}p_t \mathrm{d}p_z \mathrm{d}\phi} &\propto \exp\left(\frac{-m_t(u_0 + 2\alpha v \cos 2\phi) + p_t(u + 2\alpha \cos 2\phi)}{T}\right) = \\
&= \exp\left(\frac{-m_t u_0 + p_t u}{T}\right) \exp\left(2\frac{\alpha(p_t - m_t v)}{T} \cos 2\phi\right) = \\
&= \frac{\mathrm{d}N}{p_t \mathrm{d}p_t \mathrm{d}p_z} \left(1 + 2\frac{\alpha(p_t - m_t v)}{T} \cos 2\phi\right)
\end{aligned}
$$

and comparing with (15) we obtain the value of the elliptic flow:

$$
v_2 = \frac{\alpha(p_t - m_t v)}{T}
$$

For light particles $m_t \simeq p_t$: v_2 is linear with p_t.
For heavier particles m_t is larger for a given p_t: v_2 is smaller.
The hydrodynamic description is good up to $p_t \sim 2$ GeV.



In the coalescence picture:

mesons: $\dfrac{\mathrm{d}N_M}{\mathrm{d}p_T^2}(p_T) = \dfrac{\mathrm{d}N_M}{\mathrm{d}p_T^2}\left[1 + 2v_{2M}(p_T)\cos(2\varphi)\right]$

but, regarding mesons as composite objects:

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\frac{\mathrm{d}N_M}{\mathrm{d}p_T^2}(p_T) &= \left\{\frac{\mathrm{d}N_q}{\mathrm{d}p_T^2}(p_T/2)\right\}^2 = \left\{\frac{\mathrm{d}N_q}{p_t\mathrm{d}p_T}\left[1 + 2v_{2q}(p_T/2)\cos(2\varphi)\right]\right\}^2 \simeq \\
&\simeq \left[\frac{\mathrm{d}N_q}{p_t\mathrm{d}p_T}\right]^2 \left[1 + 4v_{2q}(p_T/2)\cos(2\varphi)\right]
\end{aligned}
$$

therefore:

$$2\,v_{2q}\left(\frac{p_T}{2}\right) = v_{2M}(p_T)$$

Analogously for baryons:

$$3\,v_{2q}\left(\frac{p_T}{3}\right) = v_{2B}(p_T)$$

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Therefore, if v_2 is originated at the parton level, we should observe quark number scaling:

$$\frac{1}{2}v_{2M}\,(2p_T) = \frac{1}{3}v_{2B}\,(3p_T)$$

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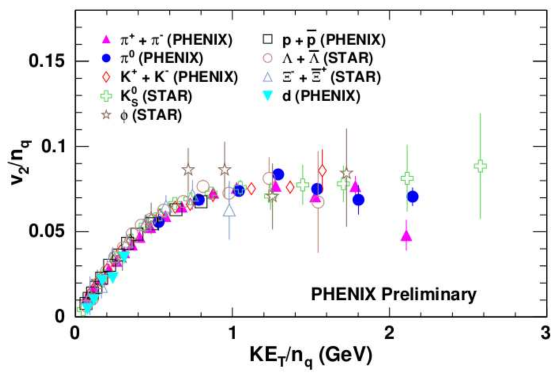

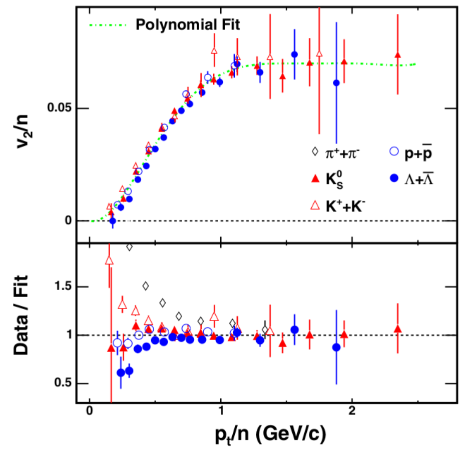

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