Hard Probes in A-A collisions: heavy-flavor

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Ph.D. Lectures
AA 2017-18

Outline

- Heavy flavor in elementary collisions as *benchmark*
  - of our understanding of pQCD,
  - to quantify medium-effects in the *AA* case;
- Heavy flavor in heavy-ion collisions:
  - From the understanding of the parton-medium interaction,
  - ... to the tomography of the produced matter ($T(x)$, $\epsilon(x)$, $\hat{q}$...)

  or *vice versa*!
- How to develop a transport calculation:
  the relativistic Langevin equation.

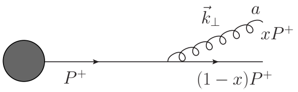
Heavy-flavor production in pQCD

The large mass M of c and b quarks makes a pQCD calculation of $Q\overline{Q}$ production possible:

- It sets a *minimal off-shellness* of the intermediate propagators (diagrams don't diverge);
- It sets a *hard scale* for the evaluation of $\alpha_s(\mu)$ (speeding the convergence of the perturbative series);
- It *prevents collinear singularities* (suppression of emission of small-angle gluon)

Both the *total cross-section* $\sigma^{\rm tot}_{\rm Q\overline{Q}}$ and the *invariant single-particle spectrum* $E(d\sigma_Q/d^3p)$ are well-defined quantities which can be calculated in pQCD

Suppression of collinear radiation



Massless case

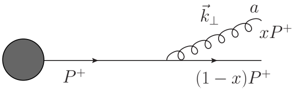
$$d\sigma^{\rm rad} = d\sigma^{\rm hard}\frac{\alpha_s}{\pi^2}C_F\frac{dx}{x}\frac{d\boldsymbol{k}_\perp}{\boldsymbol{k}_\perp^2}$$

Due to collinear gluon-radiation ($\sim d\theta/\theta$), partonic cross-sections of hard processes are not well defined, but require the introduction of a "cutoff" (*factorization scale* μ_F) to regularize collinear divergences. Only hadronic cross-section

$$d\sigma_h \equiv \sum_f d\sigma_f(\mu_F)\otimes D_f^h(z,\mu_F)$$

are *collinear-safe* observables.

Suppression of collinear radiation



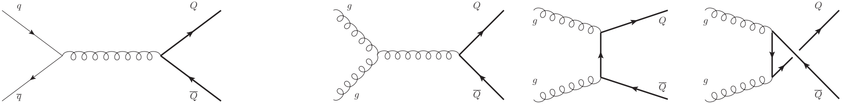
Massive case

$$d\sigma^{\rm rad} = d\sigma^{\rm hard}\frac{\alpha_s}{\pi^2}C_F\frac{dx}{x}d\boldsymbol{k}_\perp\frac{\boldsymbol{k}_\perp^2}{[\boldsymbol{k}_\perp^2+x^2M^2]^2}$$

Gluon radiation at angles $\theta < M/E$ is suppressed (*dead-cone effect*!) and heavy-quark production is well-defined even at the partonic (for what concerns the final state) level.

Leading Order contribution

- The LO processes are:

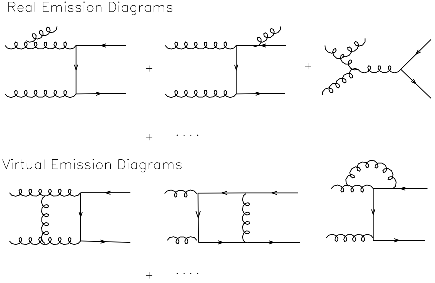


- The propagators introduce in the amplitudes the denominators:

$$
\begin{aligned}
(p_1+p_2)^2 &= 2m_T^2(1+\cosh\Delta y)\\
(p_3-p_1)^2 &= -m_T^2\left(1+e^{-\Delta y}\right)\\
(p_3-p_2)^2 &= -m_T^2\left(1+e^{\Delta y}\right)
\end{aligned}
$$

  - *Minimal off-shellness* $\sim m_T^2$;
  - Q and $\overline{Q}$ close in rapidity.

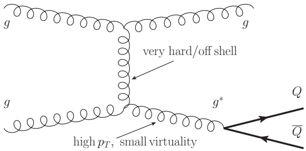
Next to Leading Order process



- Real emission: $|\mathcal{M}_{\rm real}|^2 \sim \mathcal{O}(\alpha_s^3)$
- Virtual corrections: $2{\rm Re}\mathcal{M}_0\mathcal{M}^*_{\rm virt} \sim \mathcal{O}(\alpha_s^3)$

- NLO calculation gives the $\mathcal{O}(\alpha_s^3)$ result for $\sigma^{\rm tot}_{Q\overline{Q}}$ and $E(d\sigma_Q)/d^3p$;
- It is implemented in *event generators* like POWHEG or MC@NLO;
- Output of hard event can be interfaced with a Parton Shower (PYTHIA or HERWIG)

NLO calculation: gluon-splitting contribution



It can be written in a factorized way:

$$d\sigma(gg \to Q\overline{Q}) = d\sigma(gg \to gg^*) \otimes \mathrm{Splitting}(g^* \to Q\overline{Q})$$

More explicitly (in terms of the AP *splitting function* $P_{Qg}(z)$):

$$d\sigma_{Q\overline{Q}} = d\sigma_{g^*}\frac{\alpha_s}{2\pi}P_{Qg}(z)dz\frac{dt}{t}, \longrightarrow N(Q\overline{Q}) \sim \frac{\alpha_s}{6\pi}\ln\frac{p_T^2}{M^2}$$

$Q\overline{Q}$ multiplicity in a gluon jet of transverse energy p_T: $\sim \alpha_s \ln(p_T/M)$

The NLO calculation *contains* an $\alpha_s \ln(p_T/M)$ term, *potentially large*!

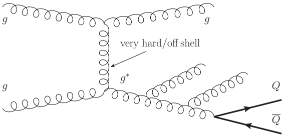
Resummation of (Next to) Leading Logs: FONLL

- Using the above result as the initial condition of the DGLAP evolution for the D_g^Q FF:

$$D_g^Q(z,\mu_0) = \frac{\alpha_s}{2\pi}\frac{1}{2}[z^2+(1-z)^2]\ln\frac{\mu_0^2}{M^2}$$

amounts to resumming all $[\alpha_s \ln(p_T/M)]^n$ terms ($\alpha_s[\alpha_s \ln(p_T/M)]^n$ with NLO splitting functions)

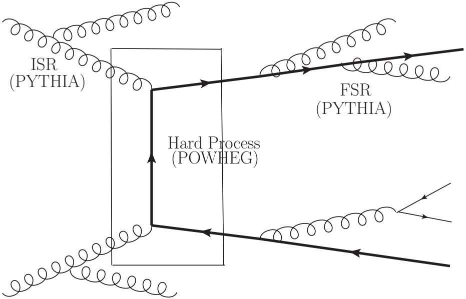
- In terms of diagrams:



$Q\overline{Q}$ from the shower of light partons produced in the hard event!

- A code like FONLL provides a calculation of $d\sigma_Q$ at this accuracy!

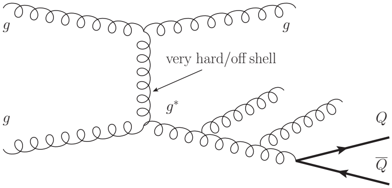
NLO calculation + Parton Shower



- A different strategy is to interface the output of a NLO event-generator for the hard process with a parton-shower describing the Initial and Final State Radiation and modeling other *non-perturbative processes* (intrinsic k_T, MPI, hadronizazion)
- This provides a *fully exclusive information on the final state*

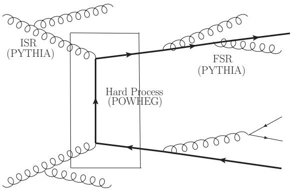
FONLL vs POWHEG+PS

FONLL



- It is a *calculation*
- It provides NLL accuracy, resumming large $\ln(p_T/M)$
- It includes processes missed by POWHEG (hard events with light partons)

POWHEG+PS



- It is an *event generator*
- Results compatible with FONLL
- It is a more flexible tool, allowing to address more differential observables (e.g. $Q\overline{Q}$ correlations)

Heavy quark production in pQCD: some references

- For a general introduction: M. Mangano, hep-ph/9711337 (lectures);
- For POWHEG: S. Frixione, P. Nason and G. Ridolfi, JHEP 0709 (2007) 126;
- For FONLL: M. Cacciari, M. Greco and P. Nason, JHEP 9805 (1998) 007.
- For a systematic comparison (POWHEG vs MC@NLO vs FONLL): M. Cacciari *et al.*, JHEP 1210 (2012) 137.

