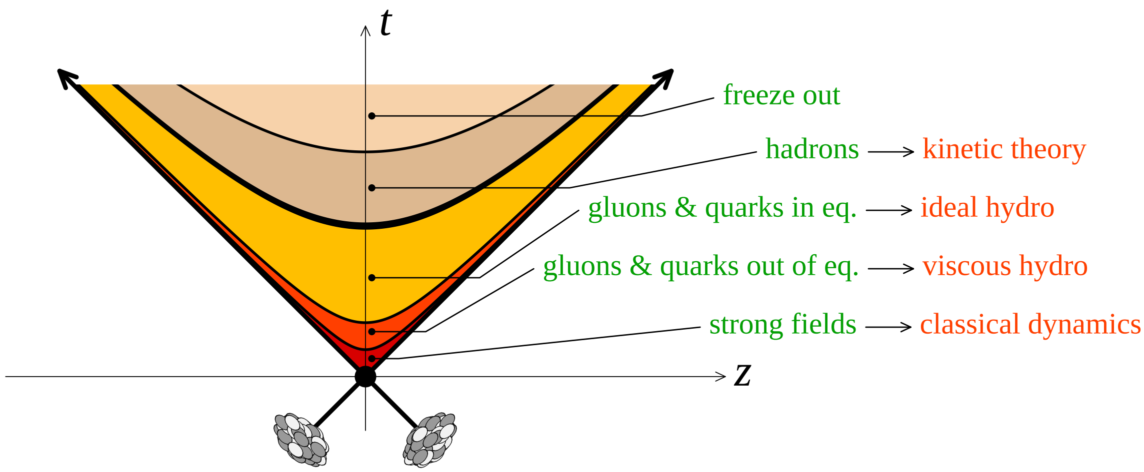
Introduction to Saturation Physics

Bibliography

- F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, Ann. Rev. Nucl. Part. Sci. **60**, 463 (2010) [arXiv:1002.0333 [hep-ph]].
- Y. V. Kovchegov, AIP Conf. Proc. **1520**, 3 (2012).
- J. L. Albacete, A. Dumitru and C. Marquet, Int. J. Mod. Phys. A 28, **1340010** (2013) [arXiv:1302.6433 [hep-ph]].
- L. McLerran, Prog. Theor. Phys. Suppl. **187**, 17 (2011) [arXiv:1011.3204 [hep-ph]].

Nucleus-nucleus collision:



F. Gelis, Nucl.Phys. **A 854** (2011) 10-17

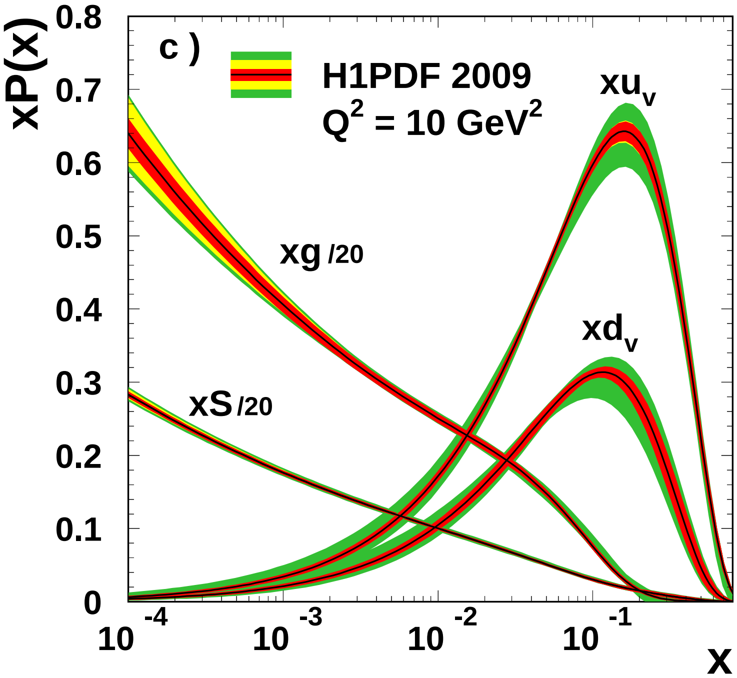
Most particles produced in nucleus-nucleus collisions have momenta in the GeV range or less, and the partons involved in the production process have momentum fraction:

$$x = \frac{m_T}{\sqrt{s}}\, e^{\pm y} \qquad m_T = \sqrt{m^2 + p_T^2}\ .$$

At RHIC energies $m_T \sim 1$ GeV correspond to $x \sim 10^{-2}$ at $y = 0$, at LHC $x \sim 10^{-4}$ (smaller values at forward rapidities).
These values are comparable to the values probed at HERA in deep inelastic collisions on protons.

