# Numerical Methods for Partial Differential Equations

Lecture 1: Introduction & Basic Concepts

# Course Outline

- Basic Concepts
- Classification: characteristic curves, Elliptic PDE, parabolic PDE, Hyperbolic PDE;
- Examples of Analytical solutions (wave eq., diffusion eq.,);
- Introduction to Finite Difference Methods;
- Elliptic PDEs: Laplace & Poisson Eq. , Gauss-Seidel, Jacobi, SOR;
- Parabolic PDEs: heat equation, explicit methods, Implicit Methods, ADI;
- Hyperbolic PDEs:
  - simple advection, systems and nonlinear equations
  - Shock, rarefactions
  - Godunov Methods Equazioni nonlineari (fluidodinamica) , metodi di Godunov

# Partial Differential Equations

- Why Partial Differential Equations ? → Physics phenomena modelled through PDE:
  - Electromagnetism → Maxwell Equations
  - Quantum Mechanics → Schrodinger Equation
  - Gravitation → Einstein Equations
  - Fluid Dynamics → Navier-Stokes Equations
  - Plasma → Vlasov Equation
- Why Numerical Methods →
- In most cases the only mean of calculating and understanding their solutions is through the design of sophisticated numerical approximation schemes. However the analytical and numerical approaches to the subject are intertwined. One cannot make progresses on the numerical solution without understanding the analytical properties.

# What is a PDE ?

- We have an unknown function $u(x_1, x_2, x_3, \ldots)$ of several independent variables $x_1, x_2, x_3, \ldots$.
- A PDE is an identity that relates the independent variables, the unknown function u and its partial derivatives:

$$F\left(x_1, x_2, x_3, \ldots\ldots, u(x_1, x_2, x_3, \ldots), \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \ldots.\right) = 0$$

- $x_1, x_2, x_3, \ldots$ are often space variables and a solution may be required in some domain $\Omega$ of space. In this case there will be some conditions to be satisfied at the boundary of the domain → boundary conditions;
- One of the independent variables may be time, in this case we have to specify the unknown function u at t=0 everywhere in $\Omega$ → initial conditions;
- We will see that the type of boundary conditions to be prescribed depend on the type of equation.
- We may also have to consider systems of PDE involving more than one unknown functions:

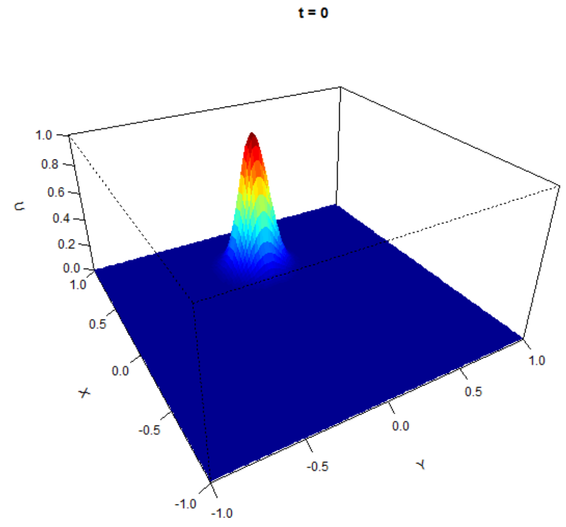
# What is a PDE ?

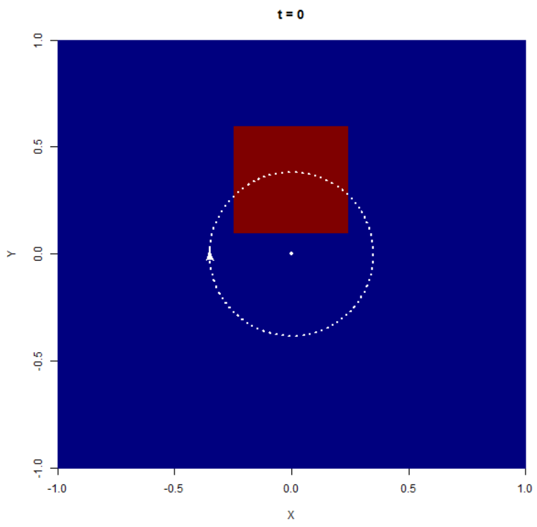
- Boundary Conditions:
  - Dirichlet conditions → u specified on the boundary
  - Neumann conditions → normal derivative $\frac{\partial u}{\partial n}$ specified on the boundary
  - Robin conditions → $\frac{\partial u}{\partial n} + au$ specified on the boundary
  - Cauchy conditions → specify both u and $\frac{\partial u}{\partial t}$ at the boundary
  - If the domain is unbounded B.C. at infinity

- Well Posed Problems: PDE in a domain with a set of initial and/or boundary conditions such that:
  - A solution exists and it is solution is unique
  - The solution depends in a continuous way on the initial/boundary conditions
  - The initial/boundary conditions may not be enough the determine a unique solution, the problem is underdetermined.
  - The initial/boundary conditions may be too many, a solution cannot exist, the problem is overdetermined.
  - Example: for the Laplace equation we cannot specify both the function and its normal derivative on the boundary (in this case the problem is overdetermined).

