Introduction to the QCD phase diagram and lattice QCD

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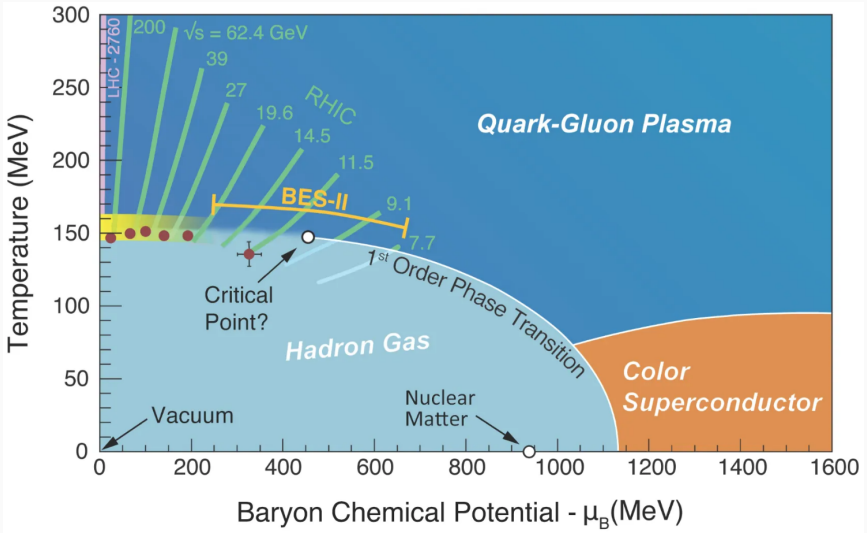
Advanced Nuclear Physics
12 Aprile 2024

The phase diagram of QCD



Study of QCD thermodynamics

Theoretical investigations of QCD thermodynamics make use of different methods and tools

From first principles:

- **Lattice QCD**
- Perturbation theory (large T and/or μ)

Models:

- Nambu-Jona-Lasinio (NJL) -type models (**Nambu and Jona-Lasinio, Phys. Rev. 122 (1961) 345, Phys. Rev. 124 (1961) 246**)
- Hadron Resonance Gas (HRG) -type models (**Hagedorn, Nuovo Cim. Suppl. 3 (1965), 147**)
- Functional methods (functional renormalization group - FRG, Dyson-Schwinger equations, etc...)
- ...

QCD Lagrangian

We know **Quantum Chromodynamics** (QCD) is a gauge theory with color $SU(3)_c$ symmetry:

$$\boxed{\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\gamma^\mu D_\mu - m_f)\psi_f - \tfrac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a}$$

where:

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_S \, f^{abc} A^b_\mu A^c_\nu$$
$$D_\mu = \partial_\mu + i \, g_S \, t^a A^a_\mu$$

Problem: perturbation theory for QCD is not feasible in the regime around the QCD transition because g_S is not small

Solution: the path integral formulation does not rely on a perturbative approach, and gives us the partition function:

$$\mathcal{Z}[A, \bar{\psi}, \psi] = \int \mathcal{D}A^a_\mu(x) \, \mathcal{D}\bar{\psi}(x) \, \mathcal{D}\psi(x) \, e^{-\int d^4x \, \mathcal{L}_E[A, \bar{\psi}, \psi]}$$

where $S_E = \int d^4x \, \mathcal{L}_E$ is the *euclidean* QCD action. Lattice QCD starts from here.

Lattice formulation of QCD

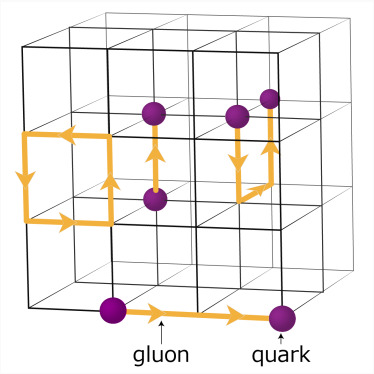
Problem: we cannot calculate the full integral for $\mathcal{Z}[A, \bar{\psi}, \psi]$.

Solution: define the theory on a discretized 3+1d lattice of size $N_s^3 \times N_\tau$, with lattice spacing a. This allows us to reduce the (otherwise infinite) dimensionality of the problem.

- The quark fields $\bar{\psi}, \psi$ are defined on the lattice sites, the gauge fields A_μ are defined on the lattice links as $U_\mu = \exp[iaA_\mu] \in SU(3)$
- Now, one can calculate a *finite* number of integrals to evaluate expressions of the like:

$$Z[U, \bar{\psi}, \psi] = \int \mathcal{D}U \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \, e^{-S_G[U,\bar{\psi},\psi] - S_F[U,\bar{\psi},\psi]}$$

where S_G and S_F are the gauge (gluonic) and fermionic actions



Lattice formulation of QCD

Actually, we can analytically perform the integral over the quark fields (the fermionic action is bilinear in the quark fields), and remain with:

$$Z[U, \bar{\psi}, \psi] = \int \mathcal{D}U \ \det M[U] \, e^{-S_G[U]}$$

and any observable $\hat{O}$ can then be calculated as:

$$\left\langle \hat{O} \right\rangle = \frac{1}{Z} \int \mathcal{D}U \, \hat{O} \, \det M[U] \, e^{-S_G[U]}$$

Problem: the integrals cannot be calculated by brute force. Even for a small 10^4 lattice, integral is 320000-dimensional!

Solution:

- Monte Carlo integration with **importance sampling**: interpret the factor $\det M[U] \, e^{-S_G[U]}$ as a weight for the configuration U, and reduce the sum only to the most "likely" configurations

Metropolis-Hastings algorithm

A **Markov-chain Monte Carlo** (MCMC) process to generate configurations according to a distribution $P(U)$. At equilibrium, we must have **detailed balance** (transition and back-transition have equal probability):

$$P(U)P(U'|U) = P(U')P(U|U') \quad \Rightarrow \frac{P(U)}{P(U')} = \frac{P(U|U')}{P(U'|U)}$$

where:

- $P(U)$ is the probability to have configuration U
- $P(U'|U)$ is the probability to transition from U to U'

We split the transition into a proposal and a (possible) acceptance:

$$P(U'|U) = pr(U'|U)ac(U'|U)$$

where:

- $pr(U'|U)$ is the probability of proposing U' when we have U
- $ac(U'|U)$ is the probability to accept the transition

Metropolis-Hastings algorithm

Substituting we get:

$$\frac{ac(U'|U)}{ac(U|U')} = \frac{P(U')pr(U|U')}{P(U)pr(U'|U)}$$

In the Metropolis-Hastings algorithm, the choice is:

$$ac(U'|U) = \min\left\{1, \frac{P(U')pr(U|U')}{P(U)pr(U'|U)}\right\}$$

If, in addition, we choose a flat proposal probability $pr(U|U') = pr(U'|U)$ (e.g. change a single link variable) we have that:

- if weight increases, change is accepted
- if weight decreases, change is accepted with probability

$$p = \exp(-\Delta S_G) \det M(U')/\det M(U)$$

This leads to correct equilibrium state, while allowing fluctuations

Lattice formulation of QCD

- *Euclidean* actions S_G and S_F only depend on coupling g (or $\beta = 6/g^2$) and fermion masses m_f. Their relationship is fixed by the so-called **line of constant physics** (lcp). The line of constant physics $\beta(m_f)$ is set such that the action reproduces the physics correctly when β is varied. It is obtained e.g. by imposing that m_π/f_π takes on its physical value

- The finiteness of the lattice spacing a serves as a *regulator* for the theory. At the end one wishes to recover the continuum theory with $\lim_{a\to 0}$ ($\lim_{N_\tau\to\infty}$): **continuum limit** $\rightarrow$ very delicate

- Calculations are done in a finite volume. When possible, one wishes to study the thermodynamic limit $\lim_{V\to\infty}$: a.k.a. **infinite volume limit**