Heavy flavour: experimental observables

- D and B mesons
- Non-prompt J/ψ's ($B \to J/\psi\, X$)
- Heavy-flavour electrons, from the decays
  - of charm (e_c)

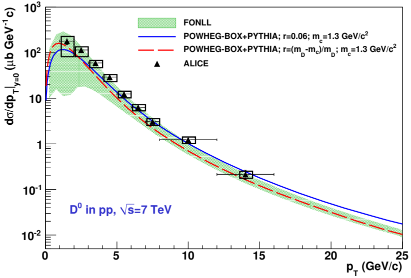
$$D \to X\nu e$$

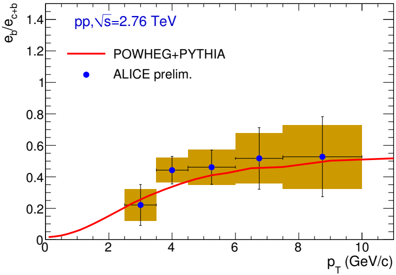
  - of beauty (e_b)

$$\begin{aligned} B &\to D\nu e \\ B &\to D\nu e \to X\nu e\nu e \\ B &\to DY \to X\nu eY \end{aligned}$$

- B-tagged jets

HF production in *pp* collisions: results





- Besides reproducing the inclusive D-meson p_T-spectra[1]
- and the heavy-flavour electrons

[1] W.M. Alberico *et al*, Eur.Phys.J. C73 (2013) 2481

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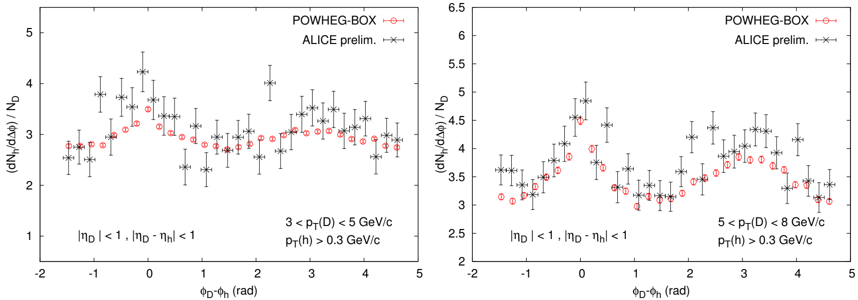

- Besides reproducing the inclusive D-meson p_T-spectra[1]
- and the heavy-flavour electrons
- ...the POWHEG+PYTHIA setup allows also the comparison with $D-h$ correlation data, which start getting available.

[1] W.M. Alberico *et al*, Eur.Phys.J. C73 (2013) 2481

HF in p-p collisions: a summary

- A setup based on a NLO pQCD event generator (POWHEG) for the hard event + a Parton-Shower stage simulated with PYTHIA is able to reproduce the experimental data;
- Such an approach provides a richer information on the final state wrt other schemes (e.g. FONLL): this can be of interest for more differential studies like azimuthal correlations

HF in AA collisions

Purpose of this lecture:

- Displaying the conceptual setup common to the different theoretical models, pointing out their nice features and limitations;
- Showing some results and compare them to the experimental data;
- Giving some hints of possible future developments.

Being a lecture I will focus mainly on one particular approach, the relativistic Langevin equation, hoping that at the end one will be able to understand the technical issues one has to face in developing a model

Heavy quarks as probes of the QGP

A realistic study requires developing *a multi-step setup*:

- Initial production: pQCD + possible nuclear effects (nPDFs, k_T-broadening) $\longrightarrow$ QCD event generators, validated on *pp* data;
- Description of the background medium (initial conditions, $T(x)$, $u^\mu(x)$) $\longrightarrow$ hydrodynamics, validated on soft hadrons;
- HQ-medium interaction $\longrightarrow$ transport coefficients, in principle from QCD, but still far from a definite answer for the relevant experimental conditions;
- Dynamics in the medium $\longrightarrow$ transport calculations, in principle rigorous *under certain kinematic conditions*, but require transport coefficients as an input;
- Hadronization: not well under control (fragmentation in the vacuum? recombination with thermal partons? validated on what?)
  - An item of interest in itself (*change of hadrochemistry* in AA)
  - However, *a source of systematic uncertainty for studies of parton-medium interaction*;
- Hadronic rescattering (e.g. $D\pi \to D\pi$), from effective Lagrangians, but no experimental data the on relevant cross-sections

Heavy Flavour in the QGP: the conceptual setup

- Description of soft observables based on hydrodynamics, assuming to deal with a system close to local thermal equilibrium (no matter why);
- Description of jet-quenching based on energy-degradation of external probes (high-p_T partons);
- Description of heavy-flavour observables requires to employ/develop a setup (transport theory) allowing to deal with more general situations and in particular to describe *how particles would (asymptotically) approach equilibrium*.

NB At high-p_T the interest in heavy flavor is no longer related to thermalization, but to the study of the mass and color charge dependence of jet-quenching (not addressed in this lecture)

Why are charm and beauty considered *heavy*?

- $M \gg \Lambda_{\rm QCD}$: their initial production (as shown!) is well described by pQCD
- $M \gg T$: thermal abundance in the plasma would be negligible; final multiplicity in the experiments (expanding fireball with lifetime $\sim$10 fm/c) set by the initial hard production
- $M \gg gT$, with gT being the *typical momentum exchange* in the collisions with the plasma particles: many soft scatterings necessary to change significantly the momentum/trajectory of the quark.

NB for realistic temperatures $g \sim 2$, so that one can wonder *whether a charm is really "heavy"*, at least in the initial stage of the evolution.

Transport theory: the Boltzmann equation

Time evolution of HQ phase-space distribution $f_Q(t, \boldsymbol{x}, \boldsymbol{p})$[2]:

$$\frac{d}{dt} f_Q(t, \boldsymbol{x}, \boldsymbol{p}) = C[f_Q]$$

- Total derivative along particle trajectory

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{F} \frac{\partial}{\partial \boldsymbol{p}}$$

Neglecting $\boldsymbol{x}$-dependence and mean fields: $\partial_t f_Q(t, \boldsymbol{p}) = C[f_Q]$

- Collision integral:

$$C[f_Q] = \int d\boldsymbol{k} [\underbrace{w(\boldsymbol{p}+\boldsymbol{k}, \boldsymbol{k}) f_Q(\boldsymbol{p}+\boldsymbol{k})}_{\text{gain term}} - \underbrace{w(\boldsymbol{p}, \boldsymbol{k}) f_Q(\boldsymbol{p})}_{\text{loss term}}]$$

$w(\boldsymbol{p}, \boldsymbol{k})$: HQ transition rate $\boldsymbol{p} \to \boldsymbol{p} - \boldsymbol{k}$

[2] Approach implemented in codes like BAMPS.

Approach to equilibrium in the Boltzmann equation

Momentum exchanges occur with light (thermal) partons *i* of the plasma. In the *classical limit* (no Pauli-blocking or Bose-enhancement) one has:

$$C[f_Q] = \int d\boldsymbol{p}' d\boldsymbol{p}_1 d\boldsymbol{p}'_1 \Big[\underbrace{\overline{w}(\boldsymbol{p},' \boldsymbol{p}'_1 | \boldsymbol{p}, \boldsymbol{p}_1) f_Q(\boldsymbol{p}') f_i(\boldsymbol{p}'_1)}_{\text{gain term}} - \underbrace{\overline{w}(\boldsymbol{p}, \boldsymbol{p}_1 | \boldsymbol{p}', \boldsymbol{p}'_1) f_Q(\boldsymbol{p}) f_i(\boldsymbol{p}_1)}_{\text{loss;term}} \Big]$$

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From *time-reversal* symmetry one has for the transition probability:

$$\overline{w}(\boldsymbol{p},' \boldsymbol{p}_1' | \boldsymbol{p}, \boldsymbol{p}_1) = \overline{w}(\boldsymbol{p}, \boldsymbol{p}_1 | \boldsymbol{p}', \boldsymbol{p}_1'),$$

hence:

$$C[f_Q] = \int d\boldsymbol{p}' d\boldsymbol{p}_1 d\boldsymbol{p}_1' \overline{w}(\boldsymbol{p},' \boldsymbol{p}_1' | \boldsymbol{p}, \boldsymbol{p}_1) \Big[f_Q(\boldsymbol{p}') f_i(\boldsymbol{p}_1') - f_Q(\boldsymbol{p}) f_i(\boldsymbol{p}_1) \Big].$$

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$C[f_Q]$ vanishes *if and only if* $f_Q(\boldsymbol{p}') f_i(\boldsymbol{p}_1') = f_Q(\boldsymbol{p}) f_i(\boldsymbol{p}_1)$, which entails:

$$f_Q(\boldsymbol{p}) = \exp\left[-E_{\boldsymbol{p}}/T\right] \quad \text{and} \quad f_i(\boldsymbol{p}_1) = \exp\left[-E_{\boldsymbol{p}_1}/T\right].$$

The Boltzmann equation *always* makes heavy quarks relax to a *thermal distribution* at the same temperature of the medium!

From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[3] (Landau)

$$C[f_Q] \approx \int d\boldsymbol{k} \left[k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\boldsymbol{p}, \boldsymbol{k}) f_Q(t, \boldsymbol{p})]$$

The Boltzmann equation reduces to the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} f_Q(t, \boldsymbol{p}) = \frac{\partial}{\partial p^i} \left\{ A^i(\boldsymbol{p}) f_Q(t, \boldsymbol{p}) + \frac{\partial}{\partial p^j} [B^{ij}(\boldsymbol{p}) f_Q(t, \boldsymbol{p})] \right\}$$

where (verify!)

$$A^i(\boldsymbol{p}) = \int d\boldsymbol{k}\, k^i w(\boldsymbol{p}, \boldsymbol{k}) \longrightarrow \underbrace{A^i(\boldsymbol{p}) = A(p)\, p^i}_{\text{friction}}$$

$$B^{ij}(\boldsymbol{p}) = \frac{1}{2} \int d\boldsymbol{k}\, k^i k^j w(\boldsymbol{p}, \boldsymbol{k}) \longrightarrow \underbrace{B^{ij}(\boldsymbol{p}) = (\delta^{ij} - \hat{p}^i \hat{p}^j) B_0(p) + \hat{p}^i \hat{p}^j B_1(p)}_{\text{momentum broadening}}$$

Problem reduced to the *evaluation of three transport coefficients*, directly derived from the scattering matrix

[3] B. Svetitsky, PRD 37, 2484 (1988)

