Hard Probes in A-A collisions: heavy-flavor

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Quark-Gluon Plasma and heavy-ion collisions: past, present and future,
9-13 July 2013, Siena

Outline

- Heavy flavor in elementary collisions as *benchmark*
  - of our understanding of pQCD,
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or *vice versa*!

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  or *vice versa*!
- How to develop a transport calculation:
  the relativistic Langevin equation.

Heavy-flavor production in pQCD

The large mass M of c and b quarks makes a pQCD calculation of $Q\overline{Q}$ production possible:

- It sets a *minimal off-shellness* of the intermediate propagators (diagrams don't diverge);
- It sets a *hard scale* for the evaluation of $\alpha_s(\mu)$ (speeding the convergence of the perturbative series);
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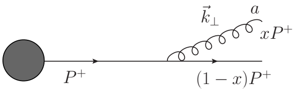
Both the *total cross-section* $\sigma^{\rm tot}_{\rm Q\overline{Q}}$ and the *invariant single-particle spectrum* $E(d\sigma_Q/d^3p)$ are well-defined quantities which can be calculated in pQCD

Suppression of collinear radiation



Massless case

$$d\sigma^{\rm rad} = d\sigma^{\rm hard}\frac{\alpha_s}{\pi^2}C_F\frac{dx}{x}\frac{d\mathbf{k}_\perp}{\mathbf{k}_\perp^2}$$

Due to collinear gluon-radiation ($\sim d\theta/\theta$), partonic cross-sections of hard processes are not well defined, but require the introduction of a "cutoff" (*factorization scale* μ_F) to regularize collinear divergences. Only hadronic cross-section

$$d\sigma_h \equiv \sum_f d\sigma_f(\mu_F)\otimes D_f^h(z,\mu_F)$$

are *collinear-safe* observables.

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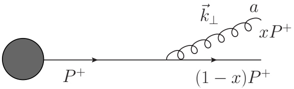


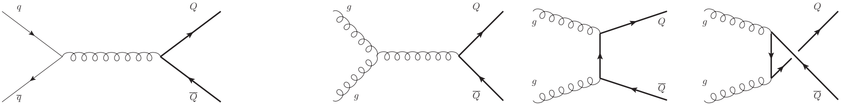
Massive case

$$d\sigma^{\rm rad} = d\sigma^{\rm hard}\frac{\alpha_s}{\pi^2}C_F\frac{dx}{x}d\mathbf{k}_\perp\frac{\mathbf{k}_\perp^2}{[\mathbf{k}_\perp^2 + x^2M^2]^2}$$

Gluon radiation at angles $\theta < M/E$ is suppressed (*dead-cone effect*!) and heavy-quark production is well-defined even at the partonic (for what concerns the final state) level.

Leading Order contribution

- The LO processes are:



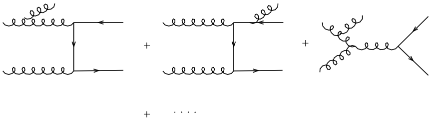
- The propagators introduce in the amplitudes the denominators:

$$(p_1 + p_2)^2 = 2m_T^2(1 + \cosh \Delta y)$$
$$(p_3 - p_1)^2 = -m_T^2 \left(1 + e^{-\Delta y}\right)$$
$$(p_3 - p_2)^2 = -m_T^2 \left(1 + e^{\Delta y}\right)$$

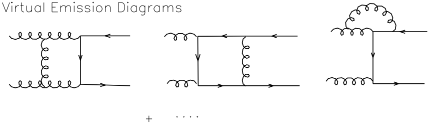
  - *Minimal off-shellness* $\sim m_T^2$;
  - Q and $\overline{Q}$ close in rapidity.

Next to Leading Order process

Real Emission Diagrams



Virtual Emission Diagrams



- Real emission: $|\mathcal{M}_{\rm real}|^2 \sim \mathcal{O}(\alpha_s^3)$
- Virtual corrections: $2{\rm Re}\mathcal{M}_0\mathcal{M}^*_{\rm virt} \sim \mathcal{O}(\alpha_s^3)$

- NLO calculation gives the $\mathcal{O}(\alpha_s^3)$ result for $\sigma^{\rm tot}_{Q\overline{Q}}$ and $E(d\sigma_Q)/d^3p$;
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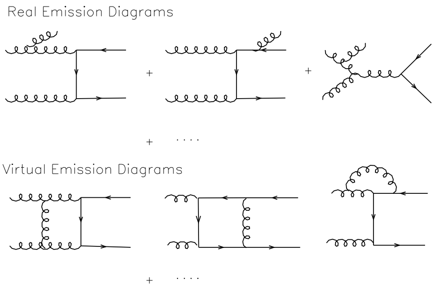

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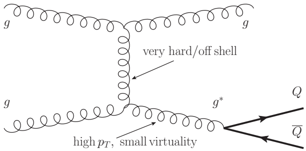
- Output of hard event can be interfaced with a Parton Shower (PYTHIA or HERWIG)

NLO calculation: gluon-splitting contribution



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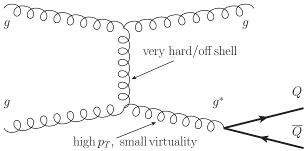

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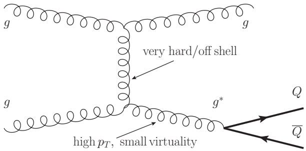

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$Q\overline{Q}$ multiplicity in a gluon jet of transverse energy p_T: $\sim \alpha_s \ln(p_T/M)$

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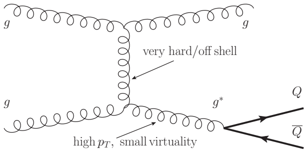

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The NLO calculation *contains* an $\alpha_s \ln(p_T/M)$ term, *potentially large*!

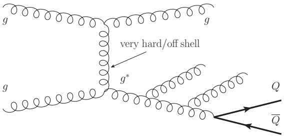
Resummation of (Next to) Leading Logs: FONLL

- Using the above result as the initial condition of the DGLAP evolution for the D_g^Q FF:

$$D_g^Q(z,\mu_0) = \frac{\alpha_s}{2\pi}\frac{1}{2}[z^2+(1-z)^2]\ln\frac{\mu_0^2}{M^2}$$

amounts to resumming all $[\alpha_s \ln(p_T/M)]^n$ terms ($\alpha_s[\alpha_s \ln(p_T/M)]^n$ with NLO splitting functions)

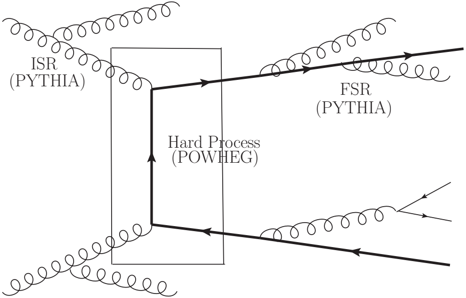
- In terms of diagrams:



$Q\overline{Q}$ from the shower of light partons produced in the hard event!

- A code like FONLL provides a calculation of $d\sigma_Q$ at this accuracy!

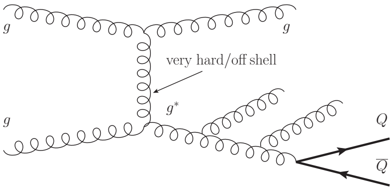
NLO calculation + Parton Shower



- A different strategy is to interface the output of a NLO event-generator for the hard process with a parton-shower describing Initial and Final State Radiation.
- This provides a *fully exclusive information on the final state*

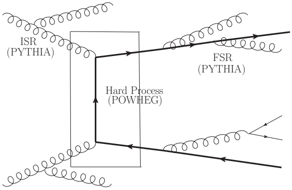
FONLL vs POWHEG+PS

FONLL



- It is a *calculation*
- It provides NLL accuracy, resumming large $\ln(p_T/M)$
- It includes processes missed by POWHEG (hard events with light partons)

POWHEG+PS



- It is an *event generator*
- Results compatible with FONLL
- It is a more flexible tool, allowing to address more differential observables (e.g. $Q\overline{Q}$ correlations)

Heavy quark production in pQCD: some references

- For a general introduction: M. Mangano, hep-ph/9711337 (lectures);
- For POWHEG: S. Frixione, P. Nason and G. Ridolfi, JHEP 0709 (2007) 126;
- For FONLL: M. Cacciari, M. Greco and P. Nason, JHEP 9805 (1998) 007.
- For a systematic comparison (POWHEG vs MC@NLO vs FONLL): M. Cacciari *et al.*, JHEP 1210 (2012) 137.

