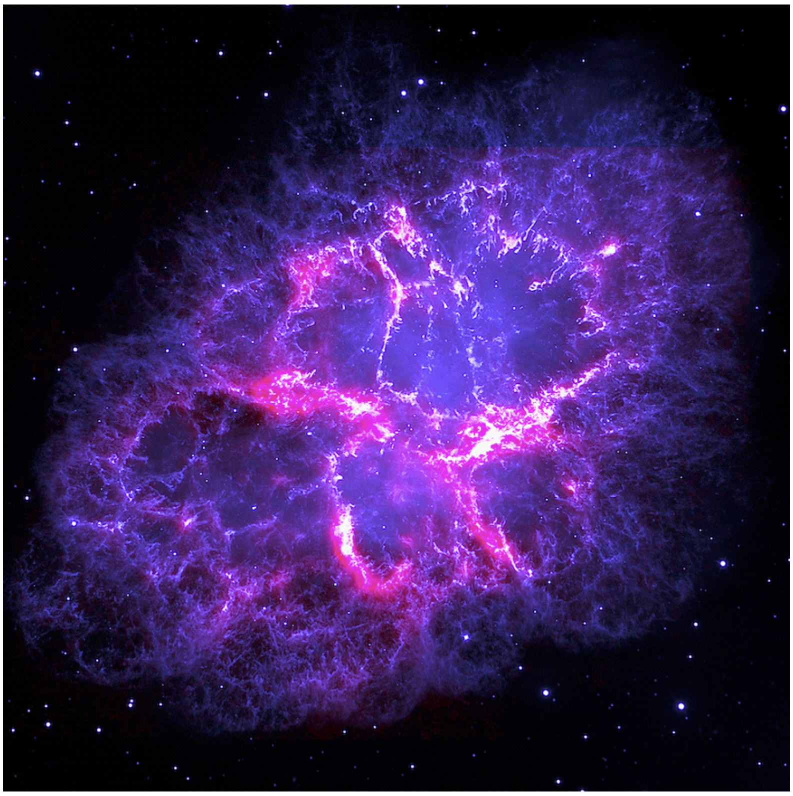

Figure 1: Herschel (red) and Hubble (blue) composite image of the Crab Nebula. *(Credit: ESA/Herschel/PACS/MESS Key Programme Supernova Remnant Team; NASA, ESA and Allison Loll/Jeff Hester - Arizona State University).*

1 Rayleigh-Taylor Instability

The Rayleigh-Taylor instability (RTI) develops at the interface between two fluids of different densities, when a lighter fluid tries to support the heavier one against gravity. In ordinary hydrodynamics, a Rayleigh-Taylor instability arises when one attempts to support a heavy fluid on top of a light fluid: the interface becomes rippled, allowing the heavy fluid to fall through the light fluid. In plasmas, a Rayleigh-Taylor instability can occur when a dense plasma is supported against gravity by the pressure of a magnetic field.

The instability can be found in many physical contexts, such as supernova explosions in which expanding core gas is accelerated into denser shell gas, instabilities in plasma fusion reactors and the common terrestrial example of a denser fluid such as water suspended above a lighter fluid such as oil in the Earth's gravitational field. Weather inversion (a deviation from the normal change of an atmospheric property with altitude) is yet another example.

In astrophysics, a typical situation where the RTI is expected to arise is in the ejecta of a supernova where a shell of dense shocked material material is gradually decelerated by lower density shocked circumstellar material. Here, in fact, the contact discontinuity propagating between the forward and the reverse shock will decelerate as it sweeps the shocked gas. An observer co-moving with the contact wave will be in an non-inertial frame where, for Newton's third law, an acceleration pointing outward is felt. This situation results in finger-like structures consisting of dense gas protruding into, and mixing with, the shocked circumstellar material.

