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“An introduction to the Statistical Hadronization Model”
Lecture 1

OUTLINE

- Clusters
- Formulation of the model. Projectors
- Interactions and resonances

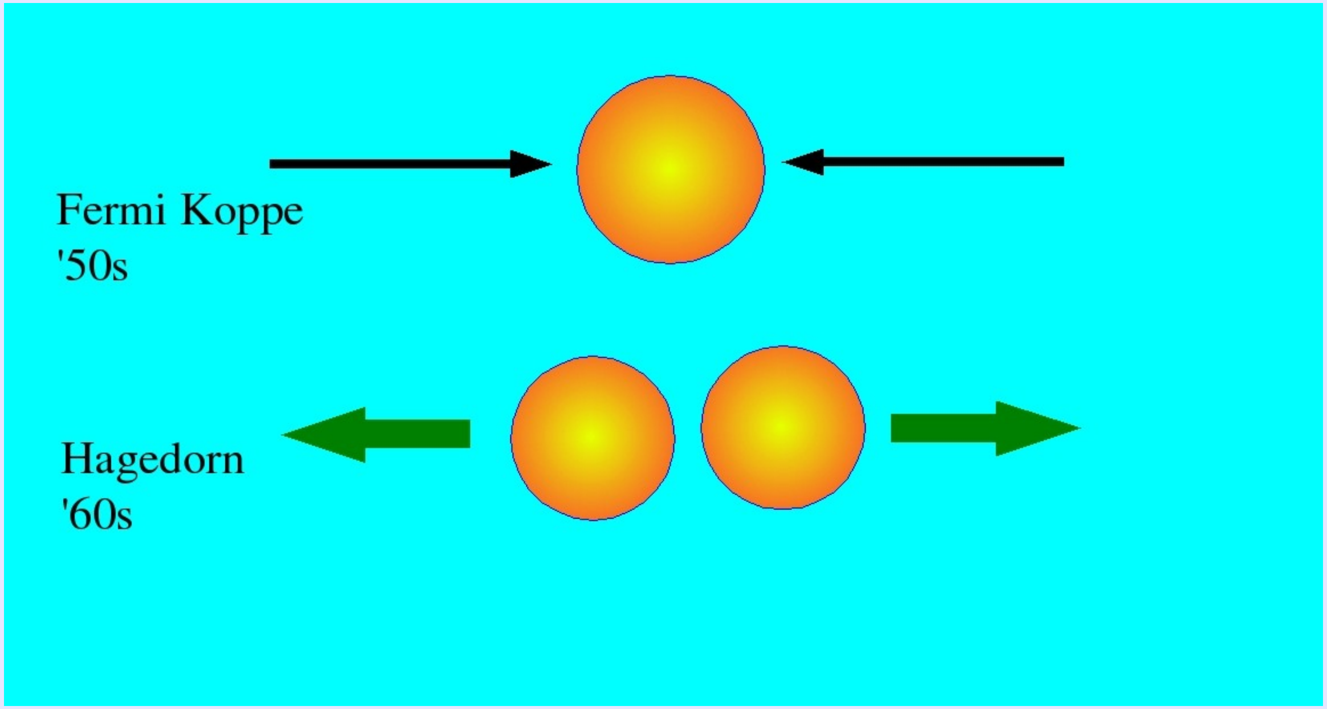
- We are not able to solve QCD in non-perturbative regime

- Non-perturbative dynamics is encoded into universal functions, PDF or fragmentation functions for the hadronization process

- We can use models to obtain this non-perturbative input

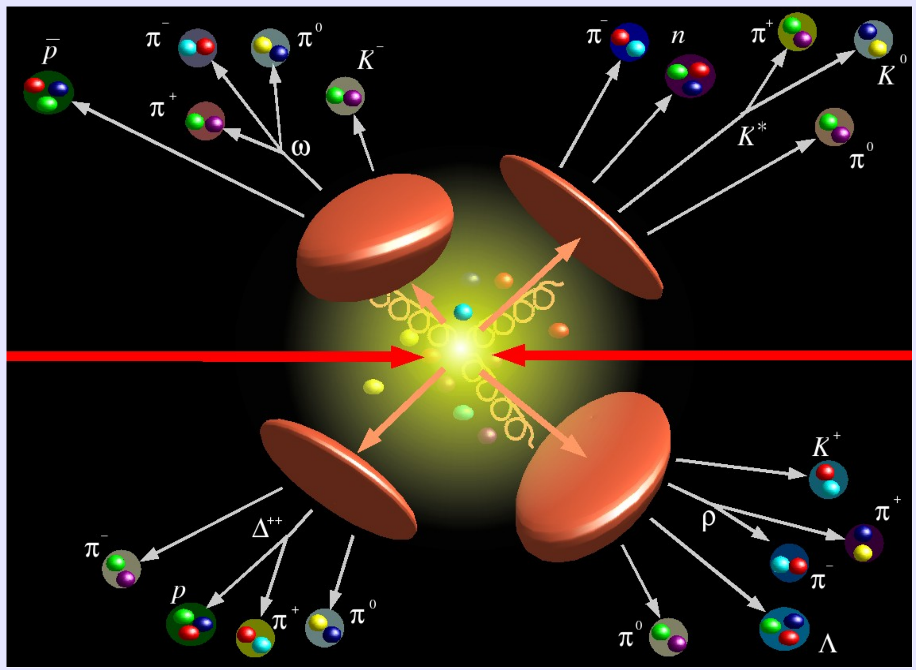
- Hadronization models: string, cluster, statistical

