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“An introduction to the Statistical Hadronization Model” Lecture 2

OUTLINE

- MC, C, GC ensembles
- Multiplicities
- Canonical suppression

“Classical” statistical ensembles

Microcanonical: energy and number of particles fixed

$$\Omega = \sum_{states} \delta(E - E_0)\delta_{N,N_0}$$

Canonical: number of particles is fixed, but energy fluctuates because the system is in contact with a reservoir

$$Z = \sum_{states} \exp[-E/T]\delta_{N,N_0}$$

Grand-canonical: both number of particles and energy fluctuate

$$Z_g = \sum_{states} \exp[-E/T]\exp[\mu N/T]$$

Relativistic statistical ensembles

$$\Omega = \sum_{h_V} \langle h_V | \delta^4(\hat{P} - P_0) \delta_{\hat{Q},Q_0} | h_V \rangle$$

$$Z = \sum_{h_V} \langle h_V | \exp[-\beta \cdot \hat{P}] \delta_{\hat{Q},Q_0} | h_V \rangle$$

$$Z_g = \sum_{h_V} \langle h_V | \exp[-\beta \cdot \hat{P}] \exp[\mu \hat{Q}/T] | h_V \rangle$$

These definitions are more rigorous. The previous ones are approximate expressions which are equivalent for sufficiently large volumes, i.e. when this replacement is possible

$$\sum_k \to \frac{V}{(2\pi)^3} \int d^3p$$

For sufficiently large volumes

Microcanonical ensemble: *energy-momentum* and *charges* (additive) fixed

$$\Omega = \sum_{states} \delta^4(P - P_0)\delta_{Q,Q_0}$$

Canonical: fixed charges, energy-momentum fluctuates because in contact with a reservoir

$$Z = \sum_{states} \exp[-\beta \cdot P]\delta_{Q,Q_0} \qquad \beta = 1/T(\gamma, \gamma \mathbf{v})$$

temperature four-vector

Grand-canonical: both charges and energy-momentum fluctuate

$$Z_g = \sum_{states} \exp[-\beta \cdot P] \exp[\mu Q/T]$$

NOTE: partition functions are Lorentz invariants

Relativistic thermodynamics

$$S = \beta \cdot \langle P \rangle + \log Z$$

$$\beta = \frac{1}{T}(\gamma, \gamma \mathsf{v})$$

In the rest frame (subtlety here!) we have the usual relation known from classical thermodynamics

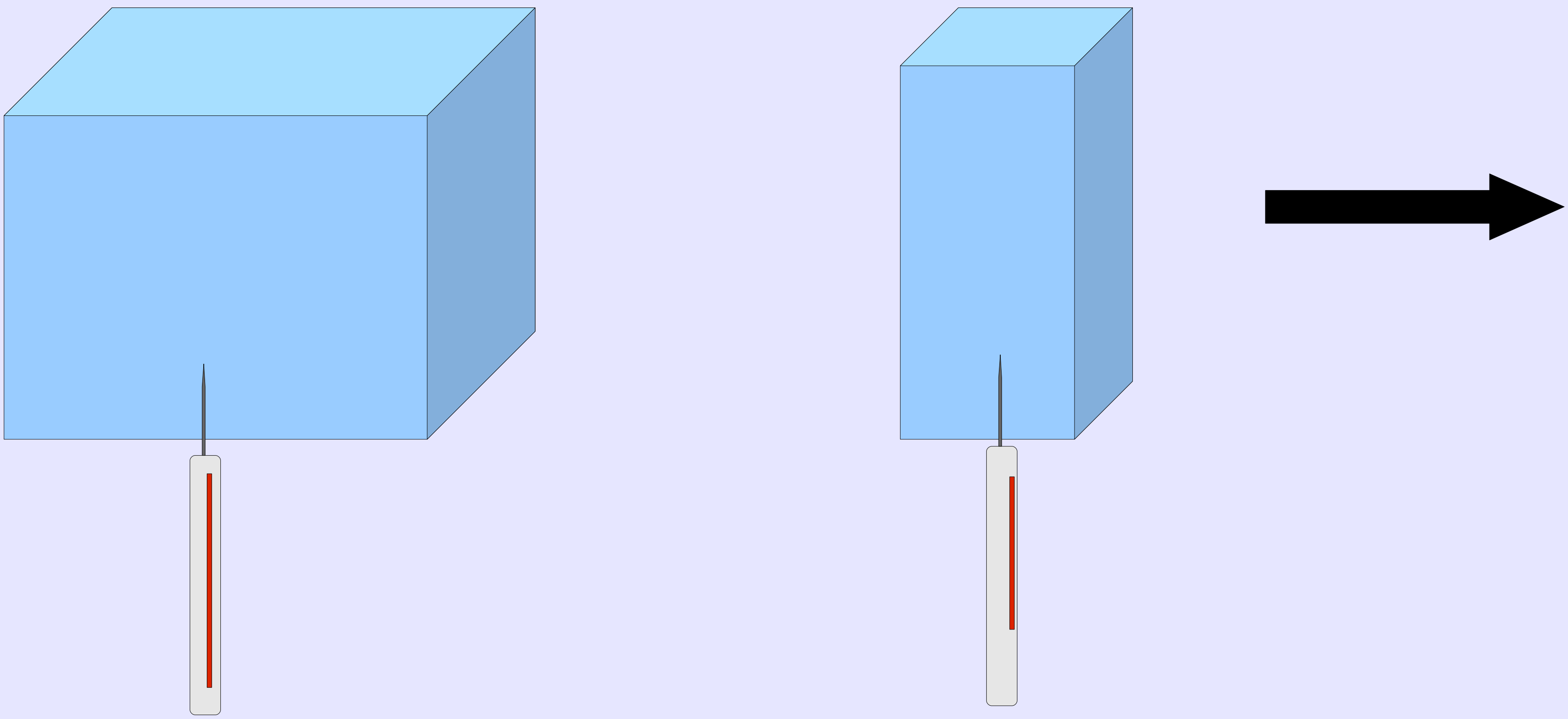
Equilibrium condition between 2 bodies in thermal contact (system and thermometer):

