Electroweak Corrections in Gluon Fusion Higgs Production

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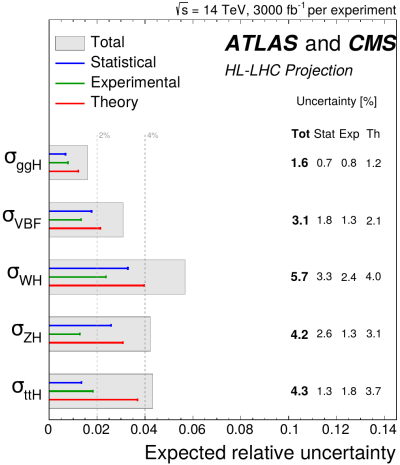
Motivation

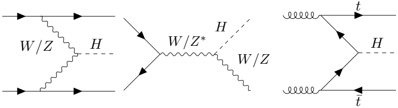
State of the Art Overview: Physics of the HL-LHC Working Group: 1902.00134

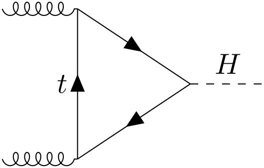


- High Luminosity phase of the LHC will reduce experimental uncertainties
- Predictions dominated my theoretical uncertainties
- Higher order and sub-dominant effects have to be computed

Higgs Production at LHC

State of the Art Overview: Physics of the HL-LHC Working Group et al.: 1902.00134





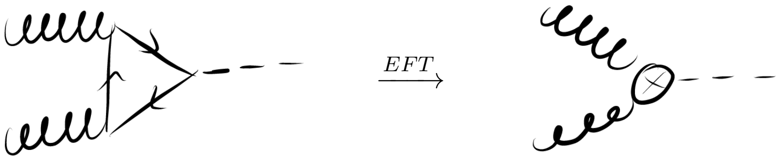
$\underline{ggF}$:

- Virtual massive particles couple to Higgs
- Information about the top Yukawa
- Precisely measured and dominant
- QCD extremely well understood
  - Quark mass effects become relevant
  - EW-correction become relevant ← this talk

Infinite Top Mass

The process that we are looking at is EW corrections to the Higgs production via gluon fusion, computed in the infinite top mass limit.

Effective theory:



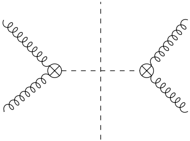
- Remove one loop!
- Work with 5 massless flavors
- Good approximation: $\delta_t^{NNLO} \sim 0.7\%$

Reminder: $\sigma_{gg\to H}$ in EFT

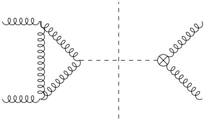
Pure QCD EFT:

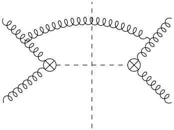
[C. Anastasiou , C. Duhr , F. Dulat , E. Furlan , T. Gehrmann , F. Herzog, , A. Lazopoulos , B. Mistlberger, 2016], [B. Mistlberger, 2018], [F. Dulat , A. Lazopoulos , B. Mistlberger, 2018], (values from iHixs 2 default setting)

- LO $\propto \alpha_s^2\alpha$: $\sigma^{\text{LO}}_{gg\to H} = 15.05$ pb $\propto$



- NLO $\propto \alpha_s^3\alpha$: $\sigma^{\text{NLO}}_{gg\to H} = 18.20$ pb $\propto$



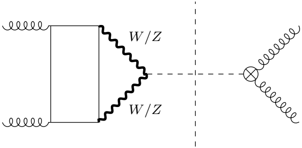


- NNLO $\propto \alpha_s^4\alpha$: $\sigma^{\text{NNLO}}_{gg\to H} = 8.11$ pb
- N^3LO $\propto \alpha_s^5\alpha$: $\sigma^{\text{N3LO}}_{gg\to H} = 1.46$ pb