Lattice formulation of QCD

- Length in "temporal" direction is given by the inverse temperature $T^{-1} = \beta = N_\tau a$ (inverse temperature $\beta = 1/T$ not to be confused with gauge coupling $\beta = 6/g^2$; unfortunate convention!)
- Summarizing

$$V = (N_s a)^3 \ , \qquad T = \frac{1}{N_\tau a}$$

- **Scale setting:** everything we calculate on the lattice is dimensionless, expressed in terms of the lattice spacing a. Eventually, we have to express a in physical units. We calculate some quantity whose value is well known, and use it to set the scale (e.g. pion decay constant, pion mass, kaon mass, etc.). For example:

$$am_\pi = \text{value} = a \cdot 135\,\text{MeV} \Rightarrow a = \frac{\text{value}}{135\,\text{MeV}} = \frac{\text{value}}{135} 197.33\,\text{fm}$$

A value for a gives us T and V in physical units, and every other dimensionful quantity

Thermodynamic description of QCD

The thermodynamics of QCD is commonly investigated in the grand canonical ensemble

- Grancanonical partition function:

$$\mathcal{Z} = \sum_N Z_N e^{\mu N}$$

  where:
  - Z_N is the canonical partition function with N particles
  - μ is the *chemical potential* associated to the particle number N

  (the chemical potential is the energy associated to a change in the number N of particles)
- In QCD, there are 3 conserved "particle numbers", which one can see as:
  - **Quark numbers:** u, d, s

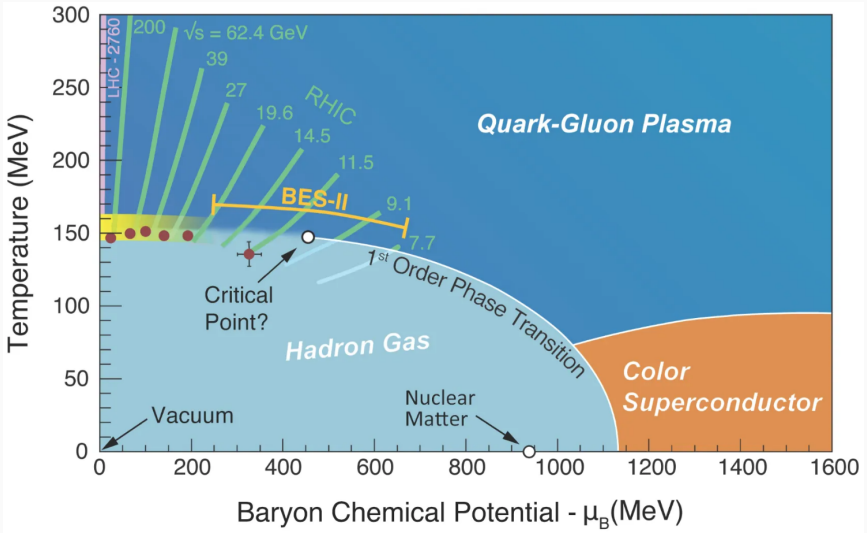
$$\mu N = \mu_u N_u + \mu_d N_d + \mu_s N_s$$

  - **Conserved charges:** B, Q, S

$$\mu N = \mu_B N_B + \mu_Q N_Q + \mu_S N_S$$

Note: weak processes ignored in heavy-ion physics, heavy flavours assumed not to thermalize

The QCD transition



Deconfinement: Polyakov loop

At the QCD transition, in reality two transitions occur at once: deconfinement and chiral symmetry restoration.

Deconfinement

The Polyakov loop is defined as:

$$P(\vec{x}) = \frac{1}{N_c} \operatorname{Tr} \prod_{i=0}^{N_\tau - 1} U_4(\vec{x}, i)$$

While the (pure) gauge action is invariant under the $\mathbb{Z}_3$ center symmetry ($\mathbb{Z}_3$ is the center of SU(3), i.e. every member of SU(3) commutes with members of $\mathbb{Z}_3$), the Polyakov loop **is not**. Hence, a finite value of $\langle P(\vec{x}) \rangle$ indicates spontaneous symmetry breaking of center symmetry. This happens at large temperatures.

$\Rightarrow$ The polyakov loop is an order parameter for the SSB of center symmetry.

Deconfinement: Polyakov loop

Moreover, one can show that

$$\langle P \rangle \sim e^{-F}$$

where F is the free energy associated to placing a free color charge in the system.

This means that:

- $\langle P \rangle = 0$ means $F \to \infty$, hence the system is **confined**
- $\langle P \rangle = 1$ means $F = 0$, hence the system is **deconfined**

$\Rightarrow$ The polyakov loop is an order parameter for confinement.

Note: because the center symmetry is exact only in the limit of infinitely heavy quarks, for physical masses the Polyakov loop is only an approximate order parameter.

Chiral symmetry restoration: chiral condensate

At the QCD transition, in reality two transitions occur at once: deconfinement and chiral symmetry restoration.

Chiral symmetry restoration

For massless quarks, the QCD action has exact chiral symmetry. The chiral condensate is defined as:

$$\langle \bar{\psi}_f \psi_f \rangle = \frac{T}{V} \frac{\partial \ln \mathcal{Z}}{\partial m_f}$$

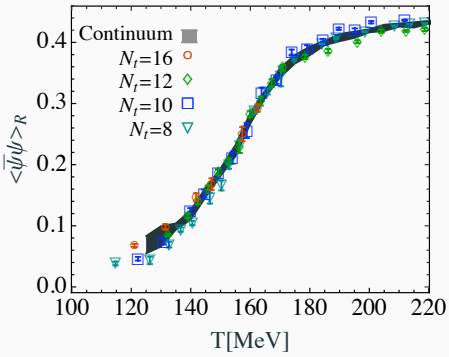
A non-zero value of the condensate indicates that the symmetry is spontaneously broken, which is the case in the low-temperature phase.

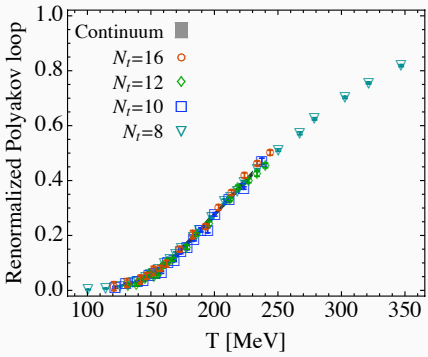
$\Rightarrow$ The chiral condensate is an order parameter for chiral symmetry restoration.

Note: because chiral symmetry is exact in the limit of massless quarks, for physical masses the chiral condensate is only an approximate order parameter (much like the Polyakov loop for confinement).

The QCD transition: observables

At the physical point (small but non-zero quark masses) both observables are able to distinguish between the two phases, but neither is a true order parameter:





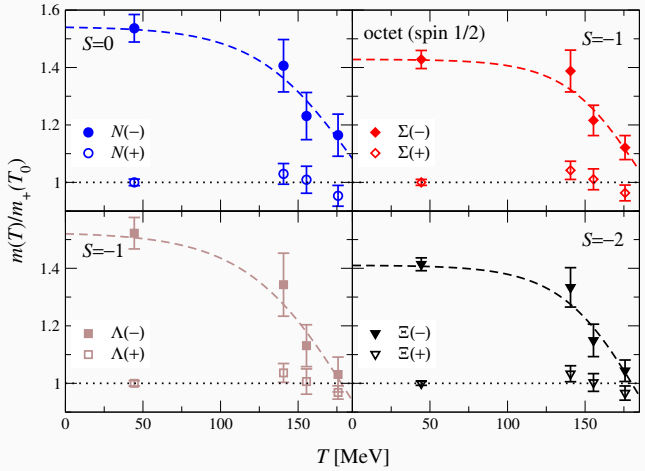
Borsanyi et al. JHEP 1009:073 (2010)

Although deconfinement and chiral transition are two "different" transitions, we have no evidence of them taking place at different temperatures. Unique T_C!

