Lattice QCD: a brief introduction

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Apr 20, 2022

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Study of QCD thermodynamics

Theoretical investigations of QCD thermodynamics make use of different methods and tools

From first principles:

- **Lattice QCD**
- Perturbation theory (large T and/or μ)
- Functional methods (functional renormalization group - FRG, Dyson-Schwinger equations, etc...)

Models:

- Nambu-Jona-Lasinio (NJL) -type models (**Nambu and Jona-Lasinio, Phys. Rev. 122 (1961) 345, Phys. Rev. 124 (1961) 246**)
- Hadron Resonance Gas (HRG) -type models (**Hagedorn, Nuovo Cim. Suppl. 3 (1965), 147**)
- ...

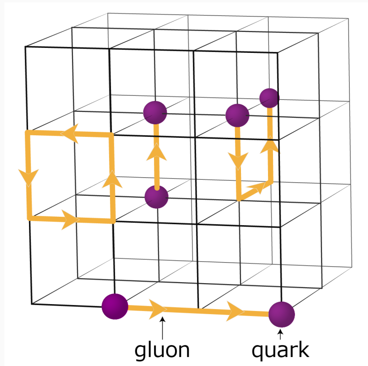
Lattice formulation of QCD

In the non-perturbative regime where the coupling g_s is not small, lattice QCD is the major tool of investigation of equilibrium (thermodynamic) properties of QCD

- Essentially, it amounts to calculating path integrals of the like:

  $$Z[U,\bar{\psi},\psi](V,T,\mu) = \int \mathcal{D}U\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-S_G[U,\bar{\psi},\psi]-S_F[U,\bar{\psi},\psi]}$$

  S_G and S_F are the *Euclidean* gauge (gluonic) and fermionic actions, U the gauge fields, $\psi, \bar{\psi}$ the fermion fields.
- The theory is defined on a discretized 3+1d lattice of size $N_s^3 \times N_\tau$, with lattice spacing a



Lattice formulation of QCD

- *Euclidean* actions S_G and S_F only depend on coupling g and fermion masses m_f
- Periodic boundary conditions for bosonic fields U, anti-periodic for fermionic fields ψ, $\bar{\psi}$ (Grassmann variables)
- The finiteness of the lattice spacing a serves as a *regulator* for the theory. At the end one wishes to recover the continuum theory with $\lim_{a\to 0}$ ($\lim_{N_\tau\to\infty}$): **continuum limit**
- Calculations are done in finite volume. When possible, one wishes to study the thermodynamic limit $\lim_{V\to\infty}$: a.k.a. **infinite volume limit**
- Simulations at finite temperature have set length in "temporal" direction $T^{-1} = \beta = N_\tau a$
- Summarizing

$$V = (N_s a)^3\ , \qquad T = \frac{1}{N_\tau a}$$

- **NOTE**: the integrals **cannot** be calculated by brute force! Even for a small 10^4 lattice, integral is 320000-dimensional! $\Rightarrow$ **Importance sampling**

Lattice QCD at $\mu_B = 0$

Thanks to asymptotic freedom (and perturbative calculations) and confinement, we expect a phase transition between *confined* and *deconfined* matter

- At low T/low μ:
  - **Confined matter** $\langle \vec{L} \rangle = 0$
  - **Chiral symmetry is spontaneously broken** by a non-zero chiral condensate

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V}\frac{\partial}{\partial m_f} \log \mathcal{Z}(T, V, \mu) \neq 0$$

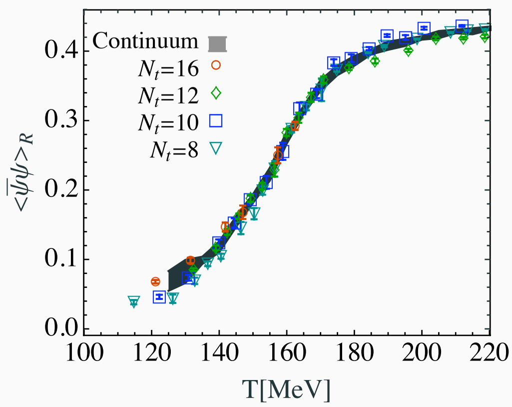
- At high T/high μ:
  - **Deconfined matter** $\langle \vec{L} \rangle \neq 0$
  - **Chiral symmetry is restored**