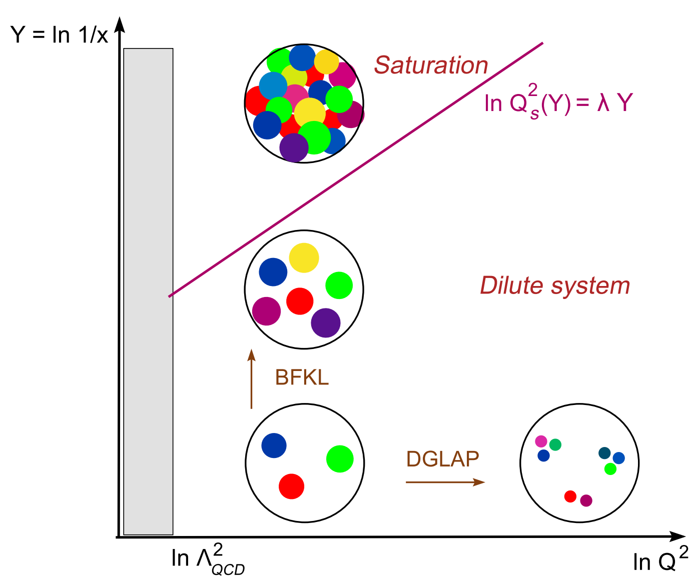
Understanding such collisions at high energies requires a good knowledge of that part of the nuclear wave functions that describes low x degrees of freedom.

It is well established that in this regime of small x, the gluons dominate the hadron wave functions. The gluon density is large and grows with lowering x, but this growth eventually saturates: one reaches then the regime of 'parton saturation',



x-evolution of the gluon, sea quark and valence quark distributions for $Q^2 = 10$ GeV^2 measured at HERA
[H1 Collab., Eur. Phys. J. **C 64** (2009) 561]

Saturation leads to a simple structure in the plane $(\log \frac{1}{x}, \log Q^2)$, a line separates dense and dilute parton systems.



each disk represents a parton with transverse area $S_T \sim 1/Q^2$ and longitudinal momentum $k^+ = xP^+$

Partons with transverse momentum $k_T \gg Q_s$ are in the dilute regime, those with $k_T \lesssim Q_s$ are in the dense, saturated regime.

In the vicinity of saturation perturbation theory breaks down: the large gluon density compensates for the weakness of the coupling, making the effective expansion parameter of order unity.

There is a balance between the inverse processes of gluon splitting and recombination:

$$g \leftrightarrow gg$$

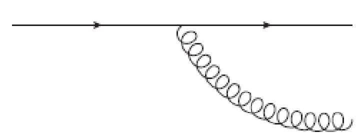
Non-linear effects are important!

The Color Glass Condensate (CGC) is a QCD-inspired effective theory, in which the classical fields are given a prominent role (high density).
The CGC aims at a complete description of the small x part of hadron wave functions that can be used to calculate many processes dominated by small x partons.

The wave function of a hadron at high energy

The wave function of a hadron is commonly characterized in terms of 'partons' carrying momentum $k = (k_z, \vec{k}_T)$, with the longitudinal momentum k_z given as a fraction x of the momentum of the parent hadron, $k_z = xP_z$.
Light cone coordinates are commonly used: $x^\pm = (t \pm z)/\sqrt{2}$
In these coordinates, a right mover parton has momentum $k^+ = xP^+$.

The basic phenomenon that controls the evolution of the wave functions is the branching of partons: $q(g) \to q(g)g$.



This process, which corresponds to the radiation of a soft gluon from either a quark or a gluon, occurs with a probability

$$dP \simeq \frac{\alpha_s C_R}{\pi^2} \frac{d^2 k_T}{k_T^2} \frac{dx}{x} \ ,$$

with $C_R = C_A = N_c$ for the radiation from a gluon, and $C_R = C_A = \frac{N_c^2 - 1}{2N_c}$ for a radiation from a quark.

The state of a hadron is built up from successive splittings of partons starting from the valence quarks.
For just a valence quark, leading order perturbation theory yields the integrated gluon distribution $xG(x; Q^2)$ as

$$xG(x; Q^2) = \frac{\alpha_s C_F}{\pi} \log\left(\frac{Q^2}{\Lambda^2_{QCD}}\right).$$

$xG(x; Q^2)$ counts the number of gluons in the hadron wave function (here the valence quark) with longitudinal momentum xP^+ and localized in the transverse plane to a region of size $\Delta x_T \sim 1/Q$.

The parton description ceases to make sense for partons that have wavelengths larger than the typical confinement scale $r_0 \sim 1/\Lambda_{QCD}$.