Slow Convergence

Reminder: $\sigma_{gg\to H}$ in EFT

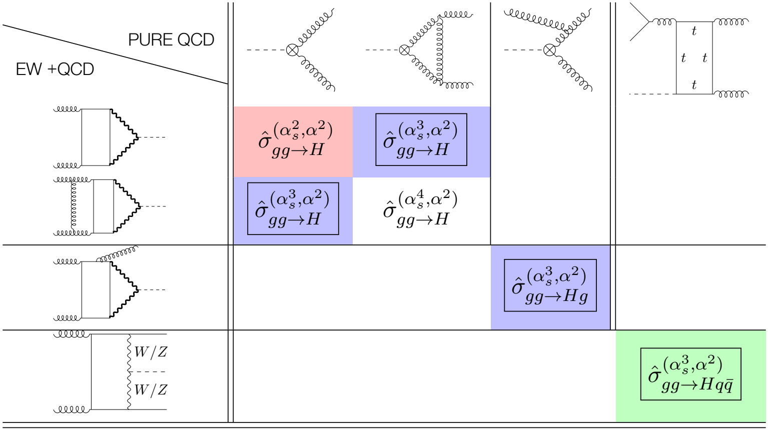
- Elektroweak contributions:
  - "LO:" $\propto \alpha_s^2\alpha^2$



  - Start at 2-loops
  - Involve weak boson masses ($\leftrightarrow$ massless QCD)
    [U. Aglietti, R. Bonciani, G. Degrassi, A. Vicini, 2004], [G. Degrassi and F. Maltoni, 2005], [S. Actis, G. Passarino, C. Sturm, S. Uccirati, 2008]
  - $\sigma_{gg\to H}^{\alpha_s^2\alpha^2} = 0.80$ pb (similar size as $1/2\sigma_{gg\to H}^{\text{N3LO}}$)

- **Do we need higher orders?** If pattern of QCD continues, yes!
  - Naive expectation (factorization): $\sigma_{gg\to H}^{\alpha_s^2\alpha^2} \sim \sigma_{gg\to H}^{\alpha_s^3\alpha^2}$

"Mixed Corrections" in ggF: $\sigma^{(m,n)}_{pp\to H+X} \propto \alpha_s^{m+2}\alpha^{n+1}$



Factorization Hypothesis

Fact: Exact computation often very hard.

Factorization Hypothesis:
NLO mixed QCD-EW~NLO QCD×LO EW

Assumptions:

- Most of the QCD and EW corrections are dominated by:
  - (universal) soft effects (factorize LO)

Task: Compute sensible multiplicative constant (k-factor)

Factorization Hypothesis

Factorization Hypothesis:
NLO mixed QCD-EW~NLO QCD×LO EW

Task: Compute sensible multiplicative constant (k-factor)

Used in $\sigma_{gg\to H}$: "Infinite Boson Mass Approximation"
[C. Anastasiou, R. Boughezal, F. Petriello 2008]

- Consider unphysical limit: $m_{W/Z} \gg m_H$

Factorization

Estimate: $\hat{\sigma}^{(\alpha_s^3,\alpha^2)}_{\text{EXACT}} \approx \frac{\hat{\sigma}^{(\alpha_s^2,\alpha^2)}_{\text{EXACT}}}{\hat{\sigma}^{(\alpha_s^2,\alpha^2)}_{m_{W/Z}\gg m_H}} \hat{\sigma}^{(\alpha_s^3,\alpha^2)}_{m_{W/Z}\gg m_H}$

- Cross-section increase of σ^{NLO}_{QCD} by 5.3% $(\pm 1\%)$
- That is the same size as the pure N^{3}LO-QCD contribution!

Mixed Corrections: Getting Confidence

Soft Gluon Approximation

[M. Bonetti, K. Melnikov, L. Tancredi, 2018]

- Take physical masses for exact virtuals
- Real radiation treated in soft limit: $\propto \alpha_s \eta_{gg}^{fact} \mathrm{d}\hat{\sigma}_{\text{EXACT}}^{(\alpha_s^2, \alpha^2)}$
- Cross-section increase of σ_{QCD}^{NLO} by 5.35%

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Small Boson Mass Approximaten

[C. Anastasiou , V. del Duca , E. Furlan , B. Mistlberger , F. Moriello , A. S. , C. Specchia, 2018]

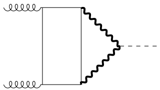
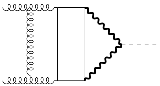
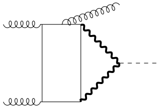
- Consider limit: $m_{W/Z} \ll m_H$
- Complete treatment of non-factorizable contributions (hard real radiation):
  - They are numerically small (non-trivial)
- Cross-section increase of σ_{QCD}^{NLO} by 5.39%

Expectation: The factorizable contributions are dominant.
Unknown: Hard real radiation with massive bosons.