$$\frac{\gamma}{T} = \frac{\partial S}{\partial E} = \frac{\partial S_T}{\partial E_T} = \frac{1}{T_T}$$

IF COMOVING ⟶

$$\frac{\partial S}{\partial \mathsf{P}} = \frac{\partial S_T}{\partial \mathsf{P}_T}$$

Temperature in relativity



Comoving thermometer:
Thermodynamic equilibrium of both energy and momentum
Proper temperature

Thermometer at rest, moving system:
Thermodynamic equilibrium of energy, but not of momentum.
Measured temperature is red-shifted

From microcanonical to grand-canonical ensemble

It can be shown that partition functions of more constrained ensemble can be approximated by those of less constrained ones better and better as extensive parameters like energy and volume increase.

Proof: asymptotic expansion obtained through the saddle point method.

Thereby, one can show that thermodynamic ensembles are equivalent in the thermodynamic limit, $V \to \infty$, but ONLY FOR FIRST MOMENTS
(i.e. quantities which can be expressed as first derivatives of partition functions).
On the other hand, for flucutations, the inequivalence persists in the thermodynamic limit
(many recent papers by Gorenstein, Hauer, Turko etc.)

Example: from micro to canonical

$$\Omega = \sum_{states} \delta(E - E_0) \quad \xrightarrow{\text{L. T.}} \quad Z(z) = \sum_{states} \exp(-zE)$$

$$\Omega = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dz\; e^{zE_0} Z(z) = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dz\; e^{zE_0 + \log Z(z)}$$

$\log Z(z) \propto$ V

Saddle point equation:

$$E_0 + \frac{d \log Z(z)}{dz} = 0$$

Solution z_0 is the inverse of temperature and:

$$\Omega \simeq A \exp[E_0/T + \log Z]$$

Saddle point equation can be read as the equality of initial energy with the average energy in the canonical ensemble

$$E_0 = -\frac{d \log Z(z)}{dz}\Big|_{z=1/T} = \langle U \rangle$$

Also

negligible in the t.d. limit

$$S = \log \Omega \sim \log A + E_0/T + \log Z(V,T)$$

The “canonical” expression of entropy is recovered

It can be shown that this equation is related to the Legendre transformation connecting entropy S to free energy F= -T log Z

From canonical to grand-canonical

$$Z = \sum_{states} \exp[-E/T]\delta_{N,N_0}$$

$$Z = \sum_{states} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, \mathrm{e}^{i(N_0 - N)\phi} \exp[-E/T] = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, \mathrm{e}^{iN_0\phi} Z_g(\mu/T = -i\phi)$$

If N_0 is large and so is the volume, another saddle point expansion is possible.

Let $z = \exp[-i\phi]$

$$\begin{aligned} Z &= \frac{1}{2\pi i} \oint \frac{dz}{z} \, z^{-N_0} Z_g(\lambda = \exp(\mu/T) = z) \\ &= \frac{1}{2\pi i} \oint \frac{dz}{z} \, \exp[-N_0 \log z + \log Z_g(z)] \end{aligned}$$

Saddle-point equation

$$-N_0 + z\frac{\partial \log Z_g}{\partial z} = 0$$

One can recognize the equality between the mean number of particles in the grand-canonical ensemble and the initial fixed number

Furthermore, being $\lambda = \mu/T$

$$Z \simeq A \exp[-N_0\mu/T] Z_g$$

Similar arguments hold for relativistic systems

Example: ideal pion gas in the canonical ensemble

$$Z = \sum_{states} \exp[-E/T]\delta_{Q,Q_0} = \sum_{states} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\ \mathrm{e}^{iQ_0\phi} \exp[-E/T - iQ\phi]$$

