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0.1 Shape function for semileptonic decays

We write the matrix element of the forward scattering amplitude for $\bar{B} \to X_u l^- \bar{\nu}_l$ as:

$$\begin{aligned} h_{\mu\nu} &= [-g_{\mu\nu} k \cdot v + k_\mu v_\nu + k_\nu v_\mu - i\epsilon_{\mu\nu\alpha\rho} k^\alpha v^\rho] \times \qquad (1) \\ & \frac{1}{2m_B} \langle B | \bar{b} \left(-\frac{2}{k^2} \right) \sum_{n=0}^{\infty} \left(-\frac{2k\pi}{k^2} \right)^n b | B \rangle \qquad (2) \end{aligned}$$

So, for each of the three structure functions:

$$h_i = \frac{c_i}{2m_B} \langle B | \bar{b} \left(-\frac{2}{k^2} \right) \sum_{n=0}^{\infty} \left(-\frac{2k\pi}{k^2} \right)^n b | B \rangle, \qquad (3)$$

$$W_i = 2 \text{ Im } h_i \qquad (4)$$

We then write:

$$\frac{1}{2m_B} \langle B | \bar{b} \pi_{(\mu_1} \ldots \pi_{\mu_n)} b - \text{ traces } | B \rangle = a_n \left(v_{\mu_1} \ldots v_{\mu_n} - \text{ traces} \right) \qquad (5)$$

and define the shape function via its moments:

$$a_n = \int \frac{dk_+}{\bar{\Lambda}} \; k_+^n \; F_i(k_+, q^2), \qquad (6)$$

where index i enumerates the three structure functions.
Now:

$$\begin{aligned} h_i &= c_i \left(-\frac{2}{k^2} \right) \sum_{n=0}^{\infty} a_n \left(-\frac{2k \cdot v}{k^2} \right)^n = \\ &= c_i \left(-\frac{2}{\bar{\Lambda} k^2} \right) \int dk_+ \; F_i(k_+, q^2) \; \sum_{n=0}^{\infty} \left(-\frac{2k_+ k \cdot v}{k^2} \right)^n = \\ &= c_i \left(-\frac{2}{\bar{\Lambda}} \right) \int dk_+ \; F_i(k_+, q^2) \; \frac{1}{k^2 + 2k_+ k \cdot v + i\epsilon} \qquad (7) \end{aligned}$$

Since $k_\mu = m_b v_\mu - q_\mu$, the denominator reads:

$$\begin{aligned} & m_b^2 + q^2 - 2m_b q_0 + 2(m_b - q_0) k_+ = \\ \sim \; & 2m_b \left(1 + \frac{k_+}{m_b} \right) \left[\frac{m_b^2 + q^2}{2m_b} + \frac{k_+}{2} \left(1 - \frac{q^2}{m_b^2} \right) - q_0 \right] \qquad (8) \end{aligned}$$

Then:

$$h_i = c_i \left(-\frac{1}{\bar{\Lambda} m_b}\right) \int dk_+ \; F_i(k_+,q^2) \; \left(1-\frac{k_+}{m_b}\right) \times \left[\frac{m_b^2+q^2}{2\; m_b} + \frac{k_+}{2}\left(1-\frac{q^2}{m_b^2}\right) - q_0 + i\epsilon\right]^{-1} \tag{9}$$

To the leading approximation:

$$\left(1-\frac{k_+}{m_b}\right) \sim 1, \tag{10}$$

And we find the form of the convolution:

$$W_i(q_0,q^2) = \int dk_+ \; F_i(k_+,q^2) \; W_i^{bare}\left[q_0 - \frac{k_+}{2}\left(1-\frac{q^2}{m_b^2}\right), q^2\right] \tag{11}$$

To the leading order we can substitute $(1-q^2/m_b^2)$ by $(1-q^2/m_b m_B)$. With such a prescription the hadronic kinematics is automatically satisfied, *i.e.* the maximum allowed value for q_0 turns out to be:

$$q_0^{max} = \frac{k_+^{max}}{2}\left(1-\frac{q^2}{m_b m_B}\right) + \frac{m_b^2+q^2}{2m_b} = \frac{m_B^2+q^2}{2\; m_B}, \tag{12}$$

which is the correct endpoint for the hadronic decay. (Remember that $k_+^{max} = m_B - m_b$). Hence, the convolution reads:

$$W_i(q_0,q^2) = \int dk_+ \; F_i(k_+,q^2) \; W_i^{bare}\left[q_0 - \frac{k_+}{2}\left(1-\frac{q^2}{m_b m_B}\right), q^2\right] \tag{13}$$

We then assume the latest equation, derived at leading order, to be valid also for higher orders.

Our method to determine the shape functions is to match their moments with (central) q_0-moments of the structure functions as predicted by the HQE:

$$\int dq_0 (q_0-a)^n \; W_i(q_0,q^2) = \int dk_+ \; F_i(k_+,q^2) \times \int dq_0 (q_0-a)^n \; W_i^{bare}(q_0 - k_+\,\Delta/2, q^2), \tag{14}$$

where:

$$a = \frac{m_b^2+q^2}{2\; m_b}, \tag{15}$$

$$\Delta = 1 - \frac{q^2}{m_b m_B} \tag{16}$$

The l.h.s. of eq. (14) can be calculated including power corrections and we denote it by:

$$I_i^{(n),\mathrm{HQE}} = \int dq_0 (q_0 - a)^n \; W_i(q_0, q^2), \tag{17}$$

While the r.h.s. can be written as:

$$\begin{aligned}
& \int dk_+ \; F_i(k_+, q^2) \int dq_0 (q_0 - a)^n \; W_i^{bare}(q_0 - k_+ \; \Delta/2, q^2) = \\
= & \int dk_+ \; F_i(k_+, q^2) \int dq_0 (q_0 - k_+ \; \Delta/2 + k_+ \; \Delta/2 - a)^n \times \\
& \qquad W_i^{bare}(q_0 - k_+ \; \Delta/2, q^2) = \\
= & \int dk_+ \; F_i(k_+, q^2) \int d\tilde{q}_0 (\tilde{q}_0 - a + k_+ \; \Delta/2)^n \; W_i^{bare}(\tilde{q}_0, q^2) = \\
= & \left(\frac{\Delta}{2}\right)^n \int dk_+ \; k_+^n \; F_i(k_+, q^2) \int d\tilde{q}_0 \; W_i^{bare}(\tilde{q}_0, q^2),
\end{aligned} \tag{18}$$

where the last step is due to vanishing of all central moments of the bare structure functions, with $n > 0$. If we denote:

$$I_i^{(0),bare} = \int dq_0 \; W_i^{bare}(q_0, q^2), \tag{19}$$

we finally find the expression for the moments of the shape functions:

$$\int dk_+ \; k_+^n \; F_i(k_+, q^2) = \left(\frac{2}{\Delta}\right)^n \frac{I_i^{(n),\mathrm{HQE}}}{I_i^{(0),bare}} \tag{20}$$

0.2 Perturbative form factors

We write the triple differential distribution using the ensemble (q_0, q^2, E_l):

$$\begin{aligned}
\left.\frac{d^3\Gamma}{dq^2 dq_0 dE_l}\right|_{b\to u} = & \frac{G_F^2 |V_{ub}|^2}{8\pi^3} \Big\{ q^2 W_1 - \left[2E_l^2 - 2q_0 E_l + \frac{q^2}{2}\right] W_2 \\
& + q^2 (2E_l - q_0) W_3 \Big\} \; \theta\left(q_0 - E_l - \frac{q^2}{4E_l}\right) \\
& \times \theta(E_l) \; \theta(q^2) \; \theta(q_0 - \sqrt{q^2}),
\end{aligned} \tag{21}$$

After introducing normalized variables:

$$\hat{q}_0 = \frac{q_0}{m_b}, \qquad \hat{q}^2 = \frac{q^2}{m_b^2} \tag{22}$$

the most general form for each structure function is:

$$W_i(q_0, q^2) = m_b^{n_i} \left[W_i^{pert}(\hat{q}_0, \hat{q}^2) + W_i^{power}(\hat{q}_0, \hat{q}^2) \right]$$

with $n_{1,2} = -1$ and $n_3 = -2$.