NOTE: this chiral condensate is renormalized subtracting the zero-temperature value

More on chiral symmetry restoration

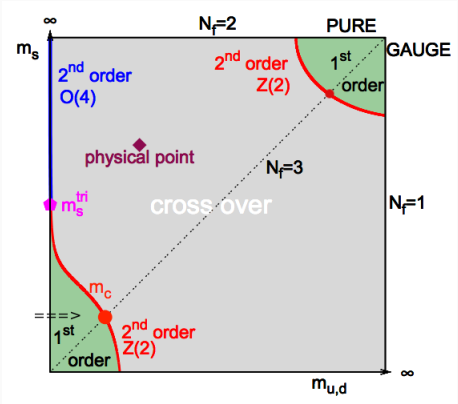
While the positive parity baryons have a mass that is stable with T, the negative parity chiral partners' masses go down at around T_c



Aarts *et al.*, PRD 99 (2019) 074503

The QCD transition: Columbia plot

As a function of the light (u,d) and strange quark masses, the order of the transition changes

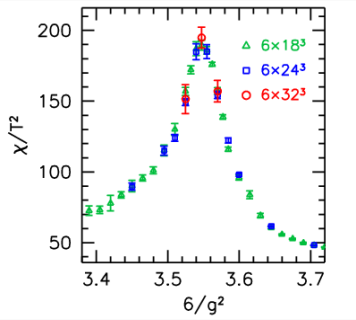


- At the physical point $m_s/m_{ud} \simeq 27$, the transition is a smooth crossover!
- In the heavy-quark limit (pure gauge), the transition is first order

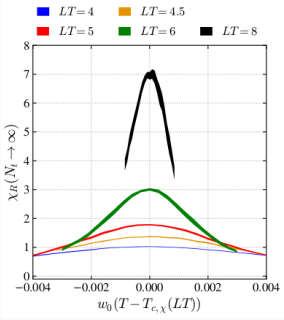
The QCD transition: crossover vs. first order

On the lattice we study the volume scaling of certain quantities to determine the order of the transition

Left: physical masses



Right: infinite masses (pure gauge)



- For a crossover (left), the peak height is independent of the volume
- For a first order transition, it scales linearly with the volume

Aoki *et al.* Nature 443 (2006), Borsányi *et al.*, arXiv:2202.05234 [hep-lat]

Thermodynamics of QCD: equation of state

The **Equation of State** (**EoS**) is extremely important since it completely describes the equilibrium properties of QCD matter.

It is one of the main inputs to hydro and several other tools for calculations in heavy-ion collisions and higher-density physics.

Thermodynamic quantities follow directly from the grancanonical partition function $\mathcal{Z}$ and the relation:

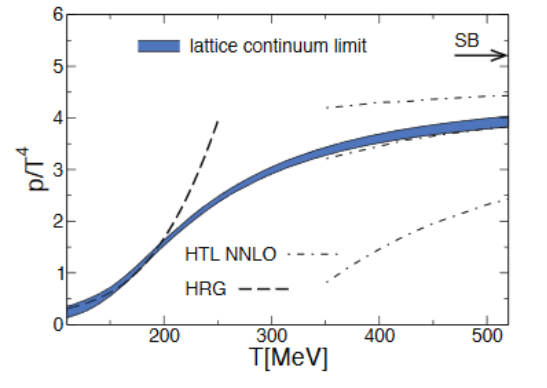
$$-k_B T \ln \mathcal{Z} = U - TS - \mu N$$

- **Pressure**: $p = -k_B T \frac{\partial \ln \mathcal{Z}}{\partial V}$
- **Entropy density**: $s = \left(\frac{\partial p}{\partial T}\right)_{\mu_i}$
- **Charge densities**: $n_i = \left(\frac{\partial p}{\partial \mu_i}\right)_{T, \mu_{j \neq i}}$
- **Energy density**: $\epsilon = Ts - p + \sum_i \mu_i n_i$
- **Speed of sound**: $c_s^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_{s/n_B}$
- More (**Fluctuations**, etc...)

Equation of state at $\mu_B = 0$

A combination of methods gives us good understanding of the EoS at $\mu_B = 0$ at all temperatures

- Perturbative QCD at high temperature
  → "pure quark-gluon phase"
- HRG model at low temperature
  → "pure hadron phase"
- **Lattice QCD bridges between regimes and captures the transition**



Borsányi *et al.*, PLB 370 (2014) 99-104

Equation of state from the lattice

The pressure cannot be determined directly on the lattice, because it is **not** a derivative of $\ln Z$ with respect to a parameter.

One alternative is the so-called integral method, whereby the pressure is calculated via an integral of the trace anomaly $I(T)$:

$$\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{dT'}{T'} \frac{I(T')}{T'^4}$$

where the **trace anomaly** $I(T)$ *can* be determined directly on the lattice, and is:

$$\frac{I(T)}{T^4} = T \left. \frac{\partial (p/T^4)}{\partial T} \right|_V = \frac{\epsilon - 3p}{T^4}$$

Why can it be determined directly on the lattice? Note that, because $T = 1/(N_\tau a)$:

$$\frac{\partial}{\partial T} = \frac{\partial a}{\partial T} \frac{\partial}{\partial a} = -\frac{1}{N_\tau T^2} \frac{\partial}{\partial a}$$

hence:

$$T \frac{\partial}{\partial T} = -\frac{1}{N_\tau T} \frac{\partial}{\partial a} = -a \frac{\partial}{\partial a}$$

Equation of state from the lattice

Now, the lattice spacing depends only on the parameters of the action, the coupling $\beta = 6/g^2$ and the fermion masses m_f, hence:

$$\frac{\partial}{\partial a} = \frac{\partial \beta}{\partial a}\frac{\partial}{\partial \beta} + \sum_f \frac{\partial m_f}{\partial a}\frac{\partial}{\partial m_f}$$

Now, in homogeneous systems one has:

$$p = \frac{T}{V}\ln Z = \frac{1}{N_\tau a (N_s a)^3}\ln Z \quad \Rightarrow \quad \frac{p}{T^4} = \left(\frac{N_\tau}{N_s}\right)^3 \ln Z$$

so the trace anomaly reads:

$$\frac{I(T)}{T^4} = -a\left(\frac{N_\tau}{N_s}\right)^3 \frac{\partial}{\partial a}\ln Z = -a\left(\frac{N_\tau}{N_s}\right)^3 \left(\frac{\partial \beta}{\partial a}\frac{\partial \ln Z}{\partial \beta} + \sum_f \frac{\partial m_f}{\partial a}\frac{\partial \ln Z}{\partial m_f}\right)$$

Equation of state from the lattice

Now, we notice that:

- $\frac{1}{N_\tau N_s^3}\frac{\partial \ln Z}{\partial \beta} = \langle -s_G \rangle$ is the gauge action
- $\frac{1}{N_\tau N_s^3}\frac{\partial \ln Z}{\partial m_f} = \langle \bar{\psi}_f \psi_f \rangle$ is the chiral condensate of flavour f

Then, we note that:

$$\frac{dT}{T} = -\frac{da}{a}$$

hence:

$$\frac{I(T)}{T^4}\frac{dT}{T} = N_\tau^4 d\beta \left(\langle -s_G \rangle + \sum_f \frac{\partial m_f}{\partial \beta} \langle \bar{\psi}_f \psi_f \rangle \right)$$

where $\partial m_f / \partial \beta$ is obtained from the line of constant physics.

To conclude, we can calculate the pressure in QCD **up to a constant** via an integral over one of the parameters of the theory:

$$\frac{p(T)}{T^4} = \frac{p(T_0)}{T_0^4} + \int_{T_0}^{T} \frac{dT'}{T'} \frac{I(T')}{T'^4}$$

Equation of state from the lattice

What about the constant? We said the pressure cannot be determined directly.

However, we can use a similar approach and integrate it in some parameter from a starting point where we know it. In particular, we know that $p = 0$ in the limit $m_q \to \infty$.