Heavy flavour: experimental observables

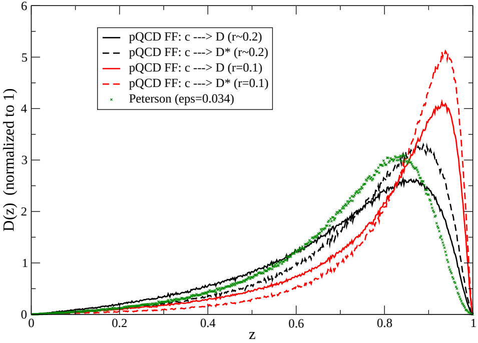
- D and B mesons;
- Non-prompt J/ψ's ($B \to J/\psi\, X$)
- Heavy-flavour electrons, from the decays
  - of charm (e_c)

$$D \to X\nu e$$

  - of beauty (e_b)

$$\begin{aligned} B &\to D\nu e \\ B &\to D\nu e \to X\nu e\nu e \\ B &\to DY \to X\nu e Y \end{aligned}$$

Fragmentation functions



FF tuned by FONLL authors to reproduce e^+e^- data[1]

- D-meson FF from HQET (Braaten *et al.*, PRD 51 (1995) 4819);

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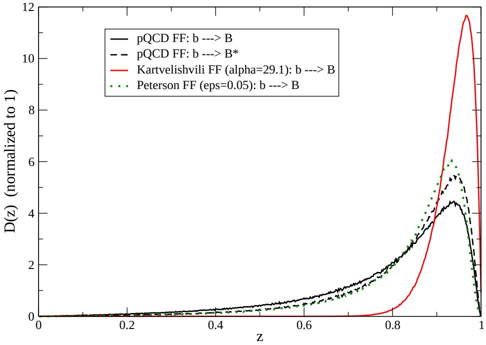

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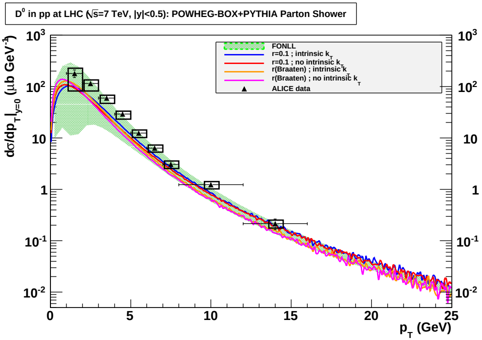
Results: D and B mesons @ 7 TeV



- Our choice (arXiv:1305.7421): POWHEG for the *hard event* interfaced with PYTHIA for the *shower* stage;
- With the same default parameters ($m_c = 1.5/1.3$ GeV, $m_b = 4.8$ GeV, $\mu_R = \mu_F = m_T$) and FF results in agreement with FONLL.

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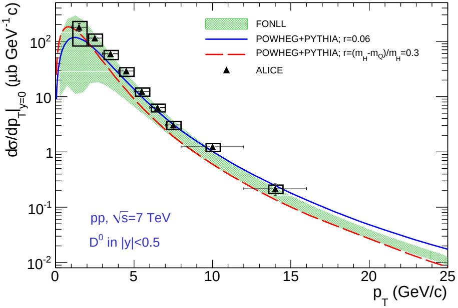

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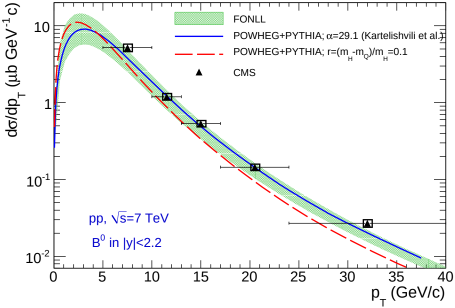

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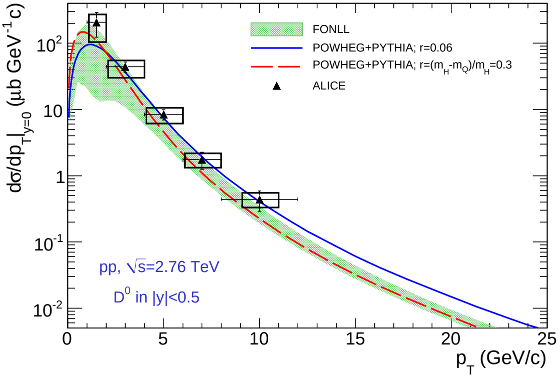
Results in p-p @ 2.76 TeV (benchmark for *AA*)



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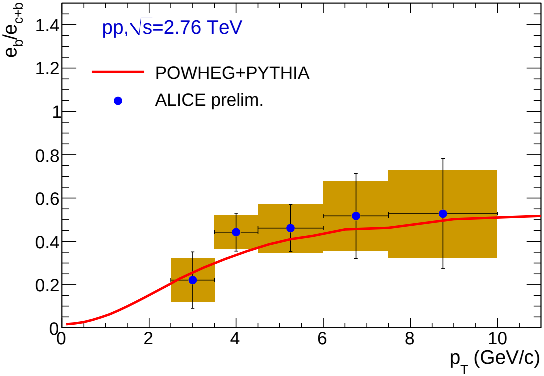

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- both for the *D*-meson spectra...
- and for the heavy-flavour electrons (e_c and e_b)

HF in p-p collisions: a summary

- A setup based on a NLO pQCD event generator (POWHEG) for the hard event + a Parton-Shower stage simulated with PYTHIA is able to reproduce the experimental data;
- Such an approach provides a richer information on the final state wrt other schemes (e.g. FONLL): this can be of interest for more differential studies like azimuthal correlations

HF in AA collisions

Purpose of this lecture:

- Displaying the conceptual setup common to the different theoretical models, pointing out their nice features and limitations;
- Showing some results and compare them to the experimental data;
- Giving some hints of possible future developments.

Being a lecture I will focus mainly on one particular approach, the relativistic Langevin equation, hoping that at the end one will be able to understand the technical issues one has to face in developing a model

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Ideally only the parton-medium interaction should be model-dependent

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In practice each model deals with the other points in a different (often rather schematic) way: difficulty in performing a systematic comparison!

Heavy Flavour in the QGP: the conceptual setup

- Description of soft observables based on hydrodynamics, assuming to deal with a system close to local thermal equilibrium (no matter why);
- Description of jet-quenching based on energy-degradation of external probes (high-p_T partons);

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- Description of heavy-flavour observables requires to employ/develop a setup (transport theory) allowing to deal with more general situations and in particular to describe *how particles would (asymptotically) approach equilibrium*.

Transport theory: the Boltzmann equation

Time evolution of HQ phase-space distribution $f_Q(t, \mathbf{x}, \mathbf{p})$[2]:

$$\frac{d}{dt} f_Q(t, \mathbf{x}, \mathbf{p}) = C[f_Q]$$

- Total derivative along particle trajectory

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{x}} + \mathbf{F}\frac{\partial}{\partial \mathbf{p}}$$

Neglecting $\mathbf{x}$-dependence and mean fields: $\partial_t f_Q(t, \mathbf{p}) = C[f_Q]$

- Collision integral:

$$C[f_Q] = \int d\mathbf{k}[\underbrace{w(\mathbf{p}+\mathbf{k}, \mathbf{k}) f_Q(\mathbf{p}+\mathbf{k})}_{\text{gain term}} - \underbrace{w(\mathbf{p}, \mathbf{k}) f_Q(\mathbf{p})}_{\text{loss term}}]$$

$w(\mathbf{p}, \mathbf{k})$: HQ transition rate $\mathbf{p} \rightarrow \mathbf{p} - \mathbf{k}$

[2] Approach implemented in codes like BAMPS.

