Università degli Studi di Torino
Scuola di Dottorato in Scienza e Alta Tecnologia

Introduction to the Physics of the Quark-Gluon Plasma

Marzia Nardi, Andrea Beraudo

Paolo Parotto
Department of Physics - Eberly College of Science
Pennsylvania State University

schedule and lecture notes:
http://personalpages.to.infn.it/∼nardi/QGPdot/

Quantum Chromo Dynamics

QCD describes interactions among quarks (q) and gluons (g).

Quarks are described in terms of Dirac fields $q_\alpha^{ir}(x)$, where:

- α = Dirac spinor index (1,2,3,4)
- i = $SU(3)$ color index (1,2,3)
- r = flavor index (u,d,s,c,b,t)

Gluons are described by vector fields $A_\mu^a(x)$, where:

- μ = Lorentz vector index (0,1,2,3)
- a = color index (1,2,. . . 8)

Gell-Mann matrices λ^a: 3×3 matrix in the color space, $(\lambda^a)_{ij}$, $i, j = 1, 2, 3$

$SU(3)$ is the special unitary group; its Lie algebra has dimension 8 and therefore it has some set with 8 linearly independent generators t_i ($i = 1, \ldots, 8$). Any element of SU(3) can be written in the form $\exp(i\theta_j t_j)$, where θ_j are real numbers and a sum over the index j is implied.

The Lie Algebra elements obey the commutation relations

$$[t_i, t_j] = if^{ijk} t_k$$

The structure constants f^{ijk} are completely antisymmetric in the three indices and have values $f^{123} = 1$, $f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$. All other f^{ijk} not related to these by permutation are zero.

The Gell-Mann matrices are one possible representation of the infinitesimal generators of $SU(3)$, involving 3×3 matrices (fundamental representation); a particolar choice is $(t_i = \frac{\lambda_i}{2})$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

λ_i are traceless, Hermitian, and obey the extra relation $\mathrm{tr}(\lambda_i \lambda_j) = 2\delta_{ij}$

Local $SU(3)$ color transformation:

$$U(x) = \mathrm{e}^{i\frac{\lambda^a}{2}\theta^a(x)} = \mathrm{e}^{\frac{i}{2}\vec{\lambda}\cdot\vec{\theta}(x)} = \mathrm{e}^{i\vec{t}\cdot\theta(x)}$$

$U(x) = 3 \times 3$ matrix in the color space

$$q(x) \to U(x) q(x) \qquad q^\dagger(x) \to q^\dagger(x) U^\dagger(x)$$
$$q_i(x) = U_{ij}(x) q_j(x) \qquad q_i^*(x) = q_j^*(x) U_{ji}^*(x)$$

QCD Lagrangian: $\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \bar{q}^r \left(i\not{D} - m_r\right) q^r$, where:

$$\begin{aligned} F^a_{\mu\nu} &\equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ \not{D} &\equiv \gamma^\mu D_\mu \\ D_\mu &\equiv \partial_\mu - igA_\mu = \partial_\mu - igt^a A^a_\mu \\ D^{ab}_\mu &= \delta_{ab}\partial_\mu - ig\left(t^c\right)_{ab} A^c_\mu \end{aligned}$$

$\mathcal{L}_{QCD}$ is invariant for local $SU(3)$ color transformations

Let us consider, for simplicity, an infinitesimal transformation:

$$
\begin{aligned}
q_r(x) &\to q'_r(x) = (1 + i t^a \theta^a) q_r(x) \\
\bar{q}_r(x) &\to \bar{q}'_r(x) = \bar{q}_r(x)(1 - i t^a \theta^a) \\
A^a_\mu &\to A'^a_\mu + \frac{1}{g}(\partial_\mu \theta^a) + f^{abc} A^b_\mu \theta^c
\end{aligned}
$$

It is easy to check (see Appendix A):

$$
\begin{aligned}
\mathcal{L}'_{QCD} &= -\frac{1}{4} F'^a_{\mu\nu} F'^{\mu\nu}_a + \bar{q}' \left(i \not{D}' - m \right) q' = \\
&= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{q} \left(i \not{D} - m \right) q = \mathcal{L}_{QCD}
\end{aligned}
$$

Renormalization $\to$ running coupling:

$$
\alpha_s(Q) = \frac{2\pi}{b_0 \log\left(\frac{Q}{\Lambda}\right)} \qquad \alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f) \log\left(\frac{Q^2}{\Lambda^2}\right)}
$$

where $\quad \Lambda^2 = \mu^2 \exp\left\{\frac{-12\pi}{(33-2n_f)\alpha_s(\mu^2)}\right\}$, $\Lambda \sim 200$ MeV , $b_0 = 11 - \frac{2}{3} n_f$.

$\alpha_s(Q)$ small for large Q ($Q \gg \Lambda$) $\quad \Longrightarrow \quad$ pQCD $\quad$ (asymptotic freedom)

For $Q \sim \Lambda$: $\alpha_s(Q)$ large $\quad \Longrightarrow \quad$ ~~pQCD~~ $\quad$ (confinement)

$\Lambda \sim 200$ MeV$\simeq 1$ fm$^{-1} \simeq$ (hadron size)$^{-1}$

Review of thermodynamics

We consider, for simplicity, single-components systems.
Thermodynamics concernes bulk properties of the system.
Fundamental law: $\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N$, $E = E(S, V, N)$

$$T = \left(\frac{\partial E}{\partial S}\right)_{VN} \qquad P = -\left(\frac{\partial E}{\partial V}\right)_{SN} \qquad \mu = \left(\frac{\partial E}{\partial N}\right)_{SV}$$

For a quantum-mechanical system in its ground state, $S = 0$ and $\mu = \left(\frac{\partial E}{\partial N}\right)_V$

Helmholtz free energy $F(T, V, N) = E - TS$, $\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N$
Gibbs free energy $G(T, P, N) = E - TS + PV$, $\mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}P + \mu\mathrm{d}N$
Thermodynamic potential $\Omega(T, V, \mu) = F - \mu N = E - TS - \mu N$,

$$\mathrm{d}\Omega = -S\mathrm{d}T - P\mathrm{d}V - N\mathrm{d}\mu$$

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V\mu} \qquad P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T\mu} \qquad N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{TV}$$

S, V, N extensive $\qquad T, P, \mu$ intensive

Review of statistical mechanics

• Microcanonical Ensemble: E, N, ... are fixed. The system is closed and isolated.
All microstates are equally probable.