NOTE:
this requires volumes larger than $\lambda^3_{\mathrm{Compton}}$

$$E = \sum_{j,k} n_{j,k}\varepsilon_{j,k}$$

$$Q = \sum_{j,k} n_{j,k} q_j$$

$$\begin{aligned} Z &= \sum_{\{n_{j,k}\}} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\ \mathrm{e}^{iQ_0\phi} \exp[-\sum_{j,k} n_{j,k}\varepsilon_k/T - in_{j,k}q_j\phi] \\ &= \sum_{\{n_{j,k}\}} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\ \mathrm{e}^{iQ_0\phi} \prod_{n_{j,k}} \exp[-n_{j,k}\varepsilon_k/T - in_{j,k}q_j\phi] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\ \mathrm{e}^{iQ_0\phi} \prod_{j,k} (1 - \exp[-\varepsilon_k/T - iq_j\phi])^{-1} \end{aligned}$$

$$
\begin{aligned}
Z &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \mathrm{e}^{iQ_0\phi} \prod_{j,k}(1-\exp[-\varepsilon_k/T - iq_j\phi])^{-1} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \mathrm{e}^{iQ_0\phi} \exp\left[\sum_{j,k}\log(1-\mathrm{e}^{-\varepsilon_k/T-iq_j\phi})^{-1}\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \mathrm{e}^{iQ_0\phi} \exp\left[\sum_{j}(2S_j+1)\frac{V}{(2\pi)^3}\int d^3p\ \log(1-\mathrm{e}^{-\varepsilon_k/T-iq_j\phi})^{-1}\right]
\end{aligned}
$$

where we have used the approximation:

$$
\sum_k \to \frac{V}{(2\pi)^3}\int d^3p
$$

which hold only for sufficiently large volumes (see lecture 1)

Boltzmann approximation

Expand the logarithm:

$$\exp\left[\sum_j \frac{(2S_j+1)V}{(2\pi)^3}\int d^3p\ \log(1-\mathrm{e}^{-\varepsilon_k/T-iq_j\phi})^{-1}\right] = \exp\left[\sum_j \frac{(2S_j+1)V}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{1}{n}\int d^3p\ \mathrm{e}^{-n\varepsilon_k/T-inq_j\phi}\right]$$

The first term corresponds to Boltzmann statistics. Holds if m > T,
i.e. if temperature is smaller than 140 MeV.

In usual applications, this approximation is not good for pions, but it is for
other hadrons (T~160 MeV).

Boltzmann approximation

$$Z \simeq \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \, \mathrm{e}^{iQ_0\phi} \exp\left[z_0 + 2z_\pm \cos\phi\right] = \mathrm{e}^{z_0} \mathrm{I}_{Q_0}(2z_\pm)$$

$$z = \frac{(2S+1)V}{(2\pi)^3}\int d^3p \, \mathrm{e}^{-\sqrt{p^2+m^2}/T} = \frac{(2S+1)V}{2\pi^2} m^2 T \mathrm{K}_2(m/T)$$

NOTA $z_\pm \cong z_0$ per T >> some MeV's

Asintoticamente $\mathrm{I}_{Q_0}(2z_\pm) \to \frac{\mathrm{e}^{2z_\pm}}{\sqrt{2z_\pm}}$

if Q_0 fixed. If, on the other hand, Q_0 grows with V being Q_0/V fixed, the limit changes

Multiplicity

In an ideal neutral pion gas in the grand-canonical ensemble, one expects that multiplicities are the same for the three pion species, as long as $\Delta m \ll T$.

But in the canonical ensemble at *finite volume* this is not the case (Canonical Suppression)

$$
\begin{aligned}
\langle n_{j,k} \rangle &= \frac{1}{Z} \sum_{\{n_{j,k}\}} n_{j,k} \exp[-\sum_{j,k} n_{j,k} \varepsilon_k / T] \delta_{Q,Q_0} \\
&= \frac{\partial}{\partial \lambda_{j,k}} \sum_{\{n_{j,k}\}} \lambda_{j,k}^{n_{j,k}} \exp[-\sum_{j,k} n_{j,k} \varepsilon_k / T] \delta_{Q,Q_0} \Big|_{\lambda_{j,k}=1} \\
&= \frac{\partial}{\partial \alpha_{j,k}} \sum_{\{n_{j,k}\}} \exp[-\sum_{j,k} n_{j,k} (\varepsilon_k / T + \alpha_{j,k})] \delta_{Q,Q_0} \Big|_{\alpha_{j,k}=0}
\end{aligned}
$$

RULE: replace, in the partition function

$$\varepsilon/T \to \varepsilon/T + \alpha(\mathsf{p})$$

and take derivatives, either normal or functional

Momentum spectrum:

$$\langle \frac{dn_j}{d^3p} \rangle = \frac{\delta}{\delta\alpha(\mathsf{p})} \log Z[\alpha(\mathsf{p})]\Big|_{\alpha(\mathsf{p})=0}$$

Integrated multiplicity:

$$\langle n_j \rangle = \frac{\partial}{\partial\alpha_j} \log Z(\alpha_j)\Big|_{\alpha_j=0}$$