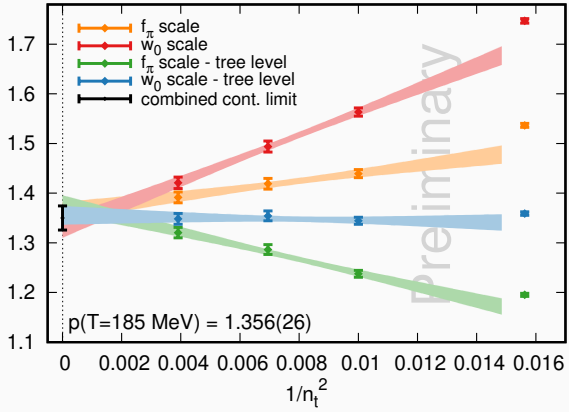
Hence, the pressure is determined as an integral in the quark masses down from infinity:

$$\frac{p(T_0)}{T_0^4} = \int_{m_s}^{m_l} dm_2 \left\langle \bar{\psi}\psi \right\rangle_{R,2} (m_2) + \int_{\infty}^{m_s} dm_3 \left\langle \bar{\psi}\psi \right\rangle_{R,3} (m_3)$$

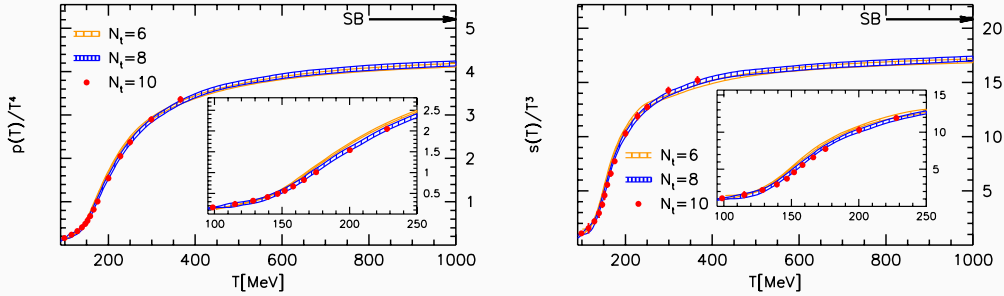
Because at the physical point quark masses differ by flavour, we need to break the integral into pieces.

This constant plus the integral of $I(T)$ gives us the pressure $p(T)$.

Equation of state: example of continuum extrapolation



Lattice QCD: equation of state at $\mu_B = 0$



Change in active degrees of freedom is evident around the transition

Note: Stefan-Boltzmann limit for free gas (large T):

$$p_{\mathrm{QCD}} = \frac{\pi^2}{45} T^4 (N_c^2 - 1) + \sum_{\text{flavours}} \frac{N_c}{3\pi^2} \left[\frac{7\pi^4 T^4}{60} + \frac{\mu_f^2 \pi^2 T^2}{2} + \frac{\mu_f^4}{4} \right]$$

Borsányi *et al.*, JHEP 11 (2010) 077

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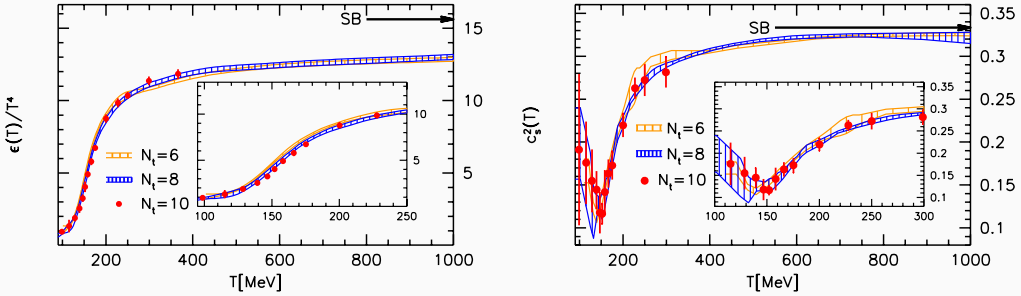

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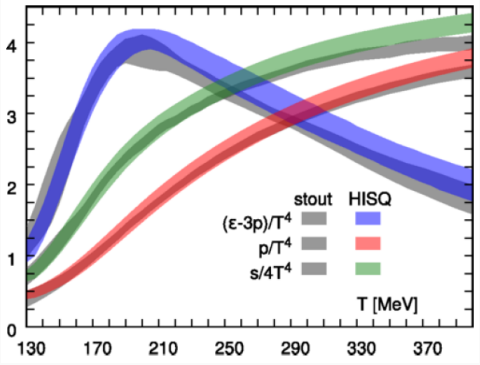
Borsányi *et al.*, JHEP 11 (2010) 077

Lattice QCD: equation of state at $\mu_B = 0$

- Current lattice simulations are performed with realistic setup (continuum limit, physical quark masses)
- Great agreement between different collaborations



Borsányi *et al.*, PLB 370 (2014) 99-104, Bazavov *et al.* PRD 90 (2014) 094503

The sign/complex action problem

We saw that euclidean path integrals are calculated with MC methods using importance sampling, with a Boltzmann weight $\det M[U]\, e^{-S_G[U]}$ for the configuration U

$$\begin{aligned} Z(V,T,\mu) &= \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-S_F(U,\psi,\bar{\psi})-S_G(U)} \\ &= \int \mathcal{D}U\ \det M(U) e^{-S_G(U)} \end{aligned}$$

When a chemical potential is introduced, a problem appears:

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

which means that in general the determinant is complex (**complex action problem**). This means that it cannot serve as a statistical weight to guide importance sampling.

However, that is not the case if:

- there is particle-antiparticle-symmetry ($\mu = 0$):
- the chemical potential is *purely imaginary* ($\mu^2 < 0$)

$$[\det M(\mu)]^* = \det M(-\mu^*) = \det M(\mu)$$

The sign/complex action problem

Because finite-μ physics is of great interest, alternatives have been widely explored:

- **Taylor expansion** around $\mu = 0$
  One calculates derivatives of the QCD pressure at $\mu = 0$, then construct the expansion:

$$\frac{p(T,\mu)}{T^4} = \sum_n c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n} , \qquad c_n(T) = \frac{1}{n!} \left.\frac{\partial^n (p/T^4)}{\partial \mu^n}\right|_{\mu=0} = \frac{1}{n!}\chi_n(T)$$

- **Reweighting**
  Because the Boltzmann weight contains $\det M(\mu)$, which is complex, move it to the observable:

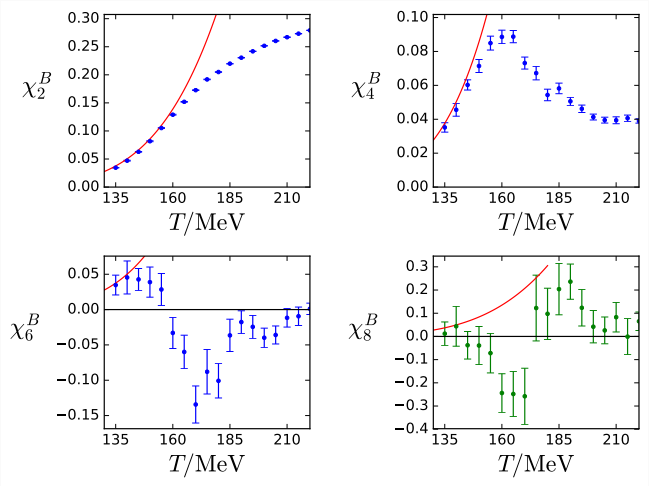
$$\det M(\mu) = \frac{\det M(\mu)}{\det M(\mu = 0)} \det M(\mu = 0)$$

- **Imaginary chemical potential**
  Simulate at imaginary μ, then (somehow) analytically continue to real μ

Lattice QCD at finite μ_B - Taylor coefficients

Results for the Taylor coefficients are currently available up to $\mathcal{O}(\hat{\mu}_B^8)$, but the reach of the equations of state is still limited to $\hat{\mu}_B \lesssim 2 - 2.5$ despite great computational effort

- The reach of Taylor expansion is limited by:
  A) the radius of convergence (unknown), linked to the closest (complex) singularity of Z
  B) in practice, computational power and the (small) number of terms
- Very computationally demanding
- Signal extraction is increasingly difficult with higher orders