1.1 Equilibrium Condition

Our equilibrium condition consists of two constant-density fluids lying on top of each other and separated by a horizontal interface lying in the xy plane,

$$\rho_0(z) = \begin{cases} \rho_T & \text{for} \quad z > 0 \\ \rho_B & \text{for} \quad z < 0 \end{cases} \tag{1}$$

where the pedicies T and B stands for the top and bottom fluids, respectively. The two fluids are immersed in a constant gravitational field and are initially in pressure equilibrium. Without loss of generality, we assume the gravitational acceleration $\boldsymbol{g}$ to be directed along the negative z direction , i.e., $\boldsymbol{g} = (0, 0, g_z) = -g\hat{\boldsymbol{e}}_z$. Force balance is then dictated by the momentum equation, which simply becomes

$$\frac{\partial}{\partial z} p_0(z) = -\rho_0(z) g \tag{2}$$

where $\rho_0(z)$ gives the density profile. Using Eq. (1), the previous equation can be easily integrated to obtain

$$p = p_0 - \int \rho_0(z) g\, dz = p_0 - \rho_0(z) g z \tag{3}$$

Note that the pressure profile obtain in this way is continuous although its derivative it's not.

A constant magnetic field $\boldsymbol{B} = (B_{0x},\, B_{0y},\, 0)$ (parallel to the interface separating the two fluids) can be introduced. Notice that this does not alter our equilibrium condition since a constant field does not exert any pressure or tension.

1.2 Linear Analysis

We start by writing the perturbed MHD equations, assuming a static equilibrium ($\boldsymbol{u}_0 = 0$):

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} &= -\boldsymbol{u}_1 \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \boldsymbol{u}_1 \\ \rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} &= -\nabla p_1 + \Big[(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 \Big] + \boldsymbol{F}_1 \\ \frac{\partial \boldsymbol{B}_1}{\partial t} &= \nabla \times \Big(\boldsymbol{u}_1 \times \boldsymbol{B}_0 \Big) \\ \frac{\partial p_1}{\partial t} &= -\boldsymbol{u}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{u}_1 \end{aligned} \tag{4}$$

where $\boldsymbol{F}_1 = \rho_1 \boldsymbol{g}$ and a factor of $\sqrt{4\pi}$ has been reabsorbed into the definition of $\boldsymbol{B}$, for convenience. The linear analysis is best carried by now introducing the displacement vector (which describes how much the plasma is displaced from an equilibrium):

$$\frac{d\boldsymbol{\xi}}{dt} = \frac{\partial \xi}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\xi} \equiv \boldsymbol{u}_1 \tag{5}$$

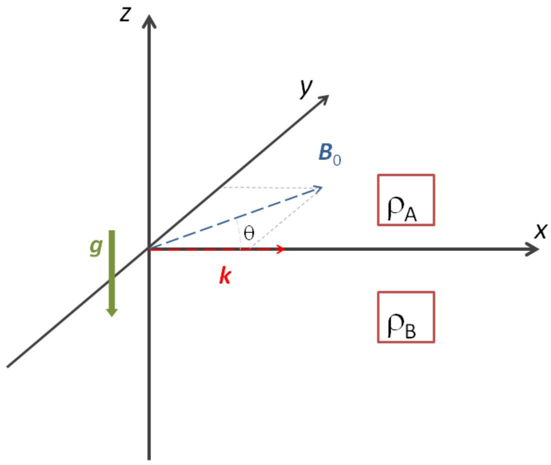

Figure 2: Initial condition and perturbation.

However, since $\boldsymbol{u}_0 = 0$ at equilibrium and $\boldsymbol{\xi}$ is a first-order quantity, we simply have $\partial_t \boldsymbol{\xi} = \boldsymbol{u}_1$ and therefore $\int \boldsymbol{u}_1 \, dt = \xi$. Using this result we now integrate the first, third and fourth Eqns. in (4) with respect to t and, assuming perturbations are initially zero, we obtain:

$$\begin{aligned} \rho_1 &= -(\boldsymbol{\xi} \cdot \nabla)\rho_0 - \rho_0(\nabla \cdot \boldsymbol{\xi}) \\ p_1 &= -(\boldsymbol{\xi} \cdot \nabla)p_0 - \gamma p_0(\nabla \cdot \boldsymbol{\xi}) \\ \boldsymbol{B}_1 &= \nabla \times \Big(\boldsymbol{\xi} \times \boldsymbol{B}_0\Big) \end{aligned} \tag{6}$$