Multiplicity of charged pions in the canonical ensemble

$$
\begin{aligned}
\langle n_+ \rangle &= \frac{\partial}{\partial \alpha_+} \log Z(\alpha_+) \Big|_{\alpha_+=0} \\
&= \frac{\partial}{\partial \alpha_+} \log \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, \mathrm{e}^{iQ_0\phi} \exp[z_0 + \mathrm{e}^{\alpha_+} z_\pm \mathrm{e}^{-i\phi} + z_\pm \mathrm{e}^{-i\phi}] \Big|_{\alpha_+=0} \\
&= z_\pm \frac{Z(Q_0 - 1)}{Z(Q_0)} \\
&= z_\pm \frac{\mathrm{I}_{Q_0-1}(2z_\pm)}{\mathrm{I}_{Q_0}(2z_\pm)}
\end{aligned}
$$

Quantum statistics effects make this expression more complicated (a series)

What about grand-canonical ensemble ?

$$Z_g = \sum_{states} \exp[-E/T + \mu/TQ] \simeq \exp\left[z_0 + z_\pm \mathrm{e}^{\mu/T} + z_\pm \mathrm{e}^{-\mu/T}\right]$$

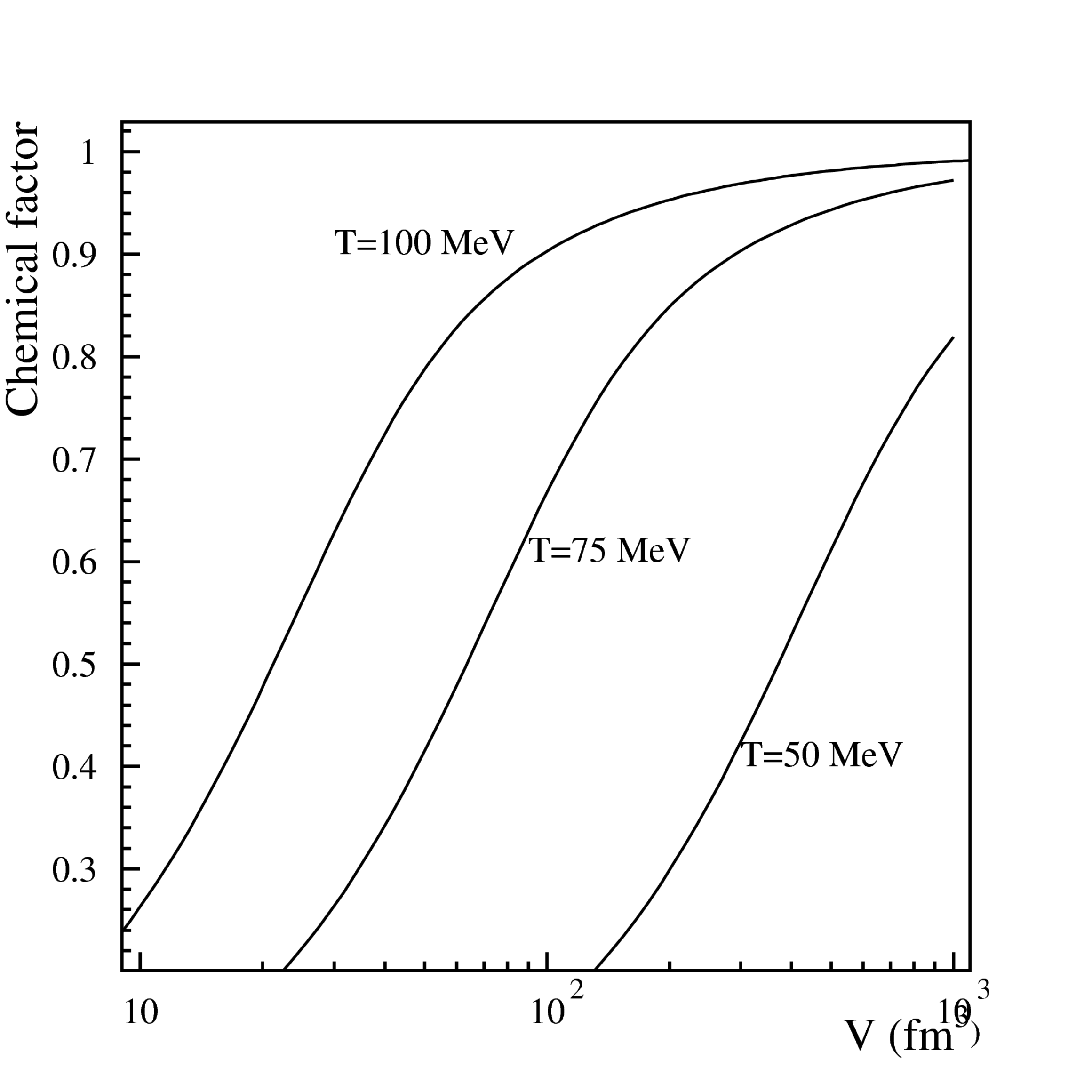
For a completely neutral gas obviously μ=0 and one readily gets:

$$\langle n_+ \rangle = z_\pm$$

whereas in the canonical ensemble

$$\langle n_+ \rangle = z_\pm \frac{\mathrm{I}_1(2z_\pm)}{\mathrm{I}_0(2z_\pm)} < z_\pm$$

Canonical suppression



Grand-canonical value
(thermdynamic limit) is recovered
for volumes of the order
di 1000 fm^3 , strongly dependent
on T

PHYSICAL INTERPRETATION
if you have to conserve a charge
in a finite system, this very fact disfavours
charged particles with respect to neutral
because you need PAIRS.