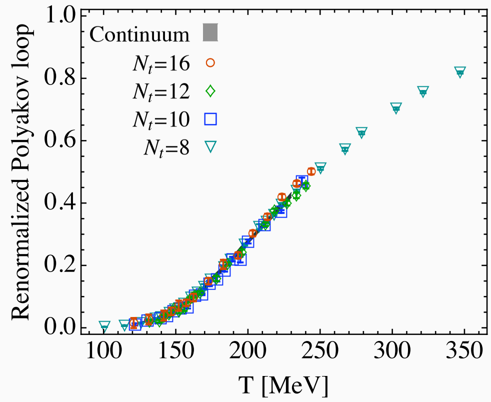
$$\langle \bar{\psi}\psi \rangle = \frac{T}{V}\frac{\partial}{\partial m_f} \log \mathcal{Z}(T, V, \mu) = 0$$

NOTE: although deconfinement and chiral transition are two “different” transitions, we have no evidence of them taking place at different temperatures. Unique T_C!

The QCD transition: observables

Both observables are able to distinguish between the two phases:



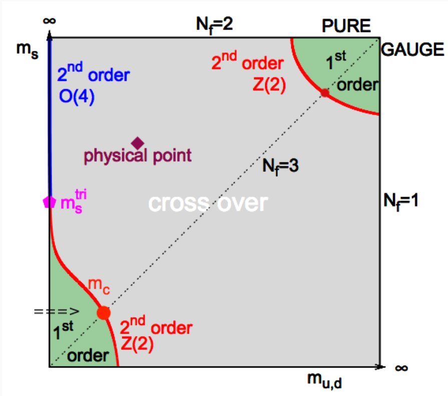


Borsanyi et al. JHEP 1009:073 (2010)

NOTE: this chiral condensate is renormalized subtracting the zero-temperature value

The QCD transition: Columbia plot

As a function of the light (u,d) ans strange quark masses, the order of the transition changes

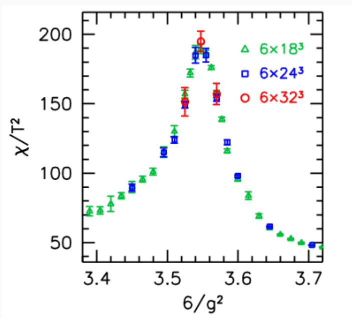


- At the physical point $m_s/m_{ud} \simeq 27$, the transition is a smooth crossover!
- In the heavy-quark limit (pure gauge), the transition is first order

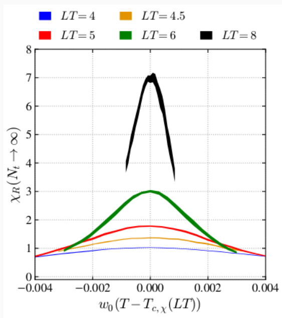
The QCD transition: crossover vs. first order

On the lattice we study the volume scaling of certain quantities to determine the order of the transition

Left: physical masses



Right: infinite masses (pure gauge)



- For a crossover (left), **the peak height is independent of the volume**
- For a first order transition, **it scales linearly with the volume**

Thermodynamic description of QCD

The thermodynamics of QCD is commonly investigated in the grand canonical ensemble

- Grancanonical partition function:

$$\mathcal{Z} = \sum_N Z_N e^{\mu N}$$

  where:
  - Z_N is the canonical partition function with N particles
  - μ is the *chemical potential* associated to the particle number N

  The chemical potential is the energy associated to a change in the number N of particles

- In QCD, there are 3 conserved "particle numbers", which one can see as:
  - **Quark numbers:** u, d, s

$$\mu N = \mu_u N_u + \mu_d N_d + \mu_s N_s$$

  - **Conserved charges:** B, Q, S

$$\mu N = \mu_B N_B + \mu_Q N_Q + \mu_S N_S$$

Thermodynamic description of QCD

The **Equation of State** (**EoS**) is extremely important since it completely describes the equilibrium properties of QCD matter.