Borsányi *et al.* JHEP 10 (2018) 205

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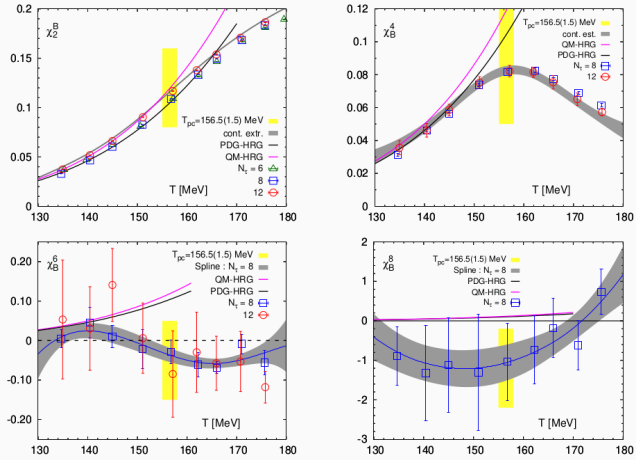
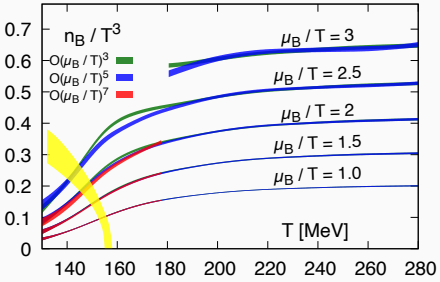
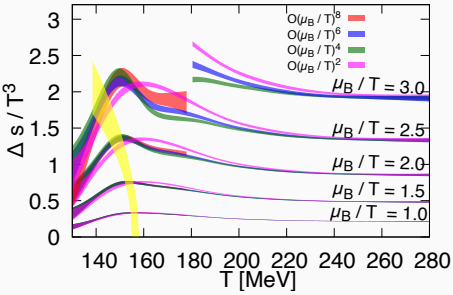


Bazavov *et al.* **PRD101 (2020), 074502**

Taylor expansion

A good coverage in μ_B is obtained, but convergence seems still far, especially at low T





Bollweg *et al.*, PRD 108 (2023) 014510

Reweighting

The idea behind reweightin is really simple. Because we have a complex weight, change the weight and move the original weight into the observable:

$$\begin{aligned}\left\langle \hat{O}(\mu) \right\rangle &= \frac{1}{Z(\mu)} \int \mathcal{D}U \, \hat{O}(\mu) \det M(\mu) e^{-S_G} = \\ &= \frac{\int \mathcal{D}U \, \hat{O}(\mu) \frac{\det M(\mu)}{\det M(0)} \det M(0) e^{-S_G}}{\int \mathcal{D}U \, \frac{\det M(\mu)}{\det M(0)} \det M(0) e^{-S_G}} = \\ &= \frac{\left\langle \hat{O}(\mu) \frac{\det M(\mu)}{\det M(0)} \right\rangle_0}{\left\langle \frac{\det M(\mu)}{\det M(0)} \right\rangle_0}\end{aligned}$$

where $\langle \cdots \rangle_0$ indicate averages over distribution $\det M(0) e^{-S_G}$.

Reweighting

Of course, this does not solve all problems!
Two problems can arise from evaluating the expression

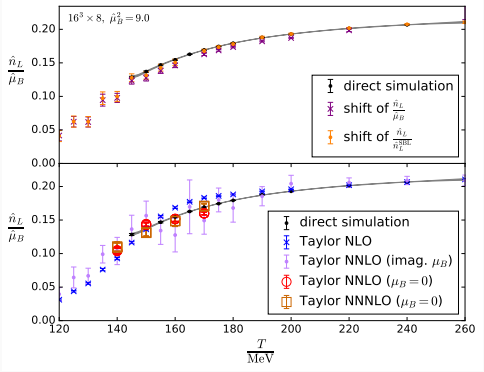
$$\left\langle \hat{O}(\mu) \right\rangle = \frac{\left\langle \hat{O}(\mu) \frac{\det M(\mu)}{\det M(0)} \right\rangle_0}{\left\langle \frac{\det M(\mu)}{\det M(0)} \right\rangle_0}$$

- **Overlap problem**: because we use importance sampling, we generated configurations according to the distribution $\det M(0) e^{-S_G}$. However, the physics we search is described by $\det M(\mu) e^{-S_G}$. If they do not *overlap* sufficiently, we are sampling the distribution in the wrong way, leading to systematic errors which are hard to quantify
- **Sign problem**: if the denominator is nearly vanishing, the error on our estimate blows up

Reweighting

Improvement over this simple version of reweighting can be made:

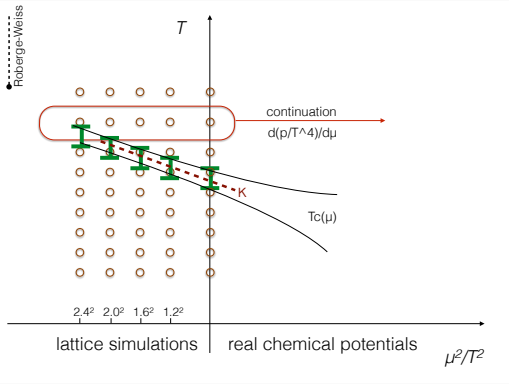
- we can reweight simultaneously in more than one parameter, e.g. (T, μ) instead of μ only, or the quark mass, etc..
- we can exploit characteristics of the observables, e.g. if we know it is real, we can disregard the phase



Borsányi *et al.*, PRD 107 (2023) L091503

Imaginary chemical potential

Simluations at imaginary chemical potential have been used extensively to calculate several quantities. We can do this because the transition at $\mu = 0$ is a crossover, so the partition function is analytic!



The actual procedure depends very much on the observable of interest

The QCD transition at finite chemical potential

One defines the transition line $T_c(\mu_B)$ as:

$$\frac{T_c(\mu_B)}{T_c(\mu_B = 0)} = 1 + \kappa_2 \left(\frac{\mu_B}{T_c(\mu_B)}\right)^2 + \kappa_4 \left(\frac{\mu_B}{T_c(\mu_B)}\right)^4$$

At physical quark masses, the chiral transition is more convenient:

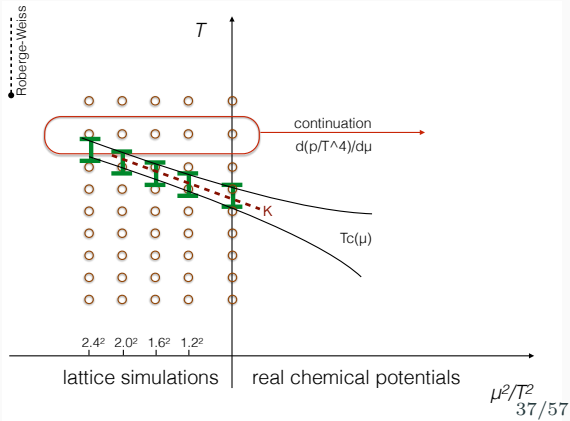
Chiral condensate

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V} \frac{\partial \ln Z}{\partial m_{ud}}$$

Chiral susceptibility

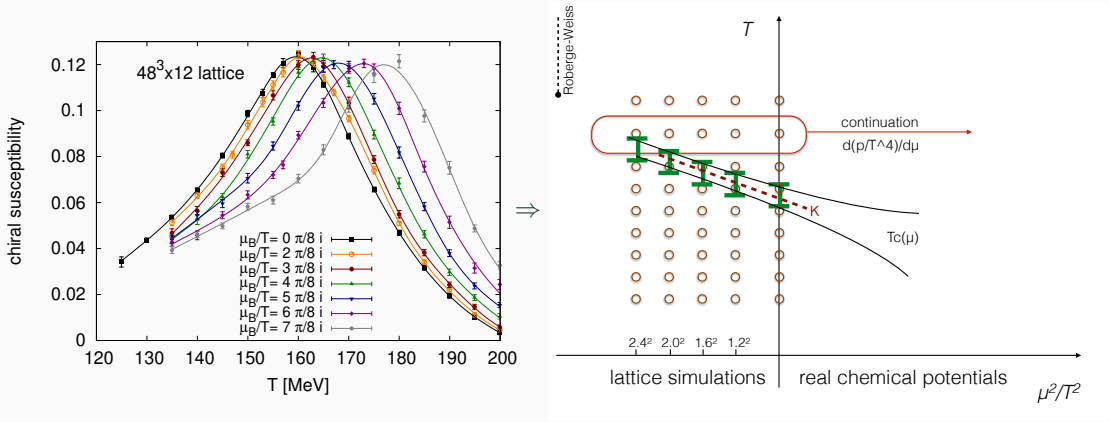
$$\chi = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial m_{ud}^2}$$

At the transition temperature T_C, the chiral condensate has an inflection point, and the chiral susceptibility has a peak.