Generalization to non-abelian groups

Generalized definition of the canonical ensemble:

If the system has some symmetry (SU(2), SU(3)c etc.), the canonical ensemble is the set of states with fixed values of the maximal set of charges (commuting observables) associated with the symmetry group.

More correctly, the canonical ensemble is the set of states which transform according to some specific irreducible vector

Example: J^2, J_z for SU(2) group. This is the maximal set of commuting observables

How can we extend the δ for the charge (U(1) group) to such case?
The simple-minded way would be to write two delta's, one for J^2, and the second for J_z but this is wrong because we would lose the crucial feature that all states should transform under SU(2) according to the irreducible vector $| J^2, J_z >$

To count ALL states transforming according to $| J^2, J_z >$ we MUST project a general state onto this vector by using the projector

$$\mathsf{P} = |J^2, J_z\rangle\langle J^2, J_z|$$

Number of states $$\sum_{states} \langle state|\mathsf{P}|state\rangle$$

Canonical ensemble

$$Z = \sum_{states} \mathrm{e}^{-E_{state}/T} \langle state|\mathsf{P}|state\rangle = \sum_{states} \langle state|\mathrm{e}^{-\hat{H}/T}\mathsf{P}|state\rangle$$

$$Z = \mathrm{tr}[\mathrm{e}^{-\hat{H}/T}\mathsf{P}]$$

General expression of the projector for compact groups (Wigner projector)

$$\mathsf{P}_{\mu,i} = \frac{d^\mu}{M(G)} \int dg\ D^\mu(g^{-1})^i_i U(g)$$

irrep μ irreducible vector $|\mu, i\rangle$ d^μ = irrep dimension

General expression of the canonical partition function

$$Z = \frac{d^\mu}{M(G)} \int dg \; D^\mu(g^{-1})^i_i \mathrm{tr}[U(g)\mathrm{e}^{-\hat{H}/T}]$$

THEOREM: This can be also written as

$$\begin{aligned} Z &= \frac{1}{M(G)} \int dg \; \mathrm{tr} D^\mu(g^{-1}) \mathrm{tr}[U(g)\mathrm{e}^{-\hat{H}/T}] \\ &= \frac{1}{M(G)} \int dg \; \chi^\mu(g^{-1}) \mathrm{tr}[U(g)\mathrm{e}^{-\hat{H}/T}] \end{aligned}$$

Full hadron gas, abelian charges

- Microcanonical ensemble

(ang mom and parity cons'n disregarded)

$$\Omega = \sum_{h_V} \langle h_V | \delta^4(P - \hat{P})\, \delta_{Q,\hat{Q}} | h_V \rangle$$

- Canonical ensemble

$$Z = \sum_{h_V} \langle h_V | \exp(-\hat{H}/T)\, \delta_{Q,\hat{Q}} | h_V \rangle$$

- Grand-canonical ensemble

$$Z_G = \sum_{h_V} \langle h_V | \exp(-\hat{H}/T) \exp(\boldsymbol{\mu} \cdot \hat{\boldsymbol{Q}}/T) | h_V \rangle$$

$$\boldsymbol{Q} = (Q, B, S, C, \ldots) \qquad \boldsymbol{\mu} = (\mu_Q, \mu_B, \mu_S, \ldots)$$

Average multiplicities (Boltzmann statistics limit)

- Grand-canonical ensemble

$$\langle n_j \rangle = \frac{V(2J_j+1)}{(2\pi)^3} \exp(\boldsymbol{\mu}\cdot\boldsymbol{q}_j/T) \int d^3\mathrm{p}\, \exp\left(-\sqrt{\mathrm{p}^2+m_j^2}/T\right)$$

$$Z(\boldsymbol{Q}) \cong Z_G \exp(-\boldsymbol{\mu}\cdot\boldsymbol{Q}/T)$$

- Canonical ensemble

$$\langle n_j \rangle = \frac{V(2J_j+1)}{(2\pi)^3} \int d^3\mathrm{p}\, \exp\left(-\sqrt{\mathrm{p}^2+m_j^2}/T\right) \frac{Z(\boldsymbol{Q}-\boldsymbol{q}_j)}{Z(\boldsymbol{Q})}$$