• Canonical Ensemble: system in thermal equilibrium with a heat bath, T and N fixed, E fluctuates. System closed but not thermally isolated.
It includes all possible microstates with N particles. $P_i \propto \mathrm{e}^{-\beta E_i}$
$(\beta \equiv \frac{1}{k_B T})$

• Grand-canonical Ensemble: system in thermal equilibrium with a heat bath, it can exchange particles with the external system. Open system.
T and μ fixed, N and E fluctuate.
It includes all possible microstates. $P_{i,N} \propto \mathrm{e}^{-\beta(E_i - \mu N)}$

In the Grand-canonical ensemble we define the Grand-partition function

$$\mathcal{Z}_G = \sum_N \sum_j \mathrm{e}^{-\beta(E_j - \mu N)} = \sum_N \sum_j \langle Nj|\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}|Nj\rangle = \mathrm{Tr}(\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})})$$

A fundamental result from statistical mechanics is: $\Omega(T, V, \mu) = -k_B T \log \mathcal{Z}_G$

Statistical operator: $\hat{\rho}_G = \frac{1}{\mathcal{Z}_G}\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})} = \mathrm{e}^{\beta(\Omega-\hat{H}+\mu\hat{N})}$

The thermal average of operator $\hat{O}$ is:

$$\langle\hat{O}\rangle = \mathrm{Tr}(\hat{\rho}_G\hat{O}) = \mathrm{Tr}(\mathrm{e}^{\beta(\Omega-\hat{H}+\mu\hat{N})}\hat{O}) = \frac{1}{\mathcal{Z}_G}\mathrm{Tr}(\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}\hat{O})$$

N is a conserved charge, e.g. B, L, S, I_3, $Q = I_3 + \frac{B+S}{2}$, C,...

Ideal gas: partition function

$$\begin{aligned}\mathcal{Z}_G &= \mathrm{Tr}(\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}) = \sum_{n_1,\dots,n_\infty}\langle n_1,\dots,n_\infty|\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}|n_1,\dots,n_\infty\rangle \\ &= \prod_{i=1}^{\infty}\mathrm{Tr}(\mathrm{e}^{-\beta(\epsilon_i-\mu)\hat{n}_i})\end{aligned}$$

✍

<u>Bosons:</u> $\mathcal{Z}_G = \prod_{i=1}^{\infty}\sum_{n=0}^{\infty}\left(\mathrm{e}^{\beta(\mu-\epsilon_i)}\right)^n = \prod_{i=1}^{\infty}\frac{1}{1-\mathrm{e}^{\beta(\mu-\epsilon_i)}}$

<u>Fermions:</u> $\mathcal{Z}_G = \prod_{i=1}^{\infty}\sum_{n=0}^{1}\left(\mathrm{e}^{\beta(\mu-\epsilon_i)}\right)^n = \prod_{i=1}^{\infty}\left(1+\mathrm{e}^{\beta(\mu-\epsilon_i)}\right)$

Boltzmann limit ($\mathrm{e}^{\beta(\epsilon_i-\mu)} \gg 1$):

$$\mathcal{Z}_G \simeq \prod_{i=1}^{\infty}\left(1+\mathrm{e}^{\beta(\mu-\epsilon_i)}\right) \qquad \log\mathcal{Z}_G \simeq \sum_{i=1}^{\infty}\mathrm{e}^{\beta(\mu-\epsilon_i)}$$

Ideal gas: thermodynamic potential

$$\Omega(T, V, \mu) = -\frac{1}{\beta} \log \mathcal{Z}_G$$

Bosons:

$$\Omega_0 = k_B T \sum_{i=1}^{\infty} \log\left(1 - \mathrm{e}^{\beta(\mu - \epsilon_i)}\right)$$

Fermions:

$$\Omega_0 = -k_B T \sum_{i=1}^{\infty} \log\left(1 + \mathrm{e}^{\beta(\mu - \epsilon_i)}\right)$$

Compact notation (upper sign for bosons, lower sign for fermions):

$$\Omega(T, V, \mu) = \pm k_B T \log\left(1 \mp \mathrm{e}^{\beta(\mu - \epsilon_i)}\right)$$

Ideal gas: number of particles

$$\langle N \rangle = \mathrm{Tr}\left(\hat{\rho}_G \hat{N}\right) = \frac{1}{\mathcal{Z}_G} \mathrm{Tr}(\mathrm{e}^{-\beta(\hat{H} - \mu\hat{N})} \hat{N})$$

Bosons:

$$\langle N \rangle = \sum_{i=1}^{\infty} n_i^0 = \sum_{i=1}^{\infty} \frac{1}{\mathrm{e}^{\beta(\epsilon_i - \mu)} - 1}$$

Fermions:

$$\langle N \rangle = \sum_{i=1}^{\infty} n_i^0 = \sum_{i=1}^{\infty} \frac{1}{\mathrm{e}^{\beta(\epsilon_i - \mu)} + 1}$$

Mean occupation number: $n_i^0 = \dfrac{1}{\mathrm{e}^{\beta(\epsilon_i - \mu)} \mp 1}$

Boltzmann limit: $n_i^0 = \mathrm{e}^{-\beta(\epsilon_i - \mu)}$

Thermodynamics of relativistic particles

System of relativistic particles in equilibrium: distribution function $f_i(\vec{p},\vec{r})$= number of particles of species i in the volume $\mathrm{d}^3r\mathrm{d}^3p$ around $(\vec{p},\vec{r})$.
$f_i(\vec{p},\vec{r})$ is a Lorentz scalar. We assume homogeneous space.
Equilibrium distribution: $f_i(\vec{p}) = \dfrac{1}{\mathrm{e}^{(\epsilon_p-\mu)/T} \mp 1}$ (+ fermions, - bosons)
$\epsilon_p = \sqrt{\vec{p}^2 + m^2}$

For 1-species particles with degeneracy g (spin,color, ...):

number density: $n = \dfrac{N}{V} = \dfrac{g}{V}\sum_{\vec{p}} f(\vec{p}) \to g\int \dfrac{\mathrm{d}^3p}{(2\pi)^3} f(p)$

energy density: $\epsilon = \dfrac{E}{V} = \dfrac{g}{V}\sum_{\vec{p}} \epsilon_p f(\vec{p}) \to g\int \dfrac{\mathrm{d}^3p}{(2\pi)^3} \epsilon_p f(\vec{p})$