We now assume that the plasma is incompressible so that $\nabla \cdot \boldsymbol{\xi} = 0$ and, substituting the previous expressions in the second of (4), it is found

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \nabla\Big(\boldsymbol{\xi} \cdot \nabla p_0\Big) + \Big\{\nabla \times \big[\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)\big]\Big\} \times \boldsymbol{B}_0 - \boldsymbol{g}(\boldsymbol{\xi} \cdot \nabla)\rho_0 \tag{7}$$

We now consider, for simplicity:

- perturbations propagating along the x direction only ($\boldsymbol{k} = k\hat{\boldsymbol{e}}_x$) so that, for any perturbed quantity q_1, we set $q_1 = f(z)e^{i(kx-\omega t)}$, as usual; The nabla operator has therefore the representation $\nabla = (ik,\, 0,\, \partial_z)$.
- a reference frame in which $\boldsymbol{\xi} = (\xi_x, 0, \xi_z)$.

Notice that picking just one of the two assumptions does not restrict the validity of our treatment. However, if both assumptions hold, propagation in the y-y is not accounted for by our analysis.

The second term in Eq. (7) can be evaluated term by term as

$$\boldsymbol{\xi} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \xi_x & 0 & \xi_z \\ B_{0x} & B_{0y} & 0 \end{vmatrix} = (-\xi_z B_{0y})\hat{\boldsymbol{e}}_x + (\xi_z B_{0x})\hat{\boldsymbol{e}}_y + (\xi_x B_{0y})\hat{\boldsymbol{e}}_z$$

$$\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ ik & 0 & \partial_z \\ -\xi_z B_{0y} & \xi_z B_{0x} & \xi_x B_{0y} \end{vmatrix} = (-\xi_z' B_{0x})\hat{\boldsymbol{e}}_x + (-\xi_z' B_{0y} - ik\xi_x B_{0y})\hat{\boldsymbol{e}}_y + (ik\xi_z B_{0x})\hat{\boldsymbol{e}}_z$$

Note that the y component of the previous expression vanishes since, using the incompribility condition, $\nabla\cdot\boldsymbol{\xi} = 0$, gives

$$ik\xi_x + \xi_z' = 0 \tag{8}$$

Taking the outermost curl,

$$\nabla\times[\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)] = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ ik & 0 & \partial_z \\ -\xi_z' B_{0x} & 0 & ik\xi_z B_{0x} \end{vmatrix} = (-\xi_z'' + k^2\xi_z)B_{0x}\hat{\boldsymbol{e}}_y$$

Finally,

$$\nabla\times[\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)]\times\boldsymbol{B}_0 = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ 0 & -\xi_z'' + k^2\xi_z & 0 \\ B_{0x} & B_{0y} & 0 \end{vmatrix} = B_{0x}^2(\xi_z'' - k^2\xi_z)\hat{\boldsymbol{e}}_z$$

Using the previous result, Eq. (7) can be written as

$$-\omega^2\rho_0\boldsymbol{\xi} = \nabla(\xi_z p_0') - \xi_z\rho_0'\boldsymbol{g} + B_{0x}^2(\xi_z'' - k^2\xi_z)\hat{\boldsymbol{e}}_z \tag{9}$$

Only the x and z component of the previous equation are non-zero:

$$\begin{aligned} -\omega^2\rho_0\xi_x &= ik\xi_z p_0' \\ -\omega^2\rho_0\xi_z &= (\xi_z p_0')' + \xi_z\rho_0' g + B_{0x}^2(\xi_z'' - k^2\xi_z) \end{aligned} \tag{10}$$

The first term on the right hand size of the z component can be rewritten using the x-component and the incompressibility condition (Eq. 8 so that $(\rho_0\xi_x)' = -(\xi_z'\rho_0)'/ik$):

$$(\xi_z p_0')' = -\frac{\omega^2}{ik}(\rho_0\xi_x)' = -\frac{\omega^2}{ik}\left(-\frac{(\xi_z'\rho_0)'}{ik}\right) = -\frac{\omega^2}{k^2}(\xi_z'\rho_0)'$$

so that the second equation in (10), after multiplication by k^2, is written as

$$\boxed{\omega^2\left[(\xi_z'\rho_0)' - k^2\rho_0\xi_z\right] = k^2\xi_z\rho_0' g + k^2 B_{0x}^2(\xi_z'' - k^2\xi_z)} \tag{11}$$

Eq. (11) is our final eigenvalue equation.