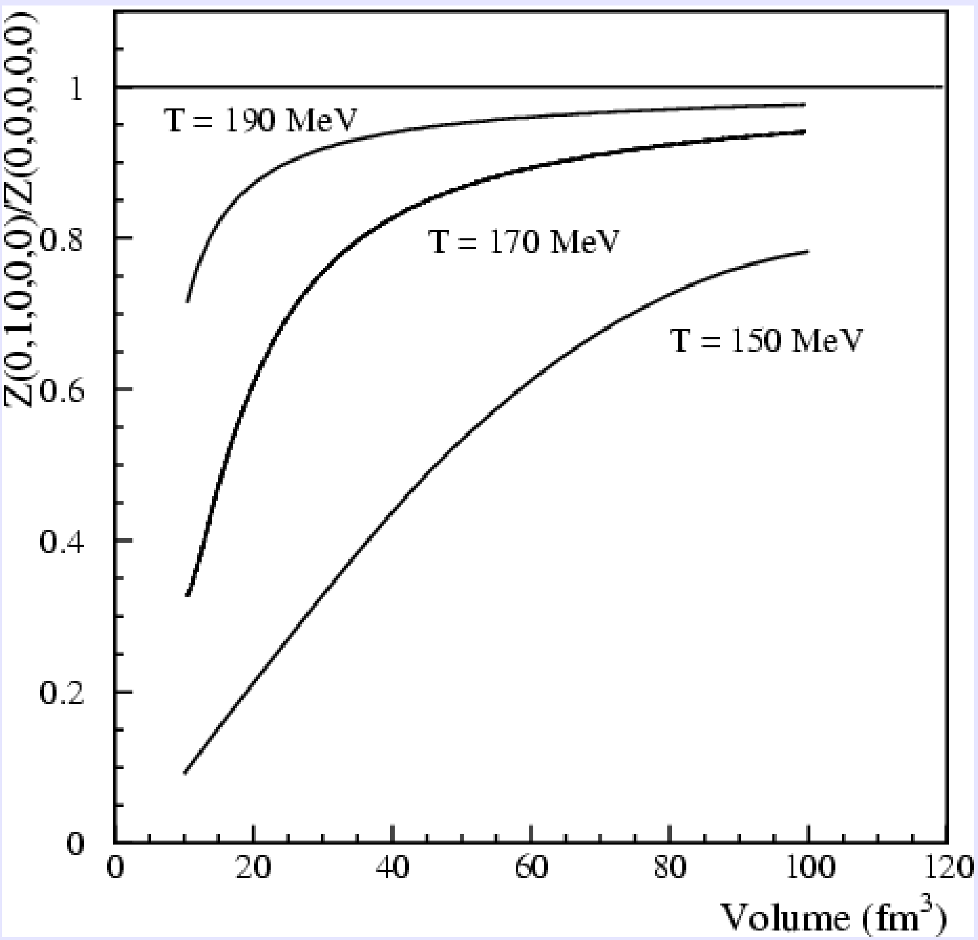
$$\Omega \cong Z(\boldsymbol{Q}) \exp(\beta \cdot P)$$

- Microcanonical ensemble

$$\langle n_j \rangle = \frac{V(2J_j+1)}{(2\pi)^3} \int d^3\mathrm{p}\, \frac{\Omega(P-p_j,\boldsymbol{Q}-\boldsymbol{q}_j)}{\Omega(P,\boldsymbol{Q})}$$

Canonical suppression

Example: neutron chemical factor in a completely neutral cluster



In GC should be:

$$\langle n_j \rangle = \frac{V(2J_j+1)}{(2\pi)^3} \int d^3\mathrm{p} \exp\left(-\sqrt{\mathrm{p}^2+m_j^2}\,/T\right)$$

Whilst in C

$$\langle n_j \rangle = \frac{V(2J_j+1)}{(2\pi)^3} \int d^3\mathrm{p} \exp\left(-\sqrt{\mathrm{p}^2+m_j^2}\,/T\right) \frac{Z(-\boldsymbol{q}_j)}{Z(0)}$$

For $V \to \infty$ $\langle n\rangle_C = \langle n\rangle_{GC}$

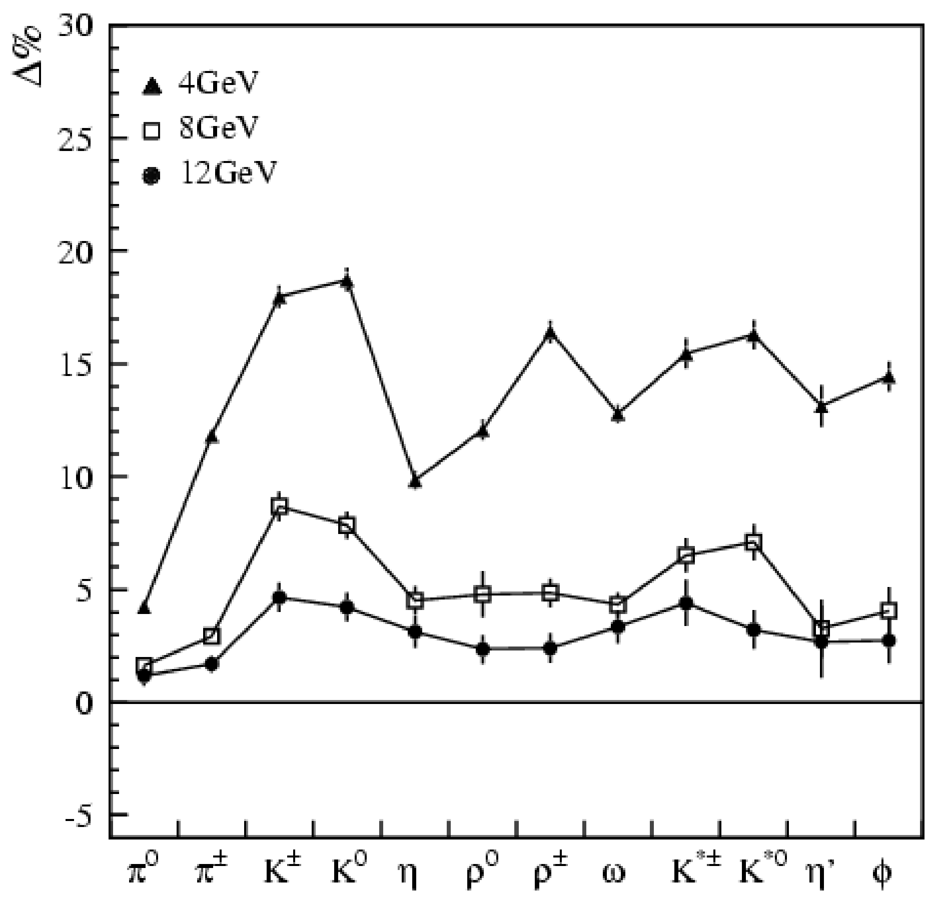
$\langle n\rangle_C < \langle n\rangle_{GC}$