We define the unintegrated parton distribution $\varphi(x, \vec{k}_T)$, which gives the density in the transverse plane of gluons with transverse momentum k_T and a definite spin and color:

$$\frac{xG(x, Q^2)}{\pi R^2} = \int^Q d^2\vec{k}_T \frac{dN}{dy d^2\vec{k}_T}$$

with $dy = dx/x$ and

$$\frac{dN}{dy d^2\vec{k}_T} = \frac{2(N_c^2 - 1)}{(2\pi)^3} \varphi(x, \vec{k}_T) \ .$$

The density of partons in the cloud of gluons that surrounds a valence quark is not a fixed quantity but a quantity that depends on the resolution with which one is probing the gluon cloud. If one increases the resolution, by increasing Q^2, one sees more and more partons, i.e., the parton density increases.

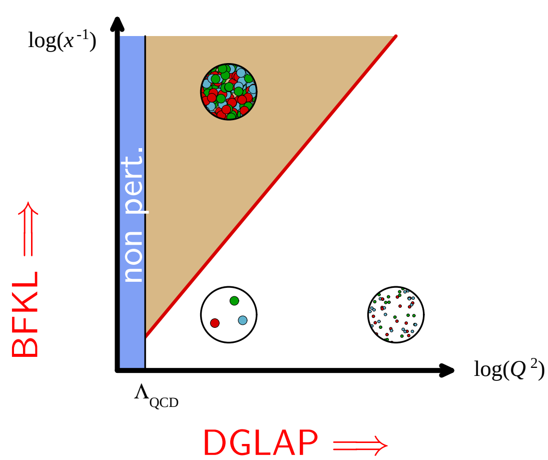
When $\log Q^2$ becomes large (even with small α_s) leading order perturbation theory is not enough: successive branchings must be included.

When $\alpha_s \log Q^2$ becomes of order unity, higher order terms become significant and must be resummed. This resummation is achieved by the DGLAP equation (Dokshitzer,Gribov,Lipatov,Altarelli,Parisi), which we write schematically as

$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} P(x|z) G(z, Q^2)$$

where $P(x|z)$ is the 'splitting function' and gives the probability that a daughter parton with momentum x is produced by the splitting of a parent parton with momentum z.

The DGLAP equation leads to an increase of the parton density with increasing Q^2. However, this increase is slow, involving typically $\log Q^2$. Since the size of the added partons decreases as $1/Q^2$, the area occupied by these new partons in the transverse plane eventually decreases with increasing Q^2. Thus, even though the density increases, the system of partons produced by the DGLAP evolution is more and more dilute with the partons effectively weakly coupled. As Q^2 grows (with x kept not too small) perturbation theory becomes more and more reliable in describing the changes in the hadron wave functions.



BFKL Equation

When one increases the rapidity (or equivalently the energy), x decreases and one eventually reaches a regime where new corrections become important: when $\alpha_s \log(1/x)$ becomes of order unity, the corresponding large logarithms need to be resummed.
This new resummation is achieved by the BFKL equation (Balitsky, Fadin, Kuraev, Lipatov): (equation for the unintegrated gluon density ($y = \log(1/x)$)):

$$\frac{\partial \varphi(y, \vec{k}_T)}{\partial y} = \bar{\alpha}_s \int \frac{d^2 \vec{p}_T}{\pi} \frac{\vec{k}_T^2}{\vec{p}_T^2 (\vec{k}_T - \vec{p}_T)^2} \left[\varphi(y, \vec{p}_T) - \frac{1}{2} \varphi(y, \vec{k}_T) \right] .$$

The most remarkable feature of the BFKL evolution is the exponential growth that it predicts for the gluon density as a function of y:

$$\varphi(y, \vec{k}_T) \sim \mathrm{e}^{\omega \bar{\alpha}_s y}$$

with $\omega = 4 \log 2$ (at leading order).

It was recognized early on that this growth of the gluon density, predicted by the linear BFKL equation, could not go on for ever, and various mechanisms leading to a 'saturation' of the process have been looked for. The early approaches to saturation invoked a non linear contribution to the evolution equation and leads (schematically) to an equation of the form

$$\frac{\partial^2}{\partial \log(1/x) \partial \log Q^2} xG(x, Q^2) = \bar{\alpha}_s xG(x, Q^2) - \frac{9}{16} \bar{\alpha}_s{}^2 \pi^2 \frac{[xG(x, Q^2)]^2}{R^2 Q^2}$$

where the second term in the r.h.s accounts for 'gluon recombination'.

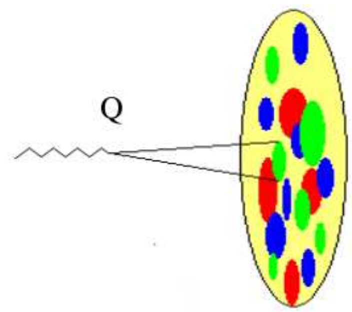
This "kinetic" vision of gluon saturation suggests immediately the existence of a characteristic momentum scale Q_s at which the processes of gluon emission and gluon recombination balance each other:

$$Q_s^2 \sim \alpha_s(Q_s^2) \frac{xG(x, Q_s^2)}{\pi R^2} .$$

At saturation the phase space density of modes with $k_T^2 \lesssim Q^2$ is large, of order $1/\alpha_s$.

