Fisica della Materia allo Stato Fluido e di Plasma

5. MHD

Magnetic Pressure & Tension

- The JxB term can be manipulated using the following the vector identity [12] on the formulary showing that

$$\nabla\left(\frac{B^2}{2}\right)=\vec{B}\times(\nabla\times\vec{B})+(\vec{B}\cdot\nabla)\vec{B} \quad \rightarrow \quad (\nabla\times\vec{B})\times\vec{B}=-\nabla\left(\frac{B^2}{2}\right)+(\vec{B}\cdot\nabla)\vec{B}$$

where $\vec{J}=\frac{c}{4\pi}\nabla\times\vec{B}$

- With this result the Lorentz force can be written as the sum of two different contributions corresponding, respectively to "magnetic pressure" and "magnetic tension":

$$\frac{1}{c}\vec{J}\times\vec{B}=-\nabla\left(\frac{B^2}{8\pi}\right)+\frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B}$$

Pressure *Tension*

Magnetic Pressure & Tension (Bellan, Ch 9.5, page 271)

9.5 Magnetic stress tensor

The existence of magnetic pressure and tension shows that the magnetic force is different in different directions, and so the magnetic force ought to be characterized by an anisotropic stress tensor. To establish this mathematically, the vector identity $\nabla B^2/2 = \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{B}\times\nabla\times\mathbf{B}$ is invoked so that the magnetic force can be expressed as

$$\begin{aligned} \mathbf{J}\times\mathbf{B} &= \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} \\ &= \frac{1}{\mu_0}\left[-\nabla\left(\frac{B^2}{2}\right)+\mathbf{B}\cdot\nabla\mathbf{B}\right] \\ &= -\frac{1}{\mu_0}\nabla\cdot\left[\frac{B^2}{2}\mathbf{I}-\mathbf{B}\mathbf{B}\right] \end{aligned} \tag{9.10}$$

where $\mathbf{I}$ is the unit tensor and the relation $\nabla\cdot(\mathbf{B}\mathbf{B}) = (\nabla\cdot\mathbf{B})\,\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{B}$ has been used. At any point $\mathbf{r}$ a local Cartesian coordinate system can be defined with z axis parallel to the local value of $\mathbf{B}$ so that Eq.(9.1) can be written as

$$\rho\left[\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}\right] = -\nabla\cdot\begin{bmatrix} P+\dfrac{B^2}{2\mu_0} & & \\ & P+\dfrac{B^2}{2\mu_0} & \\ & & P-\dfrac{B^2}{2\mu_0} \end{bmatrix} \tag{9.11}$$

showing again that the magnetic field acts like a pressure in the directions transverse to $\mathbf{B}$ (i.e., x, y directions in the local Cartesian system) and like a tension in the direction parallel to $\mathbf{B}$.

Magnetic Pressure & Tension *(Bellan, Ch 9.5, page 271)*

While the above interpretation is certainly useful, it can be somewhat misleading because it might be interpreted as implying the existence of a force in the direction of $\mathbf{B}$ when in fact no such force exists because $\mathbf{J} \times \mathbf{B}$ clearly does not have a component in the $\mathbf{B}$ direction. A more accurate way to visualize the relation between magnetic pressure and tension is to rearrange the second line of Eq.(9.10) as

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}\left[-\nabla\left(\frac{B^2}{2}\right) + B^{\mathbf{2}}\hat{B}\cdot\nabla\hat{B} + \hat{B}\hat{B}\cdot\nabla\left(\frac{B^2}{2}\right)\right] = \frac{1}{\mu_0}\left[-\nabla_\perp\left(\frac{B^2}{2}\right) + B^2\boldsymbol{\kappa}\right] \tag{9.12}$$

or

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}\left[-\nabla_\perp\left(\frac{B^2}{2}\right) + B^2\boldsymbol{\kappa}\right]. \tag{9.13}$$