Chiral observables at imaginary μ_B

At imaginary chemical potential, the transition moves as $T_c(\mu_B)$



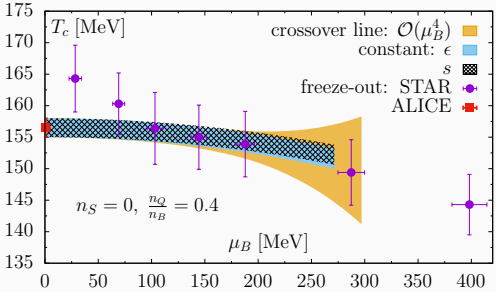
Extract the peak at each μ_B, for every N_τ, and find the transition line

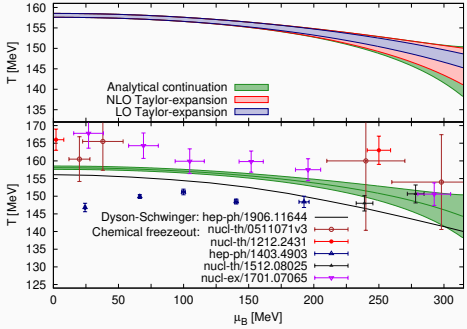
Borsányi, PP *et al.* PRL 125 (2020), 052001

The transition at finite chemical potential

Current results (different collaborations agree within errors):

$$T_c(\mu_B = 0) = 158.0 \pm 0.6 \text{ MeV} \qquad \kappa_2 = 0.0153 \pm 0.0018 \qquad \kappa_4 = 0.00032 \pm 0.00067$$



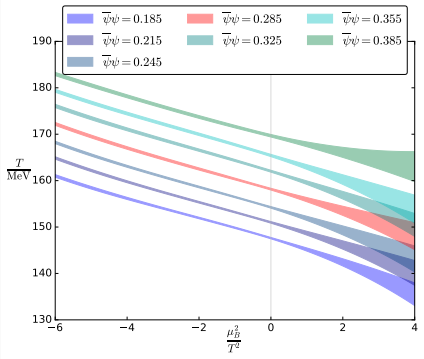


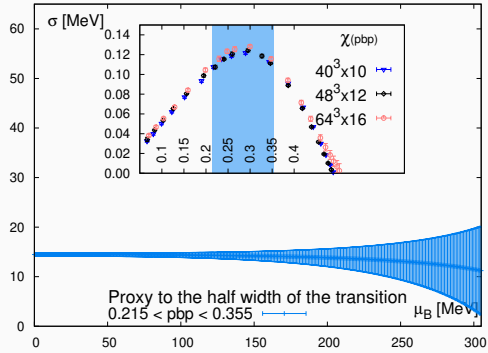
The transition line can also be determined via simulations at $\mu_B = 0$ only (left plot)

Bazavov *et al.* PLB 795 (2019) 15-21; Borsányi *et al.* PRL 125 (2020), 052001

The width of the transition at finite chemical potential

It is also interesting to see whether the **width** of the transition changes (if it vanishes, we are at a first order transition)





One gets at $\mu_B = 0$ for the width:

$$\Delta T(LT = 4, \mu_B = 0) = 15.0 \pm 1.0 \text{ MeV}$$

which also has a very mild chemical potential dependence (no critical point in sight)

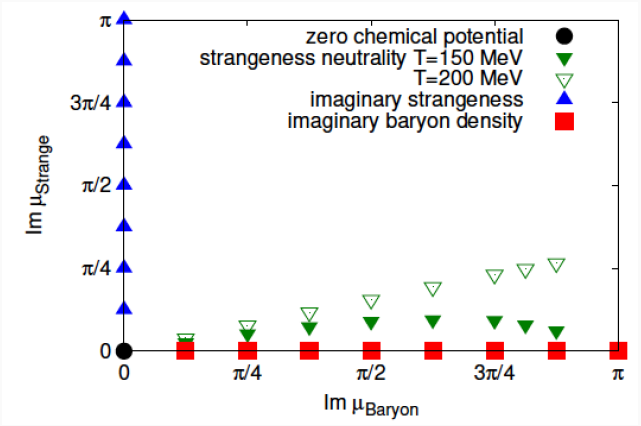
Borsányi *et al.*, PRL 125 (2020), 052001

Intermezzo: strangeness neutrality

Strangeness neutrality (or not)

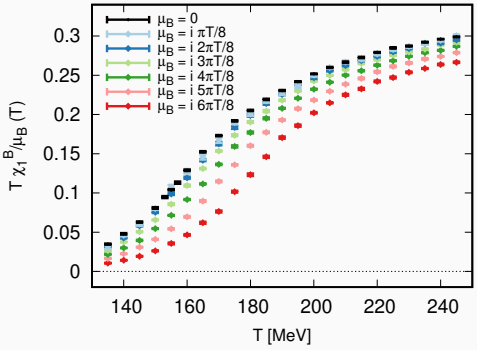
Set the chemical potentials for heavy-ion collisions scenario, or simpler setup:

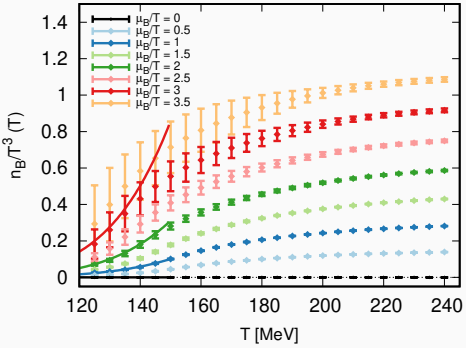
$$\langle n_S \rangle = 0 \qquad \langle n_Q \rangle = 0.4 \langle n_B \rangle \qquad \text{or} \qquad \mu_Q = \mu_S = 0$$



Imaginary chemical potential: equation of state

A similar approach is used for the equation of state. In this case, calculate baryon density at imaginary μ_B, then extrapolate it to real μ_B



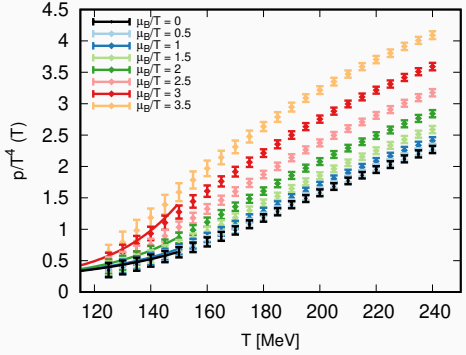


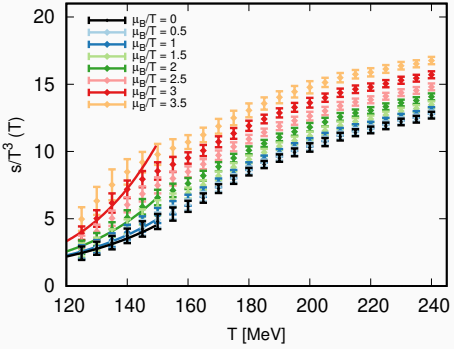
Borsányi *et al.* **PRL 126 (2021), 232001**

Thermodynamics at finite (real) μ_B

The pressure is just the integral:

$$\frac{p(T,\hat{\mu}_B)}{T^4} = \frac{p(T,0)}{T^4} + \int_0^{\hat{\mu}_B} \mathrm{d}\hat{\mu}'_B \frac{\chi_1^B(T,\hat{\mu}'_B)}{T^3}$$

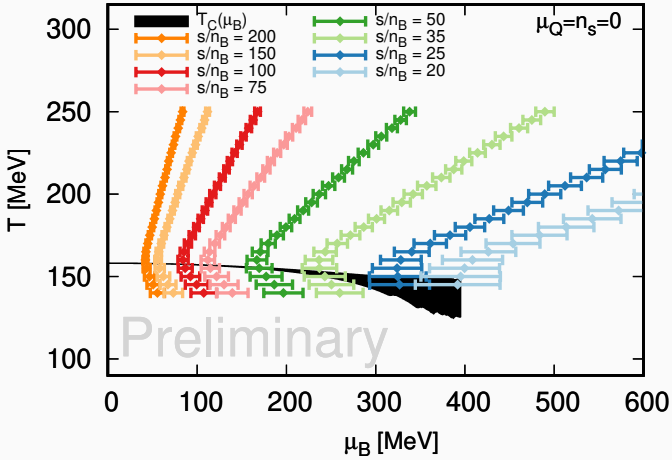




Borsányi *et al.* PRL 126 (2021), 232001; Borsányi *et al.* 2202.05574

Equation of state at finite $\hat{\mu}_B$

Currently, this method gives us the widest coverage in μ_B of the QCD phase dagram