The collision integral: a closer look

Momentum exchanges occur with light (thermal) partons *i* of the plasma.
In the *classical limit* (no Pauli-blocking or Bose-enhancement) one has:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}'_1 \Big[\underbrace{\overline{w}(\mathbf{p},' \mathbf{p}'_1|\mathbf{p},\mathbf{p}_1) f_Q(\mathbf{p}') f_i(\mathbf{p}'_1)}_{\text{gain term}} - \underbrace{\overline{w}(\mathbf{p},\mathbf{p}_1|\mathbf{p}',\mathbf{p}'_1) f_Q(\mathbf{p}) f_i(\mathbf{p}_1)}_{\text{loss;term}} \Big]$$

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From *time-reversal* symmetry one has for the transition probability:

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hence:

$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) \Big[f_Q(\mathbf{p}') f_i(\mathbf{p}_1') - f_Q(\mathbf{p}) f_i(\mathbf{p}_1) \Big].$$

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$$C[f_Q] = \int d\mathbf{p}' d\mathbf{p}_1 d\mathbf{p}_1' \overline{w}(\mathbf{p},' \mathbf{p}_1' | \mathbf{p}, \mathbf{p}_1) \Big[f_Q(\mathbf{p}') f_i(\mathbf{p}_1') - f_Q(\mathbf{p}) f_i(\mathbf{p}_1) \Big].$$

$C[f_Q]$ vanishes *if and only if* $f_Q(\mathbf{p}') f_i(\mathbf{p}_1') = f_Q(\mathbf{p}) f_i(\mathbf{p}_1)$, which entails:

$$f_Q(\mathbf{p}) = \exp\left[-E_{\mathbf{p}}/T\right] \quad \text{and} \quad f_i(\mathbf{p}_1) = \exp\left[-E_{\mathbf{p}_1}/T\right].$$

The Boltzmann equation *always* makes heavy quarks relax to a *thermal distribution* at the same temperature of the medium!

From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[3] (Landau)

$$C[f_Q] \approx \int d\mathbf{k} \left[k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\mathbf{p}, \mathbf{k}) f_Q(t, \mathbf{p})]$$

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The Boltzmann equation reduces to the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} f_Q(t,\mathbf{p}) = \frac{\partial}{\partial p^i} \left\{ A^i(\mathbf{p}) f_Q(t,\mathbf{p}) + \frac{\partial}{\partial p^j} [B^{ij}(\mathbf{p}) f_Q(t,\mathbf{p})] \right\}$$

where (verify!)

$$A^i(\mathbf{p}) = \int d\mathbf{k}\, k^i w(\mathbf{p},\mathbf{k}) \quad \longrightarrow \quad \underbrace{A^i(\mathbf{p}) = A(p)\, p^i}_{\text{friction}}$$

$$B^{ij}(\mathbf{p}) = \frac{1}{2} \int d\mathbf{k}\, k^i k^j w(\mathbf{p},\mathbf{k}) \quad \longrightarrow \quad \underbrace{B^{ij}(\mathbf{p}) = \hat{p}^i \hat{p}^j B_0(p) + (\delta^{ij} - \hat{p}^i \hat{p}^j) B_1(p)}_{\text{momentum broadening}}$$

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Problem reduced to the *evaluation of three transport coefficients*

[3] B. Svetitsky, PRD 37, 2484 (1988)

Fokker-Planck equation: solution

- *Ignoring the momentum dependence* of the transport coefficients $\gamma \equiv A(p)$ and $D \equiv B_0(p) = B_1(p)$ the FP equation reduces to

$$\frac{\partial}{\partial t} f_Q(t, \mathbf{p}) = \gamma \frac{\partial}{\partial p^i}[p^i f_Q(t, \mathbf{p})] + D\, \Delta_{\mathbf{p}} f_Q(t, \mathbf{p})$$

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- Starting from the *initial condition* $f_Q(t=0, \mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}_0)$ one gets

$$f_Q(t, \mathbf{p}) = \left(\frac{\gamma}{2\pi D[1 - \exp(-2\gamma t)]} \right)^{3/2} \exp\left[-\frac{\gamma}{2D} \frac{[\mathbf{p} - \mathbf{p}_0 \exp(-\gamma t)]^2}{1 - \exp(-2\gamma t)} \right]$$

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- Asymptotically the solution *forgets about the initial condition* and tends to a thermal distribution

$$f_Q(t, \mathbf{p}) \underset{t \to \infty}{\sim} \left(\frac{\gamma}{2\pi D} \right)^{3/2} \exp\left[-\left(\frac{\gamma M_Q}{D} \right) \frac{\mathbf{p}^2}{2M_Q} \right]$$

$\longrightarrow D = M_Q \gamma T$: Einstein *fluctuation-dissipation* relation

Fokker-Planck solution: derivation (I)

Consider (for simplicity) the 1D FP equation and start setting $D=0$:

$$\frac{\partial}{\partial t} f_Q = \gamma \frac{\partial}{\partial p}[p\, f_Q] \quad \longrightarrow \quad \frac{\partial f_Q}{\partial t} - \gamma p \frac{\partial f_Q}{\partial p} = \gamma\, f_Q \quad \longrightarrow \quad \frac{df_Q}{dt} = \gamma\, f_Q$$

viewing the LHS as the *total derivative* d/dt wrt to the motion of a particle feeling a friction force $dp/dt = -\gamma p$.

One can then write the solution as:

$$f_Q = Q(u) e^{\gamma t} \quad \text{with} \quad p(t) = u\, e^{-\gamma t}$$

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For the full equation, with $D \neq 0$ one can attempt a solution of the form

$$f_Q = Q\left(t, u = p e^{\gamma t}\right) e^{\gamma t}$$

whose partial derivatives are given by:

$$\frac{\partial f_Q}{\partial p} = e^{2\gamma t} \frac{\partial Q}{\partial u}, \qquad \frac{\partial^2 f_Q}{\partial p^2} = e^{3\gamma t} \frac{\partial^2 Q}{\partial u^2}$$

$$\frac{\partial f_Q}{\partial t} = \gamma e^{\gamma t} Q + e^{\gamma t} \left[\frac{\partial Q}{\partial t} + \gamma u \frac{\partial Q}{\partial u}\right]$$

Fokker-Planck solution: derivation (II)

Inserting it into the full FP equation

$$\frac{\partial f_Q}{\partial t} = \gamma f_Q + \gamma p \frac{\partial f_Q}{\partial p} + D \frac{\partial^2 f_Q}{\partial p^2}$$

One gets the simpler equation:

$$\frac{\partial Q}{\partial t} = D e^{2\gamma t} \frac{\partial^2 Q}{\partial u^2}$$

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Introducing the temporal variable $\theta = (e^{2\gamma t} - 1)/2\gamma \longrightarrow d\theta = e^{2\gamma t} dt$ one gets the diffusion equation:

$$\frac{\partial Q}{\partial \theta} = D \frac{\partial^2 Q}{\partial u^2} \quad \text{with} \quad Q(0,u) = Q_0(u) = \delta(u - u_0)$$

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Solution is an *superposition of plane-waves*

$$Q(\theta, u) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A_k \, e^{-i\omega_k \theta + iku}$$

with $A_k = e^{-iku_0}$ (init.cond.) and $\omega_k = -iDk^2$ (diff.eq.)

Fokker-Planck solution: derivation (III)

The integration is gaussian and can be performed exactly, getting

$$Q(\theta, u) = \left(\frac{1}{4\pi D\theta}\right) \exp\left[-\frac{(u-u_0)^2}{4D\theta}\right]$$

Going back to the original variables[4]:

$$f_Q(t, \mathbf{p}) = \left(\frac{\gamma}{2\pi D[1-\exp(-2\gamma t)]}\right)^{1/2} \exp\left[-\frac{\gamma}{2D}\frac{[p - p_0\exp(-\gamma t)]^2}{1-\exp(-2\gamma t)}\right]$$

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The generalization to the 3D case is trivial

$$f_Q(t, \mathbf{p}) = \left(\frac{\gamma}{2\pi D[1-\exp(-2\gamma t)]}\right)^{3/2} \exp\left[-\frac{\gamma}{2D}\frac{[\mathbf{p} - \mathbf{p}_0\exp(-\gamma t)]^2}{1-\exp(-2\gamma t)}\right]$$

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Fokker-Planck solution: physical meaning

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From the first moments of the momentum distribution...