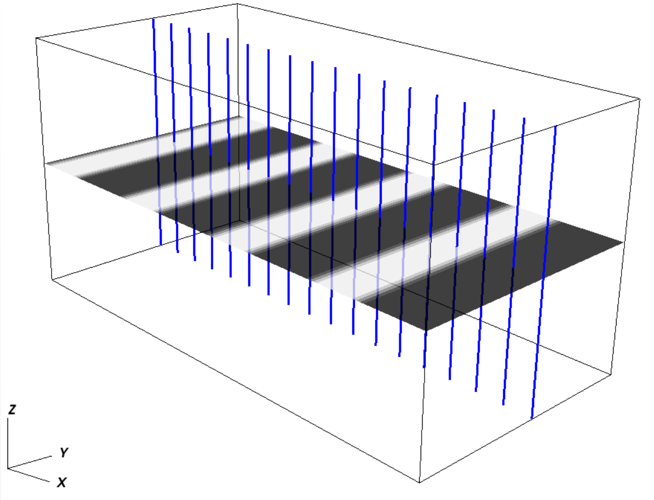
Here

$$\boldsymbol{\kappa} = \hat{B} \cdot \nabla\hat{B} = -\frac{\hat{R}}{R} \tag{9.14}$$

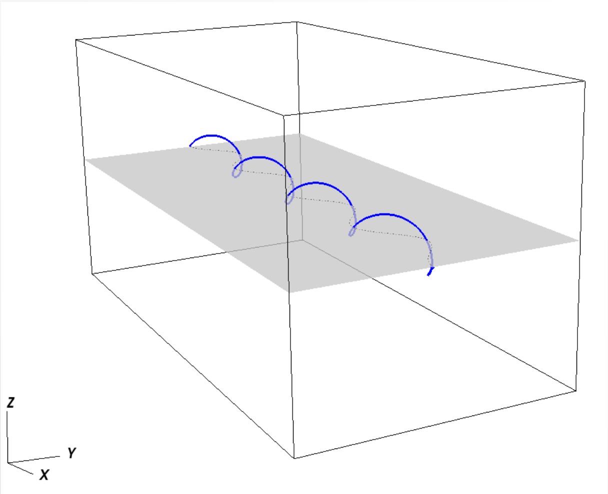
is a measure of the curvature of the magnetic field at a selected point on a field line and, in particular, $\mathbf{R}$ is the local radius of curvature vector. The vector $\mathbf{R}$ goes from the center of curvature to the selected point on the field line. The $\boldsymbol{\kappa}$ term in Eq.(9.13) describes a force which tends to straighten out magnetic curvature and is a more precise way for characterizing field line tension (recall that tension similarly acts to straighten out curvature). The term involving $\nabla_\perp B^2$ portrays a magnetic force due to pressure gradients perpendicular to the magnetic field and is a more precise expression of the hoop force.

Examples

- Consider $B = (0,0,B_0\sin(kx))$
  → this field has only pressure



- Consider $B = (B_{0x}, B_{0p}\sin(kx), B_{0p}\cos(kx))$:
  → this has only tension

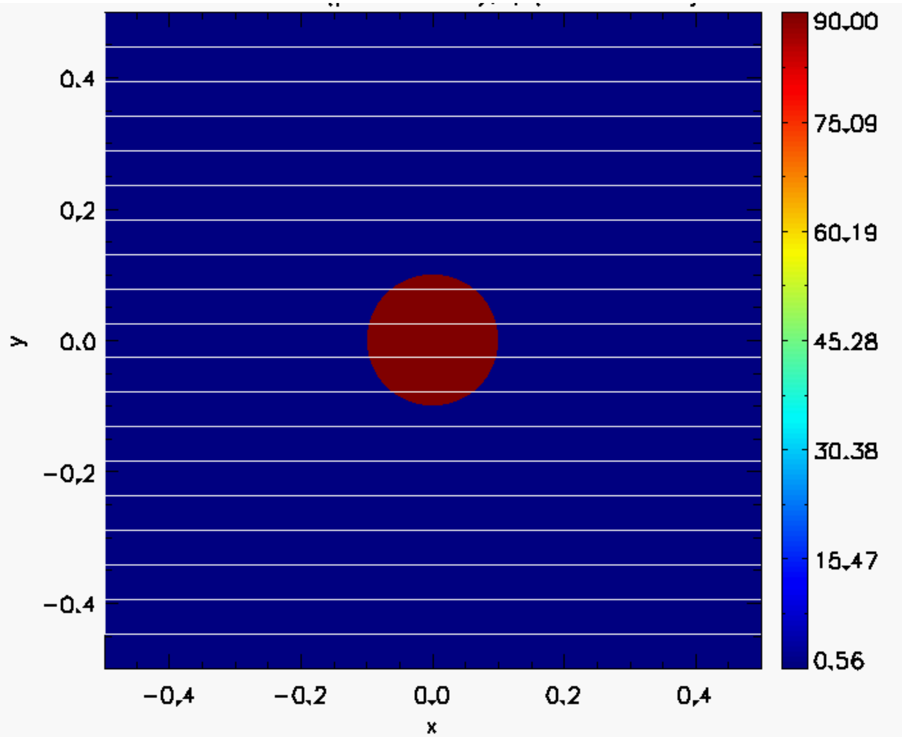


Example: the MHD Blast Wave Problem

- A highly pressurized region in a uniform medium, threaded by a uniform constant magnetic field, parallel to the X-axis.