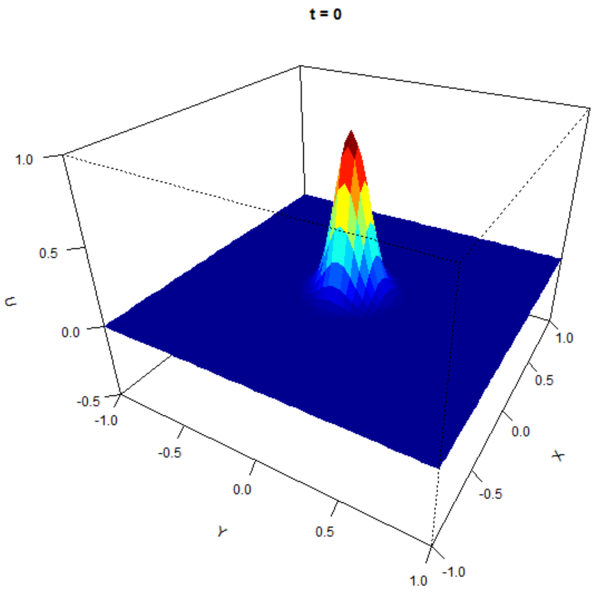
# PDE Examples:

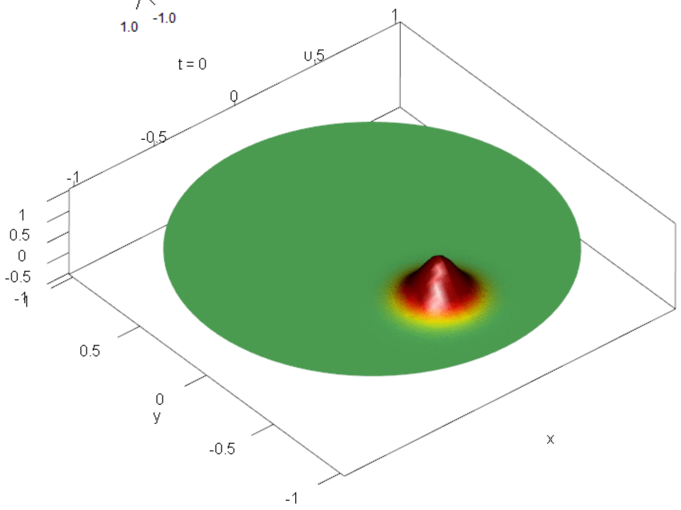
### *Advection equation in 2D*



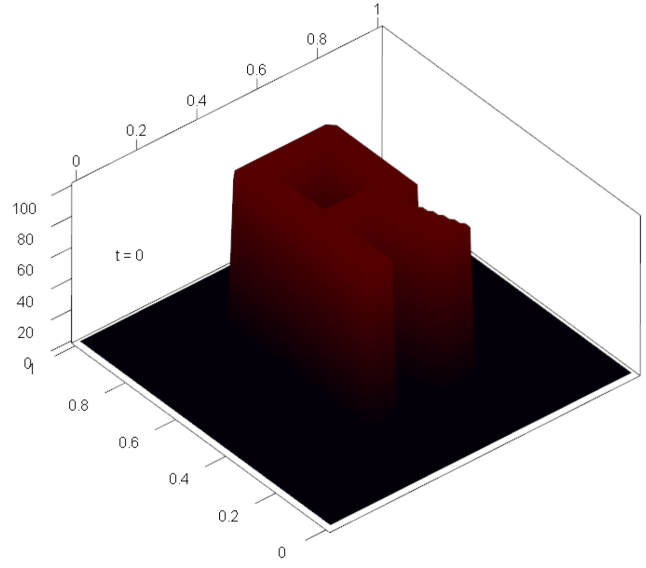




### *Wave equation in 2D*




### *Diffusion equation*



# Linear PDE

- A *linear* PDE is one in which the equations and any boundary or initial conditions do not include any product of the unknown functions or their derivatives.
- PRINCIPLE OF SUPERPOSITION: a linear equation has the useful property that if $u_1$ and $u_2$ both satisfy the equation then so does any linear combination of $u_1$ and $u_2$.
- This is often used in constructing solutions to linear equations (for example for satisfying boundary or initial conditions).
- Examples:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$ Advection equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$ Wave equation $$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0$$