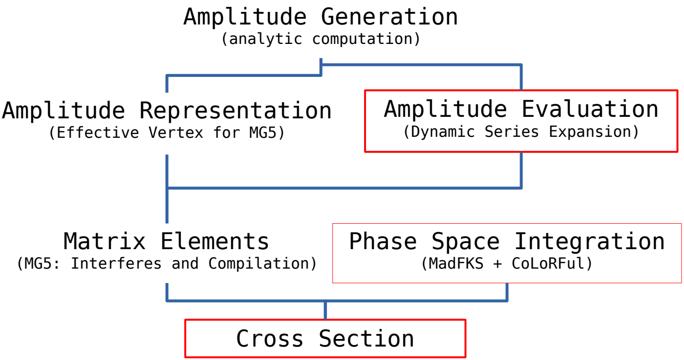
Mixed Corrections In 2020: Exact

They involve:

- Massive 2-loop to order ε^2
- Massive 3-loop (one scale)
- Massive 2-loop (three scales)

×			
Status	✓ (2017) [M. Bonetti, K. Melnikov, L. Tancredi, 2017]	✓ (2017) [M. Bonetti, K. Melnikov, L. Tancredi, 2017]	✗ (2018, planar ones) [M. Becchetti , R. Bonciani , V. Casconi , V. Del Duca , F. Moriello, 2018] ✓ (2020, also non-planar) [M. Bonetti , E. Panzer , V. A. Smirnov , L. Tancredi, 2020] [M. Becchetti , R. Bonciani , V. Del Duca , V. Hirschi , F. Moriello , A. S, 2020]

Overview of Cross Section Computation



- This talk: focus according to frame width
- You can find the code here: `https://bitbucket.org/aschweitzer/mg5_higgs_ew_plugin/`

Quick Facts About Multi-Loop Amplitudes

General Facts:

- There is only a “small” number of Feynman integrals to compute (for us: 48 planar, 61 non-planar)
  [Tkachov 1981],[Chetyrkin, Tkachov 1981],[Laporta 2000]
- Feynman Integrals fulfill (coupled) differential equations
  [A. Kotikov, 1991], [E. Remiddi, 1997],[T. Gehrmann, E. Remiddi, 2002]
- The differential equation can often be made “canonical” (ours can)
  [Henn, 2013]

$$\mathrm{d}\vec{I}^{(j)}(\vec{s}) = \mathrm{d}(A(\vec{s}))\, \vec{I}^{(j-1)}(\vec{s})$$

vector of integrals at order ε^{j-1}

vector of integrals at order ε^{j}

matrix elements are logarithms of the external scales
(NP: 61 different logs, 8 roots)

- Solution iteratively

Multi-Loop Amplitudes: Cross-section requirements

$$\mathrm{d}\vec{I}^{(j)}(\vec{s}) = \mathrm{d}(A(\vec{s}))\ \vec{I}^{(j-1)}(\vec{s})$$

vector of integrals at order ε^{j-1}

vector of integrals at order ε^{j}

matrix elements are logarithms of the external scales
(NP: 61 different logs, 8 roots)

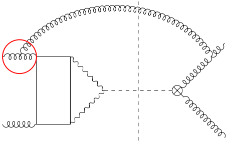
Requirements of the solution for cross-section:

- High numerical precision, (reasonable) speed, available in complete phase space
- Traditional approach
  - Solve in terms of special functions (non-algorithmic, hard)
  - Perform analytical continuation (can be very non-trivial)
  - Use specialized programs for numerical evaluation
- We use: Generalized series expansion approach
  [F. Moriello, 2019]
  - No special functions, analytic continuation trivial
  - Relatively fast, completely algorithmic, arbitrary precision
  - See appendix and Martijns Talk