- The relative strength between thermal and magnetic pressure is given by the plasma beta parameter,

$$\boxed{\beta = \frac{p}{B^2/8\pi} = \frac{8\pi p}{B^2}}$$

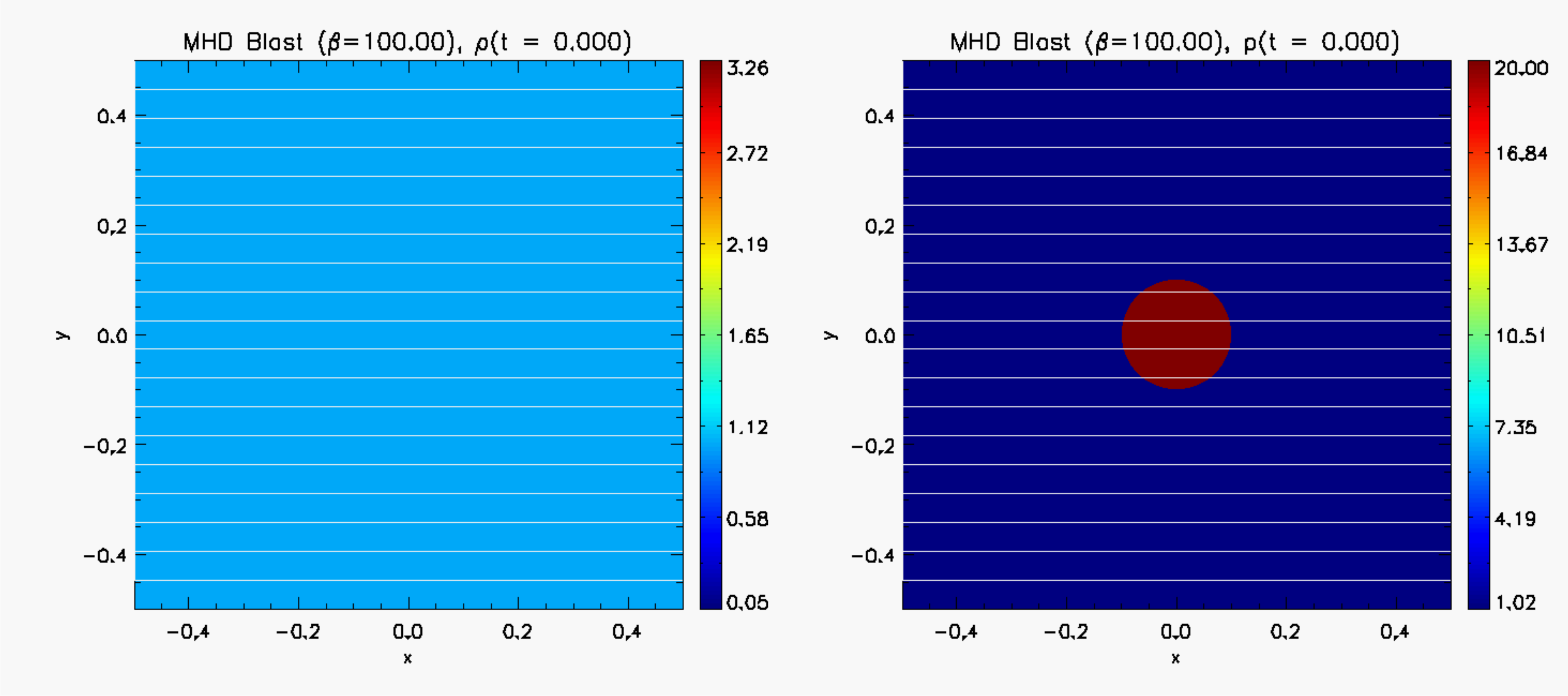


- For large values of β (β >> 1), → fluid dominated by hydrodynamics effects (magnetic field has a negligible role);

- For small values of β (β << 1), → fluid dominated by magnetic effects;