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$$ Diffusion or heat conduction equation $$\frac{\partial^2 u}{\partial t^2} - k\nabla^2 u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \nabla^2 u = 0$$ Laplace equation $$\nabla^2 u = \rho$$ Poisson equation

# Linear PDE: Maxwell Eq.

- Another important example are Maxwell's equations, assuming a linear relation for Ohm's law:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$$

# Nonlinear PDE

- Nonlinear PDE arise when at least terms of order 2 (or higher) are present:
- Incompressible hydrodynamic or magnetohydrodynamics:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \boldsymbol{B} \cdot \nabla \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

non linear terms

- Burger's eq. $$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

# Characteristic Curves

# Characteristic Curves

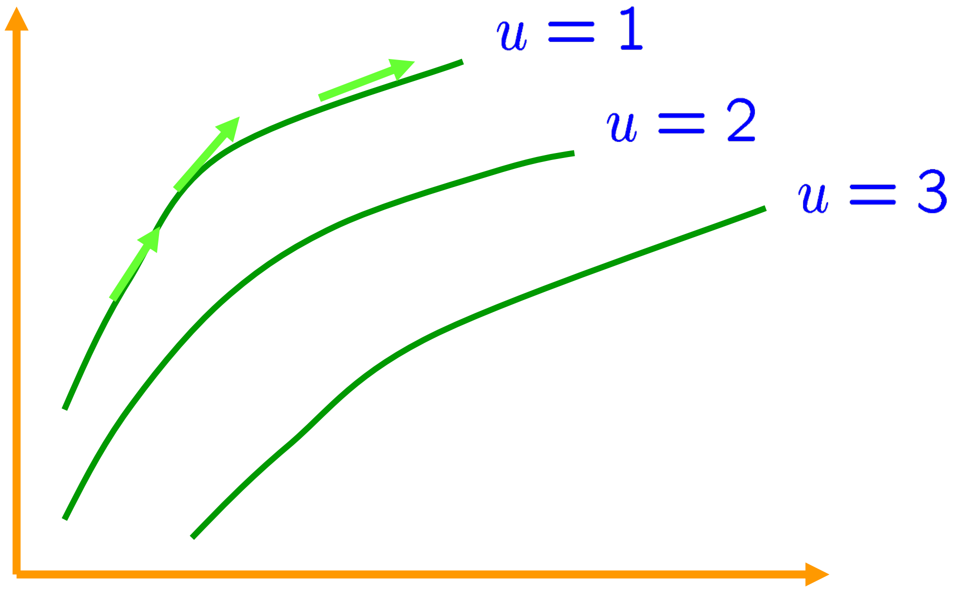
- Let's start from the linear advection equation, $$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$
- The previous equation is defined on $-\infty < x < \infty$
- For constant speed a, the solution to the previous equation is: $u(x,t) = u_0(x - at)$, where $u_0(x) - at)$ is the initial condition.
- → Initial conditions propagates at constant velocity a
- On the lines x-at = const we have the condition: $$\frac{du}{dt} = 0$$

# Characteristic Curves

- More generally, $$b(u,x,t)\frac{\partial u}{\partial t} + c(u,x,t)\frac{\partial u}{\partial x} = 0$$

  divide by b(u,x,t) → $$\frac{\partial u}{\partial t} + a(x,u,t)\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = (1,a)\cdot\left(\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}\right)$$

- The equation becomes nonlinear if a depends on u (we can still call it 'quasi-linear').
- In the (x,t) plane, it represents the derivative in the direction (1,a): the solution u(x,t) is constant in the (x,t) plane along the curves tangent to this direction !




- These curves are identified by $$\frac{dx}{dt} = a$$
- And the PDE has been reduced to a system of ODE

$$u = const \quad \text{for} \quad \frac{dx}{dt} = a$$

# An Example

- Consider the PDE $x\dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x}=0 \;\; with \;\; x \geq 0, t \geq 0$

- With IC and BC defined by $u(x,0)=f(x), \qquad u(0,t)=g(t)$

- Rewrite in characteristic form: $u = const \;\; \text{for} \;\; \dfrac{dx}{dt}=\dfrac{1}{x} \;\; \rightarrow t=\dfrac{x^2}{2}+const$

- The value of u on each characteristic should be determined from the initial or boundary conditions:

$$t \leq \frac{x^2}{2} \;\; \rightarrow characteristic\ through\ positive\ x\ axis$$
$$t \geq \frac{x^2}{2} \;\; \rightarrow characteristic\ through\ negative\ x\ axis$$

- Consider a generic point ($x_1$,$t_1$): the CC passing through this point is: $t-t_1=(x^2-x_1^2)/2$