Approach to equilibrium in the FP equation

The FP equation can be viewed as a continuity equation for the phase-space distribution of the kind $\partial_t \rho(t,\vec{p}) + \vec{\nabla}_p \cdot \vec{J}(t,\vec{p}) = 0$

$$\frac{\partial}{\partial t}\underbrace{f_Q(t,\boldsymbol{p})}_{\equiv \rho(t,\vec{p})} = \frac{\partial}{\partial p^i}\underbrace{\left\{A^i(\boldsymbol{p}) f_Q(t,\boldsymbol{p}) + \frac{\partial}{\partial p^j}\left[B^{ij}(\boldsymbol{p}) f_Q(t,\boldsymbol{p})\right]\right\}}_{\equiv -J^i(t,\vec{p})}$$

admitting a steady solution $f_{\rm eq}(p) \equiv e^{-E_p/T}$ when the current vanishes:

$$A^i(\vec{p}) f_{\rm eq}(p) = -\frac{\partial B^{ij}(\vec{p})}{\partial p^j} f_{\rm eq}(p) - B^{ij}(\boldsymbol{p})\frac{\partial f_{\rm eq}(p)}{\partial p^j}.$$

One gets

$$A(p) p^i = \frac{B_1(p)}{T E_p} p^i - \frac{\partial}{\partial p^j}\left[\delta^{ij} B_0(p) + \hat{p}^i \hat{p}^j (B_1(p) - B_0(p))\right],$$

leading to the Einstein *fluctuation-dissipation* relation

$$A(p) = \frac{B_1(p)}{T E_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p))\right]$$

Fokker-Planck equation: solution

- *Ignoring the momentum dependence* of the transport coefficients $\gamma \equiv A(p)$ and $D \equiv B_0(p) = B_1(p)$ the FP equation reduces to

$$\frac{\partial}{\partial t} f_Q(t, \boldsymbol{p}) = \gamma \frac{\partial}{\partial p^i}[p^i f_Q(t, \boldsymbol{p})] + D\, \Delta_{\boldsymbol{p}} f_Q(t, \boldsymbol{p})$$

- Starting from the *initial condition* $f_Q(t=0, \boldsymbol{p}) = \delta(\boldsymbol{p} - \boldsymbol{p}_0)$ one gets

$$f_Q(t, \boldsymbol{p}) = \left(\frac{\gamma}{2\pi D[1 - \exp(-2\gamma t)]}\right)^{3/2} \exp\left[-\frac{\gamma}{2D}\frac{[\boldsymbol{p} - \boldsymbol{p}_0 \exp(-\gamma t)]^2}{1 - \exp(-2\gamma t)}\right]$$

- Asymptotically the solution *forgets about the initial condition* and tends to a thermal distribution

$$f_Q(t, \boldsymbol{p}) \underset{t \to \infty}{\sim} \left(\frac{\gamma}{2\pi D}\right)^{3/2} \exp\left[-\left(\frac{\gamma M_Q}{D}\right)\frac{\boldsymbol{p}^2}{2M_Q}\right]$$

$\longrightarrow D = M_Q \gamma T$: Einstein *fluctuation-dissipation* relation

Fokker-Planck solution: derivation (I)

Consider (for simplicity) the 1D FP equation and start setting $D=0$:

$$\frac{\partial}{\partial t} f_Q = \gamma \frac{\partial}{\partial p}[p\, f_Q] \quad \longrightarrow \quad \frac{\partial f_Q}{\partial t} - \gamma p \frac{\partial f_Q}{\partial p} = \gamma\, f_Q \quad \longrightarrow \quad \frac{df_Q}{dt} = \gamma\, f_Q$$

viewing the LHS as the *total derivative* d/dt wrt to the motion of a particle feeling a friction force $dp/dt = -\gamma p$.
One can then write the solution as:

$$f_Q = Q(u) e^{\gamma t} \quad \text{with} \quad p(t) = u\, e^{-\gamma t}$$

For the full equation, with $D \neq 0$ one can attempt a solution of the form

$$f_Q = Q\left(t, u = p e^{\gamma t}\right) e^{\gamma t}$$

whose partial derivatives are given by:

$$\frac{\partial f_Q}{\partial p} = e^{2\gamma t} \frac{\partial Q}{\partial u}, \qquad \frac{\partial^2 f_Q}{\partial p^2} = e^{3\gamma t} \frac{\partial^2 Q}{\partial u^2}$$

$$\frac{\partial f_Q}{\partial t} = \gamma e^{\gamma t} Q + e^{\gamma t} \left[\frac{\partial Q}{\partial t} + \gamma u \frac{\partial Q}{\partial u}\right]$$

Fokker-Planck solution: derivation (II)

Inserting it into the full FP equation

$$\frac{\partial f_Q}{\partial t} = \gamma f_Q + \gamma p \frac{\partial f_Q}{\partial p} + D \frac{\partial^2 f_Q}{\partial p^2}$$

One gets the simpler equation:

$$\frac{\partial Q}{\partial t} = D e^{2\gamma t} \frac{\partial^2 Q}{\partial u^2}$$

Introducing the temporal variable $\theta = (e^{2\gamma t} - 1)/2\gamma \longrightarrow d\theta = e^{2\gamma t} dt$ one gets the diffusion equation:

$$\frac{\partial Q}{\partial \theta} = D \frac{\partial^2 Q}{\partial u^2} \quad \text{with} \quad Q(0,u) = Q_0(u) = \delta(u - u_0)$$

Solution is an *superposition of plane-waves*

$$Q(\theta, u) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} A_k \, e^{-i\omega_k \theta + iku}$$

with $A_k = e^{-iku_0}$ (init.cond.) and $\omega_k = -iDk^2$ (diff.eq.)

Fokker-Planck solution: derivation (III)

The integration is gaussian and can be performed exactly, getting

$$Q(\theta, u) = \left(\frac{1}{4\pi D\theta}\right) \exp\left[-\frac{(u-u_0)^2}{4D\theta}\right]$$

Going back to the original variables[4]:

$$f_Q(t, \boldsymbol{p}) = \left(\frac{\gamma}{2\pi D[1-\exp(-2\gamma t)]}\right)^{1/2} \exp\left[-\frac{\gamma}{2D}\frac{[p - p_0\exp(-\gamma t)]^2}{1-\exp(-2\gamma t)}\right]$$

The generalization to the 3D case is trivial

$$f_Q(t, \boldsymbol{p}) = \left(\frac{\gamma}{2\pi D[1-\exp(-2\gamma t)]}\right)^{3/2} \exp\left[-\frac{\gamma}{2D}\frac{[\boldsymbol{p} - \boldsymbol{p}_0\exp(-\gamma t)]^2}{1-\exp(-2\gamma t)}\right]$$

[4] Derivation from F. Reif, *Fundamentals of Statistical and Thermal Physics*

Fokker-Planck solution: physical meaning

$$f_Q(t,\boldsymbol{p}) = \left(\frac{\gamma}{2\pi D[1-\exp(-2\gamma t)]}\right)^{3/2} \exp\left[-\frac{\gamma}{2D}\frac{[\boldsymbol{p}-\boldsymbol{p}_0\exp(-\gamma t)]^2}{1-\exp(-2\gamma t)}\right]$$

From the first moments of the momentum distribution...

- $$\langle \boldsymbol{p}(t)\rangle = \boldsymbol{p}_0\, e^{-\gamma t}$$

  γ: friction coefficient

- $$\langle \boldsymbol{p}^2(t)\rangle - \langle \boldsymbol{p}(t)\rangle^2 = \frac{3D}{\gamma}\left(1-e^{-2\gamma t}\right) \underset{t\to 0}{\sim} 6Dt$$

  D: momentum-diffusion coefficient

Ex: derive the above results. Trivial, after setting

$$\boldsymbol{p} = \left(\boldsymbol{p}-\boldsymbol{p}_0 e^{-\gamma t}\right) + \boldsymbol{p}_0 e^{-\gamma t}$$

The challenge: addressing the experimental situation

One needs a tool, *equivalent to the Fokker-Planck equation*, but allowing to face the complexity of the experimental situation[5] in which

- heavy quarks can be relativistic, so that one must deal with the *momentum dependence*[6] *of the transport coefficients*;
- the dynamics in the medium must be *interfaced with the initial hard production*, possibly given by pQCD event generators;
- the stochastic dynamics takes place in a medium which undergoes a hydrodynamical expansion.

A proper *relativistic generalization of the Langevin equation* allows to accomplish this task

[5] A.B. et al., NPA 831 59 (2009) and EPJC 71 (2011) 1666
For a review: R. Rapp and H. van Hees, arXiv:0903.1096

[6] A.W.C. Lau and T.C. Lubensky, PRE 76, 011123 (2007)

The relativistic Langevin equation

The Fokker-Planck equation can be recast into a form suitable to follow the dynamics of each individual quark: the Langevin equation

$$\frac{\Delta p^i}{\Delta t} = -\underbrace{\eta_D(p) p^i}_{\text{determ.}} + \underbrace{\xi^i(t)}_{\text{stochastic}},$$

with the properties of the noise encoded in

$$\langle \xi^i(\boldsymbol{p}_t) \rangle = 0 \quad \langle \xi^i(\boldsymbol{p}_t) \xi^j(\boldsymbol{p}_{t'}) \rangle = b^{ij}(\boldsymbol{p}) \frac{\delta_{tt'}}{\Delta t} \quad b^{ij}(\boldsymbol{p}) \equiv \kappa_L(p) \hat{p}^i \hat{p}^j + \kappa_T(p)(\delta^{ij} - \hat{p}^i \hat{p}^j)$$

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Transport coefficients to calculate:

- *Momentum diffusion*;
- *Friction* term

In the following we will establish their link with the transport coefficients appearing in the Fokker-Planck equation. In particular, the momentum dependence of the noise term requires some care.