Historical prologue

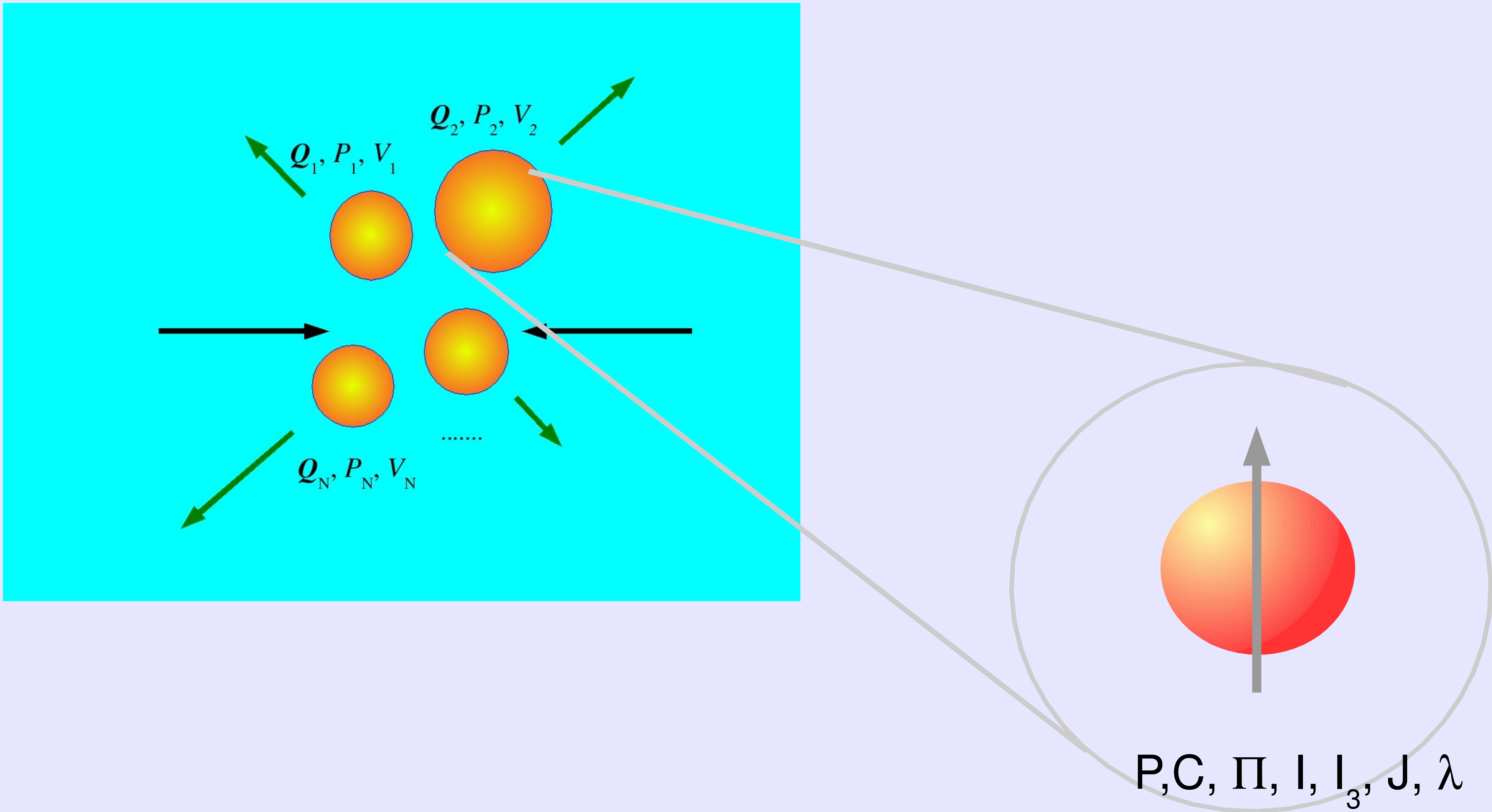


Multiple hadron production proceeds from highly excited regions (clusters or fireballs) emitting hadrons according to a pure statistical law