# An Example

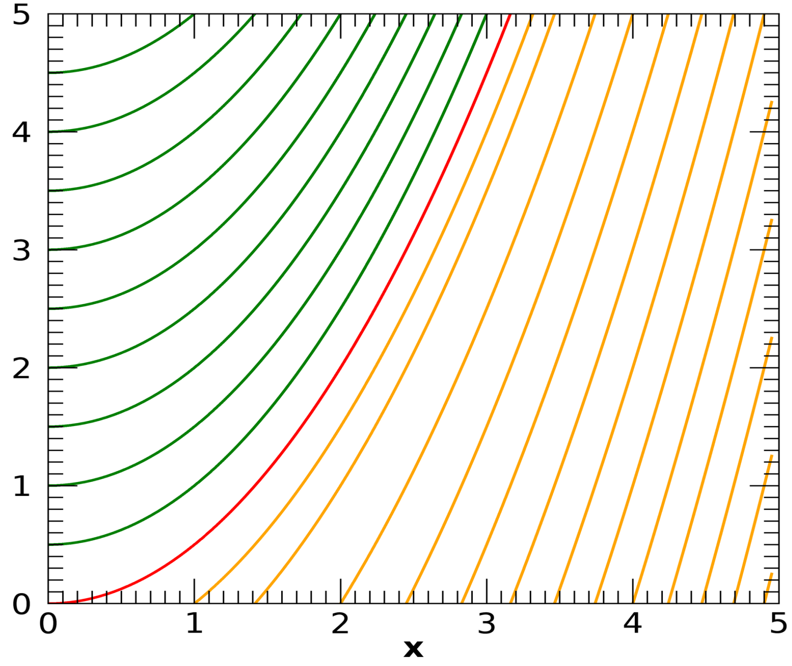
- This curve will intersect the positive x-axis or t-axis depending on the previous conditions;
- In the 1st case, the value of u on the CC will be set by the IC, in the 2nd case it will be fixed by the B.C.

- Intersections will be given by: $x_0 = \sqrt{x_1^2 - 2t_1}$ or $t_0 = t_1 - x_1^2/2$

- And therefore we will have

$$u(x_1, t_1) = f(x_0) = f\left(\sqrt{x_1^2 - 2t_1}\right) \quad (1^{st}\ case)$$

$$u(x_1, t_1) = b(t_0) = g(t_1 - x_1^2/2) \quad (2^{nd}\ case)$$



# PDE: Classification

# Classification of 2° order PDE

- We start from the general 2° order PDE: $Au_{xx} + Bu_{xy} + Cu_{yy} = \Phi(x, y, u, u_x, u_y)$

- Three classes of PDE depending on the sign of $\Delta(x_0, y_0) = \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B^2 - 4AC$

$$\begin{aligned} \Delta > 0 \quad &\rightarrow \textit{Hyperbolic} \\ \Delta = 0 \quad &\rightarrow \textit{Parabolic} \\ \Delta < 0 \quad &\rightarrow \textit{Elliptic} \end{aligned}$$

- Other PDE can be reduced to canonical form by variable transformation. The new variables are coordinates along the characteristic curves.

- Important: the nature of a PDE is local !

- Example: given the PDE (Tricomi Eq.) $\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} = 0$ it is: $\begin{cases} \text{Hyperbolic if } & x<0 \\ \text{Parabolic if } & x=0 \\ \text{Elliptic if } & x>0 \end{cases}$

# Classification of 2° order PDE

- From the physical point of view, these PDEs respectively represents the wave propagation, the time-dependent diffusion processes, and the steady state or equilibrium processes.
- Hyperbolic equations model the transport of some physical quantity, such as fluids or waves.
- Parabolic problems describe evolutionary phenomena that lead to a steady state described by an elliptic equation.
- Elliptic equations are associated to a special state of a system, in principle corresponding to the minimum of the energy.

# Reduction to Canonical Form: Change of Variables

- In 2D, introduce the transformation $\xi = \xi(x, y),\ \eta = \eta(x, y)$

- The transformation must be invertible, $J = \dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$

- Now define $w(\xi, \eta) = u\big(x(\xi, \eta), y(\xi, \eta)\big)$
- So that the original PDE becomes $a w_{\xi\xi} + b w_{\xi\eta} + c w_{\eta\eta} = \phi(\xi, \eta, w, w_\xi, w_\eta)$

  where
  $$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
  $$b = 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y$$
  $$c = A\eta_x^2 + B\eta_x\eta_y + C{\eta_y}^2$$

- One can show that the sign of **Δ** is invariant under a variable transformation. The equation maintains its nature: $b^2 - 4ac = J^2\,(B^2 - 4AC)$.