Numerical implementation

Introduce the tensor

$$\begin{aligned} C^{ij}(\boldsymbol{p}) &\equiv \sqrt{\kappa_L(p)}\hat{p}^i\hat{p}^j + \sqrt{\kappa_T(p)}(\delta^{ij} - \hat{p}^i\hat{p}^j) \\ &\equiv \sqrt{\kappa_L(p)}P_L^{ij} + \sqrt{\kappa_T(p)}P_T^{ij} \end{aligned}$$

and redefine the noise variable as

$$\xi^i \equiv C^{ik}(\boldsymbol{p})\frac{1}{\sqrt{\Delta t}}\zeta^k \quad \text{with} \quad \langle\zeta^k(t_n)\rangle = 0 \quad \text{and} \quad \langle\zeta^k(t_n)\zeta^l(t_m)\rangle = \delta_{mn}\delta^{kl}.$$

The Langevin equation becomes then

$$\Delta p^i = -\eta_D(p)p^i\Delta t + C^{ik}(\boldsymbol{p} + \xi\Delta\boldsymbol{p})\sqrt{\Delta t}\,\zeta^k,$$

where, for the sake of generality, the argument of the strength of the noise term can be evaluated ($\xi \in [0,1]$) at any point in the interval $[\boldsymbol{p}, \boldsymbol{p} + \Delta\boldsymbol{p}]$. In the following we will consider the cases $\xi = 0$ (Ito *pre-point* scheme) and $\xi = 1$ (*post-point* scheme).

The link with the Fokker-Plank equation

Consider an arbitrary function of the HQ momentum $g(\boldsymbol{p})$ and take the expectation value over the thermal ensemble of its variation, keeping only terms up to order Δt:

$$\langle g(\boldsymbol{p}+\Delta\boldsymbol{p})-g(\boldsymbol{p})\rangle=\left\langle\frac{\partial g}{\partial p^i}\Delta p^i+\frac{1}{2}\frac{\partial^2 g}{\partial p^i\partial p^j}\Delta p^i\Delta p^j\right\rangle+\dots$$

From

$$\Delta p^i=-\eta_D(p)p^i\Delta t+C^{ik}(\boldsymbol{p}+\xi\Delta\boldsymbol{p})\sqrt{\Delta t}\,\zeta^k,\quad\langle\zeta^k\rangle=0,\quad\langle\zeta^k\zeta^l\rangle=\delta^{kl}$$

one gets:

$$\langle g(\boldsymbol{p}+\Delta\boldsymbol{p})-g(\boldsymbol{p})\rangle=\left\langle\frac{\partial g}{\partial p^i}\left(-\eta_D p^i+\xi\frac{\partial C^{ik}}{\partial p^j}C^{jk}\right)+\frac{1}{2}\frac{\partial^2 g}{\partial p^i\partial p^j}C^{ik}C^{jk}\right\rangle\Delta t+\dots$$

In the above the expectation value is taken accorrding to the HQ phase-space distribution

$$\langle g(\boldsymbol{p})\rangle_t\equiv\int d\boldsymbol{p}\,g(\boldsymbol{p})f(t,\boldsymbol{p})$$

The link with the Fokker-Plank equation

Time evolution given be the differential equation

$$\frac{d}{dt}\langle g(\boldsymbol{p})\rangle_t \equiv \int d\boldsymbol{p}\, g(\boldsymbol{p}) \frac{\partial}{\partial t} f(t,\boldsymbol{p})$$

Integrating by parts:

$$\text{LHS} = \int d\boldsymbol{p}\, g(\boldsymbol{p}) \left\{ \frac{\partial}{\partial p^i} \left[\left(\eta_D p^i - \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk} \right) f(t,\boldsymbol{p}) \right] + \frac{1}{2} \frac{\partial^2}{\partial p^i \partial p^j} \left[(C^{ik} C^{jk}) f(t,\boldsymbol{p}) \right] \right\}$$

Comparing with the FP equation

$$\frac{\partial}{\partial t} f_Q(t,\boldsymbol{p}) = \frac{\partial}{\partial p^i} \left[A^i(\boldsymbol{p}) f_Q(t,\boldsymbol{p}) \right] + \frac{\partial}{\partial p^i \partial p^j} \left[B^{ij}(\boldsymbol{p}) f_Q(t,\boldsymbol{p}) \right]$$

one gets

$$A^i(\boldsymbol{p}) \equiv A(p) p^i = \eta_D p^i - \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

$$C^{ij}(\boldsymbol{p}) \equiv \sqrt{\kappa_L(p)} P_L^{ij} + \sqrt{\kappa_T(p)} P_T^{ij} = \sqrt{2B_1(p)} P_L^{ij} + \sqrt{2B_0(p)} P_T^{ij}$$

Dependence on the discretization scheme

The transport coefficients describing momentum-diffusion in the Langevin equation *always* coincide with the corresponding ones in the Fokker-Planck equation, no matter which discretization scheme is employed. In general, this is not the case for the friction term. From

$$\eta_D(p)p^i = A(p)p^i + \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

one gets

$$\eta_D(p) = A(p) + \xi \left[\frac{1}{p}\frac{\partial B_1}{\partial p} + \frac{d-1}{p^2}\sqrt{2B_0(p)}(\sqrt{2B_1(p)} - \sqrt{2B_0(p)})\right]$$

where, furthermore, $A(p)$, $B_0(p)$ and $B_1(p)$ are related by the Einstein relation.

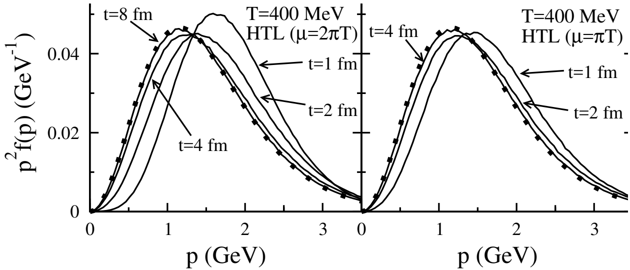
The *pre-point* Ito discretization

Actually, in the Ito *pre-point* scheme $\xi = 0$, so that the friction coeffiecients appearing in the FP and Langevin equations are the same: $A(p) = \eta_D^{\mathrm{pre}}(p)$. Furthermore, in order to approach thermal equilibrium, the Einstein relation must be satisfied:

$$\eta_D^{\mathrm{pre}}(p) = A(p) = \frac{B_1(p)}{TE_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p))\right]$$

NB: $A(p)$, $B_0(p)$ and $B_1(p)$ can be calculated from the scattering matrix. However, since the Einstein relation must satisfied, one has to calculate only two of them and fix the last one through the above equation

A first check: thermalization in a static medium



For $t \gg 1/\eta_D$ one approaches a relativistic Maxwell-Jüttner distribution[7]

$$f_{\rm MJ}(p) \equiv \frac{e^{-E_p/T}}{4\pi M^2 T\, K_2(M/T)}, \qquad \text{with} \quad \int d^3p\, f_{\rm MJ}(p) = 1$$

(Test with a sample of c quarks with $p_0 = 2$ GeV/c)

[7] A.B., A. De Pace, W.M. Alberico and A. Molinari, NPA 831, 59 (2009)

The realistic case: expanding fireball

Within our POWLANG setup (POWHEG+LANGevin) the HQ evolution in heavy-ion collisions is simulated as follows

- $Q\overline{Q}$ pairs initially produced with the POWHEG-BOX package (with nPDFs) and distributed in the transverse plane according to $n_{\rm coll}(\boldsymbol{x}_\perp)$ from (optical) Glauber model;
- update of the HQ momentum and position to be done at each step *in the local fluid rest-frame*
  - $u^\mu(x)$ used to perform the boost to the fluid rest-frame;
  - $T(x)$ used to set the value of the transport coefficients

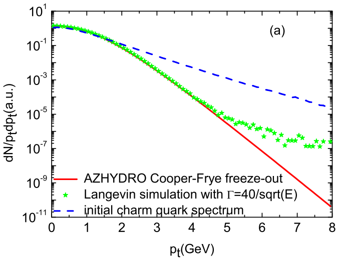
  with $u^\mu(x)$ and $T(x)$ fields taken from the output of hydro codes[8];
- Procedure iterated until hadronization

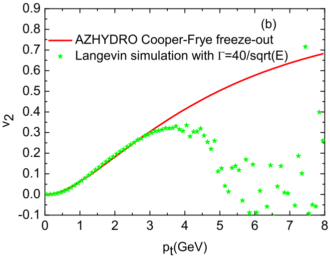
[8] P. Romatschke and U.Romatschke, Phys. Rev. Lett. **99** (2007) 172301
L. Del Zanna *et al.*, Eur.Phys.J. C73 (2013) 2524

Expanding fireball: testing the algorithm

In the limit of large transport coefficients heavy quarks should reach local thermal equilibrium and decouple from the medium as the other light particles, according to the Cooper-Frye formula:

$$E(dN/d^3p) = \int_{\Sigma_{\rm fo}} \frac{p^\mu \cdot d\Sigma_\mu}{(2\pi)^3} \exp[-p\cdot u/T_{\rm fo}]$$





This was verified to be actually the case (M. He, R.J. Fries and R. Rapp, PRC 86, 014903).

The Langevin/FP approach: a critical perspective

Although the Langevin approach is a very convenient numerical tool and allows one to establish a link between observables and transport coefficients derived from QCD... it was nevertheless derived starting from a *soft-scattering expansion* of the collision integral $\mathcal{C}[f]$ truncated at second order (friction and diffusion terms), which may be *not always justified*, in particular for charm, possibly affecting the final R_{AA} (V. Greco *et al.*, Phys.Rev. C90 (2014) 4, 044901)

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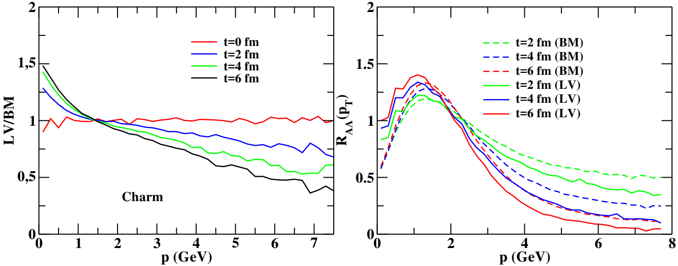

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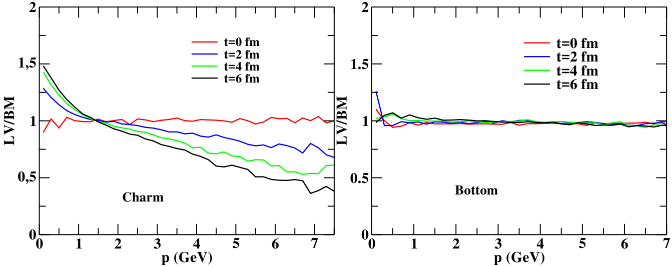


For beauty on the other hand Langevin≡Boltzmann!