In modern view, the statistical model is a model of hadronization, describing the process of hadron formation at the scale where QCD is no longer perturbative



The fundamental brick of the SHM is the CLUSTER or FIREBALL: a massive *extended* relativistic object with inner charges



Extension is the key property

Cluster ~ bag of the MIT model (relativistic extended massive object)

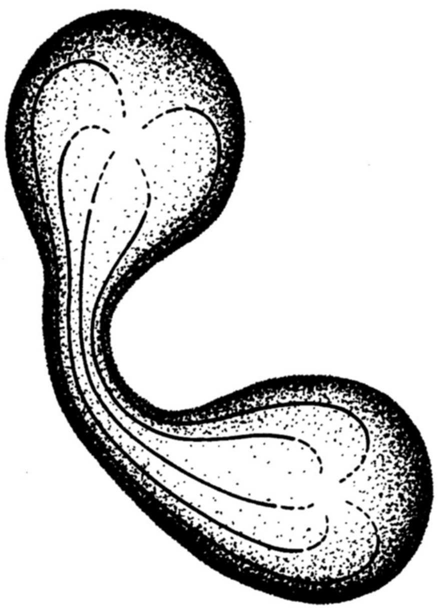


FIG. 1. A color-singlet bag attempting to fission into two bags which are not color singlets. The flux lines of the colored gluon field are shown explicitly.

The statistical hadronization model can be seen as an effective model for the "formidable task" of calculating bag decays

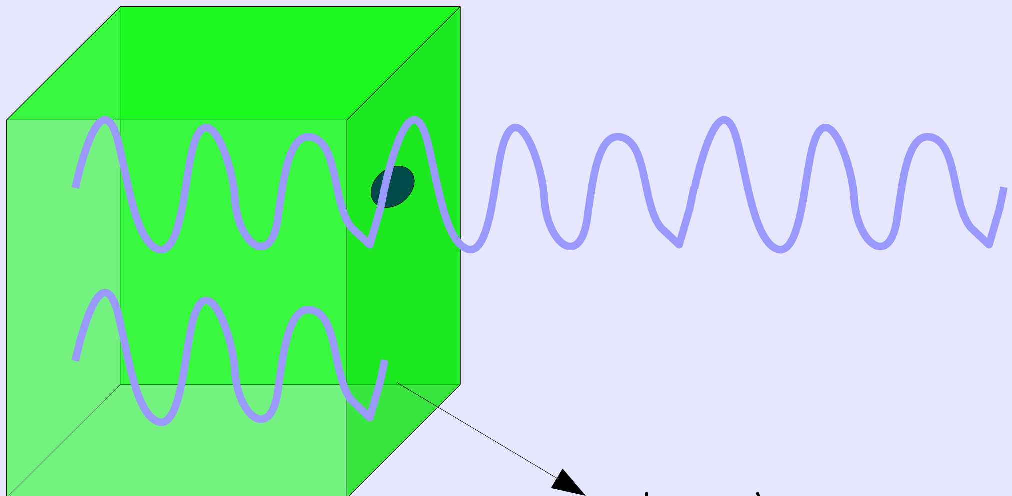
A. Chodos et al., Phys. Rev. D 12 (3471) 1974

The SHM's *urprinzip*

Every localized multihadronic state within the cluster compatible with conservation laws is equally likely

The word “localized” gives finite extension a crucial role

Localized vs asymptotic states



$$|f\rangle = |p_1, \sigma_1; p_2, \sigma_2, \ldots\rangle$$

$$|h_V\rangle$$

described by the occupation numbers of the field in the box (see e.g. bag model)

The distinction is unimportant if the volume is sufficiently larger than (cubic root of) Compton wavelenghts, but it is crucial if they are comparable

Example: one particle in non-relativistic Quantum Mechanics

$$|h_V\rangle = |\mathsf{k}\rangle = \begin{cases} \frac{1}{\sqrt{V}} \exp(i\mathsf{k}\cdot\mathsf{x}) & \text{if } \mathsf{x} \in V \\ 0 & \text{if } \mathsf{x} \notin V \end{cases} \qquad \mathsf{k} = \pi n_x/L_x \pi \hat{\mathsf{i}} + n_y/L_y \hat{\mathsf{j}} + \pi n_z/L_z \hat{\mathsf{k}}$$

$$\langle \mathsf{k}|\mathsf{p}\rangle = \frac{1}{\sqrt{(2\pi)^3 V}} \int_V d^3x \, \mathrm{e}^{i(\mathsf{p}-\mathsf{k})\cdot\mathsf{x}}$$