Summarizing Remarks

- Evaluation such that integral precision $> 10^{-16}$ (dynamic)
- Evaluation time: $< \mathcal{O}(1\text{min})$ up to $\mathcal{O}(30\text{min})$ (deep in IR) with average ~ 1min
- In total ~ 500k points evaluated

Real Radiation and Phase Space Integration



$$\hat{\sigma}^{\alpha_s^3\alpha^2} = \int_{H+g} \mathrm{d}^D\sigma^{\text{real}} + \int_{H} \mathrm{d}^D\sigma^{\text{virtual}}$$

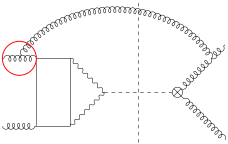
Local cancellation of singular configurations (numeric)

Global cancellation poles in the regulator (analytic)

$$\hat{\sigma}^{\alpha_s^3\alpha^2} = \int_{H+g} \left(\mathrm{d}^4\sigma^{\text{real}} - \mathrm{d}^4\sigma^{\text{CT}}\right) + \int_{H} \left(\mathrm{d}^D\sigma^{\text{virtual}} + \int_{g} \mathrm{d}^D\sigma^{\text{CT}}\right)$$

Proportional to Born

Real Radiation Matrix Element: Phase Space Integration



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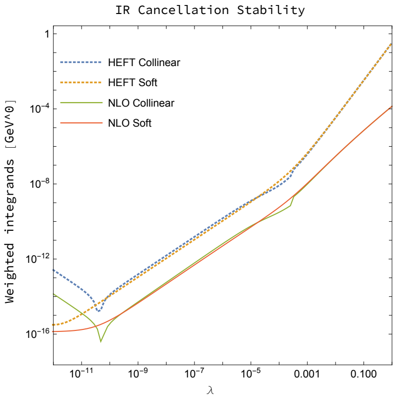
Proportional to Born

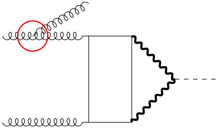
Phase Space Integration

- We use two different subtraction schemes
- FKS subtraction
  [**F**rixione, **K**unszt, **S**igner, 1996],[R. Frederix, S. Frixione, F. Maltoni, T. Stelzer, 2009]
  - Default subtraction scheme of MG5aMC: MadFKS
  - Replace matrix elements of pure HEFT computation
- Colorful Subtraction:
  [G. Somogy, 2005]
  - Implementation: MadN^{k}LO
  - Private implementation by N. Deutschmann, V. Hirschi and S. Lionetti
- Advantage of having versatile standalone evaluations

Stability Near Singularities



Numerical stability of the locally subtracted 2-loop real-emission matrix elements in IR limits. λ is approach parameter. Real-emission matrix near IR limits scales like λ^{-1}. The weighted integrand scales like λ^{α} with $\alpha > \frac{1}{2}$.



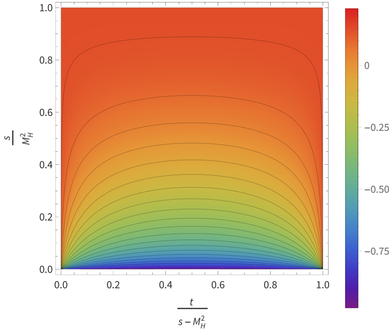
Stability of the subtraction:

- Matrix element scales λ^{-1}
- E.g.: $\lambda \sim 10^{-12} \leftrightarrow p_T \sim \mathcal{O}(\text{MeV})$
- Benchmark: HEFT, limited only by numerical precision
- We see: EW is as stable as HEFT

Result

Exact: Cross-section increase of σ_{QCD}^{NLO} by 5.1%[1]

(all other approximations $5.2 - 5.4\%$)



isolines range is $[-0.75, 0.15]$ in steps of 0.05.

Can we see why approximations work so well?