# Change of Variables: Notable cases

- Consider $$Au_{xx} + Bu_{xy} + Cu_{yy} = \Phi(x, y, u, u_x, u_y) \qquad [Original\ PDE]$$
$$\Updownarrow$$
$$aw_{\xi\xi} + bw_{\xi\eta} + cw_{\eta\eta} = \phi(\xi, \eta, w, w_\xi, w_\eta) \qquad [Transformed\ PDE]$$

Coordinate transformation may be chosen to satisfy:

1. $a = c = 0$ → Hyperbolic Eq ($\Delta > 0$) : $w_{\xi\eta} = \phi(\xi, \eta, w, w_\xi, w_\eta)/b$ [1st canonical form];
2. $b = 0,\ c = -a$ → Hyperbolic Eq ($\Delta > 0$): $w_{\xi\xi} - w_{\eta\eta} = \phi(\xi, \eta, w, w_\xi, w_\eta)/a$ [2nd canonical form]
3. $b = 0\ \ and\ \ \{a = 0\ \ or\ \ c = 0\}$ → Parabolic Eq ($\Delta = 0$): $w_{\xi\xi} = \phi(\xi, \eta, w, w_\xi, w_\eta)/c$
4. $b = 0\ \ and\ \ c = a$ → Elliptic Eq ($\Delta < 0$): $w_{\xi\xi} + w_{\eta\eta} = \phi(\xi, \eta, w, w_\xi, w_\eta)/a$

- We will now examine the kind of transformation required to reduce the PDE to its canonical form.

# Change of Variables: Hyperbolic Eq. (Characteristics)

- When *a = c = 0 (case 1)*, characteristics may be found from

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$
$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

$$\longrightarrow \quad A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$
$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

- The two equations have identical distinct roots:

$$\mu_1(x,y) = \frac{\xi_x}{\xi_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$
$$\mu_2(x,y) = \frac{\eta_x}{\eta_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\longrightarrow \quad \xi_x - \mu_1\xi_y = 0$$
$$\eta_x - \mu_2\eta_y = 0$$

- Along the coordinate line,

$$\xi(x,y) = cost. \qquad d\xi = \xi_x dx + \xi_y dy = 0$$

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} \qquad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y} \qquad A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

- Solving for dy/dx yields the characteristic equations:

$$\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A} = \lambda_1(x,y) \qquad \frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A} = \lambda_2(x,y)$$

# Change of Variables: Hyperbolic Eq. (Characteristics)

- If the coefficients A, B, and C are constants, it is easy to integrate the previous equations

$$y = \lambda_1 x + c_1 \quad \text{and} \quad y = \lambda_2 x + c_2$$

- The two families of characteristic curves associated with PDE reduces to two distinct families of parallel straight lines. Since the families of curves ξ = constant and η = constant are the characteristic curves, the change of variables are given by the following equations:

$$\xi = y - \lambda_1 x \quad \text{and} \quad \eta = y - \lambda_2 x$$

# Change of Variables: Parabolic Eq. (Characteristics)

- When $b=0$ and $a=0$ (case 3) $\quad a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \longrightarrow A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$
- One finds $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$

$$\longrightarrow \quad \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = \lambda(x,y)$$

(Since $B^2$-4AC = 0, there's only one root)

- → Only one family of real characteristic curves.
- To determine the second transformation variable ($\eta$), remember that

$$b = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0$$

which after some manipulation becomes $(B^2 - 4AC)\eta_y = 0$, meaning that $\eta_y(x,y)$ is arbitrary.

- For constant coefficients,

$$\xi = y - \frac{B}{2A}x \qquad \longrightarrow \qquad w_{\eta\eta} = \psi(\xi,\eta,w,w_\xi,w_\eta)/c \quad \text{(Canonic Form)}$$

$$\eta = x$$

# Classification of 1st-Order Systems of PDE

- Consider the system of 1st-order PDE in matrix notations,

$$\sum_{k=1}^{m} A^k \frac{\partial U}{\partial x_k} = F$$

- Where $A^k$ is a $n\ X\ n$ matrix of coefficients, $U = [u_1, u_2, \dots, u_n]^T$ is a column vector containing the dependent variables, while $x_1, x_2, \dots, x_m$ are the m independent coordinates.

- For simplicity, let m = 2: we have $(x_1, x_2) = (x, y)$ and $A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F(x, y, U)$

- If $A$ is not singular we may rewrite the system in more convenient form:

$$\frac{\partial U}{\partial x} + A^{-1}B\frac{\partial U}{\partial y} = A^{-1}F(x, y, U) \;\rightarrow\; \frac{\partial U}{\partial x} + D\frac{\partial U}{\partial y} = E(x, y, U)$$

- Just as in the case of a single partial differential equation, the important properties of solutions of the system (48) depend only on its principal part (left hand side).