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- $$\langle \mathbf{p}(t)\rangle = \mathbf{p}_0\, e^{-\gamma t}$$

  γ: friction coefficient

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Ex: derive the above results. Trivial, after setting

$$\mathbf{p} = \left(\mathbf{p}-\mathbf{p}_0 e^{-\gamma t}\right) + \mathbf{p}_0 e^{-\gamma t}$$

The challenge: addressing the experimental situation

One needs a tool, *equivalent to the Fokker-Planck equation*, but allowing to face the complexity of the experimental situation[5] in which

[5]A.B. et al., NPA 831 59 (2009) and EPJC 71 (2011) 1666
For a review: R. Rapp and H. van Hees, arXiv:0903.1096
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- heavy quarks can be relativistic, so that one must deal with the *momentum dependence*[6] *of the transport coefficients*;
- the dynamics in the medium must be *interfaced with the initial hard production*, possibly given by pQCD event generators;
- the stochastic dynamics takes place in a medium which undergoes a hydrodynamical expansion.

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A proper *relativistic generalization of the Langevin equation* allows to accomplish this task

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The relativistic Langevin equation

The Fokker-Planck equation can be recast into a form suitable to follow the dynamics of each individual quark: the Langevin equation

$$\frac{\Delta p^i}{\Delta t} = -\underbrace{\eta_D(p)p^i}_{\text{determ.}} + \underbrace{\xi^i(t)}_{\text{stochastic}},$$

with the properties of the noise encoded in

$$\langle \xi^i(\mathbf{p}_t)\xi^j(\mathbf{p}_{t'})\rangle = b^{ij}(\mathbf{p}_t)\frac{\delta_{tt'}}{\Delta t} \qquad b^{ij}(\mathbf{p}) \equiv \kappa_\parallel(p)\hat{p}^i\hat{p}^j + \kappa_\perp(p)(\delta^{ij} - \hat{p}^i\hat{p}^j)$$

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Transport coefficients to calculate:

- *Momentum diffusion* $\kappa_\perp \equiv \frac{1}{2}\frac{\langle \Delta p_\perp^2\rangle}{\Delta t}$ and $\kappa_\parallel \equiv \frac{\langle \Delta p_\parallel^2\rangle}{\Delta t}$;
- *Friction* term (dependent on the discretization scheme!)

$$\eta_D{}^{\text{Ito}}(p) = \frac{\kappa_\parallel(p)}{2TE_p} - \frac{1}{E_p^2}\left[(1-v^2)\frac{\partial \kappa_\parallel(p)}{\partial v^2} + \frac{d-1}{2}\frac{\kappa_\parallel(p) - \kappa_\perp(p)}{v^2}\right]$$

fixed in order to assure approach to equilibrium (Einstein relation):

The Langevin equation: numerical implementation (I)

- Start from the original equation

$$\frac{\Delta p^i}{\Delta t} = -\eta_D(p)p^i + \xi^i(t),$$

with

$$\langle \xi^i(\mathbf{p}_t)\xi^j(\mathbf{p}_{t'})\rangle = b^{ij}(\mathbf{p}_t)\frac{\delta_{tt'}}{\Delta t} \qquad b^{ij}(\mathbf{p}) \equiv \kappa_\parallel(p)\hat{p}^i\hat{p}^j + \kappa_\perp(p)(\delta^{ij} - \hat{p}^i\hat{p}^j)$$

- Introduce the tensor

$$\begin{aligned} g^{ij}(\mathbf{p}) &\equiv \sqrt{\kappa_L(p)}\hat{p}^i\hat{p}^j + \sqrt{\kappa_T(p)}(\delta^{ij} - \hat{p}^i\hat{p}^j) \\ &\equiv g_L(p)\hat{p}^i\hat{p}^j + g_T(p)(\delta^{ij} - \hat{p}^i\hat{p}^j) \end{aligned}$$

- Factor out the momentum dependence of the noise term (verify!)

$$\frac{dp^i}{dt} = -\eta_D(p)p^i + g^{ij}(\mathbf{p})\eta^j(t) \quad \text{with} \quad \langle \eta^i(t)\eta^j(t')\rangle = \delta^{ij}\delta(t - t')$$

The Langevin equation: numerical implementation (II)

The numerical implementation requires to set a *discretization scheme*

$$p^i_{n+1} - p^i_n = -\eta_D^{\text{Ito}}(p_n)p^i_n\Delta t + g^{ij}(\mathbf{p}_n)\zeta^i(t_n)\sqrt{\Delta t},$$

with

$$\zeta^i \equiv \eta^i\sqrt{\Delta t} \quad \text{and} \quad \langle\zeta^i(t_n)\zeta^j(t_m)\rangle = \delta_{m,n}\delta^{i,j}$$

At each time-step one has simply to extract 3 independent (δ^{ij}) random numbers from a gaussian distribution with $\sigma = 1$ ($\langle\zeta_x^2\rangle=\langle\zeta_y^2\rangle=\langle\zeta_z^2\rangle=1$): much simpler then the original Boltzmann equation!

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In the Ito discretization scheme:

- Transport coefficients evaluated at step t_n
- Friction coefficient receives a discretization correction to assure the proper continuum limit:

$$\eta_D{}^{\rm Ito}(p) = \frac{\kappa_\parallel(p)}{2TE_p} - \frac{1}{E_p^2}\left[(1-v^2)\frac{\partial \kappa_\parallel(p)}{\partial v^2} + \frac{d-1}{2}\frac{\kappa_\parallel(p) - \kappa_\perp(p)}{v^2}\right]$$

The Langevin equation as a SDE (I)

The Langevin equation, written in the general form (here in 1D)

$$\frac{dp}{dt} = f(p) + g(p)\eta(t) \quad \text{with} \quad \langle\eta(t)\rangle = 0, \quad \langle\eta(t)\eta(t')\rangle = \delta(t-t'),$$

is an example of *Stochastic Differential Equation* (applied in many domains of science). In our case $g(p) = \sqrt{\kappa(p)}$ and

$$f(p) = -\eta_D(p)p \equiv -\eta_D^{(0)}(p)p + f_1(p),$$

where $\eta_D^{(0)}$ and f_1 will be set in order to assure asymptotic thermal equilibrium independently on the discretization scheme.

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where $\eta_D^{(0)}$ and f_1 will be set in order to assure asymptotic thermal equilibrium independently on the discretization scheme.
Formally one can integrate the above equation,

$$p(t + \Delta t) - p(t) = \int_t^{t+\Delta t} ds[f(p(s)) + g(p(s))\eta(s)]$$

however, *due to the noise term* $\eta(s)$, the solution is not an ordinary Riemann integral. *Where to evaluate* f *and* g*?*

The Langevin equation as a SDE (II)

A whole family of different discretizations, labeled by a parameter $\alpha \in [0,1]$, such that

$$\Delta p = f[p(t)+\alpha\Delta p]\Delta t + g[p(t)+\alpha\Delta p] \int_t^{t+\Delta t} ds\, \eta(s)$$

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Expanding

$$g[p_0+\alpha\Delta p] = g(p_0) + g'(p_0)\,\alpha\Delta p + \dots$$

and keeping terms up to $\mathcal{O}(\Delta t)$:

$$\langle\Delta p\rangle = f(p_0)\Delta t + \alpha g(p_0)g'(p_0)\Delta t \qquad \text{and} \qquad \langle(\Delta p)^2\rangle = g^2(p_0)\Delta t$$

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The friction term has to be fixed imposing the equivalence with the Fokker-Planck equation for

$$P(p, t+\Delta t) = \int_{-\infty}^{+\infty} dp_0 \underbrace{P(p, t+\Delta t|p_0, t)}_{\text{cond. probab.}} P(p_0, t)$$

The Langevin equation as a SDE (III)