Example: a Blast Wave

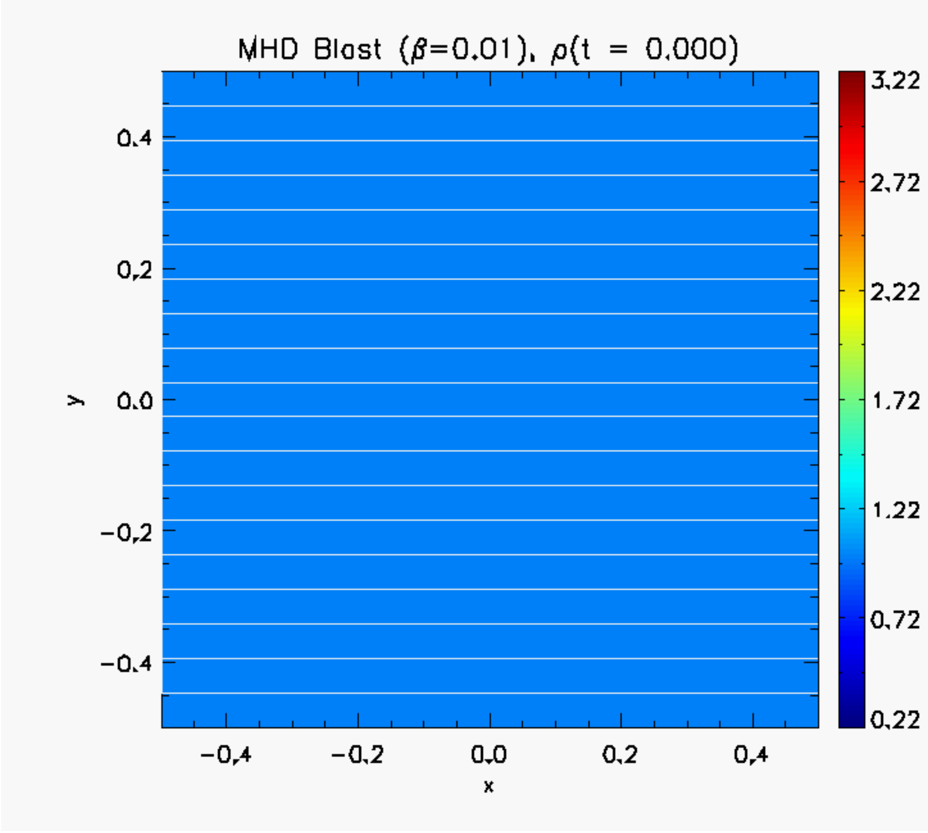
Large plasma beta

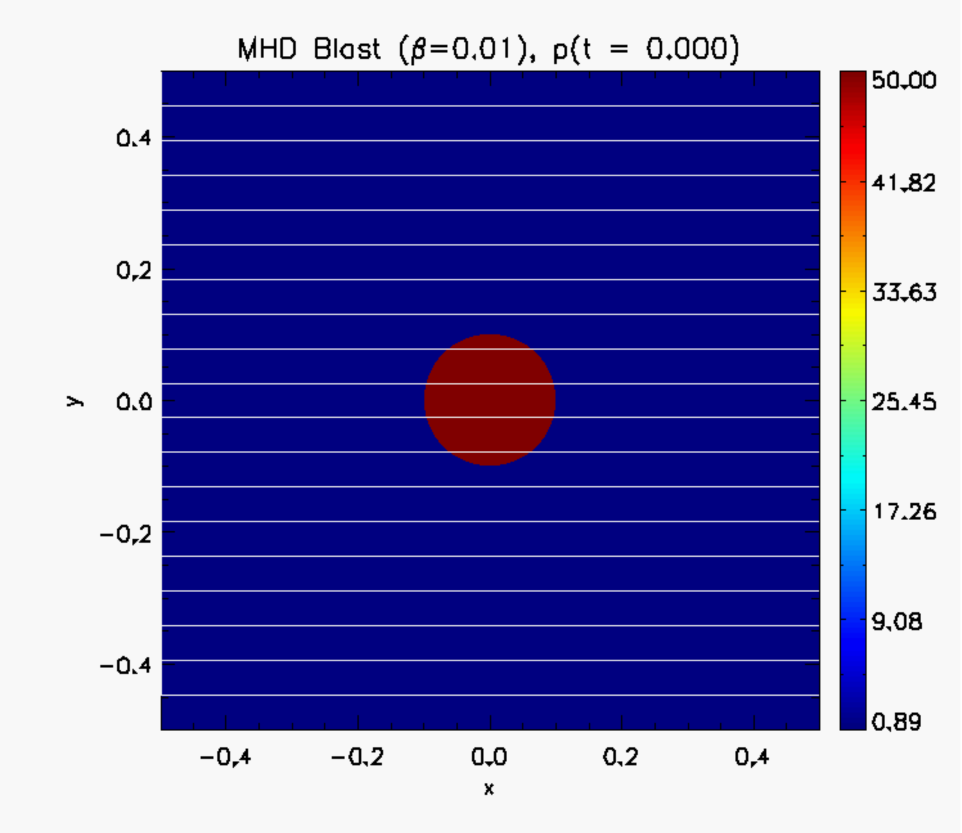


The code and the visualization scripts for this simulation (PLUTO/Test_Probblems/Educational/MHD_Blast/) can be downloaded from the webpage.

Example: a Blast Wave

Small plasma beta





The code and the visualization scripts for this simulation (PLUTO/Test_Probblems/Educational/MHD_Blast/) can be downloaded from the webpage.

Flux Freezing (ideal MHD)

Across a closed surface, the divergence-free condition gives

$$\Phi(t) = \oint \boldsymbol{B}(t)\cdot d\boldsymbol{S} = -\Phi_1(t) + \Phi_2(t) + \Phi_3 = 0$$

where

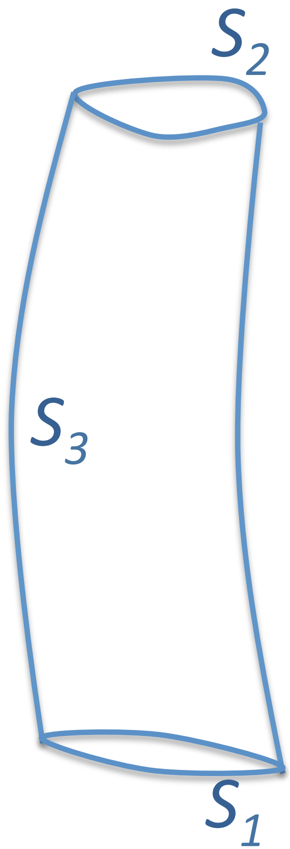
$$\Phi_3 = \int_\ell \boldsymbol{B}\cdot\delta\boldsymbol{l}\times\boldsymbol{u}\Delta t$$

Now consider

$$\begin{aligned}\Phi_2(t+\Delta t) - \Phi_1(t) &= \Phi_2(t+\Delta t) - \Phi_2(t) - \Phi_3 \\ &= \int_{S_2} [\boldsymbol{B}(t+\Delta t) - \boldsymbol{B}(t)]\cdot d\boldsymbol{S} - \Delta t\int_\ell \boldsymbol{B}\cdot\delta\boldsymbol{l}\times\boldsymbol{u}\end{aligned}$$

Taking the limit for $\Delta t \to 0$:

$$\begin{aligned}\frac{\Phi_2(t+\Delta t) - \Phi_1(t)}{\Delta t} &= \int_{S_2} \frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S} - \int_\ell \boldsymbol{B}\cdot\delta\boldsymbol{l}\times\boldsymbol{u} \\ &= \int_{S_2} \nabla\times(\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{S} - \int_\ell \boldsymbol{B}\cdot\delta\boldsymbol{l}\times\boldsymbol{u} \\ &= \int_\ell \boldsymbol{u}\times\boldsymbol{B}\cdot\delta\boldsymbol{l} - \int_\ell \boldsymbol{B}\cdot\delta\boldsymbol{l}\times\boldsymbol{u} = 0\end{aligned}$$



Diffusion Equation *(http://math.utsa.edu/~gokhman/ftp//courses/notes/heat.pdf*

In the limit S >> 1 (large resistivity), the induction equation reduces to the diffusion equation which, for constant η reads

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B}$$

This is a linear equation in the sense that if $B_1(x)$ and $B_2(x)$ are solution of the previous equation, than any linear combination of these two is also a solution. For this reason, we can write the general solution using Fourier decomposition and thus set

$$\vec{B}(\vec{x}, t) = \int B_k(t) e^{-i\vec{k}\cdot\vec{x}}\, d^3k$$

where Bk(t) is the amplitude of the k-th mode and can be found by inserting this result in the original equation,