Power corrections, up to $\mathcal{O}(1/m_b^3)$ are encoded in W_i^{power} and are quoted in Appendix B. After the divergence has been subtracted and the gluon mass has been set to zero, the perturbative part, to first order in α_s, reads:

$$\begin{aligned} \widetilde{W}_i^{pert}(q_0, q^2) &= \left[W_i^{TL}(\hat{q}^2) + C_F \frac{\alpha_s}{\pi} \widetilde{V}_i(\hat{q}^2)\right] \delta\left(1 + \hat{q}^2 - 2\hat{q}_0\right) \\ &+ C_F \frac{\alpha_s}{\pi} \widetilde{R}_i(\hat{q}_0, \hat{q}^2)\, \theta\left(1 + \hat{q}^2 - 2\hat{q}_0\right), \end{aligned} \tag{23}$$

Soft terms have been already subtracted to form factors of the real emission and inserted in $\tilde{V}(\hat{q}^2)$. The "tilde" points out that perturbative corrections are meant here in the *on-shell* scheme.

The chosen normalization is such that:

$$W_1^{TL}(\hat{q}^2) = 1 - \hat{q}^2, \qquad W_2^{TL}(\hat{q}^2) = 4, \qquad W_3^{TL}(\hat{q}^2) = 2 \tag{24}$$

The NLO soft-virtual form factors read:

$$\begin{aligned} \widetilde{V}_1(\hat{q}^2) &= -\frac{1}{4}\left(1 - \hat{q}^2\right)\left[8 \ln^2\left(1 - \hat{q}^2\right) + 2\left(\frac{1}{\hat{q}^2} - 5\right) \ln\left(1 - \hat{q}^2\right)\right. \\ &\quad \left. + 4\, \mathrm{Li}_2\left(\hat{q}^2\right) + \frac{4\pi^2}{3} + 5\right] \end{aligned} \tag{25}$$

$$\begin{aligned} \widetilde{V}_2(\hat{q}^2) &= \frac{1}{3}\Big[-24 \ln^2\left(1 - \hat{q}^2\right) + 30 \ln\left(1 - \hat{q}^2\right) \\ &\quad - 12\, \mathrm{Li}_2\left(\hat{q}^2\right) - 4\pi^2 - 15\Big] \end{aligned} \tag{26}$$

$$\begin{aligned} \widetilde{V}_3(\hat{q}^2) &= \frac{1}{2}\Big[-8 \ln^2\left(1 - \hat{q}^2\right) - 2\left(\frac{1}{\hat{q}^2} - 5\right) \ln\left(1 - \hat{q}^2\right) \\ &\quad - 4\, \mathrm{Li}_2\left(\hat{q}^2\right) - \frac{4\pi^2}{3} - 5\Big] \end{aligned} \tag{27}$$

And the real gluon emission terms:

$$\begin{aligned} \widetilde{R}_1(\hat{q}_0, \hat{q}^2) &= \frac{(\hat{q}_0 + 5)\hat{u}}{4\left(\hat{q}_0^2 - \hat{q}^2\right)} + \frac{5}{2} - 2\sqrt{\hat{q}_0^2 - \hat{q}^2}\left(\frac{\ln \hat{u}}{\hat{u}}\right)_+ \\ &+ \frac{\ln \hat{u} - 2\ln(1 - \hat{q}_0 + \sqrt{\hat{q}_0^2 - \hat{q}^2})}{8\sqrt{\hat{q}_0^2 - \hat{q}^2}}\left[\frac{(\hat{q}_0 + 5)(\hat{q}_0 - 1)^3}{\hat{q}^2 - \hat{q}_0^2} + \hat{q}_0^2 - 2\hat{q}_0 - \hat{q}^2 - 14\right] \\ &+ \left[\frac{7(\hat{q}_0 - 1)}{2} + 4\sqrt{\hat{q}_0^2 - \hat{q}^2}\, \ln(1 - \hat{q}_0 + \sqrt{\hat{q}_0^2 - \hat{q}^2})\right]\left(\frac{1}{\hat{u}}\right)_+, \end{aligned} \tag{28}$$

$$
\begin{aligned}
\widetilde{R}_2(\hat{q}_0,\hat{q}^2) &= -\frac{3(\hat{q}_0+5)\,\hat{u}\,\hat{q}_0^2}{4\left(\hat{q}_0^2-\hat{q}^2\right)^2} - \frac{6\hat{q}_0^2-3\hat{q}^2\,\hat{q}_0+41\hat{q}_0-\hat{q}^2-5}{4\left(\hat{q}_0^2-\hat{q}^2\right)} \\
&+ \frac{4(\hat{q}_0-1)}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\left(\frac{\ln\hat{u}}{\hat{u}}\right)_+ + \frac{\ln\hat{u}-2\ln(1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2})}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\left[\frac{32\hat{q}_0^4+12(\hat{u}-8)\hat{q}_0^3}{8\left(\hat{q}_0^2-\hat{q}^2\right)^2}\right. \\
&+ \left.\frac{2\left(\hat{u}^2-16\hat{u}+48\right)\hat{q}_0^2+2\left(4\hat{u}^2+7\hat{u}-16\right)\hat{q}_0+\hat{u}\left(\hat{u}^2-7\hat{u}+6\right)}{8\left(\hat{q}_0^2-\hat{q}^2\right)^2}\right] \\
&- \left[7+\frac{8(\hat{q}_0-1)}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\ln(1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2})\right]\left(\frac{1}{\hat{u}}\right)_+ ,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\widetilde{R}_3(\hat{q}_0,\hat{q}^2) &= -\frac{3\hat{q}_0+\hat{q}^2}{2\left(\hat{q}_0^2-\hat{q}^2\right)} + \frac{2(\hat{q}_0-1)}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\left(\frac{\ln\hat{u}}{\hat{u}}\right)_+ \\
&+ \frac{\ln\hat{u}-2\ln(1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2})}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\left[\frac{\hat{q}_0(2\hat{q}_0+\hat{q}^2-3)}{4\left(\hat{q}_0^2-\hat{q}^2\right)}\right] \\
&- \left[\frac{7}{2}+\frac{4(\hat{q}_0-1)}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\ln(1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2})\right]\left(\frac{1}{\hat{u}}\right)_+ ,
\end{aligned}
\tag{30}
$$

where $\hat{u} = 1-2\hat{q}_0+\hat{q}^2$.