The saturation scale Q_s

Consider a boosted nucleus interacting with an external probe
Transverse area of a parton: $1/Q^2$
Cross-section: $\sigma \sim \alpha_s/Q^2$
Parton density: $n = xG(x, Q^2)/\pi R_A^2$
Partons start to overlap when $S_A = \pi R_A^2 \simeq N_A \sigma$, $n\sigma \sim 1$ (N_A= number of partons=$xG(x, Q^2)$)



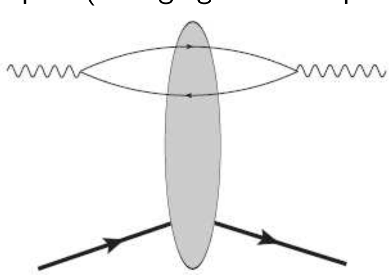
The parton density saturates at the scale: $Q_S^2 \sim \alpha_s(Q_s^2) N_A/\pi R^2 \sim A^{1/3}$

At saturation N_A is proportional to $1/\alpha_s$

Q_s^2 is proportional to the density of participating nucleons; larger for heavy nuclei.

Color dipoles

Another view of saturation is based on a picture commonly used in the analysis of lepton-hadron deep-inelastic scattering (DIS). In an appropriate frame, one can describe the interaction of the virtual photon with the hadron as the interaction of a color $q\bar{q}$ dipole (emerging form the photon) with the color field of the hadron.



The factor in the interaction cross section that is relevant is $\sigma_{dip}(x; \vec{r}_T)$, the total dipole-hadron cross-section, which can be calculated in the eikonal approximation (the size r_T of the dipole remains unchanged during the interaction). In this approximation, the S-matrix for the scattering of a quark moving in the negative z direction is given by the Wilson line

$$U(\vec{x}_T) = \mathcal{P}\exp\left\{-ig\int_{-\infty}^{\infty} dz^-\, A^+(z^-, \vec{x}_T)\right\}$$

where $\mathcal{P}$ denotes an ordering along the x^- axis, and A^+ is the classical (frozen) color field of the hadron moving close to the speed of light in the $+z$ direction.

The S-matrix for the scattering of the dipole contains another, complex conjugate, Wilson line. We can write the total dipole cross section as

$$\sigma_{dip} = 2 \int d\vec{b}(1 - S(\vec{b}, \vec{r}_T))$$

with

$$S(\vec{b}, \vec{r}_T) = \frac{1}{N_c} \mathrm{Tr} \left\langle U \left(\vec{b} + \frac{\vec{r}_T}{2} \right) U^\dagger \left(\vec{b} - \frac{\vec{r}_T}{2} \right) \right\rangle$$

It is also customary to define $S = 1 - N$, with N denoting the imaginary part of the forward scattering amplitude.

The total cross section will depend on the ratio between r_T and another length scale, call it r_s, that is determined entirely by the dynamics of the field with which the dipole interact. For the S-matrix one finds:

$$S(\vec{r}_T) = \mathrm{e}^{-r_T^2/4r_s^2} = \mathrm{e}^{-Q_s^2 r_T^2/4}$$

with $Q_s = 1/r_s$ (the factor 4 is conventional). The S-matrix exhibits a change of regime in the interaction of the dipole with the field of the hadron. A small dipole, with $r_T \ll r_s$, is little affected (color transparency), and its scattering amplitude measures directly the gluon density. A large dipole ($r_T \gg r_s$) is strongly absorbed: its cross section saturates to the black disk limit, and it is not capable to resolve the parton structure of the field in which it propagates.

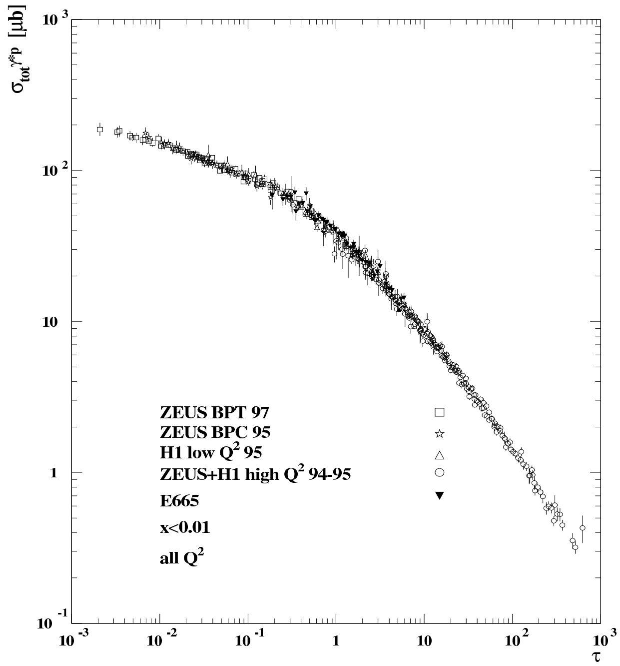
This allows one to view saturation as resulting from the multiple scatterings that the dipole undergoes as it traverses the hadron.