# Canonical Form and Classification

- Let's consider the trasformation $V = P^{-1}U$. Where P is an arbitrary, non singular nxn matrix.
- Then:

$$\frac{\partial U}{\partial x} = P\frac{\partial V}{\partial x} + \frac{\partial P}{\partial x}V, \qquad \frac{\partial U}{\partial y} = P\frac{\partial V}{\partial y} + \frac{\partial P}{\partial y}V$$

- Our PDE becomes:

$$\frac{\partial V}{\partial x} + P^{-1}DP\frac{\partial V}{\partial y} = H$$

where

$$H = P^{-1}G = P^{-1}\left[E - \left(\frac{\partial P}{\partial x} + D\frac{\partial P}{\partial y}\right)V\right]$$

- If we take P to be matrix of right eigenvectors if D. then we have exactly:

$$P^{-1}DP = \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

- Where $\lambda_k$ are the eigenvalues of $D$.

# Canonical Form and Classification

- In this form, the system of PDE is finally diagonal and thus all Equations uncouple:

$$\frac{\partial V}{\partial x} + \Lambda(x,y)\frac{\partial V}{\partial y} = H(x,y,V)$$

- The classification of the system of first-order PDEs is done based on the nature of the eigenvalues of the matrix $D = A^{-1}B$:
  - If all the n eigenvalues of D are real and distinct the system is called *hyperbolic* type;
  - If all the n eigenvalues of D are complex the system is called *elliptic* type;
  - If some of the n eigenvalues are real and other complex the system is considered as hybrid of elliptic-hyperbolic type.
  - If the rank of matrix D is less than n, i.e., there are less than n real eigenvalues (some of the eigenvalues are repeated) then the system is said to be *parabolic* type.

# Example #1

Classify the single first-order equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f$$

where $a$ and $b$ are real constants.

**Solution** In the standard matrix form the above equation may be written as

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F$$

where

$$A = \left[\, a \,\right], \qquad B = \left[\, b \,\right], \qquad U = \left[\, u \,\right], \qquad F = \left[\, f \,\right]$$

The $D$ matrix, in this case, can be easily found:

$$D = A^{-1}B = \left[\, a^{-1} \,\right]\left[\, b \,\right] = \left[\, b/a \,\right]$$

The matrix $D$ has the eigenvalue, $\lambda = b/a$. This is always real and hence, a single first-order PDE is always hyperbolic in the space $(x, y)$. Note that here we have only a single eigenvalue and thus a characteristic direction.

# Example #2

Classify the following system of first-order equation:

$$a\frac{\partial\phi}{\partial x} + c\frac{\partial\psi}{\partial y} = f_1$$
$$b\frac{\partial\psi}{\partial x} + d\frac{\partial\phi}{\partial y} = f_2$$

The matrix form of the system is:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This equation may be written as

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = F$$

where

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \qquad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Therefore, we have

$$a_{11} = a \qquad a_{12} = 0 \qquad a_{21} = 0 \qquad a_{22} = b$$
$$b_{11} = 0 \qquad b_{12} = c \qquad b_{21} = d \qquad b_{22} = 0$$

The relevant determinants can be now evaluated as

$$|A| = a_{11}a_{22} - a_{12}a_{21} = ab$$
$$|B| = b_{11}b_{22} - b_{12}b_{21} = -cd$$
$$|b| = a_{11}b_{22} - a_{21}b_{12} + a_{22}b_{11} - a_{12}b_{21} = 0$$

and the $D$ matrix is given by

$$D = A^{-1}B = \frac{1}{|A|}\begin{bmatrix} a_{22}b_{11} - a_{12}b_{21} & a_{22}b_{12} - a_{12}b_{22} \\ a_{11}b_{21} - a_{21}b_{11} & a_{11}b_{22} - a_{21}b_{12} \end{bmatrix} = \frac{1}{ab}\begin{bmatrix} 0 & bc \\ ad & 0 \end{bmatrix} = \begin{bmatrix} 0 & c/a \\ d/b & 0 \end{bmatrix}$$

so that the system (52) may be written as

$$\frac{\partial}{\partial x}\begin{bmatrix} \phi \\ \psi \end{bmatrix} + \begin{bmatrix} 0 & c/a \\ d/b & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

The eigenvalues of $D = A^{-1}B$ are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|A||B|}}{2|A|} = \pm\sqrt{\frac{cd}{ab}}$$

# Example #2

If $cd/ab > 0$ then the eigenvalues are real and distinct and the system is hyperbolic in the space $(x,y)$. For instance, $a = b = 1$; $c = d = 1$ with vanishing right-hand side, the system of equation becomes

$$\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} = 0$$
$$\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

By eliminating the variable $\psi$ and replacing $y$ by $t$, we obtain the well-known wave equation in $\phi$:

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$$

which is a hyperbolic equation as seen previously.

# Example #2

If $cd/ab < 0$ then the eigenvalues are complex and the system is elliptic in the space $(x, y)$. For instance, $a = b = 1$; $c = -d = -1$ and vanishing right-hand side, the system of equation becomes the well-known *Cauchy–Riemann equation*:

$$\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} = 0$$
$$\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

By eliminating the variable $\psi$, we obtain the Laplace equation in $\phi$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is the standard form of elliptic equations and describes steady-state diffusion phenomena. Note that we could also obtain the Laplace equation in $\psi$ by eliminating the variable $\phi$.