The Langevin/FP approach: a critical perspective

At the same time the Langevin/FP approach, although formally derived as a soft-scattering limit of the Boltzmann equation, can be considered *more general than the latter*, requiring simply the knowledge of a few transport coefficients (friction and diffusion) *meaningful even in a non-perturbative framework* and *not relying on quasi-particle picture* of the medium.
Notice that, for the light quarks/gluons of the medium one has

- Thermal de Broglie wavelength: $\lambda_{\rm th} \sim 1/T$
- Mean free path: $\lambda_{\rm mfp} \sim 1/g^2 T$

In the weak-coupling regime one has $\lambda_{\rm th} \ll \lambda_{\rm mfp}$, so that between the relatively rare scatterings one has the propagation of *localized on-shell particles*. However as the coupling gets large $\lambda_{\rm th} \sim \lambda_{\rm mfp}$, the two scales are no longer well separated and a *picture based on on-shell distribution function* may be *no longer valid*

The Langevin equation provides a link between *what is possible to calculate in QCD* (transport coefficients) and *what one actually measures* (final p_T spectra)

Evaluation of transport coefficients:

- Weak-coupling hot-QCD calculations[9]
- Non perturbative approaches
    - Lattice-QCD
    - AdS/CFT correspondence
    - Resonant scattering

[9] Our approach: W.M. Alberico *et al.*, Eur.Phys.J. C71 (2011) 1666

Transport coefficients: perturbative evaluation

It's the stage where the various models differ!

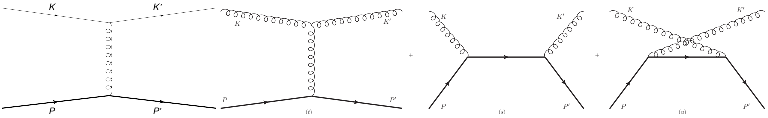
We account for the effect of $2 \to 2$ collisions in the medium

Intermediate cutoff $|t|^* \sim m_D^2$ [10] *separating the contributions of*

- hard collisions ($|t| > |t|^*$): kinetic pQCD calculation
- soft collisions ($|t| < |t|^*$): Hard Thermal Loop approximation (*resummation of medium effects*)

[10] Similar strategy for the evaluation of dE/dx in S. Peigne and A. Peshier, Phys.Rev.D77:114017 (2008).

Transport coefficients $\kappa_{T/L}(p)$: hard contribution

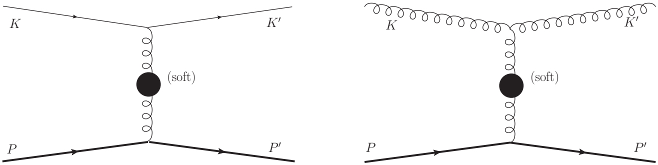


$$\kappa_T^{g/q(\mathrm{hard})} = \frac{1}{2}\frac{1}{2E}\int_k \frac{n_{B/F}(k)}{2k}\int_{k'}\frac{1\pm n_{B/F}(k')}{2k'}\int_{p'}\frac{1}{2E'}\theta(|t|-|t|^*)\times$$
$$\times(2\pi)^4\delta^{(4)}(P+K-P'-K')\left|\overline{\mathcal{M}}_{g/q}(s,t)\right|^2 q_T^2$$

$$\kappa_L^{g/q(\mathrm{hard})} = \frac{1}{2E}\int_k \frac{n_{B/F}(k)}{2k}\int_{k'}\frac{1\pm n_{B/F}(k')}{2k'}\int_{p'}\frac{1}{2E'}\theta(|t|-|t|^*)\times$$
$$\times(2\pi)^4\delta^{(4)}(P+K-P'-K')\left|\overline{\mathcal{M}}_{g/q}(s,t)\right|^2 q_L^2$$

where: $(|t|\equiv q^2-\omega^2)$

Transport coefficients $\kappa_{T/L}(p)$: soft contribution



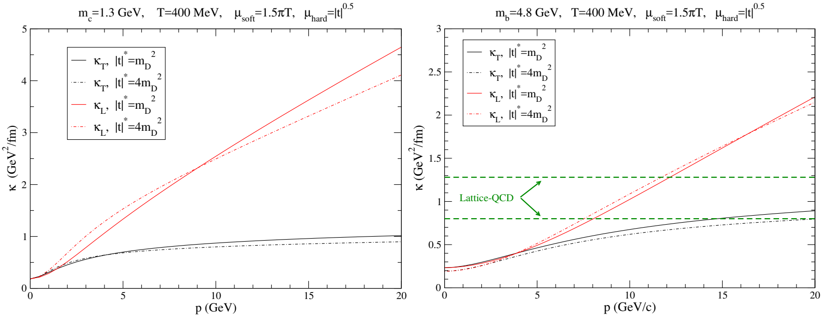
When the exchanged 4-momentum is **soft** the t-channel gluon feels the presence of the medium and requires **resummation**.
The *blob* represents the *dressed gluon propagator*, which has longitudinal and transverse components:

$$\Delta_L(z,q) = \frac{-1}{q^2 + \Pi_L(z,q)}, \quad \Delta_T(z,q) = \frac{-1}{z^2 - q^2 - \Pi_T(z,q)},$$

where *medium effects* are embedded in the HTL gluon self-energy.

Transport coefficients: numerical results

Combining together the hard and soft contributions...



...the dependence on the intermediate cutoff $|t|^*$ is very mild!

NB Notice, in the case of charm, the strong momentum-dependence of κ_L, much milder in the case of beauty, for which $\kappa_L \approx \kappa_T$ up to 5 GeV

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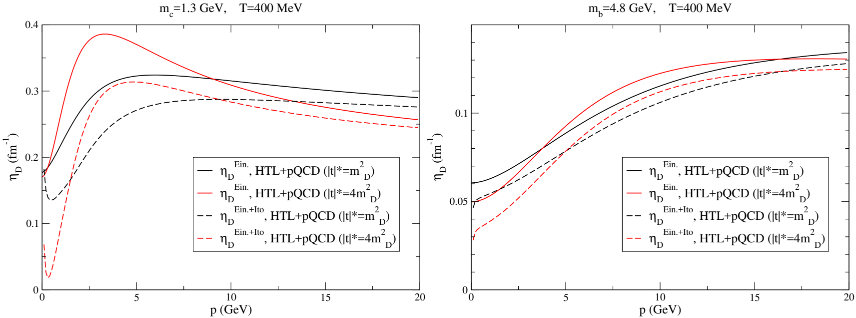

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Lattice-QCD transport coefficients

Ongoing efforts to extract transport coefficients from lattice-QCD simulations *assuming a non-relativistic Langevin dynamics of the HQs*

- κ from electric-field correlators[11];
- η_D from current-current correlators, exploiting the diffusive dynamics of conserved charges[12]

General considerations:

- In principle lattice-QCD would provide an "exact" non-perturbative result;
- Difficulties in extracting real-time quantities (transport coefficients) from euclidean ($t = -i\tau$) simulations;
- Current results limited to the static ($M = \infty$) or at most non-relativistic limit.

[11] Solana and Teaney, PRD 74, 085012 (2006)
[12] Petreczky and Teaney, PRD 73, 014508 (2006)

Lattice-QCD transport coefficients: setup

One consider the non-relativistic limit of the Langevin equation:

$$\frac{dp^i}{dt} = -\eta_D p^i + \xi^i(t), \quad \text{with} \quad \langle \xi^i(t)\xi^j(t')\rangle = \delta^{ij}\delta(t-t')\kappa$$

Hence, in the $p \to 0$ limit:

$$\kappa = \frac{1}{3}\int_{-\infty}^{+\infty} dt \langle \xi^i(t)\xi^i(0)\rangle_{\rm HQ} \approx \frac{1}{3}\int_{-\infty}^{+\infty} dt\, \underbrace{\langle F^i(t)F^i(0)\rangle_{\rm HQ}}_{\equiv D^>(t)},$$

In the static limit the force is due to the color-electric field:

$$\boldsymbol{F}(t) = g\int d\boldsymbol{x}\, Q^\dagger(t,\boldsymbol{x})t^a Q(t,\boldsymbol{x})\boldsymbol{E}^a(t,\boldsymbol{x})$$

In a thermal ensemble $\sigma(\omega) \equiv D^>(\omega) - D^<(\omega) = (1-e^{-\beta\omega})D^>(\omega)$ and

$$\kappa \equiv \lim_{\omega\to 0}\frac{D^>(\omega)}{3} = \lim_{\omega\to 0}\frac{1}{3}\frac{\sigma(\omega)}{1-e^{-\beta\omega}} \underset{\omega\to 0}{\sim} \frac{1}{3}\frac{T}{\omega}\sigma(\omega)$$

Lattice-QCD transport coefficients: results

The spectral function $\sigma(\omega)$ has to be reconstructed starting from the *euclidean electric-field correlator*

$$D_E(\tau) = -\frac{\langle \mathrm{Re}\,\mathrm{Tr}[U(\beta,\tau)gE^i(\tau,\mathbf{0})U(\tau,0)gE^i(0,\mathbf{0})]\rangle}{\langle \mathrm{Re}\,\mathrm{Tr}[U(\beta,0)]\rangle}$$

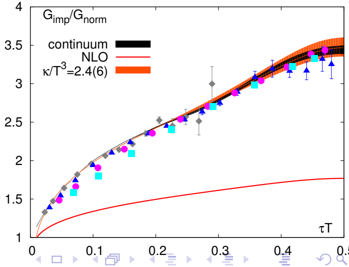
according to

$$D_E(\tau) = \int_0^{+\infty} \frac{d\omega}{2\pi} \frac{\cosh(\tau-\beta/2)}{\sinh(\beta\omega/2)}\sigma(\omega)$$

One gets (arXiv:1409.3724)

$$\kappa/T^3 \approx 2.4(6) \;(\textit{quenched QCD, cont.lim.})$$

$\sim$3-5 times larger then the perturbative result (W.M. Alberico *et al*, EPJC 73 (2013) 2481).
Challenge: approaching the continuum limit in full QCD (see Kaczmarek talk at QM14)!