- Plot of the quantity $$\frac{\left(\mathcal{M}_{gg\to Hg}^{(\alpha_s^3\alpha^2)}/\mathcal{M}_{gg\to Hg}^{(\alpha_s^3\alpha)} - R^{\text{NLO}}\right)}{R^{\text{NLO}}}$$
- $R^{\text{NLO}} = \sigma_{gg\to H+X}^{(\alpha_s^3\alpha^2)}/\sigma_{gg\to H+X}^{(\text{HEFT},\alpha_s^3\alpha)}$
- Basically ratio of matrix elements centered around 0
- Full factorization $\leftrightarrow$ constant everywhere
- $s \to M_H^2$: Ratio quickly stabilizes (factorization!)
- Mass uncertainty resolved

[1] same parameter set as the approximations

Result Overview

Exact: Cross-section increase of σ_{QCD}^{NLO} by 5.1%

What did we do?

- QCD amplitudes in HEFT
- Light Quark corrections with exact boson masses
- NLO in QCD and LO in EW
- Purely gg initiated

What is missing?

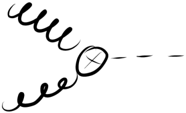
- What about heavy quarks in QCD amplitudes?
- What about the top-quark in EW amplitudes?
- What about higher orders?
- What about channels like e.g. qg?

How do we estimate missing contributions?

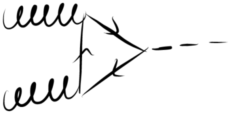
Find appropriate rescaling

Heavy Quarks in QCD amplitudes and EW-amplitudes

Reintroduce heavy quark:



$$\sim V^{\text{(LO)}}_{\text{FIN,HEFT}} \xrightarrow{\text{heavy quark } (M_Q)} V^{\text{(LO)}}_{\text{FIN},M_Q} \sim$$



Rescale cross-section with: $K^{\text{QCD}}_{\text{(N)LO},M_Q} = \dfrac{V^{\text{((N)LO)}}_{\text{FIN},M_Q}}{V^{\text{((N)LO)}}_{\text{FIN,HEFT}}}$

- We estimate with only the interference of the virtuals (V_{FIN})
- LO: rescaling gives exact result
- NLO: missing real radiation (50% uncertainty)

quark	M_Q [GeV]	$K^{\text{QCD}}_{\text{LO},M_Q}$	$K^{\text{VIRT,QCD}}_{\text{NLO},M_Q}$
c	1.3	-0.010	-0.018
b	4.2	-0.042	-0.069
t	173	1.032	1.031

K-factors for different heavy quark masses $M_Q = M_c, M_b, M_t$.

Heavy Quarks in QCD amplitudes and EW-amplitudes

QCD-estimate:

- We estimate with only the interference of the virtuals (V_{FIN})
- LO: rescaling gives exact result
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Top quark in EW amplitudes:

LO: reduction of LO cross-section by $\sim -2\%$.

[G. Degrassi and F. Maltoni, 2005], [S. Actis, G. Passarino, C. Sturm, S. Uccirati, 2008]

NLO: we don't know $\rightarrow$ pure uncertainty of same size

Higher Order QCD

Estimate with pure rescaling (assume full factorization):

$$\overbrace{\sigma^{(\alpha_s^4+\alpha_s^5)\alpha^2}_{gg\to H+X}}^{\text{Mixed NNLO and N}^3\text{LO estimate}} \simeq \frac{\overbrace{\sigma^{(\alpha_s^4+\alpha_s^5)\alpha}_{gg\to H+X}}^{\text{HEFT NNLO and N}^3\text{LO}}}{\underbrace{\sigma^{(\alpha_s^2+\alpha_s^3)\alpha}_{gg\to H+X}}_{\text{HEFT LO and NLO}}} \overbrace{\sigma^{(\alpha_s^2+\alpha_s^3)\alpha^2}_{gg\to H+X}}^{\text{Mixed LO and NLO}}$$