We note that all the dependence on the wave function is contained in the scale r_s that characterizes the change of regime between the dilute regime and the saturated one. The energy dependence of r_s is determined by the dynamics of gauge fields; we expect r_s to appear in the cross section only in the ratio $r_T^2/r_s^2 = r_T^2 Q_s^2$.

A very simple model for the dipole cross-section has been proposed by Golec-Biernat and Wüsthoff with the parameterization

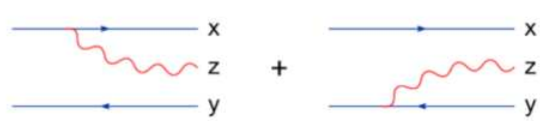
$$\sigma_{dip}(x, \vec{r}_T) = \sigma_0 \left[1 - \mathrm{e}^{-\frac{1}{4} Q_s^2(x) r_T^2} \right] \ .$$

with $Q_s^2(x) = Q_0^2 \, (x_0/x)^\lambda$. This model gives a very good description of HERA data at $x < 10^{-2}$ and moderate Q^2.
The fact that the x dependence of the cross section enters only in the definition of the basic scale r_s has been called 'geometrical scaling'.



The cross section $\sigma(\gamma^* p)$ as a function of the scaling variable $\tau = Q^2/Q_s^2$

The saturation momentum Q_s is determined by the dynamics of the gauge fields, i.e. from QCD. In particular the energy dependence of Q_s follows non-linear evolution equations.
In the dipole picture, the evolution is due to the emission of a gluon by a color dipole.



The original dipole turns into a dipole-gluon system, whose propagation, in the hadron field, is different from that of the original dipole.

Correlator of two Wilson lines:

$$\partial_Y \langle \mathrm{Tr}(U_x^\dagger U_y) \rangle_Y = -\frac{\alpha_s}{2\pi^2} \int d^2 z \frac{(x-y)^2}{(x-z)^2 (y-z)^2} \langle N_c \mathrm{Tr}(U_x^\dagger U_y) - \mathrm{Tr}(U_x^\dagger U_z) \mathrm{Tr}(U_z^\dagger U_y) \rangle$$

The Balitsky-Kovchegov (BK) equation is obtained by assuming

$$\langle \mathrm{Tr}(U_x^\dagger U_z) \mathrm{Tr}(U_z^\dagger U_y) \rangle \approx \langle \mathrm{Tr}(U_x^\dagger U_z) \rangle \langle \mathrm{Tr}(U_z^\dagger U_y) \rangle$$

justified for large N_c.
By writing $N = 1 - S$ the BK eq. can be written for the scattering amplitude:

$$\partial_Y N_{xy} = -\frac{\alpha_s N_c}{\pi} \int \frac{d^2 z}{2\pi} \frac{(x-y)^2}{(x-z)^2 (y-z)^2} (N_{xz} + N_{zy} - N_{xy} - N_{xy} N_{zy})$$

Color Glass Condensate

Near saturation the color fields are 'strong', with amplitude $A \sim 1/g$, and they provide a natural description of highly occupied, strongly coupled, modes.

The Color Glass Condensate formalism puts these classical fields at the heart of all considerations. Its goal is to provide a complete description of the small x component of the nuclear wave functions, thereby allowing the calculation of observables that control the early stages of nucleus-nucleus collisions.

The effective degrees of freedom in this framework are color sources ρ at large x and gauge fields A^μ at small x. At high energies, because of time dilation, the former are frozen color configurations on the natural time scales of the strong interactions and are distributed randomly from event to event. The latter are dynamical fields coupled to the static color sources. It is the stochastic nature of the sources, combined with the separation of time scales, that justify the "glass" appellation. The "condensate" designation comes from the fact that saturated gluons have large occupation numbers $\mathcal{O}(1/\alpha_s)$, with typical momenta peaked about a characteristic value $k_\perp \sim Q_s$.