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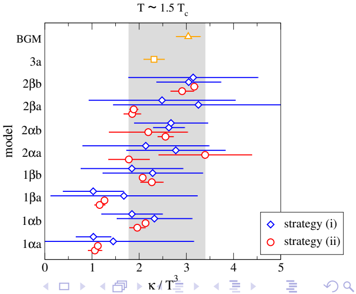

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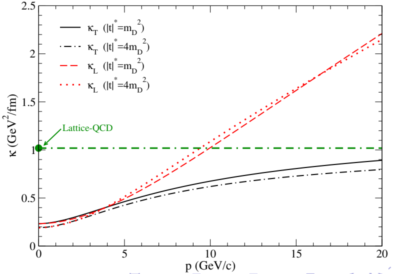

l-QCD transport coefficients: detailed derivation (I)

Derivation of κ in l-QCD done in the $M \to \infty$ limit. In this case the HQ field ψ is only coupled to the A_0 component of the colour-field:

$$\mathcal{L} = Q^\dagger(i\partial_0 + gA_0)Q, \quad \text{with} \quad \left\{Q_i(t,\boldsymbol{x}), Q_j^\dagger(t,\boldsymbol{y})\right\} = \delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y})$$

HQ evolution described by the path-ordered exponential $U(t,t_0)$

$$Q_i(t) = \mathcal{P}\exp\left[ig\int_{t_0}^{t} A_0(t')dt'\right]_{ij} Q_j(t_0) = U_{ij}(t,t_0)Q_j(t_0)$$

One needs then to evaluate the expectation value

$$\langle F^i(t)F^i(0)\rangle_{\rm HQ} \equiv \frac{\sum_s \langle s|e^{-\beta H}F^i(t)F^i(0)|s\rangle}{\sum_s \langle s|e^{-\beta H}|s\rangle}$$

taken over a thermal ensemble of states $|s\rangle$ of the environment *plus* one additional heavy quark:

$$\sum_s \langle s|\dots|s\rangle \equiv \sum_{s'}\int d\boldsymbol{x}\,\langle s'|Q_i(-T,\boldsymbol{x})\dots Q_i^\dagger(-T,\boldsymbol{x})|s'\rangle$$

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taken over a thermal ensemble of states $|s\rangle$ of the environment *plus* one additional heavy quark. In particular:

$$\sum_s \langle s|e^{-\beta H}|s\rangle = Z_{\rm HQ}$$

l-QCD transport coefficients: detailed derivation (II)

Thermal weight $e^{-\beta H} \equiv$ *imaginary-time translation operator*

$$Q(-T)e^{-\beta H} = e^{-\beta H}e^{\beta H}Q(-T)e^{-\beta H} = e^{-\beta H}Q(-T-i\beta)$$

one gets for the HQ partition function (i.e. the denominator)

$$\begin{aligned} Z_{\rm HQ} &= \sum_{s'}\int d\boldsymbol{x}\,\langle s'|Q_i(-T,\boldsymbol{x})e^{-\beta H}Q_i^\dagger(-T,\boldsymbol{x})|s'\rangle \\ &\sim \sum_{s'}\langle s'|e^{-\beta H}U_{ii}(-T-i\beta,-T)|s'\rangle = Z_0\langle {\rm Tr}\,U(-T-i\beta,-T)\rangle, \end{aligned}$$

where the last expectation values is over the environment only.
The numerator can be evaluated analogously starting from

$$\begin{aligned} \sum_s\langle s|e^{-\beta H}\boldsymbol{F}(t)\cdot\boldsymbol{F}(0)|s\rangle = \sum_{s'}\frac{1}{N_c}\int d\boldsymbol{x}\int d\boldsymbol{r}\int d\boldsymbol{r}' \\ \times\,\langle s'|Q_i(-T,\boldsymbol{x})e^{-\beta H}Q_j^\dagger(t,\boldsymbol{r})g\boldsymbol{E}_{jk}(t,\boldsymbol{r})Q_k(t,\boldsymbol{r}) \\ \times\, Q_l^\dagger(0,\boldsymbol{r}')g\boldsymbol{E}_{lm}(0,\boldsymbol{r}')Q_m(0,\boldsymbol{r}')Q_i^\dagger(-T,\boldsymbol{x})|s'\rangle \end{aligned}$$

l-QCD transport coefficients: detailed derivation (III)

The force-force correlator we need is then given by

$$\langle F^i(t)F^i(0)\rangle_{\rm HQ} = \langle {\rm Tr}[U(-T-i\beta,t)gE^i(t) \\ \times U(t,0)gE^i(0)U(0,-T)]\rangle/\langle {\rm Tr}\, U(-T-i\beta,-T)\rangle$$

Lattice-QCD simulations performed in imaginary time: one actually evaluate

$$D_E(\tau) = -\frac{\langle {\rm Re}\,{\rm Tr}[U(\beta,\tau)gE^i(\tau,\mathbf{0})U(\tau,0)gE^i(0,\mathbf{0})]\rangle}{\langle {\rm Re}\,{\rm Tr}[U(\beta,0)]\rangle}$$

and extract $\sigma(\omega)$ from

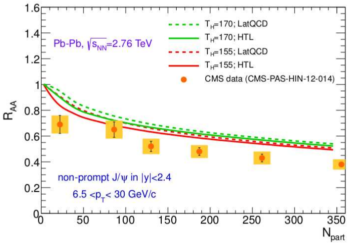
$$D_E(\tau) = \int_0^{+\infty}\frac{d\omega}{2\pi}\frac{\cosh(\tau-\beta/2)}{\sinh(\beta\omega/2)}\sigma(\omega) \quad\longrightarrow\quad \kappa \underset{\omega\to 0}{\sim} \frac{1}{3}\frac{T}{\omega}\sigma(\omega)$$

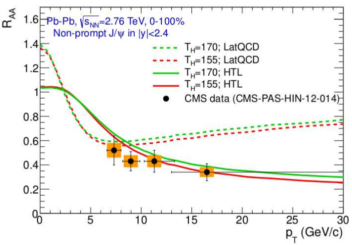
NB $D_E(\tau)$ known just for ~ 10 points makes the inversion ill-defined! Various strategies adopted: Maximum Entropy Method, χ^2 after ansatz for the functional form of $\sigma(\omega)$

The final message: look at beauty!

Measurements of beauty in AA collisions (with future *detector upgrades*) in the next years will allow one to establish a link between first-principle theoretical predictions (*continuum-extrapolated* lattice-QCD calculations) and experimental observables:

- $M \gg gT$: Langevin equation equivalent to Boltzmann equation;
- $M \gg T$: static ($M = \infty$) l-QCD results more reliable for beauty





Measurements so far limited to non-prompt J/ψ's at quite high p_T. B-meson results by CMS are getting available

HF hadronization

From quarks to hadrons

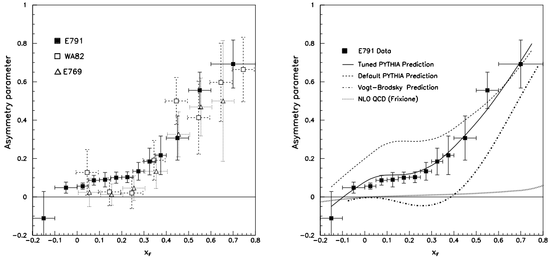
In the presence of a medium, rather then fragmenting like in the vacuum (e.g. $c \to cg \to c\overline{q}q$), HQ's can hadronize by recombining with light thermal quarks (or even *diquarks*) from the medium. This has been implemented in several ways in the literature:

- $2 \to 1$ (or $3 \to 1$ for baryon production) coalescence of partons close in phase-space: $Q + \overline{q} \to M$
- String formation: $Q + \overline{q} \to \mathrm{string} \to \mathrm{hadrons}$
- Resonance formation/decay $Q + \overline{q} \to M^{\star} \to Q + \overline{q}$

In-medium hadronization may affect the R_{AA} and v_2 of final D-mesons due to the *collective (radial and elliptic) flow* of light quarks. Furthermore, it can change the HF hadrochemistry, leading for instance to and enhanced productions of strange particles (D_s) and baryons (Λ_c): no need to excite heavy $s\overline{s}$ or diquark-antidiquark pairs from the vacuum as in elementary collisions, a lot of thermal partons available nearby! Selected results will be shown in the following.

From quarks to hadrons

The first indications of breaking of factorization in the hadronization of charm quarks and of the possibility of their recombination with nearby light partons came from π^--nucleus collisions at SPS and at Fermilab.