Plus distributions are defined in order to factorize $\delta\left(1+\hat{q}^2-2\hat{q}_0\right)$ in front of $\ln\lambda$ and $\ln^2\lambda$ singularities. This allows one to cancel it with the corresponding virtual term. Given a smooth function $G(\hat{q}_0,\hat{q}^2)$ the plus prescription is such that:

$$
\begin{aligned}
\int_a^{\frac{1+\hat{q}^2}{2}} d\hat{q}_0\,\frac{G(\hat{q}_0,\hat{q}^2)}{(1+\hat{q}^2-2\hat{q}_0)_+} &= \frac{1}{2}G\left(\frac{1+\hat{q}^2}{2},\hat{q}^2\right)\ln\left(1+\hat{q}^2-2a\right) \\
&+ \int_a^{\frac{1+\hat{q}^2}{2}} d\hat{q}_0\,\frac{G(\hat{q}_0,\hat{q}^2)-G\left(\frac{1+\hat{q}^2}{2},\hat{q}^2\right)}{1+\hat{q}^2-2\hat{q}_0}
\end{aligned}
\tag{31}
$$

and:

$$
\begin{aligned}
\int_a^{\frac{1+\hat{q}^2}{2}} d\hat{q}_0\; G(\hat{q}_0,\hat{q}^2)\left[\frac{\ln\left(1+\hat{q}^2-2\hat{q}_0\right)}{1+\hat{q}^2-2\hat{q}_0}\right]_+ &= \frac{1}{4}G\left(\frac{1+\hat{q}^2}{2},\hat{q}^2\right)\ln^2\left(1+\hat{q}^2-2a\right) \\
+ \int_a^{\frac{1+\hat{q}^2}{2}} d\hat{q}_0\;\left[G(\hat{q}_0,\hat{q}^2)-G\left(\frac{1+\hat{q}^2}{2},\hat{q}^2\right)\right]&\frac{\ln\left(1+\hat{q}^2-2\hat{q}_0\right)}{1+\hat{q}^2-2\hat{q}_0}
\end{aligned}
\tag{32}
$$

0.3 Wilsonian scale separation

The framework of the Wilsonian scale separation is based on the idea of including into the Feynman integrals only gluons with energies larger than a cutoff μ around 1 GeV. Contributions of gluons below this scale are reabsorbed in the definitions of heavy quark parameters. Physical quantities are, of course, independent of the cutoff. This ensures a better convergence and reliability of the perturbative series, as divergences coming from soft gluons are factorized.

We restrict the dissertation to $\mathcal{O}(\alpha_s)$ corrections. The extension to the BLM case is straightforward.

Inserting a cutoff on the gluon energy means integrating over the gluon momentum k with an extra factor $\theta(k_0-\mu)$). Eq. (23) is modified to:

$$\begin{aligned} W_i^{pert}(q_0,q^2,\mu) &= \left[W_i^{TL}(\hat{q}^2)+C_F\frac{\alpha_s}{\pi}V_i(\hat{q}^2,\eta)\right]\delta(1+\hat{q}^2-2\hat{q}_0) \\ &+ C_F\frac{\alpha_s}{\pi}R_i(\hat{q}_0,\hat{q}^2,\eta)\ \theta(1+\hat{q}^2-2\hat{q}_0) \\ &+ C_F\frac{\alpha_s}{\pi}B_{1,i}(\hat{q}^2,\eta)\ \delta'(1+\hat{q}^2-2\hat{q}_0), \end{aligned} \tag{33}$$

where:

$$\eta=\mu/m_b \tag{34}$$

and we have assumed $0<\eta<1/2$. The effect of the cutoff is to soften the divergence of the form factors near the endpoint, eliminating the infrared divergence in the spectrum. The growth softens but the form factors are still divergent due to collinear effects.

We were able to perform the explicit calculation in the presence of the cutoff for the real gluon emission, finding the analytic result for $R_i(\hat{q}_0,\hat{q}^2,\eta)$:

$$R_i(\hat{q}_0,\hat{q}^2,\eta)=\widetilde{R}_i(\hat{q}_0,\hat{q}^2)\ \theta\left(s-\hat{q}_0\right)+R_i^{cut}(\hat{q}_0,\hat{q}^2,\eta)\ \theta\left(\hat{q}_0-s\right) \tag{35}$$

where $s=s(\hat{q}^2,\eta)=1/2-\eta+\hat{q}^2/(2-4\eta)$. It is easy to see that for $\hat{q}^2\geq 1-2\eta$ the above relation reduces to:

$$R_i(\hat{q}_0,\hat{q}^2,\eta)=\widetilde{R}_i(\hat{q}_0,\hat{q}^2), \tag{36}$$

in the whole allowed $\hat{q}_0$ domain. Expressions of R_i^{cut} are collected in Appendix A.

The direct calculation of the η-dependence of (soft) virtual form factors would be much more cumbersome, but we can still infer their expressions from the requirement that physical quantities and, in particular, the integral over q_0 of each form factor must be independent of the cutoff. That is:

$$\int_{-\infty}^{+\infty}dq_0W_i(q_0,q^2,0)=\int_{-\infty}^{+\infty}dq_0W_i(q_0,q^2,\mu)+\mathcal{O}(\alpha_s^2), \tag{37}$$

where, as already stated:

$$W_i(q_0,q^2,\mu)=m_b^{n_i}\left[W_i^{pert}(\hat{q}_0,\hat{q}^2,\eta)+W_i^{power}(\hat{q}_0,\hat{q}^2,\eta)\right]. \tag{38}$$

Renormalization of non perturbative parameters and of m_b yield a cutoff-dependence of the form:

$$m_b(0) = m_b(\eta) + \left[\bar{\Lambda}(\eta)\right]_{\rm pert} + \frac{\left[\mu_\pi^2(\eta)\right]_{\rm pert}}{2 m_b(\eta)}, \tag{39}$$

$$\mu_\pi^2(0) = \mu_\pi^2(\eta) - [\mu_\pi^2(\eta)]_{\rm pert} \tag{40}$$

$$\rho_D^3(0) = \rho_D^3(\eta) - [\rho_D^3(\eta)]_{\rm pert}, \tag{41}$$

with:

$$\begin{aligned} \left[\bar{\Lambda}(\mu)\right]_{\rm pert} &= \frac{4}{3} C_F \frac{\alpha_s}{\pi} \eta \\ \left[\mu_\pi^2(\eta)\right]_{\rm pert} &= C_F \frac{\alpha_s}{\pi} \eta^2 \\ \left[\rho_D^3(\eta)\right]_{\rm pert} &= \frac{2}{3} C_F \frac{\alpha_s}{\pi} \eta^3, \end{aligned} \tag{42}$$

We write the perturbative contributions to form factors, with the Wilsonian scale separation, in the following way:

$$R_i(\hat{q}_0, \hat{q}^2, \eta) = \widetilde{R}_i(\hat{q}_0, \hat{q}^2) + \delta R_i(\hat{q}_0, \hat{q}^2, \eta), \tag{43}$$

$$V_i(\hat{q}^2, \eta) = \widetilde{V}_i(\hat{q}^2) + \delta V_i(\hat{q}^2, \eta). \tag{44}$$

According to def. (35), δR_i is simply:

$$\delta R_i(\hat{q}_0, \hat{q}^2, \eta) = \left[R_i^{cut}(\hat{q}_0, \hat{q}^2, \eta) - \widetilde{R}_i(\hat{q}_0, \hat{q}^2) \right] \, \theta\left(\hat{q}_0 - s\right) \tag{45}$$