It is one of the main inputs to hydro and several other tools for calculations in heavy-ion collisions and higher-density physics.

Thermodynamic quantities follow directly from the grancanonical partition function $\mathcal{Z}$ and the relation:

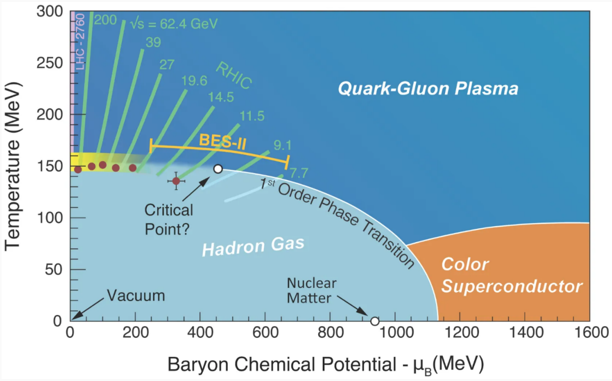
$$-k_B T \ln \mathcal{Z} = U - TS - \mu N$$

- **Pressure**: $p = -k_B T \frac{\partial \ln \mathcal{Z}}{\partial V}$
- **Entropy density**: $s = \left(\frac{\partial p}{\partial T}\right)_{\mu_i}$
- **Charge densities**: $n_i = \left(\frac{\partial p}{\partial \mu_i}\right)_{T, \mu_{j \neq i}}$
- **Energy density**: $\epsilon = Ts - p + \sum_i \mu_i n_i$
- **Speed of sound**: $c_s^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_{s/n_B}$
- More (**Fluctuations**, etc...)

The phase diagram of QCD

Different phases of QCD matter (in equilibrium) are depicted in (temperature vs baryo-chemical potential) phase diagram

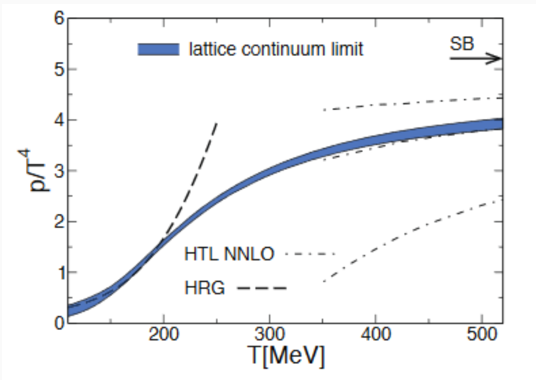
- **Hadron gas** at low-T and/or low-μ_B
- **Quark Gluon Plasma (QGP)** at large T and (possibly) at large μ_B
- **More exotic phases** proposed at low-T and high-μ_B (color superconductivity, etc...)



Equation of state at $\mu_B = 0$

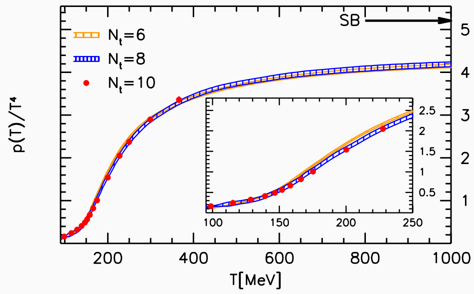
A combination of methods gives us good understanding of the EoS at $\mu_B = 0$ at all temperatures