1.3 Non-magnetized Rayleigh-Taylor Instability

We first consider the un-magnetized case $\boldsymbol{B}_0 = 0$. Upon multiplying Eq. (11) times ξ_z and integrating in $z \in [-\infty, \infty]$ we obtain:

$$\omega^2\left[\int_{-\infty}^{+\infty}(\rho_0\xi_z')'\xi_z\,dz - k^2\int_{-\infty}^{+\infty}(\rho_0\xi_z^2)\,dz\right] = k^2 g\int_{-\infty}^{+\infty}\rho_0'\xi_z^2\,dz \tag{12}$$

The first integral in square brackets can be integrated by parts assuming that $\xi_z \to \xi_z' \to 0$ at infinity. Eq. (12) gives therefore the desired dispersion relation,

$$\omega^2 = -k^2 g \frac{\displaystyle\int_{-\infty}^{+\infty} \rho_0' \xi_z^2 \, dx}{\displaystyle\int_{-\infty}^{+\infty} \rho_0 [(\xi_z')^2 + k^2 \xi_z^2] \, dz} \tag{13}$$

From Eq. (13) we immediately note that the condition for stability ($\omega^2 > 0$) depends solely on the sign of $\rho_0' = \partial_z \rho_0$. A monotonically decreasing profile of density will then give a *sufficient* condition for stability. On the contrary, a monotonically increasing profile will result in an unstable configuration ($\omega^2 < 0$). In our case, using Eq. (1), density can be represented as

$$\rho_0' = (\rho_T - \rho_B)\delta(z)$$

As a consequence, Eq. (11) for $z \neq 0$ will give an explicit expression for the displacement ξ_z:

$$\xi_z'' - k^2 \xi_z = 0 \quad \Longrightarrow \quad \xi_z = \begin{cases} \xi_z(0) e^{-kz} & \text{for} \quad z > 0 \\ \xi_z(0) e^{kz} & \text{for} \quad z < 0 \end{cases}$$

With this solution, Eq. (13) will simplify to

$$\omega^2 = -k^2 g \frac{(\rho_T - \rho_B)\xi_z(0)^2}{\rho_B \displaystyle\int_{-\infty}^{0-} () + \rho_T \int_{0+}^{\infty} ()} = -k^2 g \frac{(\rho_T - \rho_B)\xi_z(0)^2}{2(\rho_B + \rho_T)k^2 \displaystyle\int_{0+}^{\infty} e^{-2kz} \, dz}$$

And finally:

$$\boxed{\omega^2 = -kg \frac{\rho_T - \rho_B}{\rho_T + \rho_B}} \tag{14}$$

As it can be seen clearly, an instability will occur if the fluid on top (ρ_T) is heavier than the fluid on bottom (ρ_B). A typical example showing the onset of a Rayleigh-Taylor instability is shown in Fig. 3. Perturbations with small wavelenght grow faster than larger ones.

On the contrary, if $\rho_T < \rho_B$ the system will be stable and the oscillation frequency will still be given by Eq. (14):

$$\omega = \sqrt{kg} \sqrt{\frac{\rho_B - \rho_T}{\rho_T + \rho_B}} \tag{15}$$

which corresponds to the *surface gravity waves*, see Fig. 4. In this type of wave motion the restoring force is gravity (sometimes surface tension needs to be considered as well, and in some situations surface tension can dominate gravity) while ρ_B and ρ_T can be identified with the density of water and air. Note that these waves are *dispersive* since the phase speed

$$\frac{d\omega}{dk} = \frac{1}{2} \sqrt{\frac{g}{k}} \sqrt{\frac{\rho_B - \rho_T}{\rho_T + \rho_B}} = \frac{\omega}{2k}$$