Fluctuations of conserved charges

Fluctuations of B, Q, S are defined as derivatives of the free energy:

$$Z(V,T,\mu_B,\mu_Q,\mu_S) = \sum_{B,Q,S} e^{B\mu_B} e^{Q\mu_Q} e^{S\mu_S} Z_C(V,T,B,Q,S)$$

wrt the associated chemical potentials:

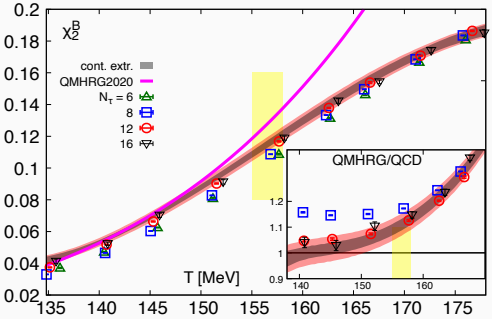
$$\chi^{BQS}_{ijk}(T,\mu_B,\mu_Q,\mu_S) = \frac{1}{VT^3} \frac{\partial^{i+j+k} \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)^i \partial\left(\mu_Q/T\right)^j \partial\left(\mu_S/T\right)^k}$$

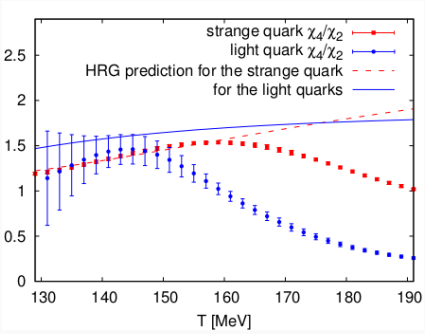
They are very important quantities to study the QCD phase diagram, for a few reasons:

- we saw that baryon fluctuations are the Taylor coefficients of the pressure
- we will see shortly that they are useful probes for critical behavior
- they give insights into the active degrees of freedom in the system
- they allow a comparison to experiment

Fluctuations of conserved charges

The value of calculated fluctuations compared to e.g. the HRG model gives info about the relevant degrees of freedom



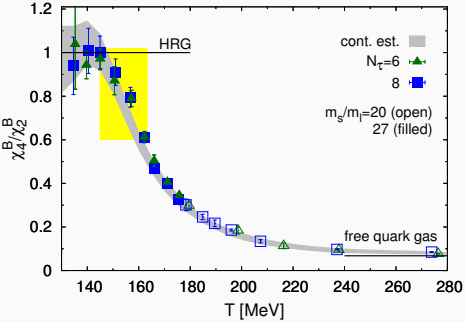


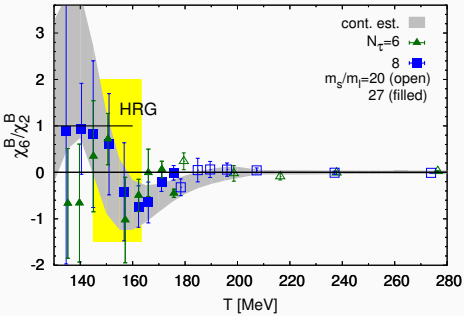
Bollweg *et al.*, PRD 104 (2021) 074512; Bellwied *et al.*, PRL 111 (2013) 202302

Disagreement with HRG model is also below T_c. Why does χ_4/χ_2 peak at different T for strange and light? Difference in hadronization temperature?

Theory-theory comparison

The value of calculated fluctuations compared to e.g. the HRG model gives info about the relevant degrees of freedom





Higher orders push the agreement to lower temperatures. Interactions, which play a larger role in higher moments, are likely the reason of the disagreement

Bazavov *et al.*, PRD 95 (2017) 054504

Fluctuations of conserved charges

One can see that $\ln Z$ is the generating function for moments of net-charge distributions:

$$\langle B\rangle = \frac{1}{VT^3}\frac{\partial \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)} = \chi_1^B$$

$$\left\langle B^2\right\rangle - \langle B\rangle^2 = \frac{1}{VT^3}\frac{\partial^2 \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)^2} = \chi_2^B$$

$$\langle BS\rangle - \langle B\rangle\langle S\rangle = \frac{1}{VT^3}\frac{\partial^2 \ln Z\left(T,\mu_B,\mu_Q,\mu_S\right)}{\partial\left(\mu_B/T\right)\partial\left(\mu_S/T\right)} = \chi_{11}^{BS}$$

In experiment, moments/cumulants $\langle(\Delta N)^n\rangle_{\text{events}}$ of net-particle distributions are measured:

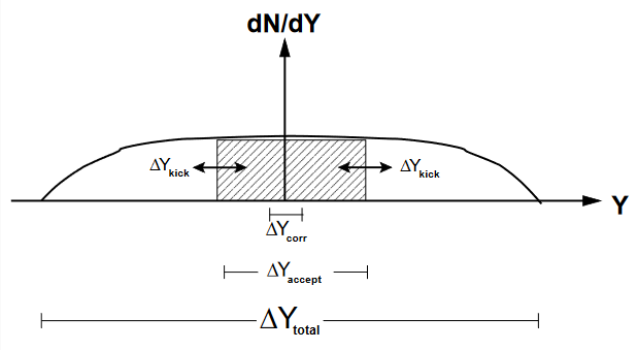
mean:	$M = \chi_1$	variance:	$\sigma^2 = \chi_2$
skewness:	$S = \chi_3/\left(\chi_2\right)^{3/2}$	kurtosis:	$\kappa = \chi_4/\left(\chi_2\right)^2$

Proxies are used for different conserved charges (proton for B, proton+kaon+pion for Q, etc..)

Fluctuations of conserved charges

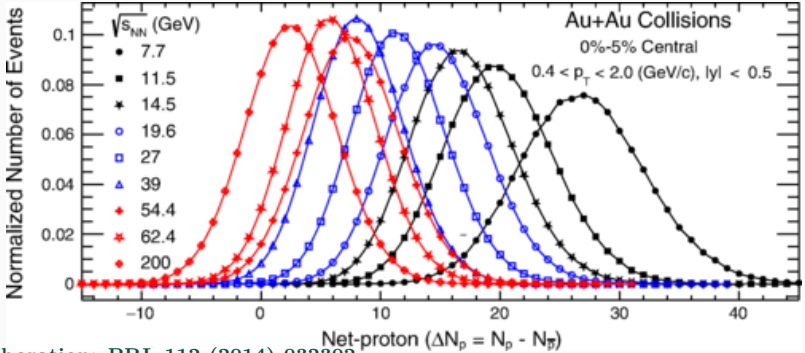
How can CONSERVED CHARGES fluctuate?

- If we could measure ALL particles in a collision, they would not
- If we look at a small enough subsystem, fluctuations occur and become meaningful



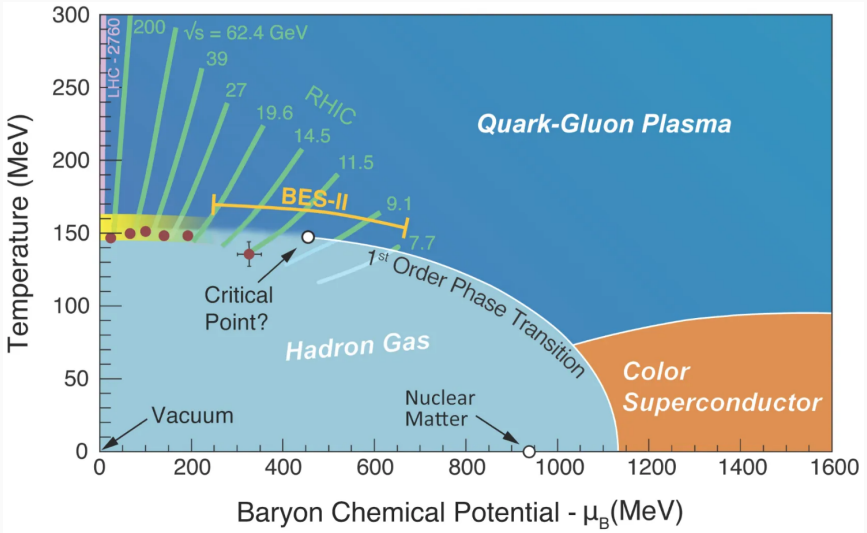
Heavy-ion collisions: event-by-event fluctuations