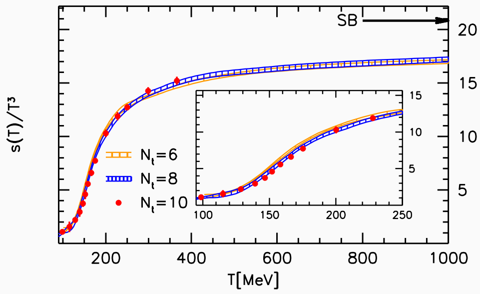
- Perturbative QCD at high temperature
  → "pure quark-gluon phase"
- HRG model at low temperature
  → "pure hadron phase"
- **Lattice QCD bridges between regimes and captures the transition**



Borsányi *et al.*, PLB 370 (2014) 99-104

Lattice QCD: equation of state at $\mu_B = 0$



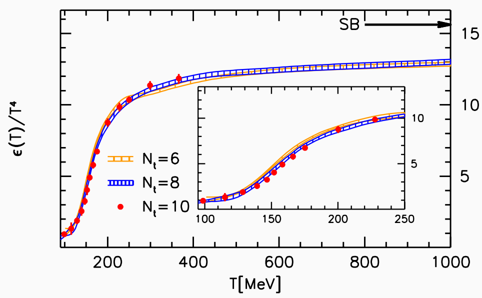


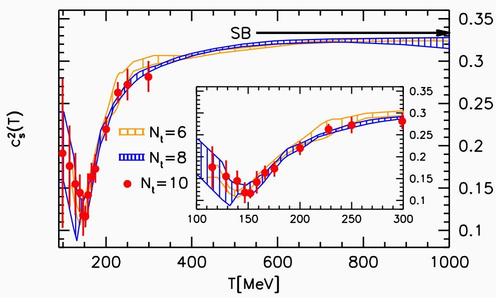
Stefan-Boltzmann limit for free gas (large T):

$$p_{\mathrm{QCD}} = \frac{\pi^2}{45} T^4 (N_c^2 - 1) + \sum_{\text{flavours}} \frac{N_c}{3\pi^2} \left[\frac{7\pi^4 T^4}{60} + \frac{\mu_f^2 \pi^2 T^2}{2} + \frac{\mu_f^4}{4} \right]$$

Borsányi *et al.*, JHEP 11 (2010) 077

Lattice QCD: equation of state at $\mu_B = 0$





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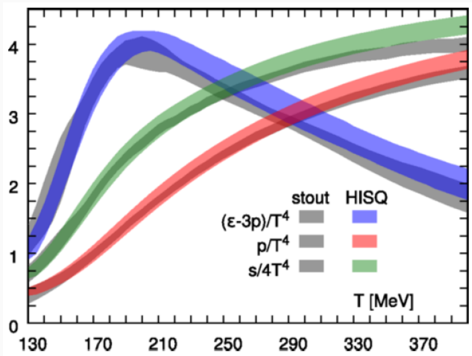
Lattice QCD: equation of state at $\mu_B = 0$

- Current lattice simulations are performed with realistic setup
- Great agreement between different collaborations
- This means systematics are well under control and results are extremely reliable



Borsányi *et al.*, PLB 370 (2014) 99-104, Bazavov *et al.* PRD 90 (2014) 094503

The sign/complex action problem

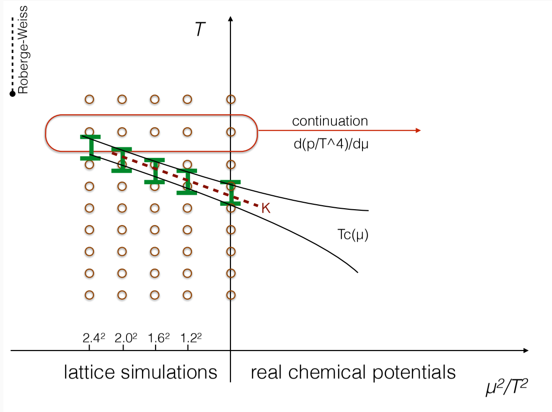
Euclidean path integrals are calculated with MC methods using importance sampling, interpreting the factor $\det M[U]\, e^{-S_G[U]}$ as the Boltzmann weight for the configuration U

$$Z(V,T,\mu) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_F(U,\psi,\bar{\psi})-S_G(U)}$$
$$= \int \mathcal{D}U\; \det M(U) e^{-S_G(U)}$$