Quantum Field Theory: localized and asymptotic states differ for the *numbers of quanta*

$$|N\rangle_V = \alpha_{0,N}|0\rangle + \alpha_{1,N}|1\rangle + \ldots + \alpha_{N,N}|N\rangle + \ldots$$

$$|0_V\rangle \neq |0\rangle$$

Casimir effect

One can write non-bijective (Bogoliubov) relations connecting creation operators for the localized and full-space field problems

$$a_{\mathsf{k}} = \int d^3p \, F(\mathsf{k},\mathsf{p}) \frac{\varepsilon_{\mathsf{k}} + \varepsilon_{\mathsf{p}}}{2\sqrt{\varepsilon_{\mathsf{k}}\varepsilon_{\mathsf{p}}}} \, a_{\mathsf{p}} + F(\mathsf{k},-\mathsf{p}) \frac{\varepsilon_{\mathsf{k}} - \varepsilon_{\mathsf{p}}}{2\sqrt{\varepsilon_{\mathsf{k}}\varepsilon_{\mathsf{p}}}} \, a^\dagger_{\mathsf{p}}$$

Translating the postulate into formulae

The cluster is described as a *mixture of states*

$$\hat{\rho} \propto \sum_{h_V} \mathsf{P}_i |h_V\rangle\langle h_V| \mathsf{P}_i \equiv \mathsf{P}_i \mathsf{P}_V \mathsf{P}_i$$

P_i is the projector onto initial cluster's quantum numbers

$$\mathsf{P}_i = \mathsf{P}_{P,J,\lambda,\Pi} \mathsf{P}_\chi \mathsf{P}_{I,I_3} \mathsf{P}_{\mathsf{Q}}$$

P 4-momentum
J spin
λ helicity
π parity
χ C-parity
Q abelian charges
I, *I*$_3$ isospin

P_i can be further factorized if worked out into the rest frame where *P = (M,**0**)*

$$\mathsf{P}_{P,J,\lambda,\pi} = \delta^4(P - \hat{P}) \mathsf{P}_{J,\lambda} \frac{\mathbf{I} + \pi\hat{\Pi}}{2}$$

Can define a probability of observing an asymptotic multi-particle state $|f\rangle$

$$p_f \propto \langle f | \mathsf{P}_i \mathsf{P}_V \mathsf{P}_i | f \rangle$$

This is positive definite and fulfills symmetry requirements.

The microcanonical partition function is recovered summing over all final states

$$\sum_f p_f \propto \sum_f \langle f | \mathsf{P}_i \mathsf{P}_V \mathsf{P}_i | f \rangle \propto \mathrm{tr} \mathsf{P}_V \mathsf{P}_i = \sum_{h_V} \langle h_V | \mathsf{P}_i | h_V \rangle \equiv \Omega$$

The simplest example: again one particle in non-relativistic Quantum Mechanics with energy conservation

$$|h_V\rangle = |\mathsf{k}\rangle = \begin{cases} \frac{1}{\sqrt{V}} \exp(i\mathsf{k}\cdot\mathsf{x}) & \text{if } \mathsf{x} \in V \\ 0 & \text{if } \mathsf{x} \notin V \end{cases} \qquad \mathsf{k} = \pi n_x/L_x \pi \hat{\imath} + n_y/L_y \hat{\jmath} + \pi n_z/L_z \hat{\mathsf{k}}$$

Therefore:

$$\Omega = \sum_{\mathsf{k}} \langle \mathsf{k} | \delta(E - \hat{H}) | \mathsf{k} \rangle = \sum_{\mathsf{k}} \int d^3p \, |\langle \mathsf{k} | \mathsf{p} \rangle|^2 \delta\left(E - \frac{p^2}{2m}\right)$$

Because of the completeness relation: $\displaystyle\sum_{\mathsf{k}} \frac{1}{V} \exp[i\mathsf{k}\cdot(\mathsf{x}-\mathsf{x}')] = \delta^3(\mathsf{x}-\mathsf{x}')$

$$\Rightarrow \quad \sum_{\mathsf{k}} |\langle \mathsf{k} | \mathsf{p} \rangle|^2 = \frac{V}{(2\pi)^3}$$

$$\Rightarrow \quad \Omega = \frac{V}{(2\pi)^3} \int d^3p \, \delta\left(E - \frac{p^2}{2m}\right)$$