Identify the conditional probability with the following expectation value over the ensemble of brownian particles:

$$P(p, t+\Delta t|p_0, t) \equiv \langle \delta[p - p(t+\Delta t)] \rangle_{p_0,t}$$

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Expand and exploit the previous results for $\langle \Delta p \rangle$ and $\langle (\Delta p)^2 \rangle$:

$$P(p, t+\Delta t | p_0, t) = \delta(p-p_0) - \langle \Delta p \rangle \frac{\partial}{\partial p} \delta(p-p_0) + \frac{1}{2} \langle (\Delta p)^2 \rangle \frac{\partial^2}{\partial p^2} \delta(p-p_0) + \ldots$$

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$$P(p, t + \Delta t | p_0, t) \equiv \langle \delta[p - p(t + \Delta t)] \rangle_{p_0, t}$$

Expand and exploit the previous results for $\langle \Delta p \rangle$ and $\langle (\Delta p)^2 \rangle$:

$$P(p, t+\Delta t | p_0, t) = \delta(p-p_0) - \langle \Delta p \rangle \frac{\partial}{\partial p} \delta(p-p_0) + \frac{1}{2} \langle (\Delta p)^2 \rangle \frac{\partial^2}{\partial p^2} \delta(p-p_0) + \dots$$

Substitute into the definition of $P(p, t + \Delta t)$ obtaining the PDE

$$\partial_t P(p,t) = \partial_p \left[-f(p) - \alpha g(p) g'(p) + (1/2) \partial_p g^2(p) \right] P(p,t)$$
$$= \partial_p \left[\eta_D^{(0)}(p) p - f_1(p) + (1/2)(1-\alpha) \partial_p \kappa(p) + (1/2) \kappa(p) \partial_p \right] P(p,t)$$

The Langevin equation as a SDE (III)

Identify the conditional probability with the following expectation value over the ensemble of brownian particles:

$$P(p, t + \Delta t | p_0, t) \equiv \langle \delta[p - p(t + \Delta t)] \rangle_{p_0, t}$$

Expand and exploit the previous results for $\langle \Delta p \rangle$ and $\langle (\Delta p)^2 \rangle$:

$$P(p, t+\Delta t | p_0, t) = \delta(p-p_0) - \langle \Delta p \rangle \frac{\partial}{\partial p} \delta(p-p_0) + \frac{1}{2} \langle (\Delta p)^2 \rangle \frac{\partial^2}{\partial p^2} \delta(p-p_0) + \dots$$

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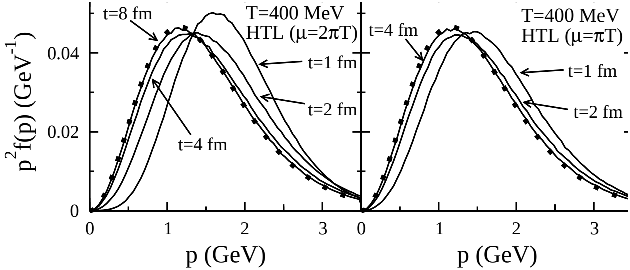
$$\begin{aligned} \partial_t P(p,t) &= \partial_p \left[-f(p) - \alpha g(p) g'(p) + (1/2) \partial_p g^2(p) \right] P(p,t) \\ &= \partial_p \left[\eta_D^{(0)}(p) p - f_1(p) + (1/2)(1-\alpha) \partial_p \kappa(p) + (1/2) \kappa(p) \partial_p \right] P(p,t) \end{aligned}$$

Equivalence with FP eq. with steady solution $\exp[-E_p/T]$ leads to

$$\eta_D^{(0)}(p) = \frac{\kappa(p)}{2 T E_p} \text{ (Einstein relation)} \quad \text{and} \quad f_1(p) = \frac{1}{2}(1-\alpha)\partial_p \kappa(p)$$

A first check: thermalization in a static medium



For $t \gg 1/\eta_D$ one approaches a relativistic Maxwell-Jüttner distribution[7]

$$f_{\rm MJ}(p) \equiv \frac{e^{-E_p/T}}{4\pi M^2 T\, K_2(M/T)}, \qquad \text{with} \int d^3p\, f_{\rm MJ}(p) = 1$$

(Test with a sample of c quarks with $p_0 = 2$ GeV/c)

[7] A.B., A. De Pace, W.M. Alberico and A. Molinari, NPA 831, 59 (2009)

The realistic case: expanding fireball

Update of the HQ momentum and position to be done at each step *in the local fluid rest-frame*

- $u^{\mu}(x)$ used to perform the boost to the fluid rest-frame;
- $T(x)$ used to set the value of the transport coefficients

[8] P.F. Kolb, J. Sollfrank and U. Heinz, Phys. Rev. C **62** (2000) 054909
P. Romatschke and U.Romatschke, Phys. Rev. Lett. **99** (2007) 172301

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- $T(x)$ used to set the value of the transport coefficients

The fields $u^\mu(x)$ and $T(x)$ can be taken from the output of hydro codes[8]. Current public codes limited to longitudinally boost-invariant ("Hubble-law") expansion ($v_z = z/t$) case:

$$x^\mu = (\tau \cosh\eta, \mathbf{r}_\perp, \tau \sinh\eta) \quad \text{with} \quad \tau \equiv \sqrt{t^2 - z^2}$$

$$u^\mu = \gamma_\perp(\cosh\eta, \mathbf{u}_\perp, \sinh\eta) \quad \text{with} \quad \gamma_\perp \equiv \frac{1}{\sqrt{1 - \mathbf{u}_\perp^2}}$$

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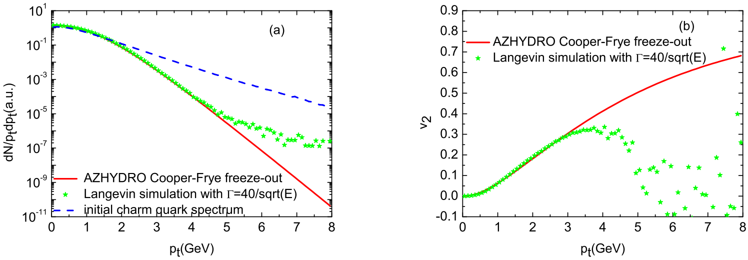
Expanding fireball: testing the algorithm

In the limit of large transport coefficients heavy quarks should reach local thermal equilibrium and decouple from the medium as the other light particles, according to the Cooper-Frye formula:

$$E(dN/d^3p) = \int_{\Sigma_{\rm fo}} \frac{p^\mu \cdot d\Sigma_\mu}{(2\pi)^3} \exp[-p\cdot u/T_{\rm fo}]$$



This was verified to be actually the case (M. He, R.J. Fries and R. Rapp, PRC 86, 014903).

The Langevin equation provides a link between *what is possible to calculate in QCD* (transport coefficients) and *what one actually measures* (final p_T spectra)

[9] Our approach: W.M. Alberico *et al.*, Eur.Phys.J. C71 (2011) 1666

The Langevin equation provides a link between *what is possible to calculate in QCD* (transport coefficients) and *what one actually measures* (final p_T spectra)

Evaluation of transport coefficients:

- Weak-coupling hot-QCD calculations[9]
- Non perturbative approaches
  - Lattice-QCD
  - AdS/CFT correspondence
  - Resonant scattering

[9] Our approach: W.M. Alberico *et al.*, Eur.Phys.J. C71 (2011) 1666

Transport coefficients: perturbative evaluation

It's the stage where the various models differ!

We account for the effect of $2 \rightarrow 2$ collisions in the medium

[10] Similar strategy for the evaluation of dE/dx in S. Peigne and A. Peshier, Phys.Rev.D77:114017 (2008).

Transport coefficients: perturbative evaluation

It's the stage where the various models differ!

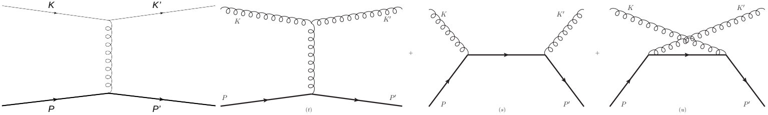
We account for the effect of $2 \to 2$ collisions in the medium

Intermediate cutoff $|t|^* \sim m_D^2$ [10] *separating the contributions of*

- hard collisions ($|t| > |t|^*$): kinetic pQCD calculation
- soft collisions ($|t| < |t|^*$): Hard Thermal Loop approximation (*resummation of medium effects*)

[10] Similar strategy for the evaluation of dE/dx in S. Peigne and A. Peshier, Phys.Rev.D77:114017 (2008).