- We know thanks to our exact computation:

$$\overbrace{\sigma^{\alpha_s^3\alpha^2}_{gg\to H+X}}^{\text{Mixed NLO estimate}} \simeq \frac{\overbrace{\sigma^{\alpha_s^3\alpha}_{gg\to H+X}}^{\text{HEFT NLO}}}{\underbrace{\sigma^{\alpha_s^2\alpha}_{gg\to H+X}}_{\text{HEFT LO}}} \overbrace{\sigma^{\alpha_s^2\alpha^2}_{gg\to H}}^{\text{Mixed LO and NLO}} \quad \text{VS} \quad \sigma^{\alpha_s^3\alpha^2}_{gg\to H+X,\text{exact}}$$

- (Naive) Factorization (overestimates) mixed NLO by $\sim 15\%$ $\leftrightarrow$ Uncertainty

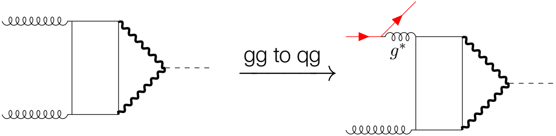
Best estimate for Gluon Fusion So Far

$$\sigma^{\text{(EW,best)}}_{gg\to H+X} = \begin{cases} 2.17 \pm 0.18\ \text{pb} & \mu_{R/F} = M_H \\ 2.02 \pm 0.14\ \text{pb} & \mu_{R/F} = \frac{1}{2} M_H \end{cases}$$

- Only including gg-initiated diagrams increases μ_F dependence
- Example of breakdown for $\mu_{R/F} = M_H$

$$\begin{aligned} \sigma^{\text{(EW,best)}}_{gg\to H+X} = \sigma^{(\text{HEFT},\alpha_s^2\alpha+\alpha_s^3\alpha)}_{gg\to H+X} \times \Big(& \\ & 4.81\ \% \quad \text{(our computation)} \\ + & \ 0.15 \pm 0.04\ \%\ \text{(top mass effects in QCD amp.)} \\ - & \ 0.27 \pm 0.09\ \%\ \text{(bottom quark effects in QCD amp.)} \\ - & \ 0.07 \pm 0.02\ \%\ \text{(charm quark effects in QCD amp.)} \\ - & \ 0.04 \pm 0.04\ \%\ \text{(top quark effects in EW)} \\ + & \ 2.5 \pm 0.4\ \%\ \text{(QCD higher orders)}\Big) \\ = & \ \sigma^{(\text{HEFT},\alpha_s^2\alpha+\alpha_s^3\alpha)}_{gg\to H+X} \times \left(7.11 \pm 0.6\ \% \right) \\ = & \ 2.17 \pm 0.18\ \text{pb} \end{aligned}$$

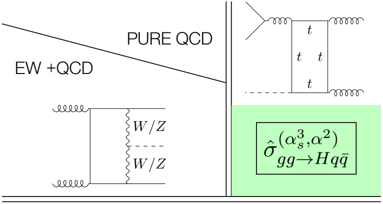
Estimate for qg-Channel



- Behavior near threshold should be well approximated by gluon initiated LO
- Factorization hypothesis of EW in qg-channel not known
- Used rescaling:

$$\sigma^{\text{(EW)}}_{qg} \simeq \frac{\overbrace{\sigma^{\alpha_s^2\alpha^2}_{gg\to H}}^{\text{LO mixed QCD-EW}}}{\underbrace{\sigma^{\alpha_s^2\alpha}_{gg\to H}}_{\text{HEFT LO}}}\,\sigma^{\text{(HEFT)}}_{qg} \to \sigma^{\text{(EW)}}_{qg} = \begin{cases} -0.10 \pm 0.05 \text{ pb}\ ;\ \mu_{R/F} = M_H \\ \ \ 0.12 \pm 0.05 \text{ pb}\ ;\ \mu_{R/F} = \frac{M_H}{2} \end{cases}$$

Other Suppressed Contributions



example of relevant one-loop contribution

- Relevant one-loop contributions
  [V. Hirschi, S. Lionetti, A.S.,2019]

$$\sigma^{(\text{EW})}_{\text{1-loop}} = \begin{cases} 0.025 \pm 0.052 \text{ pb} \; ; \; \mu_{R/F} = M_H \\ 0.031 \pm 0.062 \text{ pb} \; ; \; \mu_{R/F} = \frac{M_H}{2} \end{cases}$$