It is simple to verify that the matching condition (37) is satisfied iff:

$$\delta V_i(\hat{q}^2, \eta) = S_i(\hat{q}^2, \eta) - 2 \int_{-\infty}^{+\infty} d\hat{q}_0 \; \delta R_i(\hat{q}_0, \hat{q}^2, \eta) \; \theta(1 + \hat{q}^2 - 2\hat{q}_0), \tag{46}$$

where:

$$\begin{aligned} \frac{C_F \alpha_s}{\pi} S_i(\hat{q}^2, \eta) = 2 \; m_b^{-n_i} \left\{ \left[\bar{\Lambda}(\eta)\right]_{\rm pert} + \frac{\left[\mu_\pi^2(\eta)\right]_{\rm pert}}{2 m_b(\eta)} \right\} \frac{\partial M_i^{(0),TL}(q^2/m_b^2)}{\partial m_b} \\ - 2 \; m_b^{-1-n_i} \left\{ \left[\mu_\pi^2(\eta)\right]_{\rm pert} \frac{\partial}{\partial \mu_\pi^2} + \left[\rho_D^3(\eta)\right]_{\rm pert} \frac{\partial}{\partial \rho_D^3} \right\} M_i^{(0),power}\left(\hat{q}^2\right) \end{aligned} \tag{47}$$

The resulting factors S_i read:

$$\begin{aligned} S_1(\hat{q}^2, \eta) &= \frac{8}{3} \hat{q}^2 \eta + \frac{\hat{q}^2 - 2}{3} \eta^2, \\ S_2(\hat{q}^2, \eta) &= 0, \\ S_3(\hat{q}^2, \eta) &= -\frac{8}{3} \eta + \frac{2}{9} \eta^3. \end{aligned} \tag{48}$$

There is one point left to clarify in eq. (23), namely the presence of a δ' term. This is related to the difference that occurs between the "rest-energy" kinetic mass of the b quark and the mass that really enters in the decay. Such a difference appears as a shift in the endpoint, which is the interpretation of the derivative of the Dirac delta:

$$\Delta\, 2q_0 = \left[-m_b^{\rm kin}(\mu) + m_b^0(\mu)\right]\left(1 - \frac{q^2}{m_b^2}\right) - \frac{m_q^2}{m_b} \tag{49}$$

Again, we can infer the expressions of $B_{1,i}$ without an explicit calculation, but requiring cutoff-independence of the first q_0 central moment (which is clearly the only one affected by the presence of a δ'). A procedure very similar to the one described for the virtual corrections yields in the end:

$$\begin{aligned}
&\frac{C_F\alpha_s}{\pi} B_{1,i}(\hat{q}^2,\eta) = 2\, m_b^{-1-n_i}(1-\hat{q}^2)\Big\{\left[\bar{\Lambda}(\eta)\right]_{\rm pert} + \frac{\left[\mu_\pi^2(\eta)\right]_{\rm pert}}{2m_b(\eta)}\Big\} M_i^{(0),TL}(\hat{q}^2) \\
&- 4\, m_b^{-2-n_i}\Big\{\left[\mu_\pi^2(\eta)\right]_{\rm pert}\frac{\partial}{\partial\mu_\pi^2} + \left[\rho_D^3(\eta)\right]_{\rm pert}\frac{\partial}{\partial\rho_D^3}\Big\} I_i^{(1),{\rm HQE}}(\hat{q}^2) \\
&-\ 4\ \frac{C_F\alpha_s}{\pi}\int_{-\infty}^{+\infty} d\hat{q}_0 \left(\hat{q}_0 - \frac{1+\hat{q}^2}{2}\right)\delta R_i(\hat{q}_0,\hat{q}^2,\eta)\ \theta(1+\hat{q}^2-2\hat{q}_0)
\end{aligned} \tag{50}$$

Hence:

$$\begin{aligned}
B_{1,1}(\hat{q}^2,\eta) &= \frac{1}{3}(1-\hat{q}^2)^2\ \eta + (1-\hat{q}^2)\ \eta^2 + \frac{1}{9}(1-\hat{q}^2)(-7+5\hat{q}^2)\ \eta^3, \\
B_{1,2}(\hat{q}^2,\eta) &= \frac{4}{3}(1-\hat{q}^2)\ \eta + 4\ \eta^2 + \frac{4}{9}(1+5\hat{q}^2)\ \eta^3, \\
B_{1,3}(\hat{q}^2,\eta) &= \frac{2}{3}(1-\hat{q}^2)\ \eta + 2\ \eta^2 + \frac{2}{9}(-7+5\hat{q}^2)\ \eta^3.
\end{aligned} \tag{51}$$

0.4 Convolution with the shape function

We now want to convolute the shape functions with the perturbative kernel, including corrections up to second order BLM. It is instructive to see how the convolution is performed in the simpler case of $\bar{B} \to X_s\gamma$, before moving to semileptonic decays.

0.4.1 $\bar{B} \to X_s\gamma$

The expression of the convolution for the radiative decay can be derived from eq. (13), by taking the $q^2 \to 0$ limit and replacing the form factor with the photon energy spectrum:

$$\frac{d\Gamma\,(E_\gamma)}{dE_\gamma} = \int_{-\infty}^{+\infty} dk_+\ F(k_+)\ \frac{d\Gamma^{pert}\left(E_\gamma - \frac{k_+}{2}\right)}{dE_\gamma} \tag{52}$$

Which, in terms of a-dimensional variables ($\xi = 2E_\gamma/m_b$, $\kappa = k_+/m_b$), reads:

$$\frac{d\Gamma(\xi)}{d\xi} = m_b \int_{-\infty}^{+\infty} d\kappa\ F(\kappa)\ \frac{d\Gamma^{pert}(\xi-\kappa)}{d\xi} \tag{53}$$

Moments of the shape function are then constrained to be:

$$\int d\kappa\ \kappa^n\ F(\kappa) = \int d\xi\ (\xi-1)^n \frac{1}{\Gamma^{\rm OPE}}\frac{d\Gamma^{\rm OPE}}{d\xi}, \tag{54}$$

where:

$$\frac{1}{\Gamma^{\rm OPE}}\frac{d\Gamma^{\rm OPE}}{d\xi} = \delta(1-\xi) + A_1\delta'(1-\xi) + A_2\delta''(1-\xi), \tag{55}$$

$$A_1 = \frac{\mu_\pi^2-\mu_G^2}{2m_b^2} - \frac{5\rho_D^3 - 7\rho_{LS}^3}{6m_b^3} \tag{56}$$

$$A_2 = \frac{\mu_\pi^2}{6m_b^2} - \frac{2\rho_D^3 - \rho_{LS}^3}{6m_b^3} \tag{57}$$

Note that eq. (54) is only valid for $n \geq 1$, as in this context, opposite to the case of semileptonic decays, we don't need the exact normalization of the shape function (namely $n = 0$). We will simply normalize to the convoluted spectrum to its integral over the whole domain.
Eq. (55) implies:

$$\int d\kappa\ \kappa\ F(\kappa) = A_1 \tag{58}$$

$$\int d\kappa\ \kappa^2\ F(\kappa) = 2A_2 \tag{59}$$

Once the shape function is determined, one can proceed in two equivalent ways to obtain the hadronic spectrum. The common starting point is the perturbative kernel, that we will now treat in the on-shell scheme. Eventually, modifications due to the introduction of a Wilsonian cutoff will be pointed out. At NLO the perturbative spectrum has the general expression:

$$\frac{1}{\Gamma^{\rm pert}}\frac{d\Gamma^{\rm pert}}{d\xi} = \frac{C_F\alpha_s}{\pi}\ f_1(\xi)\ \theta(1-\xi)\ \theta(\xi) + A\ \delta(1-\xi), \tag{60}$$

where $A = 1 + (C_F\alpha_s/\pi)V_1$ and can be fixed after regularization and cancellation of infrared and collinear divergences (*e.g.* introducing *plus-distributions* as done in the previous chapter).