Transport coefficients $\kappa_{T/L}(p)$: hard contribution

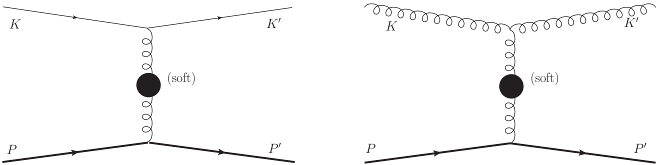


$$\kappa_T^{g/q(\mathrm{hard})} = \frac{1}{2}\frac{1}{2E}\int_k \frac{n_{B/F}(k)}{2k}\int_{k'}\frac{1 \pm n_{B/F}(k')}{2k'}\int_{p'}\frac{1}{2E'}\theta(|t|-|t|^*)\times$$
$$\times\,(2\pi)^4\delta^{(4)}(P+K-P'-K')\left|\overline{\mathcal{M}}_{g/q}(s,t)\right|^2 q_T^2$$

$$\kappa_L^{g/q(\mathrm{hard})} = \frac{1}{2E}\int_k \frac{n_{B/F}(k)}{2k}\int_{k'}\frac{1 \pm n_{B/F}(k')}{2k'}\int_{p'}\frac{1}{2E'}\theta(|t|-|t|^*)\times$$
$$\times\,(2\pi)^4\delta^{(4)}(P+K-P'-K')\left|\overline{\mathcal{M}}_{g/q}(s,t)\right|^2 q_L^2$$

where: $(|t| \equiv q^2 - \omega^2)$

Transport coefficients $\kappa_{T/L}(p)$: soft contribution



When the exchanged 4-momentum is **soft** the t-channel gluon feels the presence of the medium and requires **resummation**.

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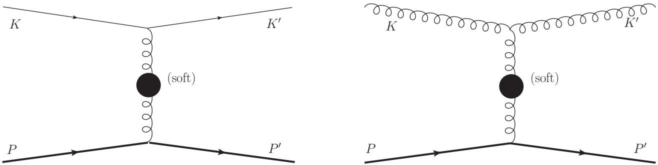


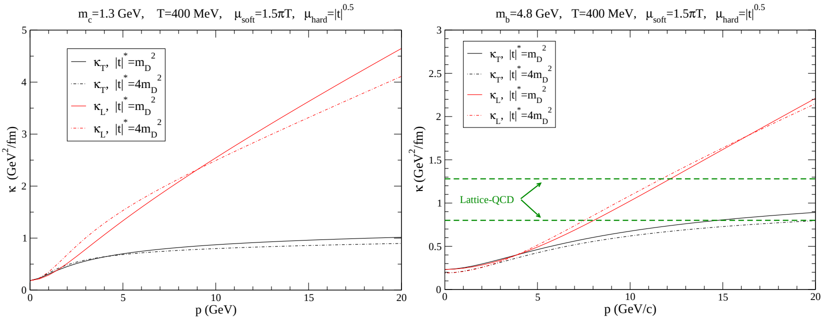
When the exchanged 4-momentum is **soft** the t-channel gluon feels the presence of the medium and requires **resummation**.
The *blob* represents the *dressed gluon propagator*, which has longitudinal and transverse components:

$$\Delta_L(z,q) = \frac{-1}{q^2 + \Pi_L(z,q)}, \quad \Delta_T(z,q) = \frac{-1}{z^2 - q^2 - \Pi_T(z,q)},$$

where *medium effects* are embedded in the HTL gluon self-energy.

Transport coefficients: numerical results

Combining together the hard and soft contributions...



...the dependence on the intermediate cutoff $|t|^*$ is very mild!

Lattice-QCD transport coefficients

Ongoing efforts to extract transport coefficients from lattice-QCD simulations *assuming a non-relativistic Langevin dynamics of the HQs*

- κ from electric-field correlators[11];
- η_D from current-current correlators, exploiting the diffusive dynamics of conserved charges[12]

[11] Solana and Teaney, PRD 74, 085012 (2006)
[12] Petreczky and Teaney, PRD 73, 014508 (2006)

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General considerations:

- In principle lattice-QCD would provide an "exact" non-perturbative result;
- Difficulties in extracting real-time quantities (transport coefficients) from euclidean $(t = -i\tau)$ simulations;
- Current results limited to the static $(M = \infty)$ or at most non-relativistic limit.

[11] Solana and Teaney, PRD 74, 085012 (2006)
[12] Petreczky and Teaney, PRD 73, 014508 (2006)

Lattice-QCD transport coefficients: setup

One consider the non-relativistic limit of the Langevin equation:

$$\frac{dp^i}{dt} = -\eta_D p^i + \xi^i(t), \quad \text{with} \quad \langle \xi^i(t)\xi^j(t')\rangle = \delta^{ij}\delta(t-t')\kappa$$

Hence, in the $p \longrightarrow 0$ limit:

$$\kappa = \frac{1}{3}\int_{-\infty}^{+\infty} dt \langle \xi^i(t)\xi^i(0)\rangle_{\rm HQ} \approx \frac{1}{3}\int_{-\infty}^{+\infty} dt \underbrace{\langle F^i(t)F^i(0)\rangle_{\rm HQ}}_{\equiv D^>(t)},$$

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In the static limit the force is due to the color-electric field:

$$\mathbf{F}(t) = g\int d\mathbf{x} Q^\dagger(t,\mathbf{x}) t^a Q(t,\mathbf{x}) \mathbf{E}^a(t,\mathbf{x})$$

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In a thermal ensemble $\sigma(\omega) \equiv D^>(\omega) - D^<(\omega) = (1-e^{-\beta\omega})D^>(\omega)$ and

$$\kappa \equiv \lim_{\omega\to 0}\frac{D^>(\omega)}{3} = \lim_{\omega\to 0}\frac{1}{3}\frac{\sigma(\omega)}{1-e^{-\beta\omega}} \underset{\omega\to 0}{\sim} \frac{1}{3}\frac{T}{\omega}\sigma(\omega)$$

Lattice-QCD transport coefficients: results

The spectral function $\sigma(\omega)$ has to be reconstructed starting from the *euclidean electric-field correlator*

$$D_E(\tau) = -\frac{\langle \mathrm{Re}\,\mathrm{Tr}[U(\beta,\tau)gE^i(\tau,\mathbf{0})U(\tau,0)gE^i(0,\mathbf{0})]\rangle}{\langle \mathrm{Re}\,\mathrm{Tr}[U(\beta,0)]\rangle}$$

according to

$$D_E(\tau) = \int_0^{+\infty} \frac{d\omega}{2\pi} \frac{\cosh(\tau-\beta/2)}{\sinh(\beta\omega/2)} \sigma(\omega)$$

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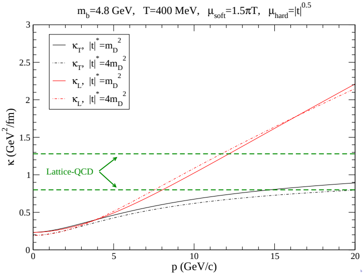
One gets[a]:

$$\kappa \approx 2.5T^3 - 4T^3$$

$\sim$3-5 times larger then the $p=0$ perturbative result

[a] A. Francis *et al.*, PoS LATTICE2011 202; D. Banerjee *et al.*, Phys.Rev. D85 (2012) 014510



l-QCD transport coefficients: detailed derivation (I)

Derivation of κ in l-QCD done in the $M \to \infty$ limit. In this case the HQ field ψ is only coupled to the A_0 component of the colour-field:

$$\mathcal{L} = Q^\dagger (i\partial_0 + gA_0) Q, \quad \text{with} \quad \left\{ Q_i(t, \mathbf{x}), Q_j^\dagger(t, \mathbf{y}) \right\} = \delta_{ij}\delta(\mathbf{x} - \mathbf{y})$$