All the thermodynamic observables can be deduced from the partition function $\mathcal{Z} = \mathrm{Tr}\,\mathrm{e}^{-\beta F}$ ($\mathcal{Z}_G = \mathrm{Tr}\,\mathrm{e}^{-\beta(F-\mu N)}$)

energy density $\quad \epsilon = -\dfrac{1}{V}\dfrac{\partial}{\partial\beta}\log\mathcal{Z}$

pressure $\quad P = \dfrac{1}{\beta}\dfrac{\partial}{\partial V}\log\mathcal{Z}$

free energy $\quad F = -\dfrac{1}{\beta}\log\mathcal{Z}$

Example: 1-species massless particles, $\mu = 0$

$$n = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\mathrm{e}^{q/T} \mp 1} = \nu \frac{\zeta(3)}{\pi^2} T^3 \qquad \nu = \begin{cases} 1 & \textit{bosons} \\ \frac{3}{4} & \textit{fermions} \end{cases}$$ ✍

where $\zeta(3) = 1.202$ (Riemann ζ function)

$$\epsilon = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{q}{\mathrm{e}^{q/T} \mp 1} = \nu' \frac{\pi^2}{30} T^4 \qquad \nu' = \begin{cases} 1 & \textit{bosons} \\ \frac{7}{8} & \textit{fermions} \end{cases}$$ ✍

pressure: $P = \dfrac{\epsilon}{3}$

entropy density: $Ts = \epsilon + P = \dfrac{4}{3}\epsilon \implies s = \dfrac{4}{3}\dfrac{\epsilon}{T} = 2\nu' \dfrac{\pi^2}{45} T^3$

Finite temperature calculations $\leftrightarrow$ Quantum mechanics

Partition function: $\mathcal{Z} = \mathrm{Tr}\left(\mathrm{e}^{-\beta \hat{H}}\right)$

Thermal average: $\langle A \rangle = \mathrm{Tr}\left(\mathrm{e}^{-\beta \hat{H}} \hat{A}\right)$

In QM there is a time evolution operator: $\mathrm{e}^{-iHt/\hbar}$

Example: $|\psi_S(t)\rangle = \mathrm{e}^{-iHt/\hbar} |\psi_S(0)\rangle$

$$\frac{t}{\hbar} \to -i\beta \qquad \mathrm{e}^{-iHt/\hbar} \to \mathrm{e}^{-\beta H}$$

$\mathrm{e}^{-\beta H}$ is an evolution operator in imaginary time.

Connection to finite temperature: $\beta \to \dfrac{1}{k_B T}$

Techniques developed in quantum mechanics are applicable to compute $\mathcal{Z}$.

Example: Path Integral.
Single particle in 1 dimension:

$$\langle x_2|\mathrm{e}^{-iHt/\hbar}|x_1\rangle = \int_{x(0)=x_1}^{x(t)=x_2} \mathcal{D}(x(t)) \exp\left\{\frac{i}{\hbar}\int_0^t \mathrm{d}t' \left[\frac{m\dot{x}^2}{2} - V(x)\right]\right\}$$

$t \to -i\tau$, $\quad x_1 = x_2$ (trace), $\quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} \to i\frac{\mathrm{d}x}{\mathrm{d}\tau}$, $\qquad (\hbar = 1)$

$$\mathcal{Z} = \langle x|\mathrm{e}^{-H\tau}|x\rangle = \int_{x(0)=x(\beta)} \mathcal{D}(x(\tau)) \exp\left\{-\int_0^\beta \mathrm{d}\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]\right\}$$

Generalize to field theory (scalar field):

$$\mathcal{Z} = \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi \exp\left\{-\int_0^\beta \mathrm{d}\tau \mathrm{d}^3x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]\right\}$$

Boundary conditions for scalar fields: ϕ periodic in imaginary time $\Longrightarrow \phi$ can be expanded in Fourier series, with discrete frequencies $\omega_n = 2n\pi T$ (Matsubara frequencies).

Let us calculate the free energy of a static q in a gluonic background.
We start from the Dirac eq. in imaginary time (✍):

$$(i\not{D} - M)\psi = 0 \overset{t\to -i\tau}{\Longrightarrow} \left[\partial_\tau - gA_0 + \vec{\alpha}\cdot\left(\frac{\vec{\nabla}}{i} + g\vec{A}\right) + M\gamma_0\right]\psi(\vec{r},\tau) = 0$$

Heavy quark: M large, $\quad \gamma_0 \to 1$, $\quad \vec{\alpha}\cdot(\dots)$ negligible in the NR limit (see Appendix B)

$$[\partial_\tau - gA_0 + M]\psi(\vec{r},\tau) = 0$$

Solution:

$$\psi(\vec{r},\tau) = \mathrm{e}^{-M\tau}\, T\exp\left\{g\int_0^\tau \mathrm{d}\tau' A_0(\vec{r},\tau')\right\}\psi(\vec{r},0) \qquad (1)$$

$T=$ imaginary time ordering.

Free energy: $F = -k_B T \log \mathcal{Z}, \quad \mathcal{Z} = \mathrm{e}^{-\beta F}$

$$\mathrm{e}^{-\beta F} = \frac{1}{N_c} \sum_{i,n} \langle n | \psi_i(\vec{r}) \, \mathrm{e}^{-\beta H} \, \psi_i^\dagger(\vec{r}) | n \rangle \qquad (2)$$

the summation is over gluonic states $|n\rangle$ and quark colors i
$\psi_i^\dagger(\vec{r})$ ($\psi_i(\vec{r})$) creates (destroys) a q of color i at point $\vec{r}$

evolution operator: $\mathrm{e}^{\beta H} \psi_i(\vec{r}) \mathrm{e}^{-\beta H} = \psi_i(\vec{r}, \beta), \quad \psi_i(\vec{r}) \equiv \psi_i(\vec{r}, 0)$

$$\begin{aligned}
\mathrm{e}^{-\beta F} &= \frac{1}{N_c} \sum_{i,n} \langle n | \mathrm{e}^{-\beta H} \underbrace{\mathrm{e}^{\beta H} \psi_i(\vec{r}) \, \mathrm{e}^{-\beta H}}_{\psi_i(\vec{r}, \beta)} \psi_i^\dagger(\vec{r}) | n \rangle \\
&= \frac{1}{N_c} \sum_{i,n} \langle n | \mathrm{e}^{-\beta H} \psi_i(\vec{r}, \beta) \psi_i^\dagger(\vec{r}) | n \rangle = \\
&= \frac{1}{N_c} \sum_{i,n} \mathrm{e}^{-\beta E_n} \langle n | \psi_i(\vec{r}, \beta) \psi_i^\dagger(\vec{r}) | n \rangle
\end{aligned}$$