Using eq. (53) one immediately finds the convoluted spectrum to be:

$$\frac{1}{m_b}\frac{d\Gamma}{d\xi} = A\ F(\xi-1) + \frac{C_F\alpha_s}{\pi}\int_{\xi-1}^{min(\hat\Lambda,\xi)} d\kappa\ G(\kappa)\ f_1(\xi-\kappa), \tag{61}$$

where we have explicited the θ-function contained in $F(\kappa)$. Namely:

$$F(\kappa) = G(\kappa)\ \theta(\hat{\Lambda} - \kappa). \tag{62}$$

The latter equation requires explicit knowledge of A, thus the introduction of a subtraction-scheme for divergences. The method we are now showing allows one to bypass the direct calculation of A.

We define, for $x \leq 1$:

$$\phi(x) = \int_x^\infty d\xi\ \frac{d\Gamma^{\rm pert}(\xi)}{d\xi} = \Gamma^{\rm pert}\left[1 - \frac{C_F\alpha_s}{\pi}\int_0^x d\xi\ f_1(\xi)\theta(1-\xi)\right], \tag{63}$$

which, by definition, is such that:

$$-\frac{d\phi(x)}{dx} = \frac{d\Gamma^{\rm pert}(x)}{dx}. \tag{64}$$

After a shift of the integration variable ($\kappa = \xi - t$), eq. (53) can be re-written as:

$$\frac{1}{m_b}\frac{d\Gamma}{d\xi} = -\int_{-\infty}^{+\infty} dt\ F(\xi - t)\ \frac{d\phi(t)}{dt}. \tag{65}$$

We then integrate the latter by parts. All surface terms vanish and the final expression for the convolution reads:

$$\frac{1}{m_b\Gamma^{\rm pert}}\frac{d\Gamma}{d\xi} = -\int_{\xi-\hat{\Lambda}}^{+\infty} dt\ G'(\xi - t)\Big[1 - \frac{C_F\alpha_s}{\pi}\int_0^t dz\ f_1(z)\theta(1-z)\Big], \tag{66}$$

The lower limit of integration over t is fixed by the θ-function in (62). The pre-factor $1/m_b\Gamma^{\rm pert}$ cancels in the normalization:

$$\int_{-\infty}^{1+\hat{\Lambda}} d\xi\ \frac{1}{\Gamma}\frac{d\Gamma}{d\xi} = 1. \tag{67}$$

Note that the domain is now extended to the hadronic kinematics.

0.4.2 $\bar{B} \to X_u e^- \bar{\nu}_e$ in the on-shell scheme

The analog of eq. (61) for each form factor of the $b \to u$ decay is obtained, in the on-shell scheme, from eqs. (13) and (23). It reads:

$$\begin{aligned} W_i(q_0, q^2) &= \frac{m_b^{1+n_i}}{\Delta}\Big[W_i^{TL}(\hat{q}^2) + \frac{C_F\alpha_s}{\pi}\widetilde{V}_i(\hat{q}^2)\Big] G_i\left(\frac{2\hat{q}_0 - 1 - \hat{q}^2}{\Delta}, \hat{q}^2\right) \\ &+ m_b^{1+n_i}\frac{C_F\alpha_s}{\pi}\int_{\frac{2\hat{q}_0-1-\hat{q}^2}{\Delta}}^{\hat{\Lambda}} d\kappa\ G_i(\kappa, \hat{q}^2)\ \widetilde{R}_i\left(\hat{q}_0 - \frac{\Delta}{2}\kappa, \hat{q}^2\right), \end{aligned} \tag{68}$$

where $\Delta = \Delta(\hat{q}^2) = (1 + \hat{\Lambda} - \hat{q}^2)/(1 + \hat{\Lambda})$, as defined in (16).

Equivalently, one can employ the second method of convolution, namely the one involving the derivative of the shape function, which yields:

$$\begin{aligned} W_i(q_0, q^2) &= -\frac{4m_b^{1+n_i}}{\Delta^2} \int_{\hat{q}_0 - \frac{\Delta\hat{\Lambda}}{2}}^{+\infty} dr\ G_i'\left[\frac{2(\hat{q}_0 - r)}{\Delta}, \hat{q}^2\right] \\ &\times \left[K_i(\hat{q}^2) - \frac{C_F\alpha_s}{\pi}\int_{\sqrt{\hat{q}^2}}^{r} dz\ \widetilde{R}_i(z, \hat{q}^2)\ \theta(1 + \hat{q}^2 - 2z)\right], \end{aligned} \tag{69}$$

where $r = \hat{q}_0 - \Delta\ \kappa/2$ and:

$$K_i(\hat{q}^2) = \int_{\sqrt{\hat{q}^2}}^{+\infty} dz\ W_i^{\text{pert}}(z, \hat{q}^2). \tag{70}$$

Note that the support of the form factors goes down to $-\infty$, but in eqs. (69) and (70) we chose, for convenience, $\sqrt{\hat{q}^2}$ as the lower limit of integration. This seems to be the most reasonable value, as it is the minimal kinematical value allowed for the energy of the W boson, but it is important to stress that it is just a reference value, unsignificant in the subtraction of (69). Using the normalization of eq. (21):

$$\begin{aligned} K_1(\hat{q}^2) &= \frac{(1-\hat{q}^2)}{2} + \frac{C_F\alpha_s}{\pi}\Big\{\frac{1}{24\hat{q}^2}\Big[\left(6+\pi^2\right)\hat{q}^4 - 12(\hat{q}^2)^{3/2} \\ &+ 12(3\hat{q}^2 - 1)\mathrm{Li}_2\left(\sqrt{\hat{q}^2}\right)\hat{q}^2 \\ &- 6(\hat{q}^2+1)\mathrm{Li}_2(\hat{q}^2)\hat{q}^2 - 3\left(-12+\pi^2\right)\hat{q}^2 - 30\sqrt{\hat{q}^2} \\ &+ 6\left(4\hat{q}^4 + \hat{q}^2 - 5\right)\log\left(1 - \sqrt{\hat{q}^2}\right) \\ &- 12\left(\hat{q}^4 + \hat{q}^2 - 2\right)\log(1-\hat{q}^2)\Big]\Big\}, \end{aligned} \tag{71}$$

$$\begin{aligned} K_2(\hat{q}^2) &= 2 + \frac{C_F\alpha_s}{\pi}\Big\{\frac{1}{12}\Big[\hat{q}^2 + 8\sqrt{\hat{q}^2} - 24\log\left(1-\sqrt{\hat{q}^2}\right) \\ &-48\mathrm{Li}_2\left(\sqrt{\hat{q}^2}\right) - 4\pi^2 + 27\Big]\Big\}, \\ K_3(\hat{q}^2) &= 1 + \frac{C_F\alpha_s}{\pi}\Big\{\frac{1}{12\hat{q}^2}\Big[\hat{q}^2\left(-18\log\left(1-\sqrt{\hat{q}^2}\right) + 6\sqrt{\hat{q}^2} - \pi^2 + 3\right) \\ &+ 6(\hat{q}^2-1)\log(1-\hat{q}^2) + 6\hat{q}^2\left(\mathrm{Li}_2(\hat{q}^2) - 6\mathrm{Li}_2\left(\sqrt{\hat{q}^2}\right)\right)\Big]\Big\}. \end{aligned} \tag{72}$$

An important difference from the case of the radative decay is that now we want each form factor to have the exact normalization, thus we also need a constraint on the zeroth moment of the shape function (namely $n = 0$ in eq. (20)) and all pre-factors in eqs. (68) and (69) must be properly taken into account.