- "Squared" electroweak contributions

$$\sigma^{(\text{squared EW})}_{gg\to H+X} = \begin{cases} 0.018 \pm 0.005 \text{ pb} \; ; \; \mu_{R/F} = M_H \\ 0.020 \pm 0.005 \text{ pb} \; ; \; \mu_{R/F} = \frac{M_H}{2} \end{cases}$$

Conclusion

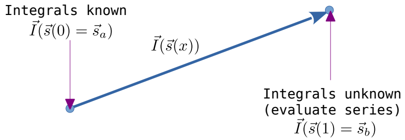
- Shown the possibility to compute a cross-section without using special functions
- Removed the uncertainty due to weak boson mass effects
- Verified the factorization hypothesis
- Included estimates for suppressed effects

The effort may be summarized by:

$$\sigma^{\text{(EW,best)}}_{pp\to H+X} = \begin{cases} (6.91 \pm 0.9\%) \times \sigma^{(\text{HEFT},\alpha_s^2\alpha+\alpha_s^3\alpha)}_{gg\to H+X} \; ; \; \mu_{R/F} = M_H \\ (6.43 \pm 0.8\%) \times \sigma^{(\text{HEFT},\alpha_s^2\alpha+\alpha_s^3\alpha)}_{gg\to H+X} \; ; \; \mu_{R/F} = \frac{M_H}{2} \end{cases} = \begin{cases} 2.11 \pm 0.28 \text{ (theory) pb} \; ; \; \mu_{R/F} = M_H \\ 2.19 \pm 0.26 \text{ (theory) pb} \; ; \; \mu_{R/F} = \frac{M_H}{2} \end{cases} .$$

Series Expansion Idea: F. Moriello, ’19

Key Ideas: Translate multi-scale to single scale problem and solve by series expansion.



- At order ε^j:

$$\mathrm{d}\vec{I}^{(j)}(\vec{s}) = \mathrm{d}(A(\vec{s}))\,\vec{I}^{(j-1)}(\vec{s}) \qquad (\vec{s}\text{: } n \text{ kinematic invariants})$$

$$\downarrow \vec{s} \mapsto \vec{s}(x) = \vec{s}_a + (\vec{s}_b - \vec{s}_a)\,x \qquad (\text{parametrization: } x \in [0,1])$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{I}^{(j)}(x) = \underbrace{\left(\sum_{i=1}^{n} \partial_{s_i} A(\vec{s}(x))\frac{\mathrm{d}s_i}{\mathrm{d}x}\right)}_{=:A(x)} \vec{I}^{(j-1)}(x)$$

expand around x_k and integrate

Series Expansion Idea: F. Moriello, '19

Key Ideas: Translate multi-scale to single scale problem and solve by series expansion.

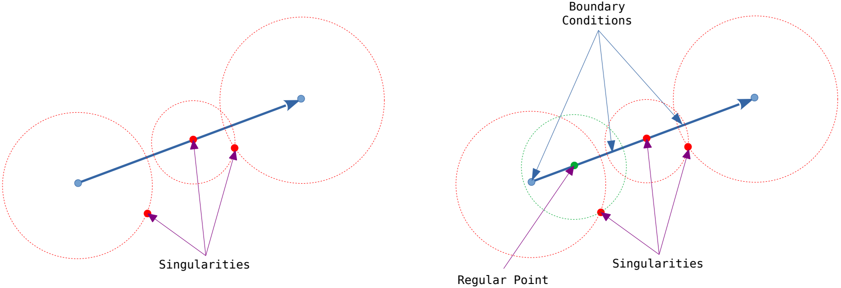
- At order ε^j: $\frac{\mathrm{d}}{\mathrm{d}x}\vec{I}^{(j)}(x) = A(x)\vec{I}^{(j-1)}(x)$ → expand around x_k and integrate
  - $\vec{I}_k^{(0)} = \overrightarrow{\text{const}}$ → first order in ε-expansion is constant
  - $A(x) \simeq \sum_{i=-2}^{i_{max}} A_{(i,k)}(x-x_k)^{\frac{i}{2}}$ → $A_{(i,k)}$ constant matrices
  - $\vec{I}_k^{(j)}(x) \simeq \sum_{i=-2}^{i_{max}} A_{(i,k)} \underbrace{\int (x-x_k)^{\frac{i}{2}} \vec{I}_k^{(j-1)}(x)\mathrm{d}x}_{\text{trivial: } \int (x-x_k)^{\frac{i}{2}} \log^j(x-x_k)\mathrm{d}x} + \vec{c}_{(k)}$
- Along the line **every** integral is **only** a generalized power series