We use solution (1): $\psi_i(\vec{r}, \beta) = \sum_j \mathrm{e}^{-M\beta} \, T \exp \left\{ g \int_0^\beta \mathrm{d}\tau \, A_0(\vec{r}, \tau) \right\}_{ij} \psi_j(\vec{r}, 0)$

$$\mathrm{e}^{-\beta F} = \mathrm{e}^{-M\beta} \sum_n \mathrm{e}^{-\beta E_n} \langle n | \left[\frac{1}{N_c} \sum_{ij,n} T \exp \left\{ g \int_0^\beta \mathrm{d}\tau A_0(\vec{r}, \tau) \right\}_{ij} \underbrace{\psi_j(\vec{r}, 0) \psi_i^\dagger(\vec{r}, 0)}_{\delta_{ij}} \right] | n \rangle$$

$$\begin{aligned}
\mathrm{e}^{-\beta F} &= \mathrm{e}^{-M\beta} \sum_n \mathrm{e}^{-\beta E_n} \langle n | \frac{1}{N_c} \underset{\text{(color)}}{\mathrm{Tr}} \; T \exp \left\{ g \int_0^\beta \mathrm{d}\tau A_0(\vec{r}, \tau) \right\} | n \rangle = \\
&= \mathrm{e}^{-M\beta} \sum_n \mathrm{e}^{-\beta E_n} \langle n | L(\vec{r}) | n \rangle \qquad \text{Polyakov line}
\end{aligned}$$

By subtracting the free energy of gluons: $\mathrm{e}^{-\beta(F - F_0 - M)} = \langle L(\vec{r}) \rangle$

$L(\vec{r})$ = Polyakov line = order parameter for the deconfinement transition:

$$F = F_0 + M - \frac{\log\langle L(\vec{r})\rangle}{\beta}$$

• if $\langle L(\vec{r})\rangle = 0 \quad \Longrightarrow \quad F \to \infty$, it costs an infinite amount of energy to put an isolated q in the system.
• if$\langle L(\vec{r})\rangle > 0 \quad \Longrightarrow \quad F$ is finite, states with a single quark are possible.

Free energy of two isolated, massive quarks ($q\bar{q}$):

$$\mathrm{e}^{-\beta(F_{q\bar{q}} - F_0 - 2M)} = \langle L(0)L^\dagger(\vec{r})\rangle$$

in the confining regime, for $|\vec{r}| \to \infty$:

$$\langle L(0)L^\dagger(\vec{r})\rangle \sim \mathrm{e}^{-\beta\sigma|\vec{r}|}$$

where σ is the string tension; in the deconfined regime: $\sigma \to 0$

Lattice QCD

Approximation scheme introduced by Wilson.
Discrete statistical mechanic system on a 4-dim Euclidean lattice.
QCD partition function in the grand-canonical ensemble:

$$\mathcal{Z}(V, T, \mu) \int \mathcal{D}A_\nu \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \mathrm{e}^{-S_E(V,T,\mu)}$$

S_E= Euclidean action, depends on g, m_f
A_ν= bosonic fields (periodic boundary conditions)
$\bar{\psi}, \psi$= fermionic fields (Grassmann variables, antiperiodic bound. cond.)

The path integral is regularized by introducing a four-dimensional space-time lattice of size $N_\sigma^3 \times N_\tau$ with a lattice spacing a. $V = (N_\sigma a)^3$, $T^{-1} = N_\tau a$.

Physical results in the continuum limit: $a \to 0$, $N_\tau \to \infty$ ($T = 1/(N_\tau a)$ fixed).

The phase structure of *QCD* is studied by analyzing observables which are suitable order parameters:

- for chiral symmetry restoration ($m_f \to 0$):

$$\text{chiral condensate } \langle \bar{\psi}_f \psi_f \rangle = \frac{T}{V} \frac{\partial}{\partial m_f} \log \mathcal{Z}(T, V, \mu_f)$$

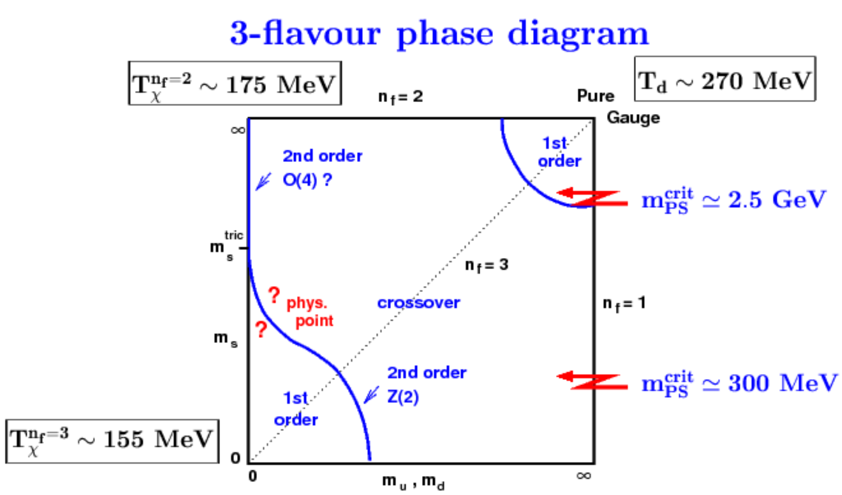
$\langle \bar{\psi}_f \psi_f \rangle > 0$: symmetry broken phase, $T < T_c$
$\langle \bar{\psi}_f \psi_f \rangle = 0$: symmetric phase, $T > T_c$

- for deconfinement:

expectation value of the trace of the Polyakov Loop $\langle L \rangle$.