- If there is particle-antiparticle-symmetry ($\mu = 0$) $\det M(U)$ is real
- For real chemical potential ($\mu^2 > 0$) $\rightarrow$ $\det M(U)$ is complex (**complex action problem**) and has wildly oscillating phase (**sign problem**)
  $\Rightarrow$ It cannot serve as a statistical weight
- For *purely imaginary* chemical potential ($\mu^2 < 0$) $\rightarrow$ $\det M(U)$ is real again, simulations can be made!

Quick note: simulations at imaginary chemical potential

- While for real chemical potential ($\mu^2 > 0$) $\det M(U)$ is complex, for **imaginary** chemical potential ($\mu^2 < 0$) $\det M(U)$ is real
- Perform simulations at imaginary chemical potentials:
- **Analytically continue** to $\mu^2 > 0$

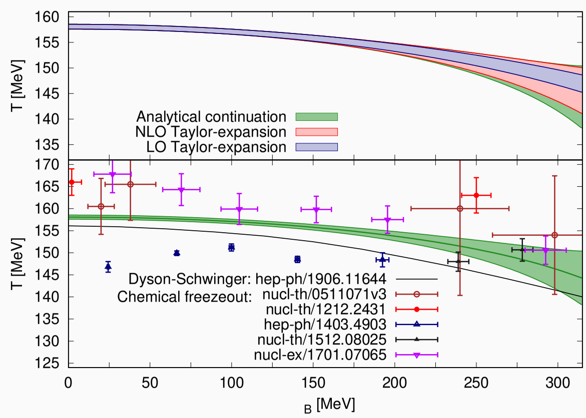


We can do this because the transition at $\mu = 0$ is a crossover, so the partition function is analytic!

The transition at finite chemical potential

Current results for transition temperature give:

$$T_c(\mu = 0) = 156 - 158\,\mathrm{MeV}$$



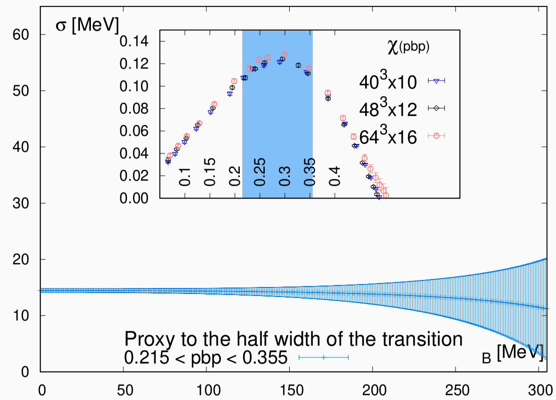
The width of the transition at finite chemical potential

We get at $\mu_B = 0$ for T_c:

$$T_c(LT = 4, \mu_B = 0) = 158.0 \pm 0.6 \text{ MeV}$$

while for the width

$$\Delta T(LT = 4, \mu_B = 0) = 15.0 \pm 1.0 \text{ MeV}$$



The width of the transition has a very mild chemical potential dependence

Summary

- Lattice QCD is a regularization scheme for QCD that does not rely on a perturbative expansion
- Thermodynamic results show that chiral restoration and deconfinement apparently at same temperature $T_c \simeq 156 - 158\,\text{MeV}$
- At physical quark masses, transition is a smooth crossover (at infinite quark masses, it is of first order)
- Equation of state of QCD known to high precision, shows liberation of degrees of freedom and approach to Stefan-Boltzmann limit
- Lattice QCD cannot at the moment give *direct* results at non-zero chemical potential (sign problem), but..
- ..thanks to extrapolations, we know the location of the phase transition line in the phase diagram (among other things)