# Example #2

Finally, if one of the coefficients is equal zero, say $c$, then there is only one real eigenvalue and the system is parabolic. For instance, with $a = -b = 1$, $c = 0$, $d = 1$ and $f_1 = \psi$, $f_2 = 0$, the system of equation becomes

$$\frac{\partial \phi}{\partial x} = \psi$$
$$\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} = 0$$

which on eliminating the variable $\phi$ and replacing $y$ by $t$ leads to the standard form for a parabolic equation:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

This is recognizable by the fact that the equation presents a combination of first and second-order derivatives.

# Example #3

Classify the Euler equations for unsteady, one-dimensional flow:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x}$$

where the speed of sound $c$ is given by the isentropic relation between pressure and density as

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Since $A$ is a unit matrix, the inverse of $A$ is $A$ itself. We now compute the matrix $D$:

$$D = A^{-1}B = B = \begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix}$$

$$\frac{\partial U}{\partial t} + D\frac{\partial U}{\partial x} = 0$$

The eigenvalues of $D = A^{-1}B$ are given by

$$\lambda_{1,2} = \frac{|b| \pm \sqrt{|b|^2 - 4|A||B|}}{2|A|} = \frac{2u \pm \sqrt{4u^2 - 4(u^2 - c^2)}}{2} = u \pm c$$

# Example #3

Therefore, the two characteristics of this hyperbolic system are given by

$$\frac{dx}{dt} = u \pm c$$

The canonical form is then given by

$$\frac{\partial V}{\partial t} + \Lambda \frac{\partial V}{\partial x} = 0$$

or

$$\frac{\partial}{\partial t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} u+c & 0 \\ 0 & u-c \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or in component form

$$\frac{\partial v_1}{\partial t} + (u+c)\frac{\partial v_1}{\partial x} = 0$$
$$\frac{\partial v_2}{\partial t} + (u-c)\frac{\partial v_2}{\partial x} = 0$$

Each equation involves only one unknown and can be easily solved by the standard methods of characteristics. The general solution of the system is

$$\begin{aligned} v_1 &= f_1[x-(u+c)t] \\ v_2 &= f_2[x-(u-c)t] \end{aligned} \tag{65}$$

where $f_1$ and $f_2$ are arbitrary functions of a single variable.

# Example #3

The vectors $U$ and $V$ are related through the following transformation:

$$U = PV \qquad \Rightarrow \qquad \begin{bmatrix} h \\ u \end{bmatrix} = \begin{bmatrix} \rho & -\rho \\ c & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \rho(v_1 - v_2) \\ c(v_1 + v_2) \end{bmatrix}$$

Now the general solution can be obtained using the relation (65) as

$$\begin{aligned} \rho &= \rho[f_1(x-(u+c)t) - f_2(x-(u-c)t)] \\ u &= c[f_1(x-(u+c)t) + f_2(x-(u-c)t)] \end{aligned}$$