$\langle L \rangle = 0$: confined phase, $T < T_c$
$\langle L \rangle > 0$: deconfined phase, $T > T_c$

Lattice QCD results: Phase transition



F.Karsch, Lect. Notes Phys. 583 (2002) 209, hep-lat/0106019

Lattice QCD results: order parameters

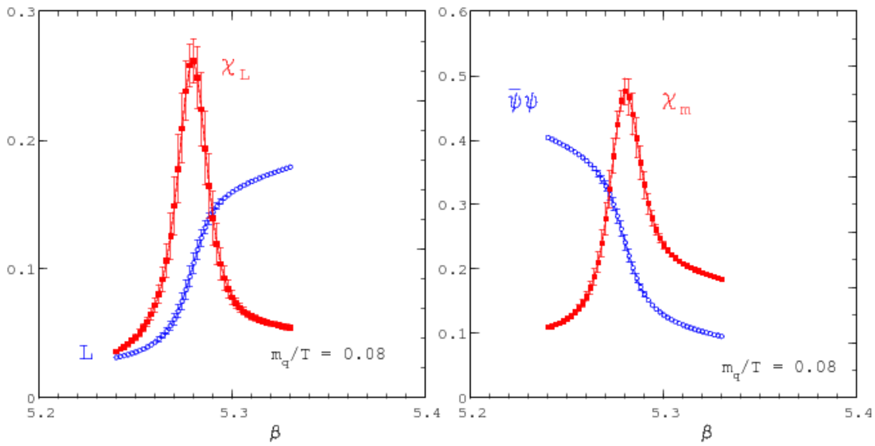
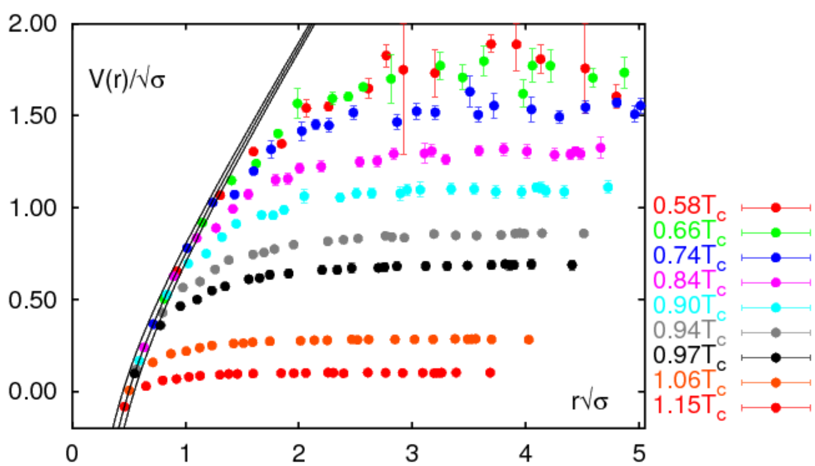


Fig. 2. Deconfinement and chiral symmetry restoration in 2-flavour QCD: Shown is $\langle L \rangle$ (left), which is the order parameter for deconfinement in the pure gauge limit ($m_q \to \infty$), and $\langle \bar\psi\psi \rangle$ (right), which is the order parameter for chiral symmetry breaking in the chiral limit ($m_q \to 0$). Also shown are the corresponding susceptibilities as a function of the coupling $\beta = 6/g^2$.

F.Karsch, Lect. Notes Phys. 583 (2002) 209, hep-lat/0106019

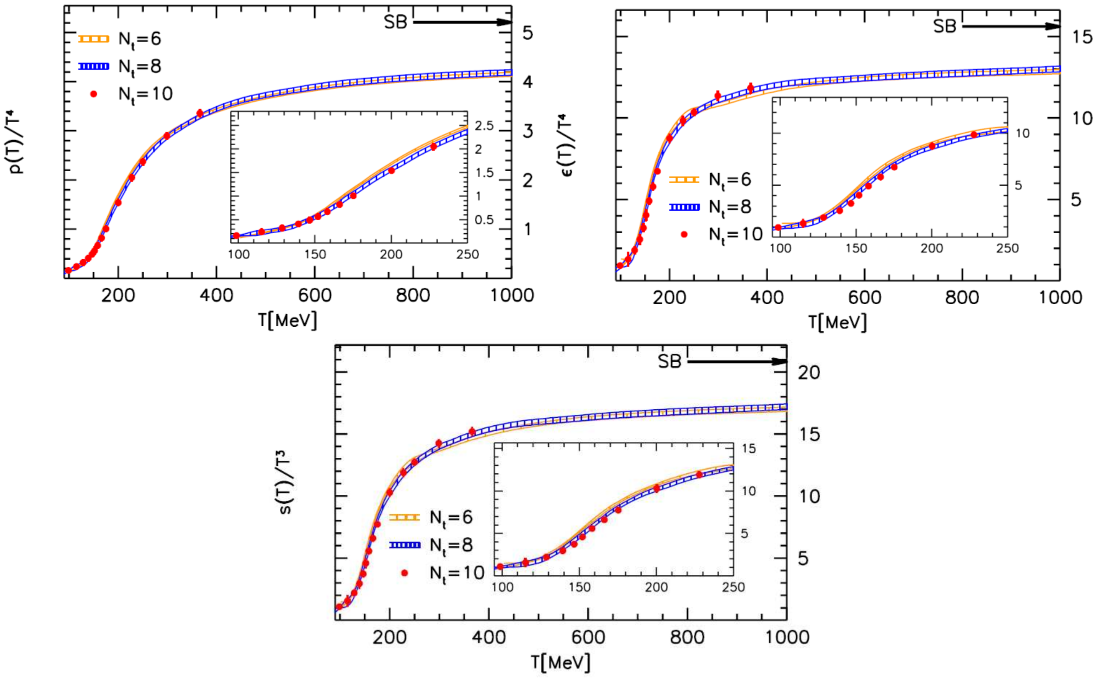
Lattice QCD results: interquark potential



F.Karsch, E. Laermann, “Quark-Gluon Plasma 3”, World Scientific, hep-lat/0305025

Lattice QCD results

Dynamical quarks, 2+1 flavors (S. Borsanyi et al., JHEP1011 (2010))