Comparison between μC and C hadron multiplicities

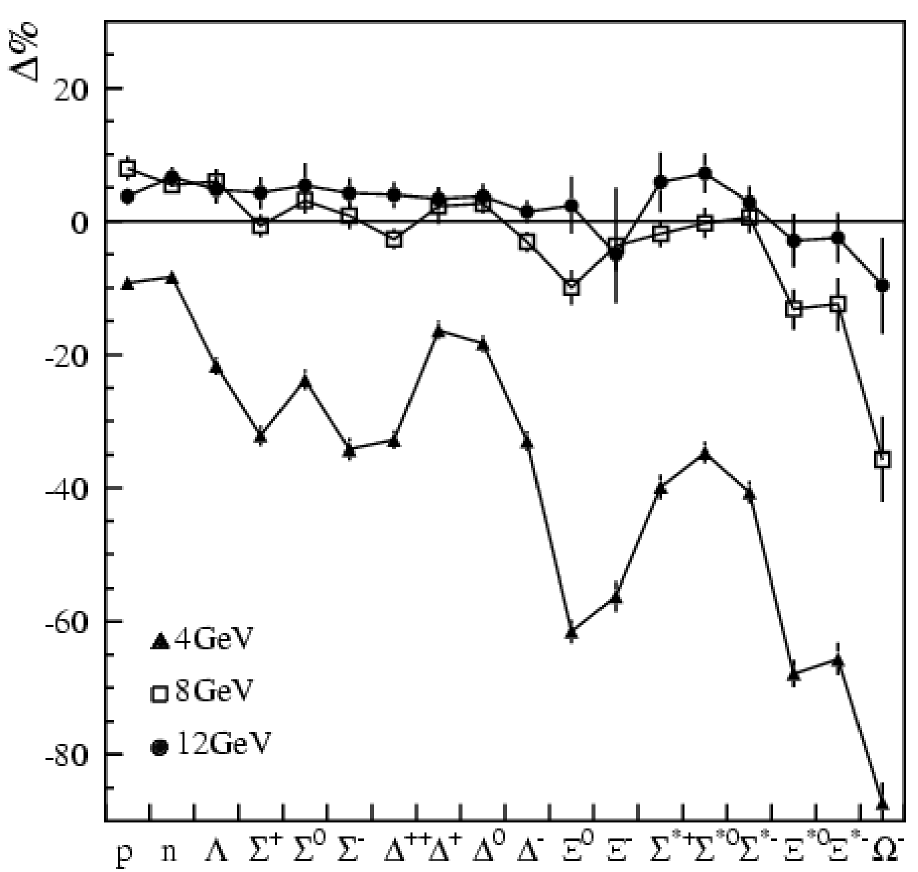
F. B., L. Ferroni, Eur. Phys. J. C 38 (2004) 225