Diffusion Equation (http://math.utsa.edu/~gokhman/ftp//courses/notes/heat.pdf

the diffusion equation, a general solution can be built by adding together many such modes at different frequencies with the right 'strength' $A_k(t)$. First we note that

$$c(\bar{x},t) = \int c_k(\bar{x},t)\mathrm{d}\bar{k} \;\; \text{where} \;\; c_k(\bar{x},t) = A_k(t)e^{-i\bar{k}\cdot\bar{x}}. \tag{47}$$

Taking the time derivative gives

$$\frac{\partial}{\partial t}c_k = e^{-i\bar{k}\cdot\bar{x}}\frac{\partial}{\partial t}A_k, \tag{48}$$

and taking the spatial derivative gives

$$\nabla^2 c_k = -A_k(\bar{k}\cdot\bar{k})e^{-i\bar{k}\cdot\bar{x}}. \tag{49}$$

Thus the linear operator $\mathcal{L}$ acting on c_k gives a differential equation for the coefficient A_k

$$\mathcal{L}\left[c_k(t)\right] = e^{-i\bar{k}\cdot\bar{x}}\frac{\partial}{\partial t}A_k + DA_k|k|^2e^{-i\bar{k}\cdot\bar{x}} = 0, \tag{50}$$

which simplifies to

$$\frac{\partial}{\partial t}A_k + DA_k|k|^2 = 0. \tag{51}$$

This first-order differential equation in time is easily solved, yielding

$$A_k(t) = A_k(0)e^{-D|k|^2t} \tag{52}$$

Diffusion Equation (http://math.utsa.edu/~gokhman/ftp//courses/notes/heat.pdf

where $A_k(0)$ is an initial condition that we will address momentarily. The first interesting fact emerges here, that higher frequencies are damped quadratically, hence any sharp variations in probability density 'smooth out' very quickly, while longer wavelength variations persist on a longer time-scale. Using $A_k(t)$ we construct the general solution as

$$c(\bar{x}, t) = \int A_k(0) e^{-D|k|^2 t} e^{-i\bar{k}\cdot\bar{x}} \mathrm{d}\bar{k}, \tag{53}$$

but still need to find the $A_k(0)$'s using the initial condition $c(\bar{x}, 0)$. The first way to proceed is simply by taking the Fourier Transform on the initial condition, namely

$$A_k(0) = \frac{1}{(2\pi)^n} \int c(\bar{x}, 0) e^{i\bar{k}\cdot\bar{x}} \mathrm{d}\bar{x}, \tag{54}$$

where n is the number of dimensions. In some sense, this is the answer, but a more useful result can be had if we study the impulse initial condition $c(\bar{x}, 0) = \delta(\bar{x} - \bar{x}')$, where

$$A_k(0) = \frac{1}{(2\pi)^n} \int \delta(\bar{x} - \bar{x}') e^{i\bar{k}\cdot\bar{x}} \mathrm{d}\bar{x} = \frac{1}{(2\pi)^n} e^{i\bar{k}\cdot\bar{x}'}. \tag{55}$$

We create a solution to the PDE known as the *Green's Function*, by studying the response to this initial δ-function impulse

$$G(\bar{x}, \bar{x}', t) = \frac{1}{(2\pi)^n} \int e^{-D|k|^2 t} e^{-i\bar{k}\cdot(\bar{x} - \bar{x}')} \mathrm{d}\bar{k}. \tag{56}$$

Diffusion Equation (http://www.maths.bris.ac.uk/~macpd/apde2/chap4.pdf)

4.5.1 Delta Function

If $u(x,0) = \delta(x)$ (e.g. all the heat/chemical initially dumped at the origin – a decent mathematical model), then $\phi(x) = \delta(x)$

and

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \delta(\xi) e^{-\frac{(x-\xi)^2}{4Dt}} \, d\xi = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

Defn: This is the **fundamental solution** of the diffusion equation.

Fourier Transform (http://www.maths.bris.ac.uk/~macpd/apde2/chap4.pdf)

- Definition $$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\, dk$$
- Example 1: Gaussian $$f(x) = \frac{e^{-x^2/a^2}}{a\sqrt{\pi}} \qquad \tilde{f} = e^{-a^2k^2/4}.$$
- Delta function: $$\int_{-\infty}^{\infty} \delta(x-c) f(x)\, dx = f(c)$$

$$\mathscr{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = 1$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$$