Lattice: Conclusions

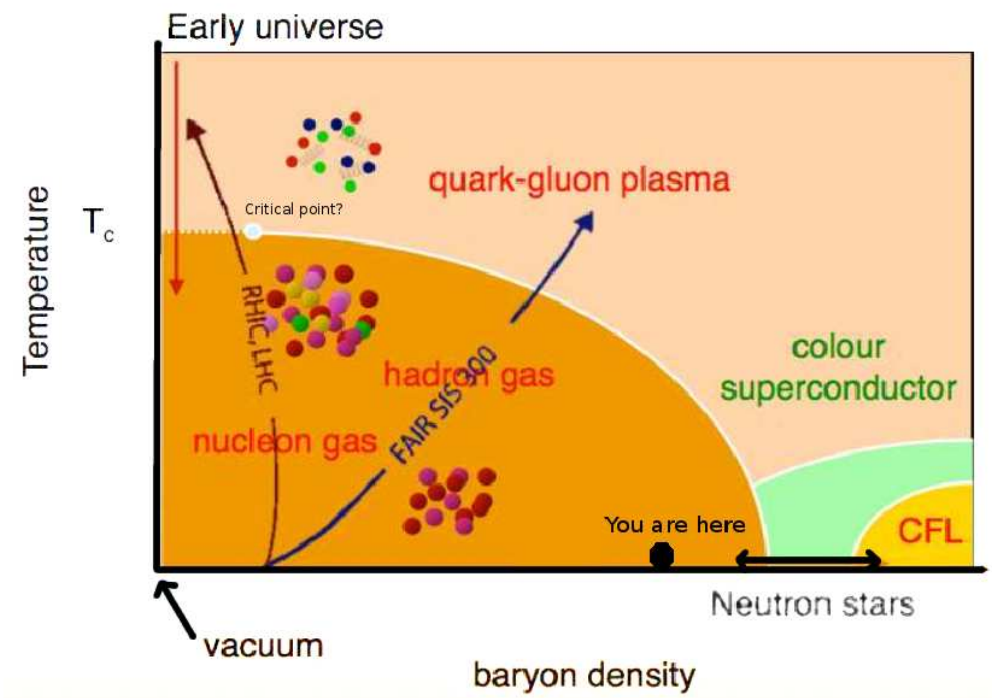
Lattice *QCD* shows that strongly interacting matter (i.e. matter interacting with *QCD*) exists in the high temperature and/or high density limit in a deconfined phase.
Still under debate:

- *Order of the phase transition: 1st, 2nd, cross over*
- *Critical temperature:* $150 \div 170$ *MeV*
- *Critical energy density:* $\sim 1 \div 1.5$ GeV/fm^3

In heavy ion collisions experiments at SPS/RHIC/LHC the extimated T and ϵ reached in the collisions are well above the critical values: it is very likely that QGP is produced. We have to identify the suitable experimental observables to study it.
This will allow

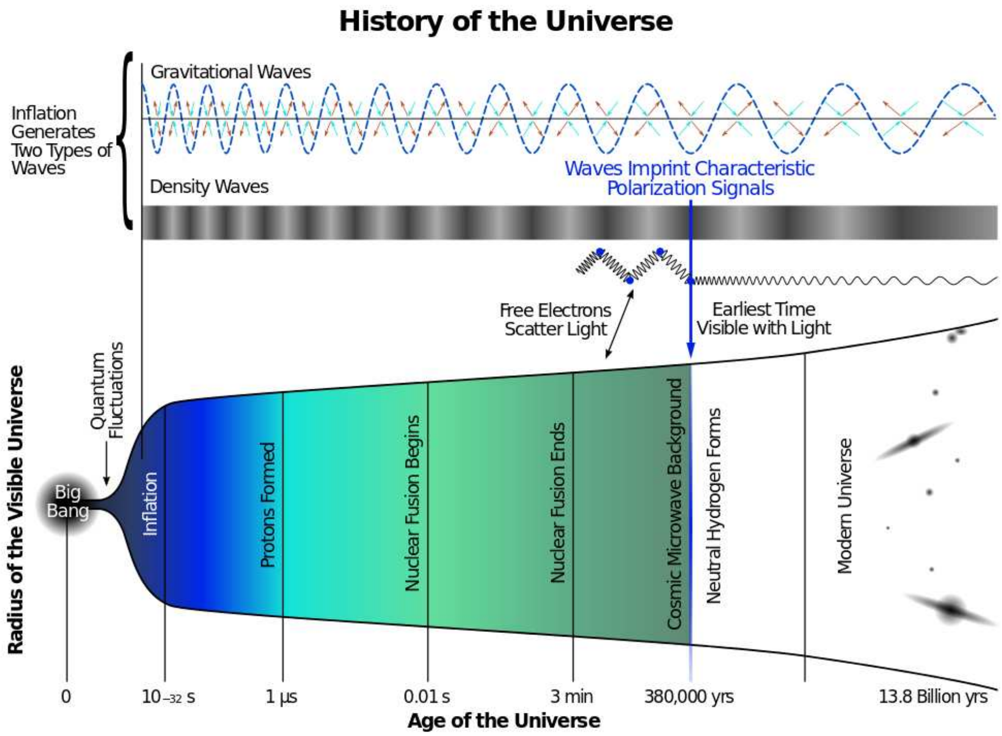
- Deeper understanding of *QCD*
- Connection with other fields (cosmology)

Phase diagram



Universe formation

- Electroweak transition and generation of masses, $T > 200$ GeV: particles are coupled to a non-vanishing expectation value of a scalar field (Higgs).
- Quark-Hadron transition, $T \sim 200$ MeV, $t \sim 10^{-6}$ s: quarks bind and form hadrons, color confinement.
- Primordial nucleosyntesis, $T \sim 0.07$ MeV: weakly bound deuterons survive collisions with other particles; p and n are not in β-equilibrium, free neutrons decay.
- Decoupling of matter from radiation, $T \sim$ eV: electrons bind to protons and form hydrogen atoms. Photons become free, black-body radiation $T \sim 2.7$ K (red-shifted)



Models of quark-hadron transitions

Hagedorn Temperature

Boltzmann statistic for 1 species ($\mu = 0$ for simplicity, natural units):

$$\log \mathcal{Z}(T,V) = \frac{V}{(2\pi)^3} \int \mathrm{d}^3 p\, \mathrm{e}^{-\frac{\sqrt{p^2+m^2}}{T}} = \frac{VTm^2}{(2\pi)^2} K_2\left(\frac{m}{T}\right)$$

in the limit $T \ll m$: $\log \mathcal{Z}(T,V) \simeq V \left(\frac{Tm}{2\pi}\right)^{3/2} \mathrm{e}^{-m/T}$