Q=0 cluster, *M/V=0.4* GeV/fm^3

Mesons



Baryons

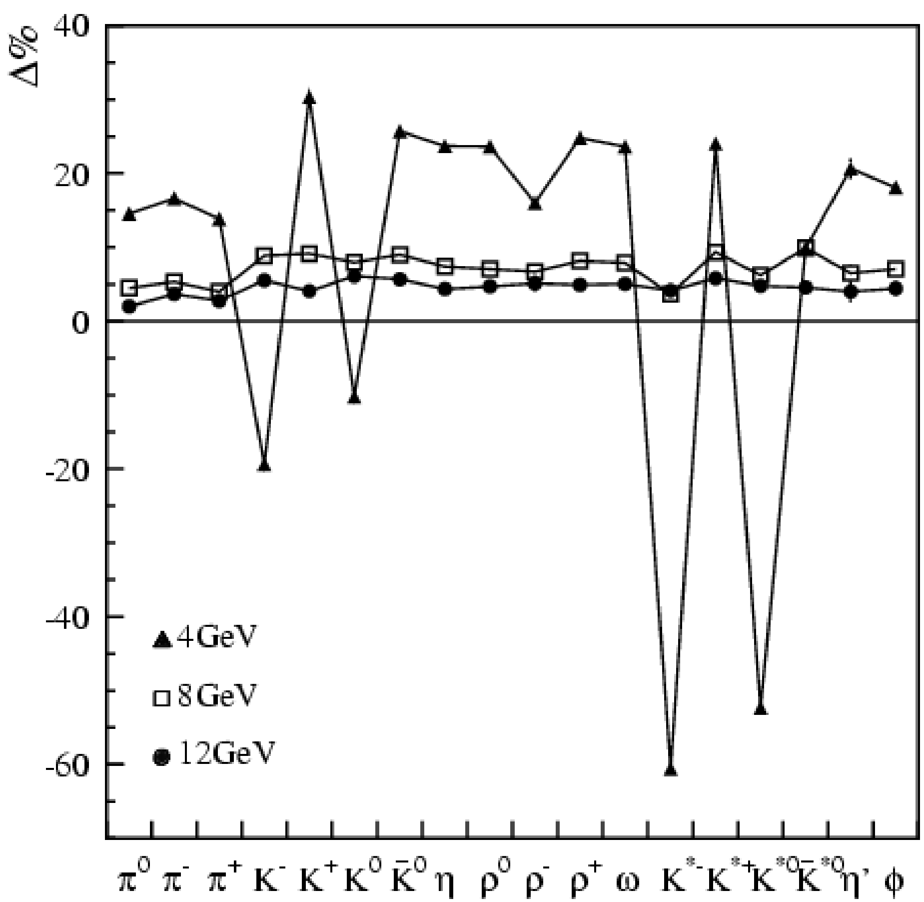


Comparison between μC and C hadron multiplicities

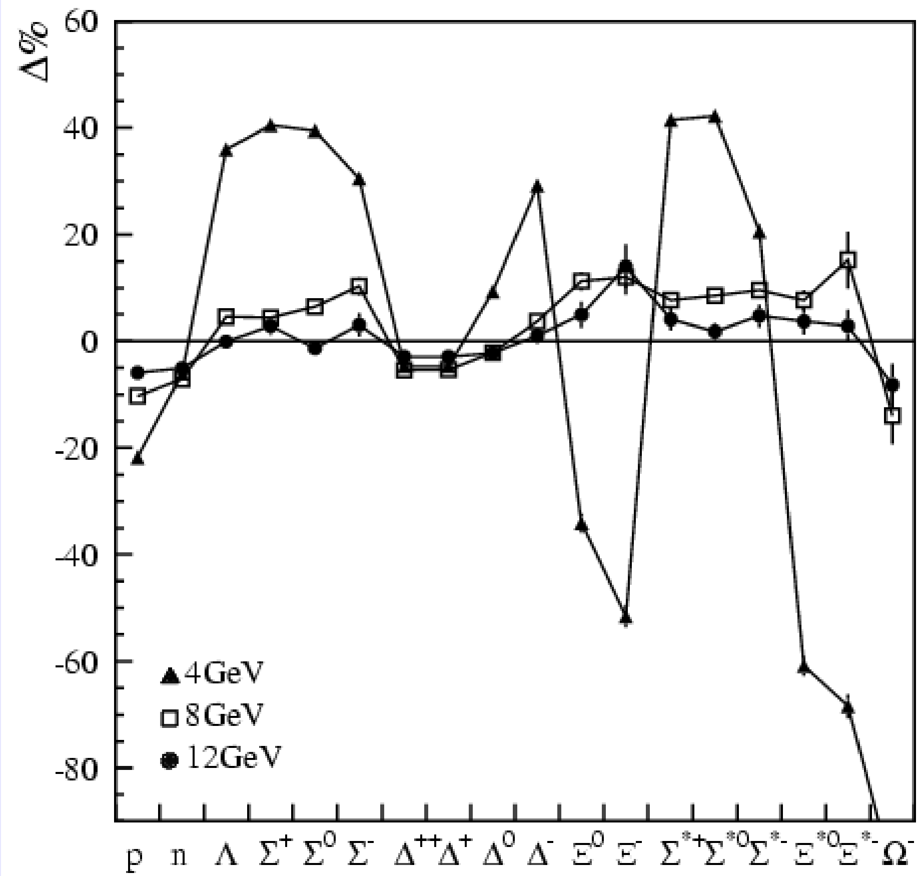
F. B., L. Ferroni,Eur. Phys. J. C 38 (2004) 225

pp-like cluster, *M/V=0.4* **GeV/fm^3**

Mesons



Baryons



Size (Mass, Volume)

Microcanonical ensemble. All conservation laws including energy-momentum (angular momentum, parity), charges enforced.

$V > 20$ fm^3, $M > 10$ GeV (F. Liu et al., Phys. Rev. C 68 (2003) 024905)

F. B., L. Ferroni,Eur. Phys. J. C 38 (2004) 225

Canonical ensemble. Energy and momentum conserved on average, charges exactly. Temperature is introduced

$V > 100$ fm^3, $M > 50$ GeV (A. Keranen, F.B., Phys. Rev. C 65 (2002) 044901)

Grand-canonical ensemble. Also charges are conserved on average. Chemical potentials are introduced

Difficulty of computing