Mixture vs pure state

Proper quantum description of a cluster as a *pure state* superposition of localized states

$$|\psi\rangle = \sum_{h_V} c_{h_V} \mathsf{P}_i |h_V\rangle$$

From the postulate: $|c_{h_V}|^2$ independent of h_V

$$\begin{aligned} |\langle f|\psi\rangle|^2 &= |\sum_{h_V} \langle f|\mathsf{P}_i|h_V\rangle c_{h_V}|^2 = \text{const} \sum_{h_V} |\langle f|\mathsf{P}_i|h_V\rangle|^2 \\ &+ \sum_{h_V \neq h'_V} \langle f|\mathsf{P}_i|h_V\rangle\langle h'_V|\mathsf{P}_i|f\rangle c_{h_V} c^*_{h'_V} \end{aligned}$$

same p_f as before

If coefficients c_{hV} have random phases, the interference term vanishes and an effective mixture description is recovered

General formulae for a multi-particle channel

Take $\mathsf{P}_i = \delta^4(P_0 - \hat{P})\delta_{Q_0,\hat{Q}}$

Without quantum statistics

$$\Omega_N = \frac{V^N}{(2\pi)^{3N}} \left(\prod_j \frac{(2S_j+1)^{N_j}}{N_j!} \right) \int d^3p_1 \ldots \int d^3p_N \; \delta^4(P_0 - \sum_i p_i) \langle 0 | \mathsf{P}_V | 0 \rangle$$

With quantum statistics

$$\Omega_{\{N_j\}} = \int d^3\mathrm{p}_1 \ldots d^3\mathrm{p}_N \; \delta^4(P_0 - P_f) \prod_j \underbrace{\sum_{\{h_{n_j}\}}} \frac{(\mp 1)^{N_j+H_j} (2J+1)^{H_j}}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \prod_{l_j=1}^{H_j} F_{n_{l_j}} \langle 0 | \mathsf{P}_V | 0 \rangle$$

partitions of Nj

$$\sum_{n_j=1}^{N_j} n_j h_{n_j} = N_j \qquad \sum_{n_j=1}^{N_j} h_{n_j} = H_j \qquad \sum_j N_j = N \qquad F_{n_l} = \prod_{i_l=1}^{n_l} \frac{1}{(2\pi)^3} \int_V d^3\mathrm{x} \, \mathrm{e}^{i\mathsf{x}\cdot(\mathsf{p}_{c_l(i_l)} - \mathsf{p}_{i_l})}$$

Cluster expansion of the microcanonical partition function

Relativistic invariance and dynamical content

$$\Gamma_N \propto \frac{V^N}{(2\pi)^{3N}} \left(\prod_j \frac{(2S_j+1)^{N_j}}{N_j!} \right) \int d^3p_1 \ldots \int d^3p_N \; \delta^4(P_0 - \sum_i p_i)$$

$$= \frac{1}{(2\pi)^{3N}} \left(\prod_j \frac{(2S_j+1)^{N_j}}{N_j!} \right) \left[\prod_{i=1}^{N} \int d^4p_i \; \Upsilon \cdot p_i \delta(p_i^2 - m_i^2)\theta(p_i^0) \right] \delta^4(P_0 - \sum_i p_i)$$

Four-volume $\Upsilon = (\gamma V, \gamma \mathbf{v} V)$ M. Chaichian, R. Hagedorn, M. Hayashi, Nucl. Phys. B92 (1975) 445