$$\vec{I}_k^{(j)}(x) \simeq \sum_{i_1=0}^{i_{max}} \sum_{i_2=0}^{N_{j,k}} (x-x_k)^{\frac{i_1}{2}} \log^{i_2}(x-x_k)\, \underbrace{\vec{c}_k^{(j,i_1,i_2)}}_{\text{const}}$$

- i_{max}: expansion order which determines precision
- Completely algorithmic, arbitrary precision
- Dynamic: store points, they become new boundary points → grid

Series Expansion: Convergence

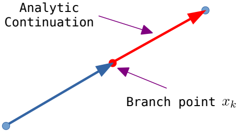
Key Idea: Patch the complete line with multiple expansions.



- Introduce additional (regular) expansion points

Series Expansion: Analytic Continuation

Key Ideas: **Define** physical thresholds, **Verify** spurious (or anomalous) thresholds



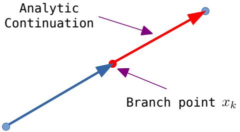
“Physical threshold”: Root or Logarithm

- Linear in one Mandelstam variable: $c_1 - c_2 s_n$
- E.g.: $\log\left(4m_W^2 - s\right)$ or $\sqrt{4m_W^2 - s}$
- Inherit Feynman prescriptions $+i\delta$: $s_n(x) = \alpha_n + \beta_n x \to \alpha_n + \beta_n x + i\delta$
- Implies: $x \to x + \text{sign}(\beta_n) i\delta$

$$\vec{I}_k^{(j)}(x) \simeq \sum_{i_1=0}^{i_{max}} \sum_{i_2=0}^{N_{j,k}} (x - x_k)^{\frac{i_1}{2}} \log^{i_2}(x - x_k) \underbrace{\vec{c}_k^{(j,i_1,i_2)}}_{\text{const}}$$

Series Expansion: Analytic Continuation

Key Ideas: **Define** physical thresholds, **Verify** spurious (or anomalous) thresholds



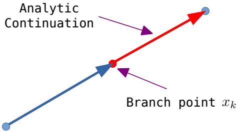
"Spurious Threshold": Logarithm

- Algebraic function in one or multiple Mandelstam variables
- Real life example:
  log $\left(\sqrt{4m_W^2\left(m_H^2-s\right)\left(m_H^2-t\right)+m_H^4\left(-m_H^2+s+t\right)}-m_H^2(s+t)+m_H^4+2st\right)$
- **Verify:** vanishes upon integration
  (similar for root, but there we verify for pre-canonical integrals)

$$\vec{I}_k^{(j)}(x) \simeq \underbrace{\sum_{i_1=0}^{i_{max}} (x-x_k)^{i_1}\vec{c}_k^{\,(j,i_1)}}_{\text{regular}} + \sum_{i_2=1}^{N_{j,k}} \log^{i_2}(x-x_k)\underbrace{\vec{c}_k^{\,(j,i_2)}}_{\text{zero}} \rightarrow \text{work with exact numbers}$$

Series Expansion: Analytic Continuation

Key Ideas: **Define** physical thresholds, **Verify** spurious (or anomalous) thresholds



"Anomalous Threshold":

- Algebraic function in one or multiple Mandelstam variables
- Real life example (planar): $\log\left(\frac{m_H^4\left(-m_H^2+s+t\right)}{m_W^6}+\frac{4\left(s-m_H^2\right)\left(t-m_H^2\right)}{m_W^4}\right)$
- Verify vanishes upon integration: **FAILURE** $\rightarrow$ **ABORT**
- We did not encounter any in our physical region.