0.4.3 $\bar{B} \to X_u e^- \bar{\nu}_e$ in the kinetic scheme

The procedure for the convolution is exactly the same, provided the perturbative form factors in (23) are substituted with those of (33). The analogue of (68) is slighlty more complicated:

$$
\begin{aligned}
W_i(q_0, q^2) &= \frac{m_b^{1+n_i}}{\Delta}\left[W_i^{TL}(\hat{q}^2) + \frac{C_F\alpha_s}{\pi}V_i(\hat{q}^2,\eta)\right] G_i\left(\frac{2\hat{q}_0 - 1 - \hat{q}^2}{\Delta}, \hat{q}^2\right) \\
&- \frac{m_b^{1+n_i}}{\Delta^2}\frac{C_F\alpha_s}{\pi}\, B_{1,i}(\hat{q}^2,\eta)\; G_i'\left(\frac{2\hat{q}_0 - 1 - \hat{q}^2}{\Delta}, \hat{q}^2\right) \\
&+ m_b^{1+n_i}\frac{C_F\alpha_s}{\pi}\int_{\frac{2}{\Delta}(\hat{q}_0 - s)}^{\hat{\Lambda}} d\kappa\; G_i(\kappa, \hat{q}^2)\; \widetilde{R}_i\left(\hat{q}_0 - \frac{\Delta}{2}\kappa, \hat{q}^2\right)\; \theta\left[\frac{\Delta\hat{\Lambda}}{2} + s - \hat{q}_0\right] \\
&+ m_b^{1+n_i}\frac{C_F\alpha_s}{\pi}\int_{\frac{2\hat{q}_0 - 1 - \hat{q}^2}{\Delta}}^{min\left(\hat{\Lambda}, \frac{2}{\Delta}(\hat{q}_0 - s)\right)} d\kappa\; G_i(\kappa, \hat{q}^2)\; R_i^{cut}\left(\hat{q}_0 - \frac{\Delta}{2}\kappa, \hat{q}^2, \eta\right). \quad (73)
\end{aligned}
$$

Moreover, expression (69) is modified to:

$$
\begin{aligned}
W_i(q_0, q^2) = &-\frac{4m_b^{1+n_i}}{\Delta^2}\int_{\hat{q}_0 - \frac{\Delta\hat{\Lambda}}{2}}^{+\infty} dr\; G_i'\left[\frac{2(\hat{q}_0 - r)}{\Delta}, \hat{q}^2\right] \times \\
&\Big\{K_i(\hat{q}^2,\eta) - \frac{C_F\alpha_s}{\pi}\int_{\sqrt{\hat{q}^2}}^{min(r,s)} dz\; \widetilde{R}_i(z, \hat{q}^2)\; \theta(1 + \hat{q}^2 - 2z) \\
&- \frac{C_F\alpha_s}{\pi}\int_s^r dz\; R_i^{cut}(z, \hat{q}^2, \eta)\; \theta(r - s)\; \theta(1 + \hat{q}^2 - 2z)\Big\} \\
&- \frac{m_b^{1+n_i}}{\Delta^2}\frac{C_F\alpha_s}{\pi}\, B_{1,i}(\hat{q}^2,\eta)\; G_i'\left(\frac{2\hat{q}_0 - 1 - \hat{q}^2}{\Delta}, \hat{q}^2\right), \quad (74)
\end{aligned}
$$

Both (73) and (74) are valid for $\hat{q}^2 < 1 - 2\eta$. For higher $\hat{q}^2$ one has to employ the corresponding expressions derived in the on-shell scheme, but still adding the term proportional to $B_{1,i}(\hat{q}^2, \eta)$.

The cutoff-dependence of $K_i(\hat{q}^2, \eta)$ can again be derived, to the third order in η, from power corrections imposing the zeroth-moment to be cutoff-independent, namely:

$$
K_i(\hat{q}^2, \eta) = K_i(\hat{q}^2) + \frac{C_F\alpha_s}{\pi}S_i(\hat{q}^2, \eta). \quad (75)
$$

Appendix A

Perturbative corrections with a Wilsonian scale separation

Real emission form factors with a Wilsonian cutoff at NLO, introduced in eq. (35), read, for $\hat{q}_0 > 1/2 - \eta + \hat{q}^2/(2-4\eta)$:

$$
\begin{aligned}
R_1^{cut}(\hat{q}_0, \hat{q}^2, \eta) &= \frac{(\hat{q}_0 - 1)\left(2\hat{q}_0^2 + (\hat{q}^2+3)\hat{q}_0 - 5\hat{q}^2 - 1\right)}{8\left(\hat{q}_0^2 - \hat{q}^2\right)^{3/2}\eta} \\
&+ \left[\frac{2(\hat{q}_0-1)^2}{\hat{u}_+} + \frac{(\hat{q}_0+5)(\hat{q}_0-1)^3}{8\,(\hat{q}_0^2-\hat{q}^2)} - \frac{\hat{q}_0^2}{8} + \frac{\hat{q}_0}{4} + \frac{\hat{q}^2}{8} - \frac{1}{4}\right] \frac{\ln \frac{1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2}}{2\,\eta}}{\sqrt{\hat{q}_0^2-\hat{q}^2}} \\
&+ \frac{5-\hat{q}_0}{8} + \frac{-\hat{q}_0^3 - 3\hat{q}_0^2 + 9\hat{q}_0 - 5}{8\left(\hat{q}^2 - \hat{q}_0^2\right)} + \frac{7(\hat{q}_0-1)}{4\hat{u}_+} - \frac{(\hat{q}^2-1)\eta^2}{4\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}} \\
&- \frac{\left(2\hat{q}_0^2 + (\hat{q}^2-3)\hat{q}_0 - 2\hat{q}^2 + 2\right)\eta}{2\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}} \\
&+ \frac{2\hat{q}_0^3 - (3\hat{q}^2+1)\hat{q}_0^2 + (4\hat{q}^2-6)\hat{q}_0 + \hat{q}^2 + 3}{16\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}} \\
&+ \frac{\hat{u}\,(2+\eta)}{16\eta\sqrt{\hat{q}_0^2-\hat{q}^2}} + \frac{(\hat{q}_0-1)^2 - 2\eta^2 - 8(\hat{q}_0-1)\eta}{4\sqrt{\hat{q}_0^2-\hat{q}^2}\,\hat{u}_+}
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
R_2^{cut}(\hat{q}_0,\hat{q}^2,\eta) &= \frac{\ln\frac{1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2}}{2\,\eta}}{\sqrt{\hat{q}_0^2-\hat{q}^2}}\Big[\frac{4-4\hat{q}_0}{\hat{u}_+}+\frac{4-4\hat{q}_0^2-2\hat{q}_0-\hat{q}^2}{8} \\
&+ \frac{-7\hat{q}_0^4-8\hat{q}_0^3+56\hat{q}_0^2-46\hat{q}_0+5}{8\left(\hat{q}^2-\hat{q}_0^2\right)}-\frac{3\left(\hat{q}_0^6+2\hat{q}_0^5-12\hat{q}_0^4+14\hat{q}_0^3-5\hat{q}_0^2\right)}{8\left(\hat{q}^2-\hat{q}_0^2\right)^2}\Big] \\
&- \frac{3\hat{q}_0+1}{8}+\frac{-6\hat{q}_0^3-10\hat{q}_0^2+41\hat{q}_0-5}{8\left(\hat{q}^2-\hat{q}_0^2\right)}-\frac{7}{2\hat{u}_+}-\frac{3\left(\hat{q}_0^2-2\hat{q}_0+1\right)\left(\hat{q}_0^3+5\hat{q}_0^2\right)}{8\left(\hat{q}^2-\hat{q}_0^2\right)^2} \\
&+ \frac{\left(2\hat{q}_0^3+2\hat{q}_0^2+\hat{q}^2(3\hat{q}^2-5)\hat{q}_0-3\hat{q}^4+\hat{q}^2\right)\eta^2}{4(\hat{q}_0-1)\left(\hat{q}_0^2-\hat{q}^2\right)^{5/2}} \\
&+ \frac{\left(4\hat{q}_0^3+2(\hat{q}^2+2)\hat{q}_0^2+\hat{q}^2(3\hat{q}^2-13)\hat{q}_0-2(\hat{q}^2-1)\hat{q}^2\right)\eta}{2\left(\hat{q}_0^2-\hat{q}^2\right)^{5/2}} \\
&+ \frac{6\hat{q}_0^5-(3\hat{q}^2+7)\hat{q}_0^4+(8-12\hat{q}^2)\hat{q}_0^3+6\left(2\hat{q}^4+\hat{q}^2-1\right)\hat{q}_0^2-2\hat{q}^2(6\hat{q}^2-5)\hat{q}_0}{16\left(\hat{q}_0^2-\hat{q}^2\right)^{5/2}} \\
&+ \frac{(\hat{q}^2-3)\hat{q}^2}{16\left(\hat{q}_0^2-\hat{q}^2\right)^{5/2}}+\frac{\hat{u}\,(3\eta-2)}{16\eta\sqrt{\hat{q}_0^2-\hat{q}^2}}+\frac{\frac{\eta^2}{(\hat{q}_0-1)}+4\eta+\frac{1-\hat{q}_0}{2}}{\sqrt{\hat{q}_0^2-\hat{q}^2}\,\hat{u}_+}-\frac{\hat{q}^2(9\hat{q}^2+1)}{8\left(\hat{q}_0^2-\hat{q}^2\right)^{5/2}\eta} \\
&+ \frac{-2\hat{q}_0^5+(\hat{q}^2+1)\hat{q}_0^4+4\hat{q}^2\hat{q}_0^3-2\left(2\hat{q}^4+5\hat{q}^2+1\right)\hat{q}_0^2+2\hat{q}^2(5\hat{q}^2+6)\hat{q}_0}{8\left(\hat{q}_0^2-\hat{q}^2\right)^{5/2}\eta}
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
R_3^{cut}(\hat{q}_0,\hat{q}^2,\eta) &= \left[\frac{2-2\hat{q}_0}{\hat{u}_+}-\frac{\hat{q}_0(2\hat{q}_0+\hat{q}^2-3)}{4\left(\hat{q}_0^2-\hat{q}^2\right)}\right]\frac{\ln\frac{1-\hat{q}_0+\sqrt{\hat{q}_0^2-\hat{q}^2}}{2\,\eta}}{\sqrt{\hat{q}_0^2-\hat{q}^2}} \\
&+ \frac{\hat{q}_0^2+3\hat{q}_0}{4\left(\hat{q}^2-\hat{q}_0^2\right)}-\frac{7}{4\hat{u}_+}+\frac{1}{4}+\frac{\hat{q}_0\eta^2}{2(\hat{q}_0-1)\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}}+\frac{(2\hat{q}_0+\hat{q}^2)\eta}{2\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}} \\
&+ \frac{\frac{\eta^2}{2(\hat{q}_0-1)}+2\eta+\frac{1-\hat{q}_0}{4}}{\sqrt{\hat{q}_0^2-\hat{q}^2}\;\hat{u}_+}-\frac{(2\hat{q}_0-1)(\hat{q}_0-\hat{q}^2)}{8\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}}-\frac{\hat{q}^2\hat{q}_0+\hat{q}_0-2\hat{q}^2}{4\left(\hat{q}_0^2-\hat{q}^2\right)^{3/2}\eta},
\end{aligned}
\tag{A.3}
$$

where $\hat{u}=1-2\hat{q}_0+\hat{q}^2$.

Appendix B

Form factors in the HQE and their q_0-moments

In the adopted normalization, power corrections to the form factors read:

$$
\begin{aligned}
W_1^{power}(\hat{q}_0,\hat{q}^2) &= \frac{\mu_G^2}{3m_b^2}\Big\{2\delta_1\left(2\hat{q}^2-5\hat{q}_0^2+7\hat{q}_0-4\right)-\delta_0\Big\}\\
&+\frac{\mu_\pi^2}{3m_b^2}\Big\{\delta_0-4\delta_2\left(\hat{q}_0-1\right)\left(\hat{q}_0^2-\hat{q}^2\right)+2\delta_1\left(\hat{q}_0\left(5\hat{q}_0-3\right)-2\hat{q}^2\right)\Big\}\\
&+\frac{\rho_D^3}{9m_b^3}\Big\{-3\delta_0+6\delta_1\left(-\hat{q}^2+\left(\hat{q}_0-1\right)\hat{q}_0-2\right)\\
&\quad+\ 4\left(3\delta_2-2\delta_3\left(\hat{q}_0-1\right)\right)\left(\hat{q}_0-1\right)\left(\hat{q}_0^2-\hat{q}^2\right)\Big\}\\
&+\frac{\rho_{LS}^3}{3m_b^3}\Big\{-\delta_0+2\delta_1\left(-\hat{q}^2+\left(\hat{q}_0-1\right)\hat{q}_0+2\right)+4\delta_2\left(\hat{q}_0-1\right)\left(\hat{q}_0^2-\hat{q}^2\right)\Big\}
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
W_2^{power}(\hat{q}_0,\hat{q}^2) &= \frac{2\mu_G^2}{3m_b^2}\Big\{2\delta_1\left(5\hat{q}_0-2\right)-5\delta_0\Big\}\\
&+\frac{2\mu_\pi^2}{3m_b^2}\Big\{5\delta_0-14\delta_1\hat{q}_0+4\delta_2\left(\hat{q}_0^2-\hat{q}^2\right)\Big\}\\
&+\frac{2\rho_D^3}{9m_b^3}\Big\{3\delta_0+6\delta_1\left(\hat{q}_0-4\right)+8\left(\hat{q}_0-1\right)\left(\delta_3\left(\hat{q}_0^2-\hat{q}^2\right)-3\delta_2\hat{q}_0\right)\Big\}\\
&+\frac{2\rho_{LS}^3}{3m_b^3}\Big\{\delta_0+2\delta_1\left(\hat{q}_0-2\right)-2\delta_2\left(\hat{q}^2+2\left(\hat{q}_0-1\right)\hat{q}_0\right)\Big\}
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
W_3^{power}(\hat{q}_0,\hat{q}^2) &= \frac{2\mu_G^2}{3m_b^2}\delta_1\left(5\hat{q}_0-6\right) \\
&+ \frac{2\mu_\pi^2}{3m_b^2}\left\{2\delta_2\left(\hat{q}_0^2-\hat{q}^2\right)-5\delta_1\hat{q}_0\right\} \\
&+ \frac{2\rho_D^3}{9m_b^3}\left\{2\left(\hat{q}_0-1\right)\left(2\delta_3\left(\hat{q}_0^2-\hat{q}^2\right)-3\delta_2\hat{q}_0\right)-3\delta_1\hat{q}_0\right\} \\
&- \frac{2\rho_{LS}^3}{3m_b^3}\left\{2\delta_2\left(\hat{q}_0-1\right)^2+\delta_1\hat{q}_0\right\}
\end{aligned}
\tag{B.3}
$$