Comparing with the well-known formula

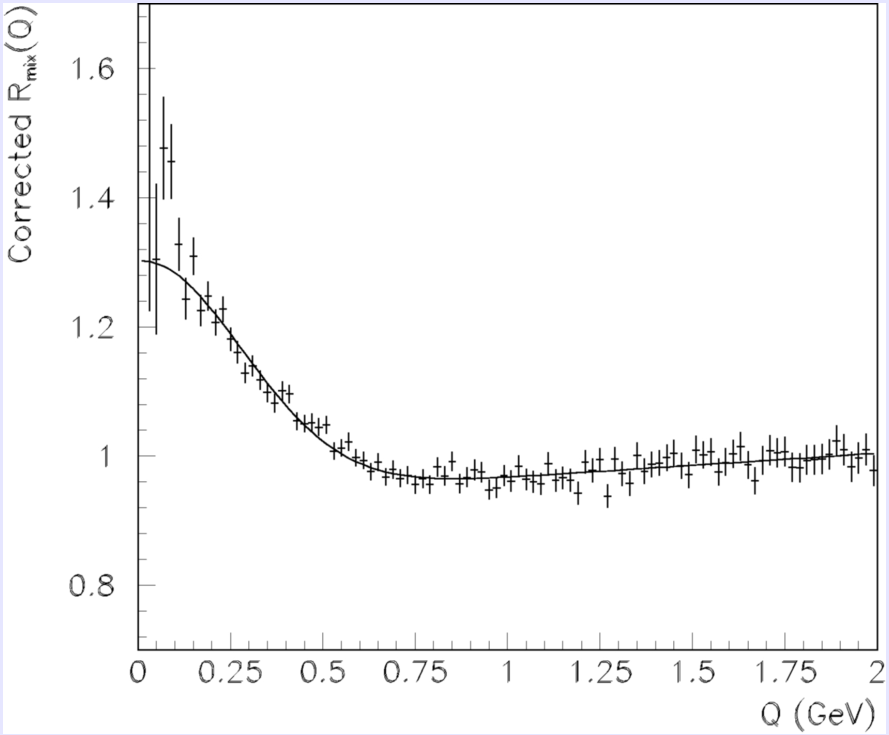
$$\Gamma_N \propto \sum_{\sigma_1,\ldots,\sigma_N} \frac{1}{(2\pi)^{3N}} \left(\prod_j \frac{1}{N_j!} \right) \int \frac{d^3p_1}{2\varepsilon_1} \cdots \int \frac{d^3p_N}{2\varepsilon_N} \; |M_{fi}|^2 \delta^4(P_0 - \sum_i p_i)$$

$$\Rightarrow \quad |M_{fi}|^2 = \prod_{i=1}^{N} \Upsilon \cdot p_i = \frac{1}{\rho^N} \prod_{i=1}^{N} P_0 \cdot p_i$$

Finite volume and quantum correlations

$$\Gamma_{\{N_j\}} \propto \int d^3\mathrm{p}_1 \ldots d^3\mathrm{p}_N \; \delta^4(P_0 - P_f) \prod_j \sum_{\{h_{n_j}\}} \frac{(\mp 1)^{N_j + H_j} (2J+1)^{H_j}}{\prod_{n_j=1}^{N_j} n_j^{h_{n_j}} h_{n_j}!} \prod_{l_j=1}^{H_j} F_{n_{l_j}}$$

Terms beyond leading in the cluster expansion account for BE and FD correlations.
Vanish for $V \to \infty$



These observations support the idea of a finite volume in high energy collisions

e+e- collisions at $\sqrt{s}$ = 91.2 GeV
ALEPH coll., Phys. Rep. 294 (1998) 1

Interactions: hadron-resonance gas

While asymptotic states can only be strongly stable hadrons the energy-momentum projector must include interactions among them