A positive asymmetry $A \equiv \frac{D^- - D^+}{D^- + D^+} > 0$ was observed at large $x_F \equiv p^z / p^z_{\max}$, reflecting an enhanced production of *leading hadrons*, i.e. the ones sharing a light valence quarks from the beam. The color connection with the beam remnants lead to this asymmetry (beam-drag effect)

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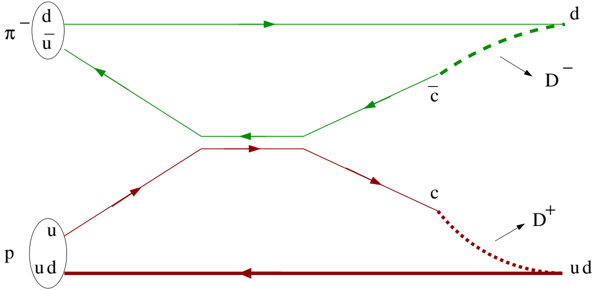

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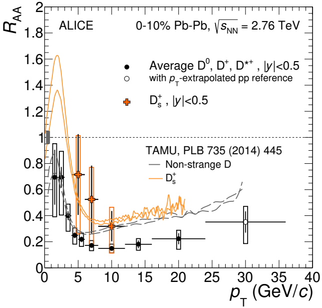
From quarks to hadrons: HF hadrochemistry

The abundance of strange quarks in the plasma can lead e.g. to an enhanced production of D_s mesons wrt p-p collisions via $c + \overline{s} \to D_s$



ALICE data for D and D_s mesons (JHEP 1603 (2016) 082) compared with TAMU-model predictions (M- He et al., PLB 735 (2014) 445)

Langevin transport simulation in the QGP + hadronization modeled via

$$\left(\partial_t + \vec{v}\cdot\vec{\nabla}\right) F_M(t,\vec{x},\vec{p}) = -\underbrace{(\Gamma/\gamma_p)F_M(t,\vec{x},\vec{p})}_{M\to Q+\overline{q}} + \underbrace{\beta(t,\vec{x},\vec{p})}_{Q+\overline{q}\to M}$$

$$\text{with} \quad \sigma(s) = \frac{4\pi}{k^2}\frac{(\Gamma m)^2}{(s-m^2)^2+(\Gamma m)^2}$$

HF in the POWLANG setup: recent developments

(Eur.Phys.J. C75 (2015) no.3, 121)

The major novelty concerns the simulation of heavy-quark hadronization, which now can be performed via

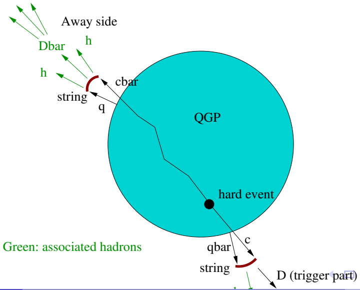
- standard vacuum Fragmentation Functions
- recombination with thermal light partons

From quarks to hadrons

In-medium hadronization may affect the R_{AA} and v_2 of final D-mesons due to the *collective flow* of light quarks. We tried to estimate the effect through this model interfaced to our POWLANG transport code:

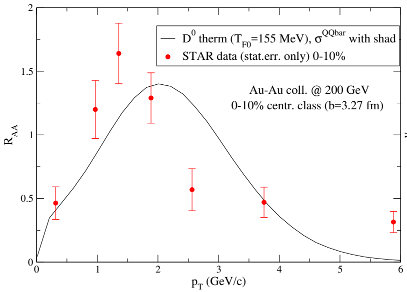
- At $T_{\rm dec}$ c-quarks coupled to light $\overline{q}$'s from a local *thermal distribution*, eventually *boosted* ($u^\mu_{\rm fluid} \neq 0$) to the lab frame;
- Strings are formed and given to PYTHIA 6.4 to simulate their fragmentation and produce the final hadrons ($D + \pi + \dots$)

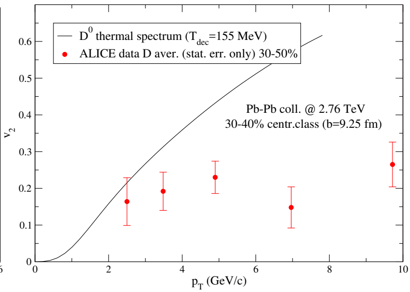
One can address the study of $D-h$ and $e-h$ correlations in AA collisions



From quarks to hadrons: effect on R_{AA} and v_2

Experimental data display a peak in the R_{AA} and a sizable v_2 one would like to interpret as a signal of *charm radial flow and thermalization*





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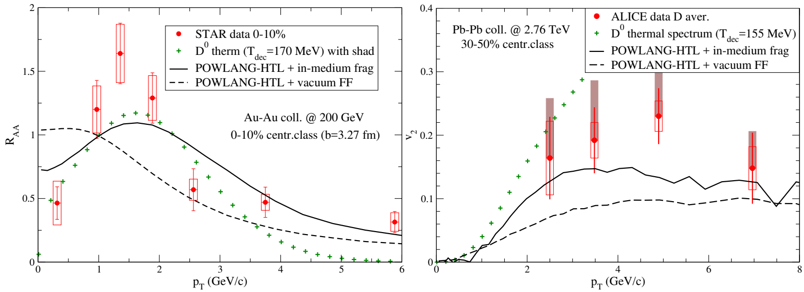
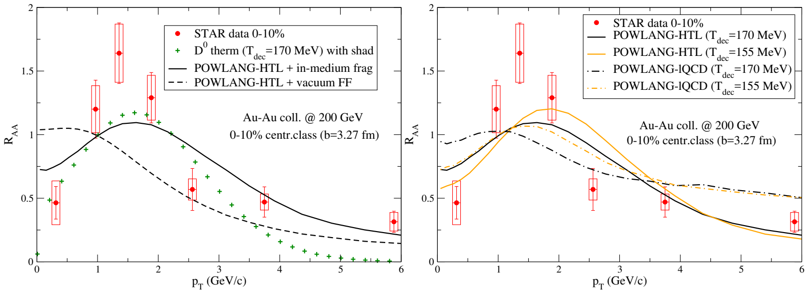


However, *comparing transport results with/without the boost* due to u^{μ}_{fluid}, at least part of the effect might be due to the radial and elliptic flow of the light partons from the medium picked-up at hadronization.

Rescattering in the hadronic phase and its effect on v_2 should be also investigated (in progress)!

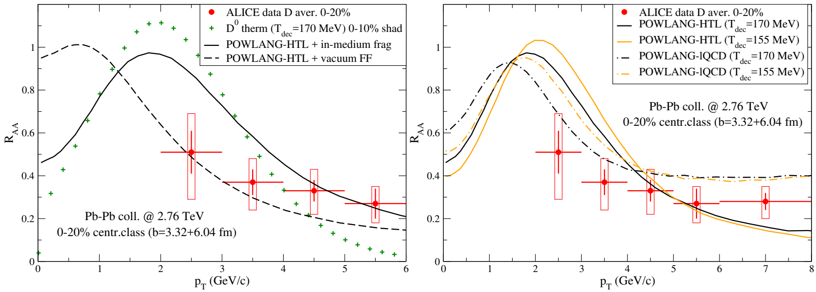
D-meson R_{AA} at RHIC



It is possible to perform a systematic study of different choices of

- Hadronization scheme (left panel)
- Transport coefficients (weak-coupling pQCD+HTL vs non-perturbative l-QCD) and decoupling temperature (right panel)

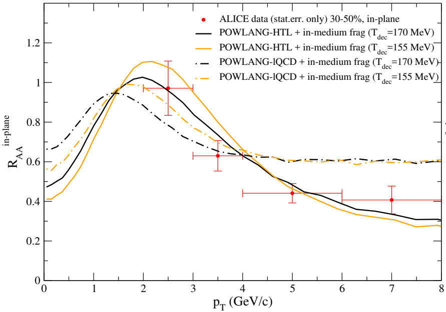
D-meson R_{AA} at LHC

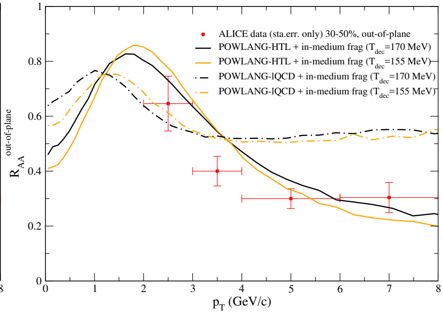


Experimental data for central $(0-20\%)$ Pb-Pb collisions at LHC display a strong quenching, but – at least with the present bins and p_T range – don't show strong signatures of the bump from radial flow predicted by "thermal" and "transport + $Q\overline{q}_{\rm therm}$-string fragmentation" curves.

D meson R_{AA}: in-plane vs out-of-plane

One can study di R_{AA} in- and out-of-plane in non-central $(30-50\%)$ Pb-Pb collisions at LHC:





- Data better described by weak-coupling (pQCD+HTL) transport coefficients;

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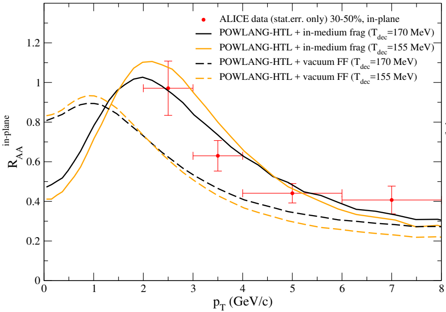
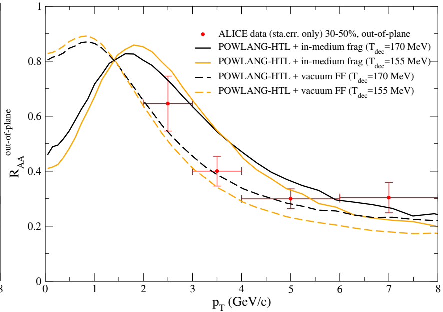
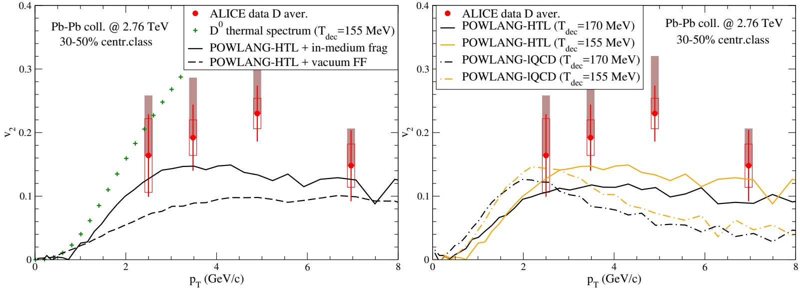




- Data better described by weak-coupling (pQCD+HTL) transport coefficients;
- $Q\overline{q}_{\rm therm}$-string fragmentation describes data slightly better than in-vacuum independent Fragmentation Functions.