Mathematical definition of the CGC

It is based on the observation that the separation between charges and fields involves a dividing scale between degrees of freedom: the field describes degrees of freedom with some particular value of x, while the color charge, and their correlations, are determined from degrees of freedom with x-values $x' > x$.
The CGC is defined mathematically by a path integral

$$\mathcal{Z} = \int_{X_0} \mathcal{D}A\, \mathcal{D}\rho\, \exp\{iS[A,\rho] - W_Y[\rho]\}$$

where X_0 is the cut-off.
Small-x gluons (A) are described as the classical colour fields radiated by colour sources at higher rapidity. Partons with large-x (ρ) act as a source.
As we move to lower and lower values of x more and more degrees of freedom are treated as random charges, and the corresponding correlators are modified.
The requirement that observables should remain independent of the dividing scale can be implemented (to leading logarithm accuracy) as a renormalization group equation for the distribution $W_Y[\rho]$.
This renormalization group equation is the JIMWLK[1] equation.
In the low-density limit it reduces to DGLAP and BFKL.

[1] JIMWLK ≡ Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner.

Phenomenology

The saturation scale depends on energy and on impact parameter:

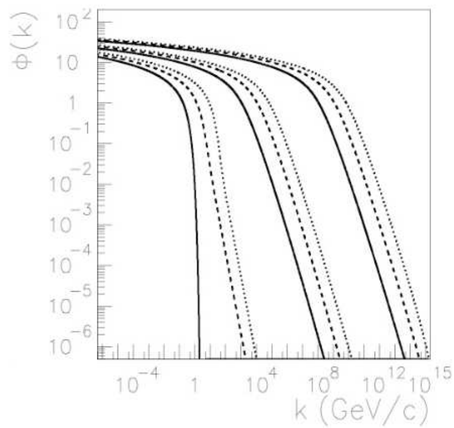
$Q_s^2 = Q_{s0}^2(x/x_0)^\lambda$, $\quad Q_{s0}^2(b) = Q_{s0}^2(0)T_A(b)$

where $T_A(b) = \int dz\rho(b,z)$, $\lambda \simeq 0.28$.

The energy dependence of Q_s can be obtained by solving the BK eq.
The unintegrated gluon distribution, obtained as solution of BK, has the tipical shapes shown in the figure.

A crude approximation consists in assuming:

$$\varphi(k_T) \simeq \begin{cases} 1/\alpha_s & k_T < Q_s(x) \\ Q_s^2(x)/k_T^2 & k_T > Q_s(x) \end{cases}$$



J.L. Albacete et al, Phys. Rev. Lett **92** (2004) 082001, hep-ph/0307179

[Initial conditions: GBW (solid lines), MV with $Q_S^2 = 4\,\mathrm{GeV}^2$ (dashed lines), MV with $Q_S^2 = 100\,\mathrm{GeV}^2$ (dotted lines); The three sets corresponds to $y = 0, 5, 10$ (left to right)]

The Structure Function F_2

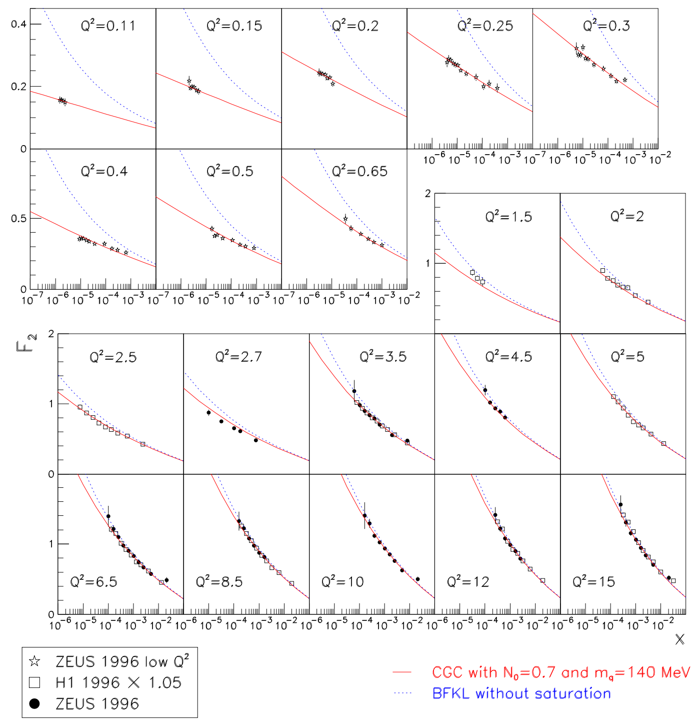
Using the dipole description of the virtual photon wavefunction, the structure function F_2 can be related to the gluon distribution function which arises from the CGC.