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HQ evolution described by the path-ordered exponential $U(t, t_0)$

$$Q_i(t) = \mathcal{P}\exp\left[ig\int_{t_0}^{t} A_0(t')dt'\right]_{ij} Q_j(t_0) = U_{ij}(t,t_0)Q_j(t_0)$$

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One needs then to evaluate the expectation value

$$\langle F^i(t)F^i(0)\rangle_{\rm HQ} \equiv \frac{\sum_s \langle s|e^{-\beta H}F^i(t)F^i(0)|s\rangle}{\sum_s \langle s|e^{-\beta H}|s\rangle}$$

taken over a thermal ensemble of states $|s\rangle$ of the environment *plus* one additional heavy quark:

$$\sum_s \langle s|\dots|s\rangle \equiv \sum_{s'}\int d\mathbf{x}\, \langle s'|Q_i(-T,\mathbf{x})\dots Q_i^\dagger(-T,\mathbf{x})|s'\rangle$$

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taken over a thermal ensemble of states $|s\rangle$ of the environment *plus* one additional heavy quark. In particular:

$$\sum_s \langle s|e^{-\beta H}|s\rangle = Z_{\rm HQ}$$

l-QCD transport coefficients: detailed derivation (II)

Thermal weight $e^{-\beta H}$ $\equiv$ *imaginary-time translation operator*

$$Q(-T)e^{-\beta H} = e^{-\beta H}e^{\beta H}Q(-T)e^{-\beta H} = e^{-\beta H}Q(-T-i\beta)$$

one gets for the HQ partition function (i.e. the denominator)

$$\begin{aligned} Z_{\rm HQ} &= \sum_{s'}\int d\mathbf{x}\,\langle s'|Q_i(-T,\mathbf{x})e^{-\beta H}Q_i^\dagger(-T,\mathbf{x})|s'\rangle \\ &\sim \sum_{s'}\langle s'|e^{-\beta H}U_{ii}(-T-i\beta,-T)|s'\rangle = Z_0\langle {\rm Tr}\,U(-T-i\beta,-T)\rangle, \end{aligned}$$

where the last expectation values is over the environment only.

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where the last expectation values is over the environment only.
The numerator can be evaluated analogously starting from

$$\begin{aligned} \sum_s \langle s| e^{-\beta H} \mathbf{F}(t)\cdot\mathbf{F}(0)|s\rangle = \sum_{s'} \frac{1}{N_c} \int d\mathbf{x} \int d\mathbf{r} \int d\mathbf{r}' \\ \times \langle s'| Q_i(-T,\mathbf{x}) e^{-\beta H} Q_j^\dagger(t,\mathbf{r}) g\mathbf{E}_{jk}(t,\mathbf{r}) Q_k(t,\mathbf{r}) \\ \times Q_l^\dagger(0,\mathbf{r}') g\mathbf{E}_{lm}(0,\mathbf{r}') Q_m(0,\mathbf{r}') Q_i^\dagger(-T,\mathbf{x})|s'\rangle \end{aligned}$$

l-QCD transport coefficients: detailed derivation (III)

The force-force correlator we need is then given by

$$\langle F^i(t)F^i(0)\rangle_{\rm HQ} = \langle {\rm Tr}[U(-T-i\beta,t)gE^i(t) \\ \times U(t,0)gE^i(0)U(0,-T)]\rangle/\langle {\rm Tr}\, U(-T-i\beta,-T)\rangle$$

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Lattice-QCD simulations performed in imaginary time: one actually evaluate

$$D_E(\tau) = -\frac{\langle {\rm Re\,Tr}[U(\beta,\tau)gE^i(\tau,\mathbf{0})U(\tau,0)gE^i(0,\mathbf{0})]\rangle}{\langle {\rm Re\,Tr}[U(\beta,0)]\rangle}$$

and extract $\sigma(\omega)$ from

$$D_E(\tau) = \int_0^{+\infty}\frac{d\omega}{2\pi}\frac{\cosh(\tau-\beta/2)}{\sinh(\beta\omega/2)}\sigma(\omega) \quad\longrightarrow\quad \kappa \underset{\omega\to 0}{\sim} \frac{1}{3}\frac{T}{\omega}\sigma(\omega)$$

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NB $D_E(\tau)$ known just for ~ 10 points makes the inversion ill-defined! Various strategies adopted: Maximum Entropy Method, χ^2 after ansatz for the functional form of $\sigma(\omega)$

POWLANG: results

In the following we will show results obtained within our POWHEG+Langevin setup

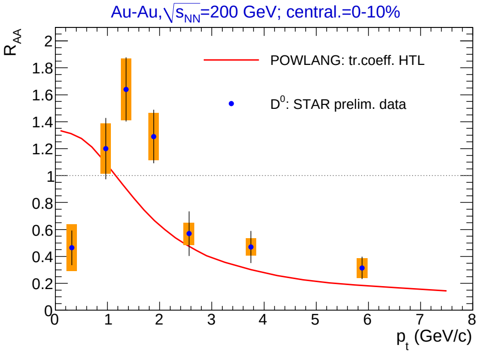
- Formalism developed in Nucl.Phys. A831 (2009) 59 and Eur.Phys.J. C71 (2011) 1666;
- Some results for LHC @ 2.76 TeV presented in J.Phys. G38 (2011) 124144 and arXiv:1208.0705;
- All the following plots taken from arXiv:1305.7421.

Initialization and cross-sections

Nuclei	$\sqrt{s_{\rm NN}}$	τ_0 (fm/c)	s_0 (fm^{-3})	T_0 (MeV)
Au-Au	200 GeV	1.0	84	333
Pb-Pb	2.76 TeV	0.6	278	475
Pb-Pb	2.76 TeV	0.1	1668	828

Collision	$\sqrt{s_{\rm NN}}$	$\sigma_{c\overline{c}}$ (mb)	$\sigma_{b\overline{b}}$(mb)
p-p	200 GeV	0.405	1.77×10^{-3}
Au-Au	200 GeV	0.356	2.03×10^{-3}
p-p	2.76 TeV	2.425	0.091
Pb-Pb	2.76 TeV	1.828	0.085

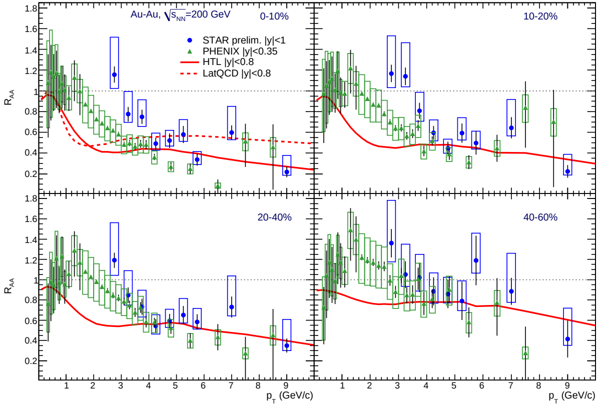
D mesons R_{AA} at RHIC



- Quenching of p_T-spectra nicely reproduced for $p_T \gtrsim 2$ GeV;
- Sharp peak around $p_T \approx 1.5$ GeV: coming from coalescence?

