Fisica della Materia allo Stato Fluido e di Plasma

Two Fluids, Single Fluid

Single Fluid: momentum equation

- Decompose with respect to

$$v^{(s)} = U + w'^{(s)} \quad \Longrightarrow \quad u^{(s)} = U + < w'^{(s)} >$$

- Momentum Eq: write the flux term as

$$< \rho^{(s)} v_i v_k > = \rho^{(s)} < (U_i + w_i'^{(s)})(U_k + w_k'^{(s)}) > =$$
$$= \rho^{(s)} \left[U_i U_k + U_i (u_k^{(s)} - U_k) + U_k (u_i^{(s)} - U_i) \right] + P_{ik}^{(s)} =$$
$$\rho^{(s)} \left(U_i u_k^{(s)} + U_k u_i^{(s)} - U_i U_k \right) + P_{ik}^{(s)}$$

Single Fluid: Energy equation

- Remembering $< \rho^{(s)} v_i v_k >= \rho^{(s)} \left(U_i u_k^{(s)} + U_k u_i^{(s)} - U_i U_k \right) + P_{ik}^{(s)}$
- The energy term is $< \frac{1}{2} \rho^{(s)} v^2 >= \frac{1}{2} \rho^{(s)} \left(2 u_i^{(s)} U_i - U^2 \right) + \frac{3}{2} P^{(s)}$
- While the flux term can be split into two parts:

$$< \frac{1}{2} \rho^{(s)} v_i^{(s)} v_k^{(s)} v_k^{(s)} >= \frac{1}{2} \rho^s \left[U_i < v_k^{(s)} v_k^{(s)} > + < w_i'^{(s)} v_k^{(s)} v_k^{(s)} > \right]$$

- Where the two flux terms can be written as:

$$\begin{cases} U_i < v_k^{(s)} v_k^{(s)} > & = U_i \left(2 u_k^{(s)} U_k - U^2 + 3 P^{(s)} / \rho^{(s)} \right) \\ < w_i'^{(s)} v_k^{(s)} v_k^{(s)} > & = \left[< w_i'^{(s)} U_k U_k > + 2 < w_i'^{(s)} w_k'^{(s)} U_k > + < w_i'^{(s)} w_k'^{(s)} w_k'^{(s)} > \right] \\ & = U_k U_k \left(u_i^{(s)} - U_i \right) + 2 \frac{P_{ik}^{(s)}}{\rho^{(s)}} U_k + 2 q_i^{(s)} / \rho^{(s)} \\ & = U_k U_k u_i^{(s)} - U_i U^2 + 2 \frac{P_{ik}^{(s)}}{\rho^{(s)}} U_k + 2 q_i^{(s)} / \rho^{(s)} \end{cases}$$

Single Fluid: Energy equation

- Summing the two energies gives

$$\sum_s \frac{1}{2}\rho^{(s)} < v_k^{(s)} v_k^{(s)} >= \sum_s \left[\frac{1}{2}\rho^{(s)}\left(2u_k^{(s)}U_k - U^2\right) + \frac{3}{2}P^{(s)}\right] = \frac{1}{2}\rho U^2 + \frac{3}{2}P$$

- Summing the fluxes (1° term):

$$\begin{aligned}\sum_s \frac{1}{2}\rho^{(s)} U_i < v_k^{(s)} v_k^{(s)} > &= \sum_s U_i \left[\frac{1}{2}\rho^{(s)}\left(2u_k^{(s)}U_k - U^2\right) + \frac{3}{2}P^{(s)}\right] \\ &= \left[\frac{1}{2}\rho U^2 + \frac{3}{2}P\right] U_i\end{aligned}$$

- Summing the fluxes (2° term):

$$\begin{aligned}\sum_s \frac{1}{2}\rho^{(s)} < w_i'^{(s)} v_k^{(s)} v_k^{(s)} > &= \sum_s \left\{\frac{1}{2}\rho^{(s)}\left(U_k U_k u_i^{(s)} - U_i U^2\right) + P_{ik}^{(s)} U_k + q_i^{(s)}\right\} \\ &= P_{ik}U_k + q_i = PU_i + \Pi_{ik}U_k + q_i\end{aligned}$$