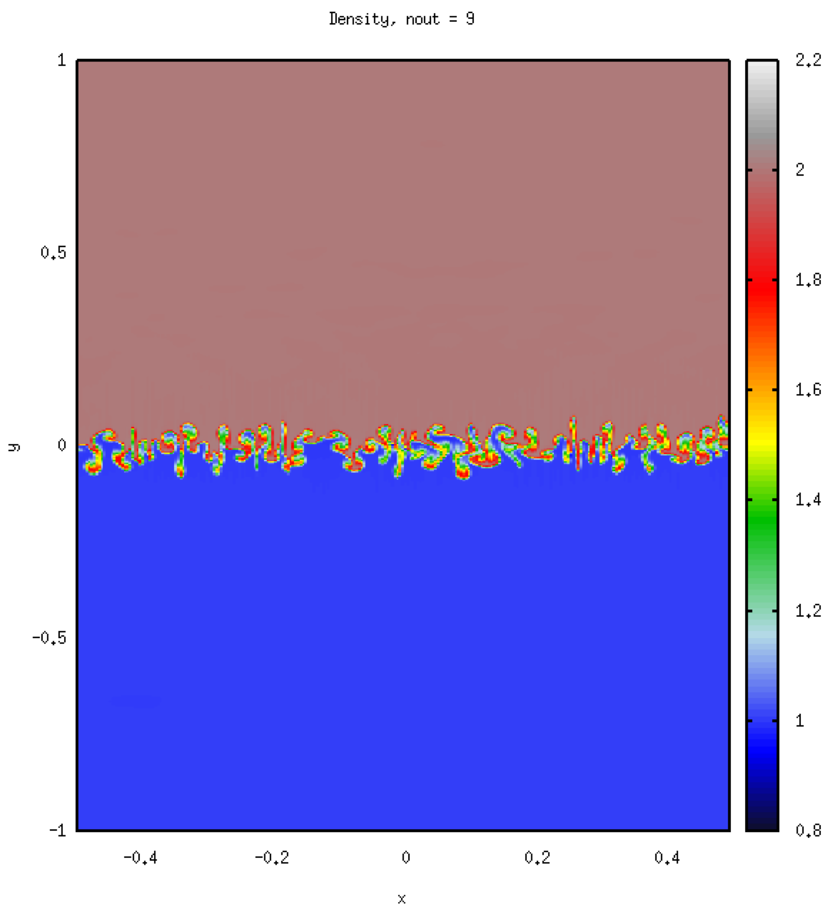

Figure 3: The onset of a Rayleigh-Taylor instability in absence of magnetic field.

is a function of k: hence a pattern of waves of different wavelengths will disperse since each harmonic will propagate at a different speed (large wavelengths will propagate faster). It is important, however, to note that the relation is valid for deep water since we have assumed $\xi_z = 0$ at large distance from the interface. For shallow water, the dispersion relation is modified.

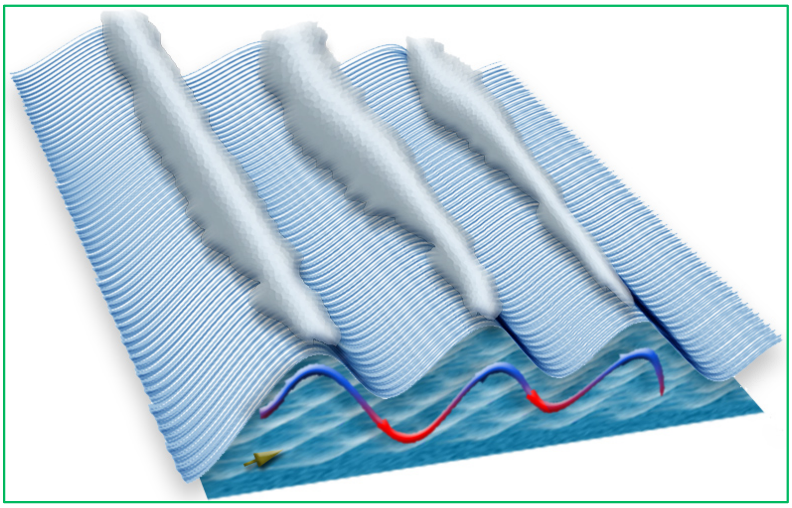

Figure 4: A typical surface gravity wave settling at an air-sea interface in the ocean.