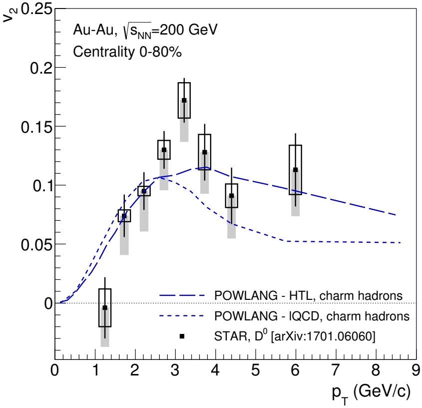
D-meson v_2 at LHC

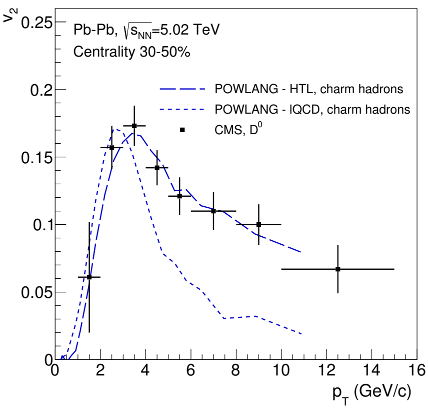


Concerning *D*-meson v_2 in non-central (30 − 50%) Pb-Pb collisions:

- $Q\overline{q}_{\rm therm}$-string fragmentation routine significantly improves our transport model predictions compared to the data;
- HTL curves with a *lower decoupling temperature* display the best agreement with ALICE data

Some recent results





Some recent results

HF in small systems

(p-Pb and d-Au collisions)

Hard and soft probes: different sensitivity to the medium

The quenching of a high-energy parton is described by the pocket formula

$$\langle \Delta E \rangle \sim C_R \alpha_s \hat{q} L^2 \sim T^3 L^2$$

with a strong dependence on the temperature and medium thickness.
If one believes that also in p-A collisions soft physics is described by hydrodynamics ($\lambda_{\rm mfp} \ll L$), then starting from an entropy-density profile

$$s(x,y) \sim \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right]$$

and employing the Euler equation (for $v \ll 1$) and $Tds = d\epsilon$

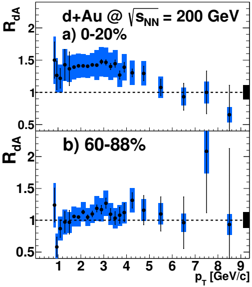
$$(\epsilon + P)\frac{d}{dt}\vec{v} = -\vec{\nabla}P \quad \underset{\vec{\nabla}P = c_s^2\vec{\nabla}\epsilon}{\longrightarrow} \quad \partial_t \vec{v} = -c_s^2 \vec{\nabla} \ln s$$

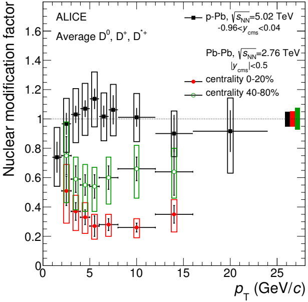
whose solution and mean square value over the transverse plane is

$$v^i = c_s^2 \frac{x^i}{\sigma_i^2} t \quad \longrightarrow \quad \overline{v}^{x/y} = c_s^2 \frac{t}{\sigma_{x/y}}$$

The result has a much milder temperature dependence ($c_s^2 \approx 1/3$) wrt $\hat{q}$ and, although the medium has a ($\approx$3 times) shorter lifetime, radial flow develops earlier, due to the larger pressure gradient

HF in small systems: experimental indications





So far, experimental data don't allow one to draw firm conclusions

- HF electrons in central d-Au collisions at RHIC: $R_{\rm AA}\gtrsim 1$
- D-meson in p-Pb at LHC: $R_{\rm AA}\approx 1$ over a wide p_T-range;

How to reconcile the two observations?

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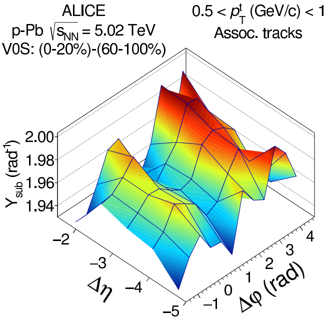

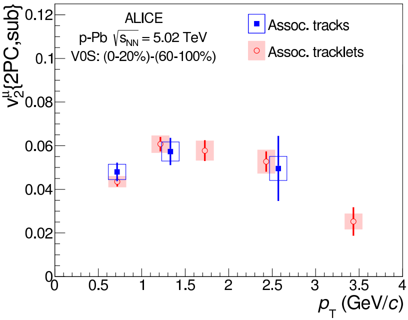

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- HF electrons in central d-Au collisions at RHIC: $R_{AA} \gtrsim 1$
- D-meson in p-Pb at LHC: $R_{AA} \approx 1$ over a wide p_T-range;
- e-h and μ-h correlations provide *hints* of a double-ridge structure

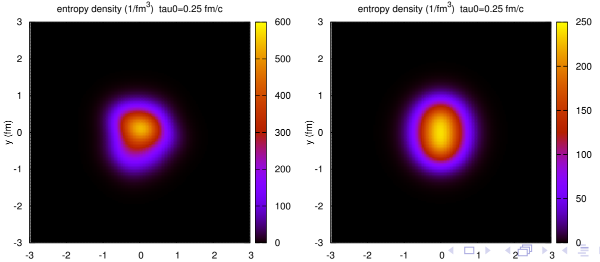
How to reconcile the two observations?

Medium modeling: event-by-event hydrodynamics

Event-by-event fluctuations (e.g. in the nucleon positions) leads to an initial *eccentricity* (responsible for a non-vanishing elliptic flow)

$$s(\boldsymbol{x}) = \frac{K}{2\pi\sigma^2}\sum_{i=1}^{N_{\rm coll}} \exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x}_i)^2}{2\sigma^2}\right] \quad \longrightarrow \quad \epsilon_2 = \frac{\sqrt{\{y^2-x^2\}^2+4\{xy\}^2}}{\{x^2+y^2\}}$$

A full event-by event hydro+transport study requires huge computing resources (time and storage). One can exploit the strong correlation $v_2 \sim \epsilon_2$ considering an *average background* obtained summing all the events of a given centrality class rotated by the *event-plane* angle ψ_2



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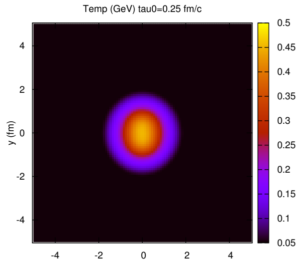
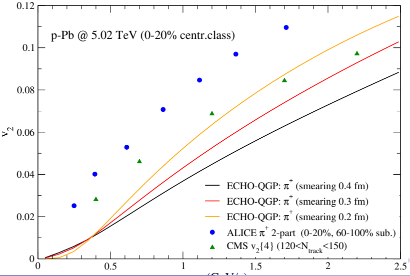
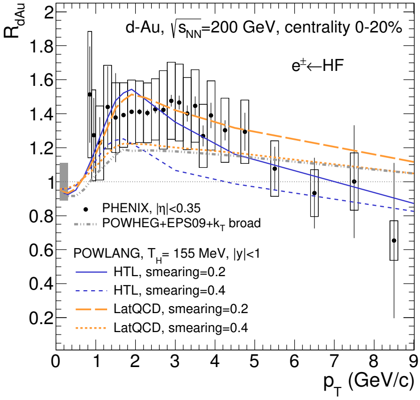
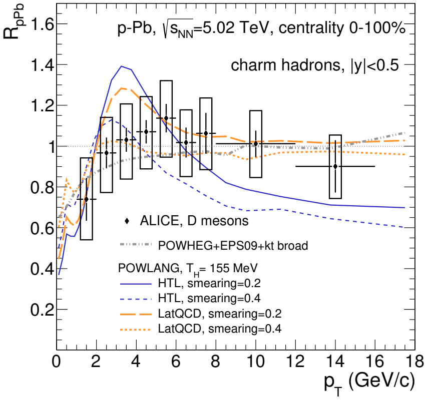




Transport-model predictions





More details can be found in A.B. *et al.*, JHEP 1603 (2016) 123.

Conclusions and future perspectives

Theory-to-experiment comparison allows one to draw some robust qualitative conclusions: c-quarks interact significantly with the medium formed in heavy-ion collision, which affects both their propagation in the plasma and their hadronization. As a result, HF-hadron spectra are quenched at high-p_T, while at low-p_T they display signatures of radial, elliptic and triangular flow.

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- Charm measurements down to $p_T \to 0$: flow/thermalization and total cross-section (of relevance for charmonium supression!)
- D_s and Λ_c measurements: change in hadrochemistry and total cross-section
- Beauty measurements in AA via exclusive hadronic decays: better probe, due to $M \gg \Lambda_{\mathrm{QCD}}, T$ (initial production, evaluation of transport coefficients and Langevin dynamics under better control)
- Charm in p-A collisions: which relevance for high-energy atmospheric muons/neutrinos (Auger and IceCube experiments)? Possible initial/final-state nuclear effects?

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The challenge is to become more quantitative, with the extraction of HF transport coefficients from the data (like η/s in hydrodynamics), goal for which beauty is the golden channel