- Putting all toghether:

$$\partial_t \left[\frac{1}{2}\rho U^2 + \frac{3}{2}P\right] + \partial_i \left[\left(\frac{1}{2}\rho U^2 + \frac{5}{2}P\right) U_i + \Pi_{ik}U_k + q_i\right] - J_k E_k = 0$$

Single Fluid: Lorentz force in Ohm's Law

- Write
$$\begin{cases} J &= n_p e u^{(p)} - n_e e u^{(e)} \approx ne(u^{(p)} - u^{(e)}) \\ \rho U &= n_p m_p u^{(p)} + n_e m_e u^{(e)} \approx n\left(m_p u^{(p)} + m_e u^{(e)}\right) \end{cases}$$
- Assuming charge neutrality, $n_e = n_p$, we invert the previous system giving:
$$u^{(p)} = \left(\frac{J}{ne} + \frac{\rho U}{n m_e}\right)\frac{1}{1 + m_p/m_e}; \quad u^{(e)} = -\left(\frac{J}{ne} - \frac{\rho U}{n m_p}\right)\frac{1}{1 + m_e/m_p}$$
- One can safely neglect the term containing J in $u^{(p)}$ when compared to the same term in $u^{(e)}$. On the contrary, the \rho U terms are comparable. However, since these two appear in the combination
$$\frac{n_p}{m_p}u^{(p)} + \frac{n_e}{m_e}u^{(e)} \approx \frac{n_e}{m_e}u^{(e)} \approx \frac{n}{m_e}\left[\frac{\rho U}{n m_p} - \frac{J}{ne}\right] \approx \frac{n}{m_e}\left(U - \frac{J}{ne}\right)$$
- Finally the expressions for the Lorentz force becomes
$$\frac{e^2}{c}\left[\frac{n_p}{m_p}u^{(p)} + \frac{n_e}{m_e}u^{(e)}\right] \times B \approx \frac{e^2 n}{c m_e}\left(U \times B - \frac{J}{ne} \times B\right)$$

Summary of single fluid equations

- Summarizing, the two fluid equations are 15:

$$\partial_t \rho + \partial_k (\rho U_k) = 0 \qquad \text{(Continuity)}$$

$$\partial_t q + \partial_k J_k = 0 \qquad \text{(Charge)}$$

$$\partial_t (\rho U_i) + \partial_k (\rho U_i U_k + P_{ik}) = qE_i + \tfrac{1}{c}\left(\vec{J} \times \vec{B}\right)_i \qquad \text{(Momentum)}$$

$$\partial_t \left(\tfrac{1}{2}\rho U^2 + \tfrac{3}{2}P\right) + \partial_k \left[\left(\tfrac{1}{2}\rho U^2 + \tfrac{5}{2}P\right) U_k + \Pi_{ik} U_i + \tilde{q}_k + S_k\right] = 0 \qquad \text{(Energy)}$$

$$E_i + \frac{1}{c}\left(\vec{U} \times \vec{B}\right)_i - \frac{J_i}{\sigma} = \frac{m_e}{e^2 n_e}\left[\partial_t J_i + \partial_j (J_i U_k + J_k U_i)\right] - \frac{1}{n_e}\partial_k P_{ik}^{(e)} + \frac{1}{e n_e c}\left(\vec{J} \times \vec{B}\right) \qquad \text{(Ohm)}$$

$$\tfrac{1}{c}\partial_t \vec{B} = -\nabla \times \vec{E} \qquad \text{(Faraday)}$$

$$\frac{1}{c}\partial_t \vec{E} = \nabla \times \vec{B} - \frac{4\pi}{c}\vec{J} \qquad \text{(Maxwell – Ampere)}$$

- The number of unknowns is 21: $\rho, P, q, \vec{U}, \vec{J}, \vec{E}, \vec{B}, \Pi, \vec{\tilde{q}}$
- Closure must be found to express $\Pi, \vec{\tilde{q}}$ in terms of macroscopic quantities.