A very nice and powerful theorem established by Dashen-Ma-Bernstein states

R. Dashen, S. K. Ma, H. Bernstein, Phys. Rev. 187 (1969) 345

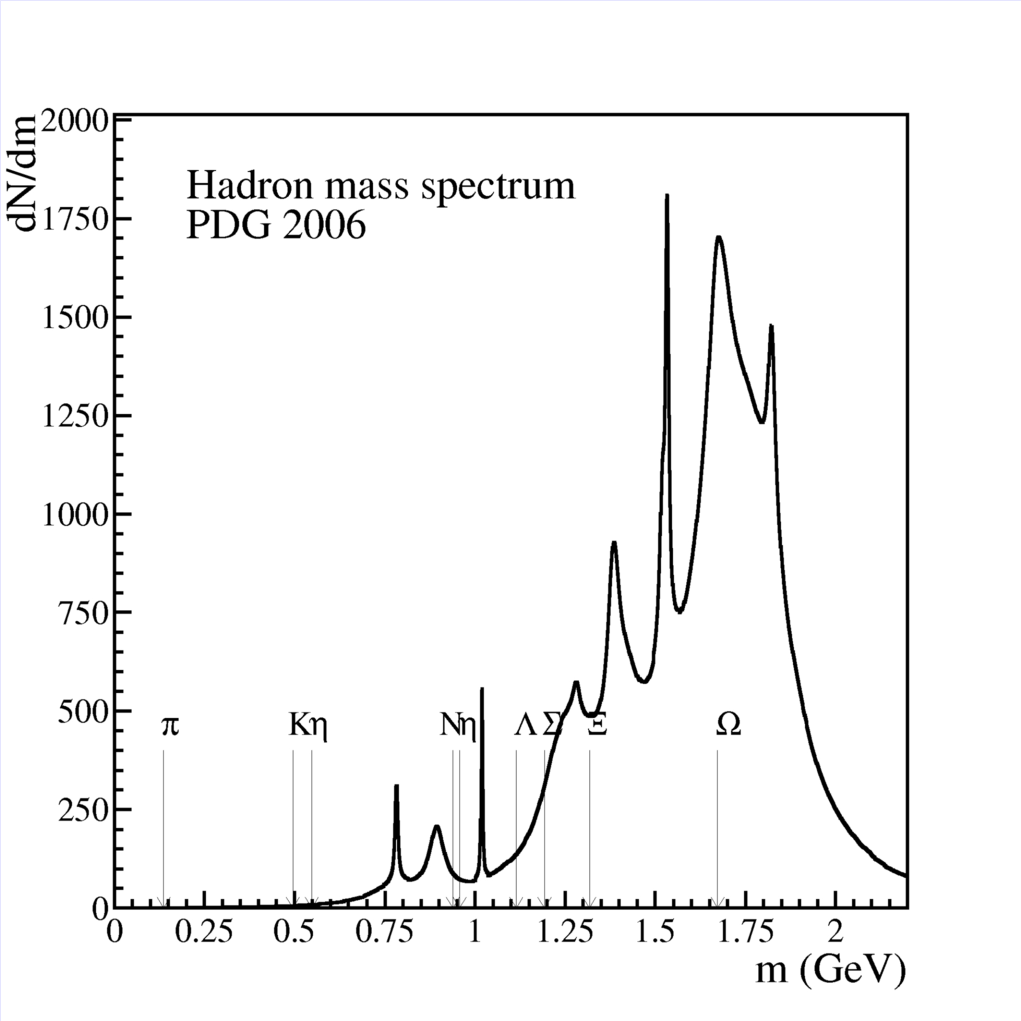
$$\mathrm{tr}\delta^4(P-\hat{P}) = \mathrm{tr}\delta^4(P-\hat{P}_0) + \frac{1}{4\pi i}\mathrm{tr}\left[\delta^4(P-\hat{P}_0)\hat{\mathcal{S}}^{-1}\frac{\overset{\leftrightarrow}{\partial}}{\partial E}\hat{\mathcal{S}}\right]$$

$\hat{\mathcal{S}}$ is the reduced scattering matrix on the energy-momentum P shell

CAVEAT: Such a theorem requires the thermodynamic limit

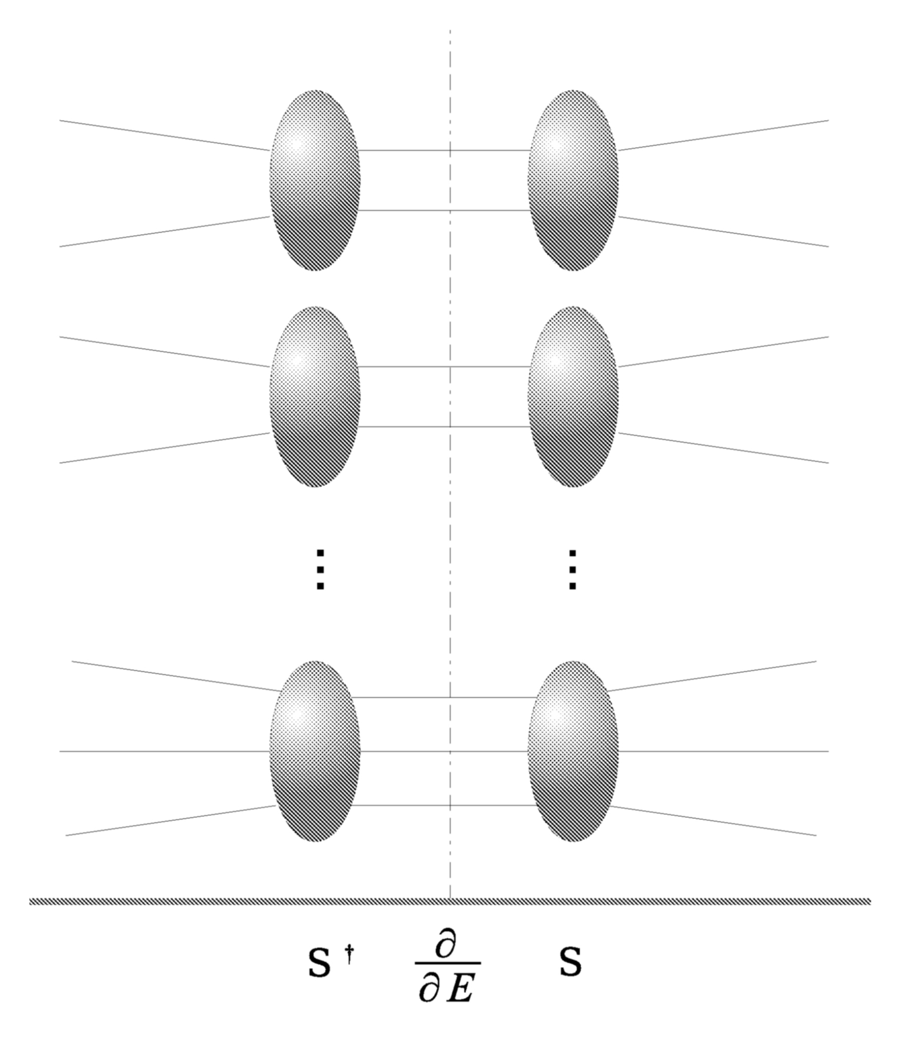
It can be proved that retaining only the resonant part of the interaction (and neglecting resonance interference) the microcanonical partition function reduces to that of a gas of free hadrons and resonances with distributed mass