1.4 Magnetized Rayleigh-Taylor Instability

In presence of a magnetic field, we can repeat the previous derivation. The displacement will be same as in the non-magnetized case and the dispersion relation will contain an additional term related to the magnetic field. The final result is

$$\omega^2 = -kg\frac{\rho_T - \rho_B}{\rho_T + \rho_B} + 2\frac{k^2 B_{0x}^2}{\rho_T + \rho_B} = \frac{-kg(\rho_T - \rho_B) + 2(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2}{\rho_T + \rho_B} \tag{16}$$

and shows that a magnetic field parallel to the interface has always a *stabilizing* effect. Note that in the limit of small magnetic fields ($\boldsymbol{B}_0 \to 0$) we obtain again Eq. (14). However, by increasing the field strength we see that small wavelengths (large k) are stabilized earlier. Complete stabilization takes place when the magnetic field is large enough so that the second term becomes larger than the first one. This takes place when

$$B_0^2 \geq \frac{g}{2}\frac{\rho_T - \rho_B}{k\cos^2\theta}$$

where θ denotes the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$. Stabilization is best achieved when $\boldsymbol{k}$ and $\boldsymbol{B}_0$ are parallel since the field line curvature induced by the perturbation will produce a restoring force.

In the particular case of a uniform medium, $\rho_T = \rho_B$, the dispersion relation (16) gives the dispersion relation for an Alfven wave.

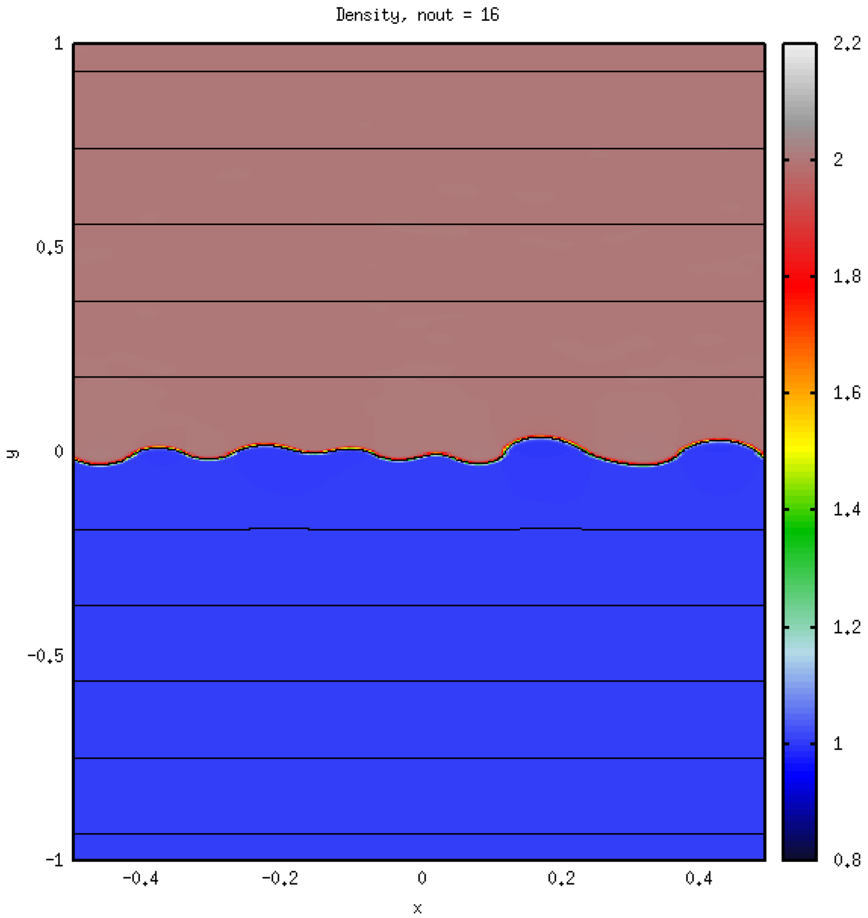

Figure 5: The onset of a Rayleigh-Taylor instability in presence of magnetic field $B = \chi B_c$ with $\chi = 0.3$.