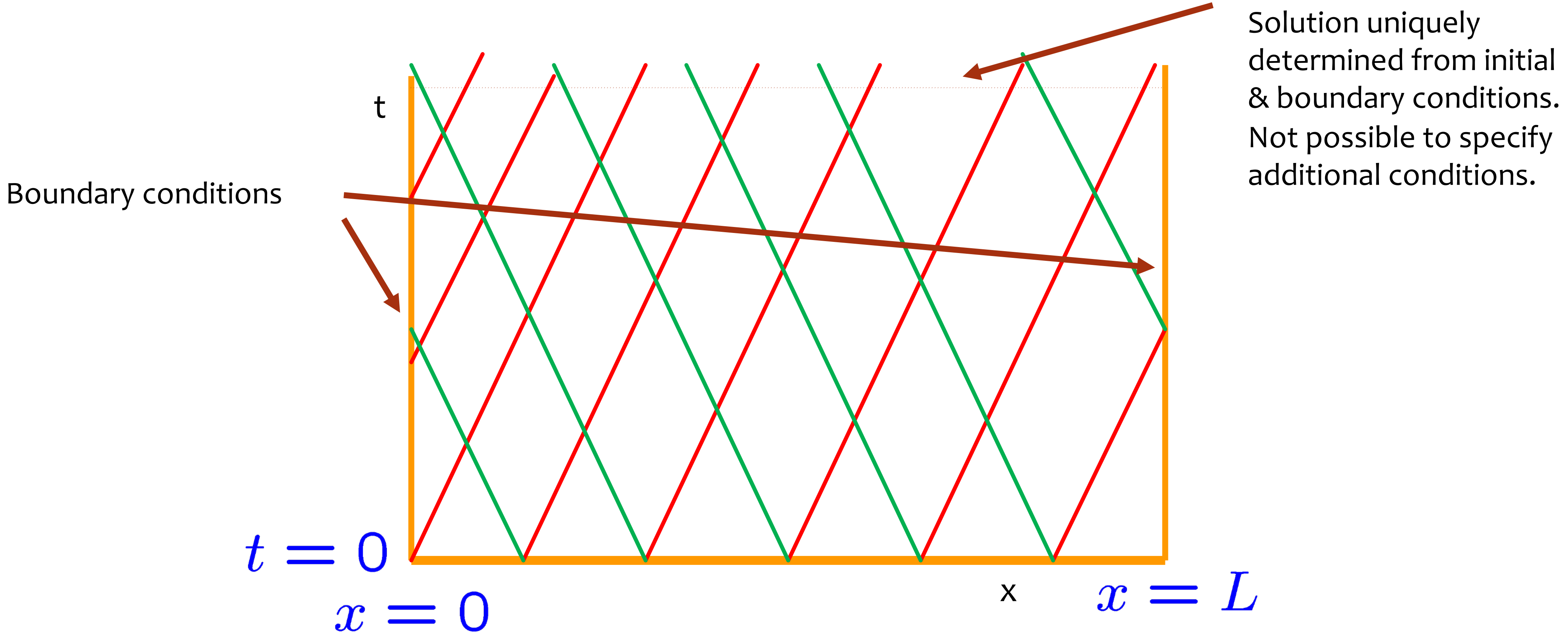
Note: Since both the eigenvalues are real, for *all* values of the velocity $u$, the system is always hyperbolic in space and time. This is an extremely important property that the steady isentropic Euler equations are elliptic in the space $(x,y)$ for subsonic velocities and hyperbolic in the space $(x,y)$ for supersonic velocities.

Here, in space and time, the inviscid isentropic equations are always hyperbolic independently of the subsonic or supersonic state of the flow. As a consequence, the same numerical algorithms can be applied for all flow velocities. On the other hand, dealing with the steady state equations, the numerical algorithms will have to adapt to the flow regime, as the mathematical nature of the system of equations is changing when passing from subsonic to supersonic, or inversely.

# Boundary Conditions

# Boundary Conditions: Hyperbolic case

- As discussed, in the hyperbolic case, initial & boundary values propagates along characteristics.
- Any point is influenced by neighbours
- The problem will be well posed if we specify the conditions described in the following diagram:



# Boundary Conditions: Hyperbolic case

- Consider the 2nd order wave equation: you'll need to specify two initial conditions (function and its derivative) + one b.c. for each side.

- Since information is brought by characteristic, it is important (for a well-posed problem) that each characteristic passes through either a initial condition or a boundary one.

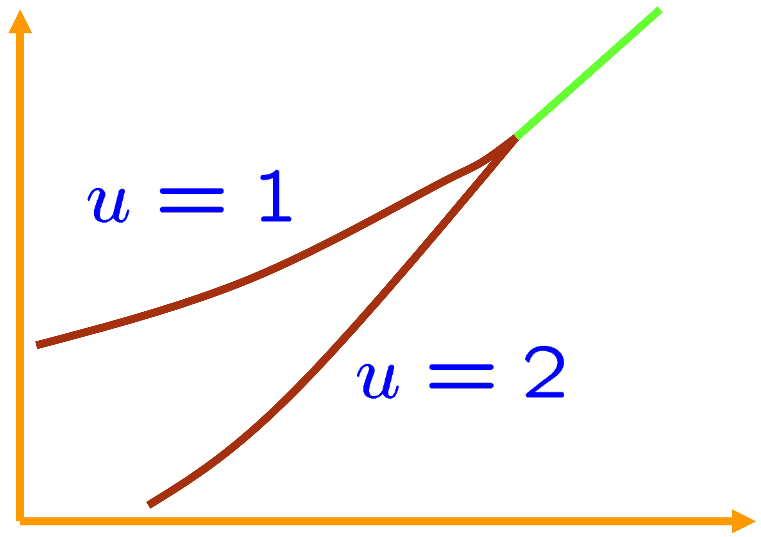
- Otherwise, the problem would be over-specified or under-specified.

- The parabolic case, from this point of view is similar to the hyperbolic case.

- If the nature of the equation changes → matching conditions at the interface.

# Boundary Conditions: Hyperbolic case

What happens if two CC intersects ?




- We end up in a situation like the one depicted: the problem is solved by the birth of a discontinuous wave governed by appropriate jump-conditions.
- These discontinuous waves (e.g. shocks) may form also when the solution is initially continuous.

# Boundary Conditions: Elliptic Case

- Need to specify on the whole boundary. Global coupling