hadron-resonance gas



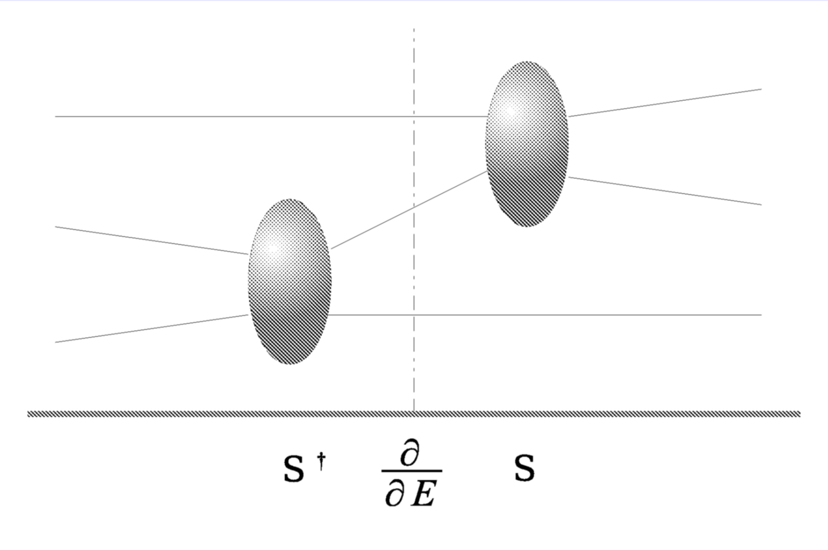
The hadron gas is "the" system where this method applies owing to the very large number of resonances

Energy density (temperature) should be large enough to excite most resonances



The hadron-resonance gas only includes the contribution of symmetric diagrams from the cluster decomposition of the scattering matrix

Non-symmetric diagrams are neglected.
They depend on unknown complex phases
(resonance interference parameters)
and might give an overall vanishing contribution



The theorem states that the trace can be decomposed into two simple terms but it has never been proved whether it applies to single trace terms

$$\mathrm{tr}\delta^4(P-\hat{P}) = \mathrm{tr}\delta^4(P-\hat{P}_0) + \frac{1}{4\pi i}\mathrm{tr}\left[\delta^4(P-\hat{P}_0)\hat{\mathcal{S}}^{-1}\frac{\overleftrightarrow{\partial}}{\partial E}\hat{\mathcal{S}}\right]$$

It is assumed

$$\mathrm{tr}_{\{N_j\}}\delta^4(P-\hat{P}) = \mathrm{tr}_{\{N_j\}}\delta^4(P-\hat{P}_0) + \frac{1}{4\pi i}\mathrm{tr}_{\{N_j\}}\left[\delta^4(P-\hat{P}_0)\hat{\mathcal{S}}^{-1}\frac{\overleftrightarrow{\partial}}{\partial E}\hat{\mathcal{S}}\right]$$

In the resonating approximation all above terms are positive

SUMMARY OF HYPOTHESES FOR HadResGas

- Thermodynamic limit
- Overall vanishing contribution from non-symmetric diagrams
- Validity of DMB theorem for single channels

Summary

Statistical model in general requires calculating quantities in the microcanonical ensemble(s) of the hadron-resonance gas

Take into account finite volume effect in a QFT framework.

Under many circumstances (that we will see), we can confine ourselves to the more manageable canonical and grand-canonical ensembles

NEXT: how to go from microcanonical to grand-canonical ensembles and calculation of inclusive multiplicities