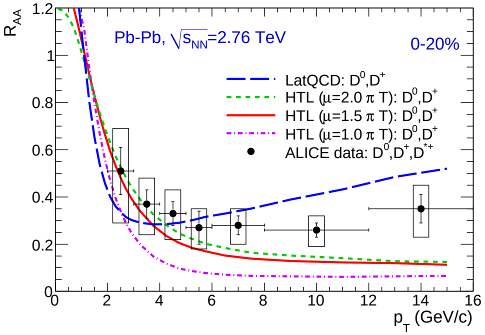
NB peak visible thanks to very fine binning at low-p_T

Heavy-flavour electrons R_{AA} at RHIC



- Rough agreement with the data for $p_T \gtrsim 4$ GeV;
- Langevin results underestimate the data at lower p_T

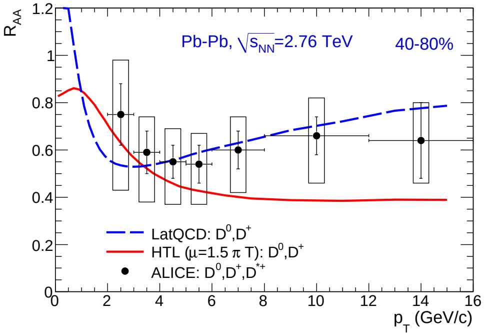
D-meson R_{AA} at LHC



Possibility to discriminate HTL (with $\mu = \pi T - 2\pi T$) and l-QCD results at high-p_T, where however:

- Langevin approach becomes questionable
- No info on momentum dependence of $\kappa_{T/L}$ is available from l-QCD

D-meson R_{AA} at LHC



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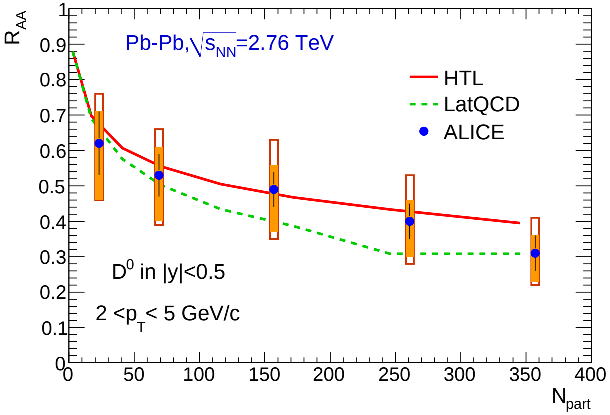

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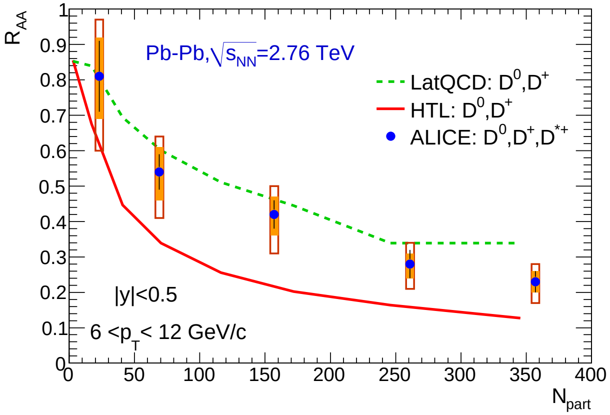
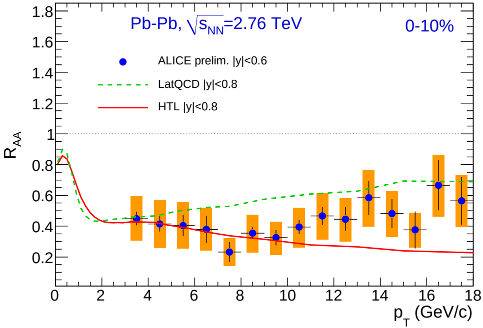


Heavy-flavour electrons R_{AA} at LHC



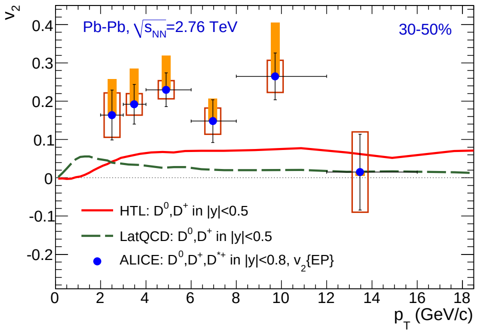
- Good agreement between HTL-Langevin and ALICE data up to $\sim$ 10 GeV;
- For larger p_T data stays between HTL and l-QCD predictions.

General considerations

Experimental heavy-flavour data at high-p_T always stay between the Langevin results with HTL and l-QCD transport coefficients, suggesting for $\kappa_L(p)$ a mild rise with the quark momentum, different from

- the strong rise foreseen by the HTL+pQCD result;
- the constant behaviour assumed for the l-QCD case.

Elliptic-flow: D-meson v_2 at LHC



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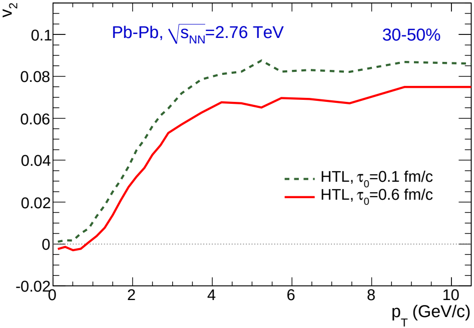
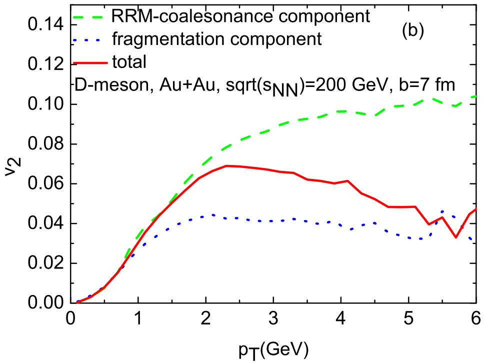


- Langevin outcomes undershoot the data, both with HTL and l-QCD transport coefficients;
- Even assuming a very short thermalization time is not sufficient to reproduce the observed flow at low-moderate p_T.

Possible role played by coalescence?



Hadronization via coalescence with light thermal partons from the medium might provide a contribution to the *D*-meson v_2, part of the flow coming from the momentum of the light quark.

Beauty in AA collisions

Beauty: a golden probe of the medium

- Clean theoretical setup, due to its *large mass*
  - Description via independent random collisions working over an extended p_T-range;
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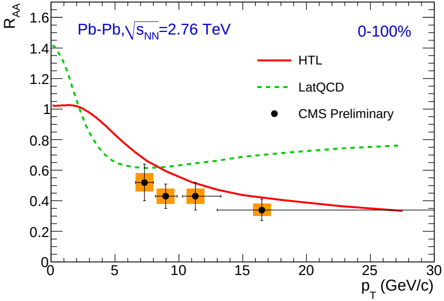
Beauty provides clean information on what happens in the partonic phase!

R_{AA} of displaced J/ψ's at LHC



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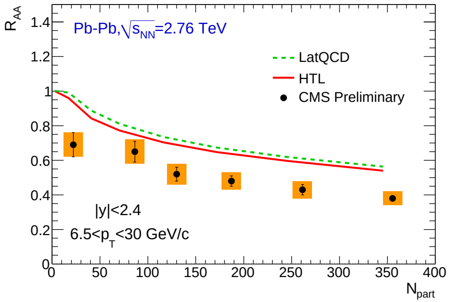

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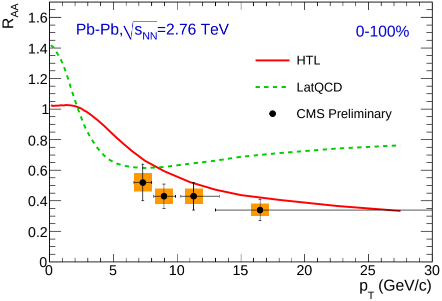
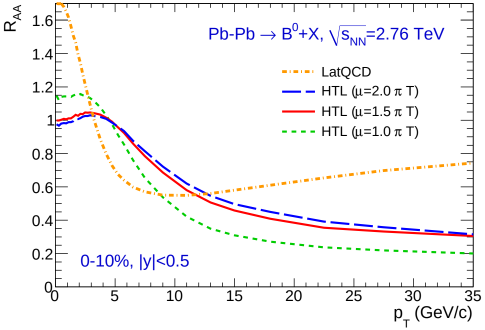


- l-QCD transport coefficients provide a *larger suppression at moderate p_T* wrt perturbative predictions;
- Ignoring momentum-dependence of l-QCD transport coefficients leads to milder suppression at high-p_T wrt HTL results;

B-meson R_{AA} at LHC



Measurements of *B*-mesons at low-p_T potentially able to discriminate the two scenarios in a regime in which the uncertainty on the momentum dependence of the transport coefficients shouldn't play a big role

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