There are 3 unknown parameters in this description: the hadron size, the scale x_0 and the quark mass. In addition, the parameter λ which controls the energy dependence of the saturation momentum is determined by experiment but it is in good agreement with theoretical calculations:

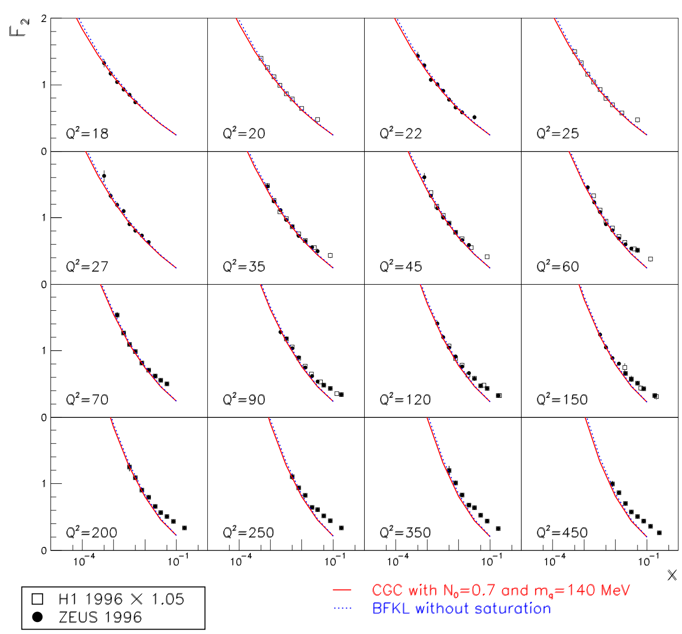
$$Q_s^2 = Q_{s0}^2(x/x_0)^\lambda \qquad\qquad \lambda \simeq 0.28$$

The results for the description of the data are remarkably good for $x \le 10^{-2}$ and $Q^2 \le 45\ GeV^2$, as shown in the next plots.

One should note that this description includes both the high and low Q^2 data. Descriptions based on DGLAP evolution can describe the large Q^2 points. The CGC description is very economical in the number of parameters which are used.



E. Iancu, K. Itakura, S. Munier, Phys.Lett. B590 (2004) 199-208

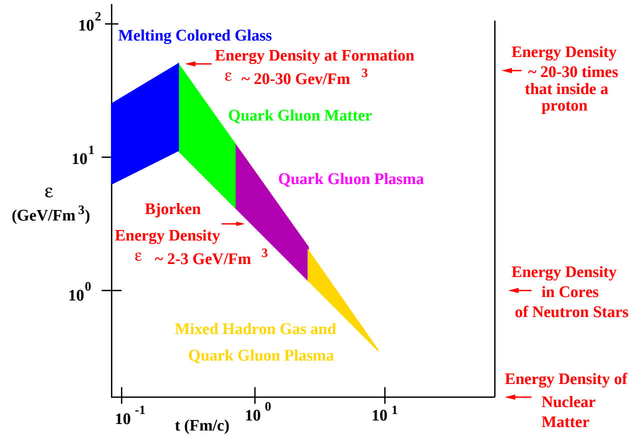


E. Iancu, K. Itakura, S. Munier, Phys.Lett. B590 (2004) 199-208

CGC in Heavy Ion Collisions

The collision of two ultrarelativistic heavy ions can be visualized as the scattering of two sheets of colored glass.

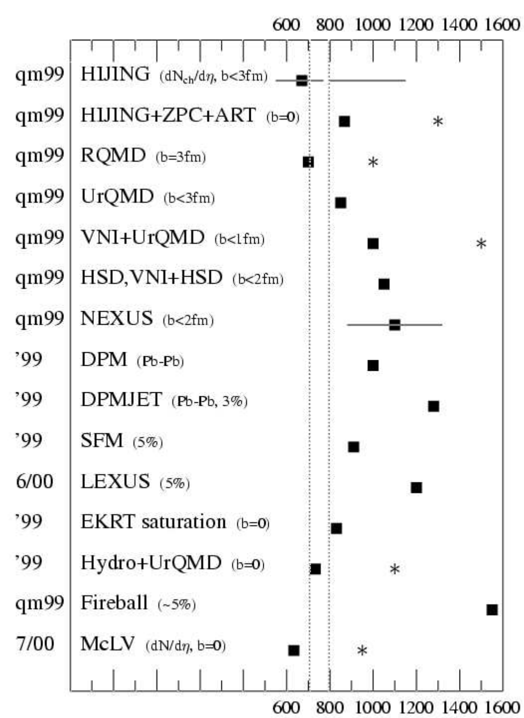
At very early times after the collision the matter is at very high energy density and in the form of a CGC. As time goes on, the matter expands. As it expands the density of gluons decreases, and gluons begin to propagate with little interaction. At later times, the interaction strength increases and there is sufficient time for the matter to thermalize and form a Quark Gluon Plasma.