We denoted by $\delta_n = \delta^{(n)}(1+\hat{q}^2-2\hat{q}_0)$, namely the n-th derivative of the Dirac delta w.r.t. its argument. We give explicit expressions for the zeroth, first and second q_0-moments, at fixed q^2, of the three form factors, up to $\mathcal{O}(1/m_b^3)$ corrections in the HQE:

$$
M_i^{(j)}(q^2) = \int dq_0\ q_0^j\ W_i^{\mathrm{HQE}}(q_0,q^2), \tag{B.4}
$$

with:

$$
W_i^{\mathrm{HQE}}(q_0,q^2) = m_b^{n_i}\left[W_i^{TL}(\hat{q}^2)+W_i^{power}(\hat{q}_0,\hat{q}^2)\right]. \tag{B.5}
$$

For convenience, we separate the tree-level and power-corrections contributions:

$$
M_i^{(j)}(q^2) = M_i^{(j),TL}(q^2) + M_i^{(j),power}(q^2) \tag{B.6}
$$

Functions I's defined in Sec. 0.1 are linear combinations of such moments:

$$
\begin{aligned}
I_i^{(0),bare} &= M_i^{(0),TL}, \\
I_i^{(1),\mathrm{HQE}} &= M_i^{(1)} - \frac{1+\hat{q}^2}{2}\ M_i^{(0)} \\
I_i^{(2),\mathrm{HQE}} &= M_i^{(2)} - \left(1+\hat{q}^2\right)\ M_i^{(1)} + \left(\frac{1+\hat{q}^2}{2}\right)^2\ M_i^{(0)}
\end{aligned}
\tag{B.7}
$$

Zeroth moments:

$$
M_1^{(0),TL} = \frac{\left(1-\hat{q}^2\right)}{2},\quad M_2^{(0),TL} = 2,\quad M_3^{(0),TL} = 1/m_b. \tag{B.8}
$$

$$
\begin{aligned}
M_1^{(0),power}(\hat{q}^2) &= \frac{\left(1-5\hat{q}^2\right)\mu_G^2}{6m_b^2} + \frac{\left(\hat{q}^2+1\right)\mu_\pi^2}{3m_b^2} + \frac{2\hat{q}^2\rho_{LS}^3}{3m_b^3}, \\
M_2^{(0),power}(\hat{q}^2) &= 0, \\
M_3^{(0),power}(\hat{q}^2) &= \frac{5\mu_G^2}{6m_b^3} - \frac{\mu_\pi^2}{2m_b^3} - \frac{\rho_D^3}{6m_b^4} - \frac{\rho_{LS}^3}{2m_b^4}.
\end{aligned}
\tag{B.9}
$$

First moments:

$$\begin{aligned}
M_1^{(1),TL} &= -\frac{m_b}{24}\left(6\hat{q}^4-6\right), \\
M_2^{(1),TL} &= \frac{m_b}{6}\left(6\hat{q}^2+6\right), \\
M_3^{(1),TL} &= \frac{1}{2}\left(\hat{q}^2+1\right), \qquad \text{(B.10)} \\
M_1^{(1),power}(\hat{q}^2) &= -\frac{\left(15\hat{q}^4-4\hat{q}^2+5\right)\mu_G^2}{24m_b}-\frac{\left(-3\hat{q}^4-8\hat{q}^2-5\right)\mu_\pi^2}{24m_b} \\
&\quad -\frac{\left(5\hat{q}^4+7\right)\rho_D^3}{24m_b^2}-\frac{\left(-15\hat{q}^4-5\right)\rho_{LS}^3}{24m_b^2}, \\
M_2^{(1),power}(\hat{q}^2) &= \frac{\left(5\hat{q}^2+1\right)\mu_G^2}{6m_b}-\frac{\left(\hat{q}^2+1\right)\mu_\pi^2}{2m_b} \\
&\quad +\frac{\left(-\hat{q}^2-5\right)\rho_D^3}{6m_b^2}+\frac{\left(-3\hat{q}^2-3\right)\rho_{LS}^3}{6m_b^2}, \\
M_3^{(1),power}(\hat{q}^2) &= -\frac{\left(1-5\hat{q}^2\right)\mu_G^2}{6m_b^2}-\frac{\left(\hat{q}^2+1\right)\mu_\pi^2}{3m_b^2}-\frac{2\hat{q}^2\rho_{LS}^3}{3m_b^3}.
\end{aligned}$$

(B.11)

Second moments:

$$\begin{aligned}
M_1^{(2),TL} &= -\frac{m_b^2}{8}\left(\hat{q}^2-1\right)\left(\hat{q}^2+1\right)^2, \\
M_2^{(2),TL} &= \frac{m_b^2}{2}\left(\hat{q}^2+1\right)^2, \\
M_3^{(2),TL} &= \frac{m_b}{4}\left(\hat{q}^2+1\right)^2, \qquad \text{(B.12)} \\
M_1^{(2),power}(\hat{q}^2) &= -\frac{1}{12}\left(5\hat{q}^6+\hat{q}^4-\hat{q}^2+3\right)\mu_G^2 \\
&\quad +\frac{1}{6}\left(2\hat{q}^4+\hat{q}^2+1\right)\mu_\pi^2-\frac{\left(\hat{q}^6+\hat{q}^2+1\right)\rho_D^3}{3m_b} \\
&\quad +\frac{\left(3\hat{q}^6+\hat{q}^4+\hat{q}^2+1\right)\rho_{LS}^3}{6m_b}, \\
M_2^{(2),power}(\hat{q}^2) &= \frac{1}{6}\left(5\hat{q}^4+6\hat{q}^2+1\right)\mu_G^2-\frac{1}{3}\left(\hat{q}^4+4\hat{q}^2+1\right)\mu_\pi^2 \\
&\quad +\frac{\left(-8\hat{q}^2-4\right)\rho_D^3}{6m_b}+\frac{\left(-4\hat{q}^4-8\hat{q}^2-2\right)\rho_{LS}^3}{6m_b}, \\
M_3^{(2),power}(\hat{q}^2) &= \frac{\left(5\hat{q}^4+2\hat{q}^2-3\right)\mu_G^2}{8m_b}-\frac{\left(3\hat{q}^4+14\hat{q}^2+3\right)\mu_\pi^2}{24m_b} \\
&\quad +\frac{\left(5\hat{q}^4-2\hat{q}^2+1\right)\rho_D^3}{24m_b^2}+\frac{\left(-15\hat{q}^4-6\hat{q}^2+1\right)\rho_{LS}^3}{24m_b^2}.
\end{aligned}$$

(B.13)