For many particle species: $\log \mathcal{Z} = \sum_i \log \mathcal{Z}_i \to \text{const.} \int \mathrm{d}m\, \rho(m) \log \mathcal{Z}(m)$

Hagedorn's bootstrap model ('60): any highly excited hadronic system is a resonance

$\Longrightarrow$ self-consistent condition on $\rho(m)$, solution:

$\rho(m) = Cm^{\alpha} \mathrm{e}^{m/T_0}$ $\alpha = -\frac{5}{2}$, $T_0 \simeq 160$ MeV (exp. fit)

$$\log \mathcal{Z}(T,V) = V \left(\frac{T}{2\pi}\right)^{3/2} C \int_{m_0}^{\infty} \mathrm{d}m\, m^{\alpha+\frac{3}{2}} \exp\left\{m\left(\frac{1}{T_0} - \frac{1}{T}\right)\right\}$$

The integral is well defined for $T < T_0$, it diverges for $T \to T_0$:

hadronic matter can not exist for $T > T_0$

T_0 is a limiting temperature. ($T_0 = 160$ Mev $\sim 2 \cdot 10^{12}$ K)

Bag model

Equilibrated baryon free matter ($\mu_B = 0$), non-interacting particles.

• Low temperature phase = pions (massless)

degeneracy: $g = 3$ for isospin (π^+, π^0, π^-)

$$\epsilon = 3\frac{\pi^2}{30}T^4 \qquad P = \frac{\epsilon}{3} = 3\frac{\pi^2}{90}T^4$$

• High temperature phase = g, q, $\bar{q}$, $N_q = N_{\bar{q}}$, two flavors (u, d).

degeneracy: $g = g_g + g_q + g_{\bar{q}} = 2 \cdot 1 \cdot 8 + \frac{7}{8} \cdot 2 \cdot 2 \cdot 3 + \frac{7}{8} \cdot 2 \cdot 2 \cdot 3 = 37$

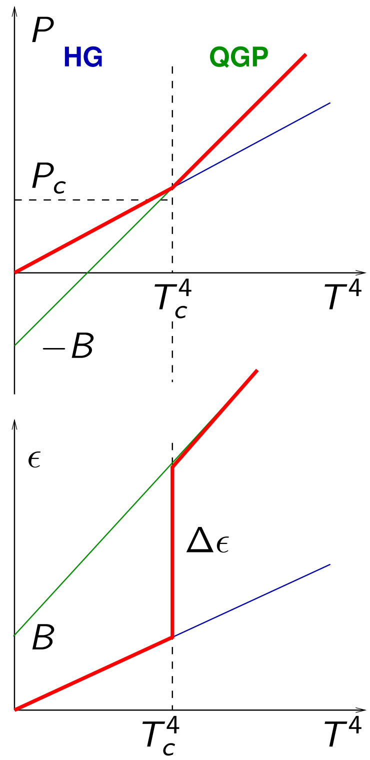
for: spin, flavor, color

$$\epsilon = 37\frac{\pi^2}{30}T^4 + B \qquad P = \frac{\epsilon}{3} = 37\frac{\pi^2}{90}T^4 - B$$

B mimics the interaction effects, force needed to equilibrate the pressure generated by the kinetic energy of the quarks inside the bag.

$B^{1/4} = 192$ MeV.

Stable phase: higher pressure (minimal internal energy)

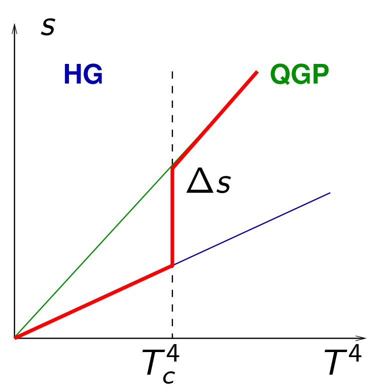


$P_{HG}(T_c) = P_{QGP}(T_c)$:

$$T_c = \left(\frac{45}{17\pi^2}B\right)^{1/4} \simeq 150 \text{ MeV}$$

This model has a 1st order phase transition *by construction*.

Latent heat: $\Delta\epsilon = \epsilon_{QGP}(T_c) - \epsilon_{HG}(T_c) = 4B$

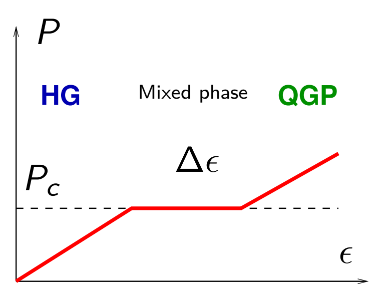


Entropy:

$$s_{HG} = \frac{4\pi^2}{30}T^3$$

$$s_{QGP} = \frac{4}{3}\cdot 37 \cdot \frac{\pi^2}{30}T^3 = \frac{148\pi^2}{90}T^3$$

$$\Delta s = \frac{68}{45}\pi^2 T_c^3 = \frac{4B}{T_c} = \frac{\Delta\epsilon}{T_c}$$



Equation of state:

Appendix A

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{q}_r \left(i \not{D} - m_r \right) q_r$$

Infinitesimal local gauge transformation:

$$\begin{aligned} q_r(x) &\rightarrow q'_r(x) = (1 + i t^a \theta^a) q_r(x) \\ \bar{q}_r(x) &\rightarrow \bar{q}'_r(x) = \bar{q}_r(x)(1 - i t^a \theta^a) \\ A^a_\mu &\rightarrow A'^a_\mu = A^a_\mu + \frac{1}{g}(\partial_\mu \theta^a) + f^{abc} A^b_\mu \theta^c \end{aligned}$$

$$\begin{aligned} \bar{q}'_r m_r q'_r &= \bar{q}_r (1 - i t^a \theta^a) m_r (1 + i t^b \theta^b) q_r = \\ &= \bar{q}_r m_r (1 - i t^a \theta^a + i t^b \theta^b + \mathcal{O}(\theta^2)) q_r \\ &= \bar{q}_r m_r q_r \end{aligned}$$