- Conserved charges in QCD are all quark numbers
  $\longrightarrow$ B (baryon number), Q (electric charge), S (strangeness)
- Weak effects are not considered (time's too short)
- Charm is ignored (might not thermalize)
- Conserved charges are conserved only on average in experiment



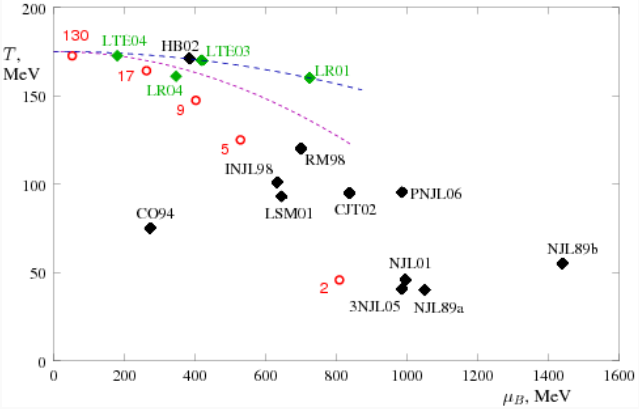
STAR Collaboration: PRL 112 (2014) 032302

The QCD critical point



The critical point of QCD: why it's "expected"

Many models of QCD predict the existence of a critical point:

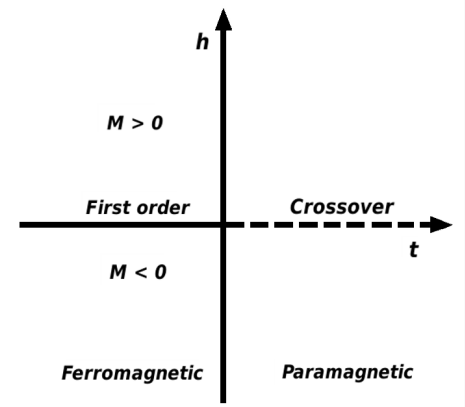


but don't help us in guessing where it may be!

The critical point of QCD: universality

The critical point of QCD is expected to be in the same universality class as the 3D Ising model. **Critical exponents** determine how different quantities diverge at the critical point

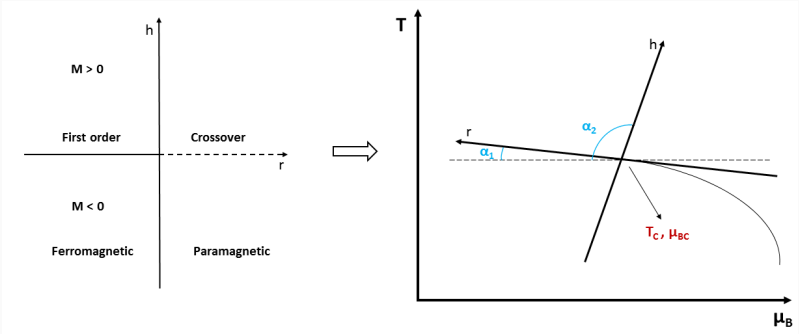
- α : specific heat at $h = 0$ behaves as $C \sim |t|^{\alpha}$;
- β : spontaneous magnetization (i.e. in the limit $h \to 0^+$) scales as $M \sim (-t)^{\beta}$;
- γ : zero-field susceptibility $\chi \equiv (\partial M/\partial H)_{H=0} \sim |t|^{-\gamma}$;
- δ : along the h axis, i.e. for $T = T_C$, the magnetization follows $M \sim \mathrm{sign}(h)\,|h|^{1/\delta}$;
- and so on...
- The following holds: $2 = \alpha + \beta(1 + \delta)$



Because the critical exponents of QCD will be the same, *if* there is a critical point, we might know what to expect!

The critical point of QCD: universality

If there is a critical point, I can locally map *linearly* its surroundings in the Ising phase diagram onto the QCD phase diagram.

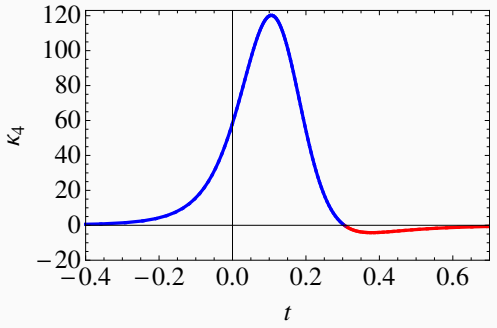


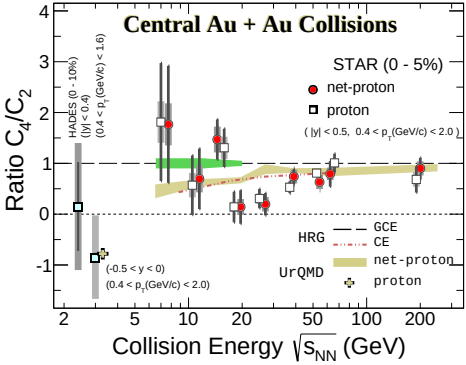
Derivatives with respect to t or h in Ising are translated (and mixed) in toderivatives with respect to T and μ_B in QCD, hence criticality appears in derivatives of $\ln Z$ wrt μ_B.

$\Rightarrow$ **baryon fluctuations will show critical behaviour!**

Looking for critical behavior: experiment

It can be shown that baryon fluctuations diverge at the critical point with increasing powers of the correlation length → higher order net-proton fluctuations are most promising





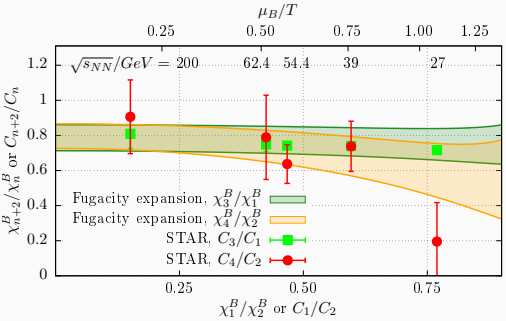
Experimental results are promising, though errors are still large

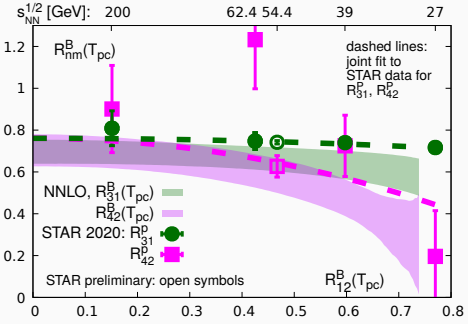
STAR, PRL 128 (2022), 202303; Stephanov, PRL 107 (2011) 052301

Looking for critical behavior: extrapolations

Extrapolations from lattice can be made for fluctuations too, e.g.:

$$\chi_2^B(T,\mu_B) = \chi_2^B(T) + \frac{1}{2}\chi_4^B(T) + \frac{1}{24}\chi_6^B(T) + \cdots$$





Lattice results similar to experiment, hard to draw conclusions for now

Borsányi *et al.*, PRD 104 (2021) 094508; Bazavov *et al.*, PRD101 (2020), 074502

Summary

- Lattice QCD is a regularization scheme for QCD that does not rely on a perturbative expansion, and the most robust tool to explore QCD thermodynamics. Most of what we know on the QCD phase diagram comes from lattice
- Chiral restoration and deconfinement occur at the same temperature $T_c \simeq 156 - 158\,\mathrm{MeV}$. At physical quark masses, this transition is a smooth crossover (at infinite quark masses, it is of first order)
- The equation of state of QCD is known to high precision, and shows liberation of degrees of freedom and approach to Stefan-Boltzmann limit
- While lattice QCD cannot at the moment give *direct* results at non-zero chemical potential (sign problem), thanks to extrapolations the phase diagram can be studied for (not too large) finite μ_B
- At finite μ_B, we know the location of the phase transition line and the equation of state
- The search for the critical point is in full swing from theory and experiment alike. Here lattice has a harder time, because μ_B might be very large at the critical point