Multiplicity

The CGC allows for a direct computation of the particle multiplicity in hadronic collisions. If one naively tries to compute jet production, the total multiplicity is infrared divergent. This follows because of the $1/p_T^4$ nature of the perturbative formula for gluon production.
In the CGC, when $p_T \leq Q_{sat}$ the perturbative divergence is suppressed.
The total gluon multiplicity goes as

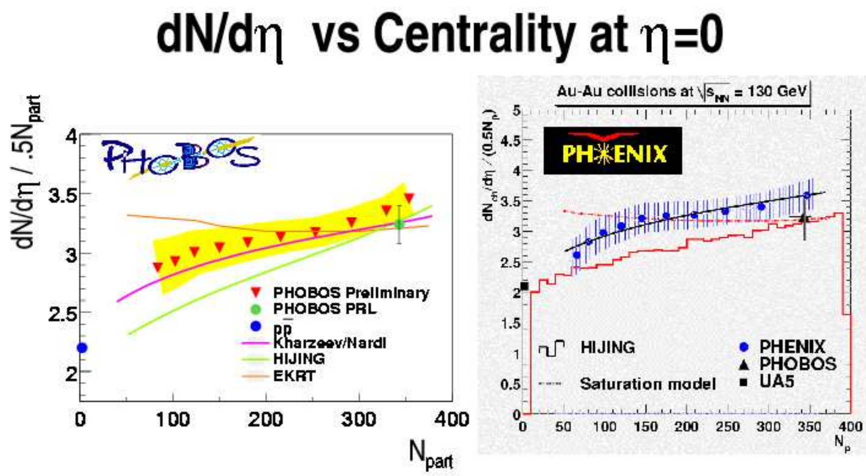
$$\frac{1}{\pi R^2}\frac{dN}{dy} \sim \frac{1}{\alpha_s} Q_{sat}^2$$



Also, the dependence of the multiplicity on the number of participants can also be computed, realizing that the saturation momentum should be (for not too small x) proportional $N_{part}^{1/3}$. This leads to

$$\frac{dN}{dy} \sim \frac{1}{\alpha_S} \tag{1}$$

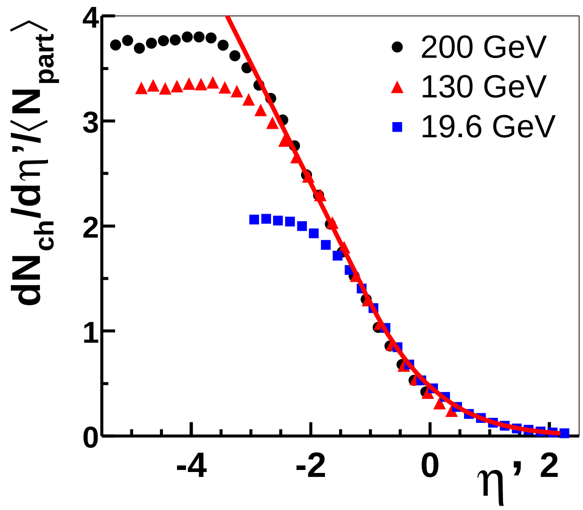
so that we have a very slow logarithmic dependence on the number of participants. This was a prediction of the CGC [KLN model] and it agreed with experiment.



Limiting Fragmentation

The rapidity distribution dN/dY in the vicinity of the beam rapidity $Y_{beam} = \log(\sqrt{s}/m)$ (i.e. in the fragmentation region) is independent of the collision energy.

In the CGC picture: $\frac{dN}{dy} \propto x_1 G(x_1, Q^2)$ (the projectile is dilute, the target is dense).



Initial condition for Hydrodynamical evolution

The multiplicity density (in the transverse plane) is estimated as

$$\frac{dN}{d^2s\,dy} \propto \min\left\{Q_{s1}^2, Q_{s2}^2\right\},$$

where Q_{s1}^2, Q_{s2}^2 are the saturation scales of the two nuclei, evaluated at the position $\vec{s}$:

$$Q_{s1}^2 \propto T_A(\vec{s})\ , \qquad\qquad Q_{s2}^2 \propto T_B(\vec{b}-\vec{s})\ .$$

In contrast, in the Glauber picture the local density is assumed to be proportional to the local density of participants $T_A(\vec{s}) + T_B(\vec{b}-\vec{s})$. Therefore the eccentricity predicted by the CGC initial conditions is larger than that obtained with Glauber initial conditions. This affects in particular elliptic flow calculation (v_2).

Conclusions

Saturation is a generic property of *QCD* in the regime of high parton densities.

Non linear evolution equations can be derived from *QCD*, based on weak coupling approaches, the non perturbative aspects of saturation arising from the large density of partons.

The CGC relies on a separation of degrees of freedom into color charge and color fields, and on a renormalization group equation that copes with the arbitrariness in the choice on the separation scales.

The CGC, or related approaches, have led to a systematic and successful phenomenology based on a few basic ingredients: Q_s and its dependence on energy, size of the system, centrality of the collision.

It is essential to know the initial state effects to correctly interpret the experimental data. The problem is: how to separate initial- and final-state effects? Data on p-A and e-A are extremely useful.