$$
\begin{aligned}
\bar{q}i\not{D}q &= \bar{q}i\gamma^\mu D_\mu q = \bar{q}i\gamma^\mu(\partial_\mu - igA^b_\mu t^b)q \rightarrow \\
\rightarrow &\ \bar{q}'i\not{D}'q' = \bar{q}'i\gamma^\mu(\partial_\mu - igA'^b_\mu t^b)q' = \\
&= \bar{q}(1 - it^a\theta^a)i\gamma^\mu\Big[\partial_\mu - ig\Big(A^b_\mu + \frac{1}{g}(\partial_\mu\theta^b) + f^{bde}A^d_\mu\theta^e\Big)t^b\Big] \times \\
&\quad \times(1 + it^c\theta^c)q = \\
&= \bar{q}(1 - it^a\theta^a)i\gamma^\mu\Big[\partial_\mu + it^c(\partial_\mu\theta^c) + it^c\theta^c\partial_\mu - igA^b_\mu t^b \\
&\quad +gA^b_\mu t^b t^c\theta^c - i(\partial_\mu\theta^b)t^b - igf^{bde}A^d_\mu\theta^e t^b + \mathcal{O}(\theta^2)\Big]q = \\
&= \bar{q}i\gamma^\mu\Big[\partial_\mu - it^a\theta^a\partial_\mu + it^c\theta^c\partial_\mu - igA^b_\mu t^b - gA^b_\mu t^a t^b\theta^a \\
&\quad +gA^b_\mu t^b t^a\theta^a - igf^{bde}A^d_\mu\theta^e t^b + \mathcal{O}(\theta^2)\Big]q = \\
&= \bar{q}i\gamma^\mu\Big[\partial_\mu - igA^b_\mu t^b + gA^b_\mu\theta^a[t^b, t^a] - igf^{bde}A^d_\mu\theta^e t^b\Big]q = \\
&= \bar{q}i\gamma^\mu\Big[\partial_\mu - igA^b_\mu t^b + igf^{bae}A^b_\mu\theta^a t^e - igf^{deb}A^d_\mu\theta^e t^b\Big]q = \\
&= \bar{q}i\gamma^\mu\big(\partial_\mu - igA^b_\mu t^b\big)q = \bar{q}i\not{D}q
\end{aligned}
$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

$$
\begin{aligned}
F^a_{\mu\nu} \rightarrow\ & F'^a_{\mu\nu} = \partial_\mu A'^a_\nu - \partial_\nu A'^a_\mu + gf^{abc}A'^b_\mu A'^c_\nu = \\
&= \partial_\mu\Big(A^a_\nu + \frac{1}{g}(\partial_\nu\theta^a) + f^{abc}A^b_\nu\theta^c\Big) - \partial_\nu\Big(A^a_\mu + \frac{1}{g}(\partial_\mu\theta^a) + f^{abc}A^b_\mu\theta^c\Big) \\
&\quad +gf^{abc}\Big(A^b_\mu + \frac{1}{g}(\partial_\mu\theta^b) + f^{bmn}A^m_\mu\theta^n\Big)\Big(A^c_\nu + \frac{1}{g}(\partial_\nu\theta^c) + f^{cpq}A^p_\nu\theta^q\Big) = \\
&= F^a_{\mu\nu} - f^{abc}\theta^b\Big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\Big) + g\big(f^{abc}f^{cpq} + f^{pac}f^{cbq}\big)A^b_\mu A^p_\nu\theta^q + \mathcal{O}(\theta^2)
\end{aligned}
$$

Jacobi's identity: $f^{abc}f^{cpq} + f^{pac}f^{cbq} + f^{bpc}f^{caq} = 0$

$$
\begin{aligned}
F'^a_{\mu\nu} &= F^a_{\mu\nu} - f^{abc}\theta^b\Big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\Big) - gf^{bpc}f^{caq}A^b_\mu A^p_\nu\theta^q + \mathcal{O}(\theta^2) = \\
&= F^a_{\mu\nu} - f^{abc}\theta^b F^c_{\mu\nu} + \mathcal{O}(\theta^2)
\end{aligned}
$$

$$
\begin{aligned}
F^a_{\mu\nu}F^{\mu\nu}_a &\rightarrow F'^a_{\mu\nu}F'^{\mu\nu}_a = \Big(F^a_{\mu\nu} - f^{abc}\theta^b F^c_{\mu\nu}\Big)\Big(F^{\mu\nu}_a - f^{ade}\theta^d F^{\mu\nu}_e\Big) = \\
&= F^a_{\mu\nu}F^{\mu\nu}_a - f^{ade}\theta^d F^a_{\mu\nu}F^{\mu\nu}_e - f^{abc}\theta^b F^c_{\mu\nu}F^{\mu\nu}_a + \mathcal{O}(\theta^2) \\
&= F^a_{\mu\nu}F^{\mu\nu}_a
\end{aligned}
$$

Appendix B

Dirac Eq. for a massive quark

$$(i\not{D} - M)\psi$$

$$D_\mu \equiv \partial_\mu - igA_\mu \qquad iD_\mu = i\partial_\mu + gA_\mu$$

$$i\not{D} = i\gamma^\mu D_\mu = i\gamma^\mu \partial_\mu + g\gamma^\mu A_\mu$$

$$\gamma^0 = \beta \qquad \alpha^k = \gamma^0\gamma^k \qquad (\gamma^0)^2 = 1$$

$$\gamma^0\Big(i\gamma^\mu\partial_\mu + g\gamma^\mu A_\mu - M\Big)\psi = 0$$

$$\Big(i\partial_0 + gA_0 + i\alpha^k\partial_k + g\alpha^k A_k - \gamma^0 M\Big)\psi = 0$$

Wick's rotation: $t \to -i\tau$, $\frac{\partial}{\partial t} = i\frac{\partial}{\partial \tau}$

$$\Big(-\partial_\tau + gA_0 + i\vec{\alpha}\cdot\vec{\nabla} - g\vec{\alpha}\cdot\vec{A} - \gamma^0 M\Big)\psi = 0$$

$$\left[\partial_\tau - gA_0 + \vec{\alpha}\cdot\left(\frac{\vec{\nabla}}{i} + g\vec{A}\right) + \gamma^0 M\right]\psi = 0$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad \begin{matrix} \Leftarrow \text{large components} \\ \Leftarrow \text{small components} \end{matrix}$$

In the NR limit (M is very large): $\varphi_2 \ll \varphi_1$

$$\left[\partial_\tau - gA_0 + M\right]\varphi_1 = 0$$