Chiral symmetry

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Introduction

The confinement/deconfinement phase transition is accompained by the breaking/restoration of *chiral symmetry*. Aim of this lecture is to discuss the role of two *global symmetries* of the QCD lagrangian (isospin and chiral symmetry) in understanding

- the hadron spectrum
- the origin of the *baryonic mass* of our universe

Outline

- Derivation of the QCD Lagrangian (local gauge symmetry)
- Reminder: *global symmetries* and conserved currents in QFT
- The QCD lagrangian and its global symmetries
- The QCD-vacuum and chiral symmetry breaking
- An explicit example: the linear sigma model
- Chiral symmetry restoration at finite temperature

The QCD Lagrangian: construction

Let us start from the *free quark Lagrangian* (diagonal in flavor!)

$$\mathcal{L}_q^{\mathrm{free}} = \overline{q}_f(x)[i\partial\!\!\!/ - m_f]q_f(x).$$

The *quark field* is actually a *vector in color space* ($N_c = 3$):

$$\text{e.g. for an up quark} \quad u^T(x) = [u_r(x), u_g(x), u_b(x)]$$

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The free quark Lagrangian is invariant under *global* $SU(3)$ (i.e. $V^\dagger V = 1$ and $\det(V) = 1$) color transformations, namely:

$$q(x) \longrightarrow V\, q(x) \quad \text{and} \quad \overline{q}(x) \longrightarrow \overline{q}(x)\, V^\dagger,$$

with

$$V = \exp\left[i\theta^a t^a\right] \quad \text{and} \quad [t^a, t^b] = i\, f^{abc} t^c \quad (a = 1, \ldots N_c^2 - 1).$$

f^{abc}: real, antisymmetric *structure constants* of the $su(3)$ algebra.

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We want to build a lagrangian invariant under *local color transformations*:

$$q(x) \longrightarrow V(x)\, q(x) \quad \overline{q}(x) \longrightarrow \overline{q}(x)\, V^\dagger(x),$$

where now $V(x) = \exp\left[i\theta^a(x) t^a\right]$.

Due to the derivative term, $\mathcal{L}_q^{\text{free}}$ is not invariant under local $SU(N_c)$ transformations:

$$\mathcal{L}_q^{\text{free}} \longrightarrow \mathcal{L'}_q^{\text{free}} = \mathcal{L}_q^{\text{free}} + \overline{q}(x) V^\dagger(x) \left[i\partial\!\!\!/ V(x) \right] q(x) \qquad (1)$$

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The solution is to couple the quarks to the gauge field $A_\mu \equiv A_\mu^a t^a$ through the *covariant derivative*

$$\partial_\mu \longrightarrow \mathcal{D}_\mu(x) \equiv \partial_\mu - igA_\mu(x),$$

getting:

$$\mathcal{L}_q = \overline{q}(x)[i\mathcal{D}\!\!\!/(x) - m]q(x) = \mathcal{L}_q^{\text{free}} + g\overline{q}(x)A\!\!\!/(x)q(x).$$

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The transformation of A_μ under local $SU(N_c)$ must be such to compensate the extra term in Eq. (1):

$$A_\mu \longrightarrow A'_\mu = VA_\mu V^\dagger - \frac{i}{g}(\partial_\mu V)V^\dagger.$$

Exercise: verify that $\mathcal{L}_q$ is now invariant under local $SU(N_c)$ transformations. In particular:

$$\mathcal{D}_\mu q \longrightarrow V\mathcal{D}_\mu q \quad \Longrightarrow \quad \mathcal{D}_\mu \longrightarrow V\mathcal{D}_\mu V^\dagger \tag{2}$$

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We must now construct the lagrangian for the gauge-field A_μ

Remember the ($U(1)$ invariant) QED lagrangian of the e.m. field

$$\mathcal{L}_{\text{gauge}}^{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

The field-strength $F_{\mu\nu}$ can be expressed through the covariant derivative

$$\mathcal{D}_\mu \equiv \partial_\mu + ieA_\mu \quad \longrightarrow \quad F_{\mu\nu} = \frac{-i}{e} [\mathcal{D}_\mu, \mathcal{D}_\nu]$$

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The generalization to QCD is now straightforward:

$$F_{\mu\nu} = \frac{i}{g}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] \quad \longrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right].$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \quad \text{(verify!)}$$

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$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \quad \text{(verify!)}$$

From the transformation of the covariant derivative in Eq. (2) one has

$$F_{\mu\nu} \longrightarrow V F_{\mu\nu} V^\dagger, \quad \text{not invariant!}$$

so that the proper $SU(N_c)$-invariant generation of the QED lagrangian is

$$\mathcal{L}_{\rm gauge}^{QCD} = -\frac{1}{2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a}$$

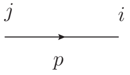
where we have used $\mathrm{Tr}(t^a t^b) = (1/2)\delta^{ab}$.

The QCD Lagrangian and Feynman rules

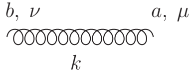
The final form of the QCD Lagrangian is then

$$\mathcal{L}^{QCD} = \sum_f \overline{q}_f [i \not{D} - m_f] q_f - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a},$$

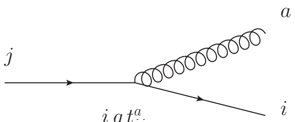
leading to the following Feynman rules (ex: derive them!)



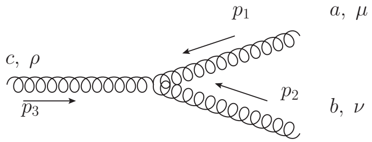
$$\delta^{ij} \frac{i(p_\mu \gamma^\mu + m)}{p^2 - m^2 + i\epsilon}$$



$$\delta^{ab} \frac{i(-g^{\mu\nu} + ...)}{k^2 + i\epsilon}$$



$i\, g\, t^a_{ij}$



$$g\, f^{abc}[g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu]$$

Global symmetries and conserved currents

Consider a Lagrangian *invariant* under the infinitesimal transformation

$$\phi \longrightarrow \phi + \delta\phi$$

i.e.

$$\begin{aligned} 0 = \mathcal{L}(\phi + \delta\phi) - \mathcal{L}(\phi) &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \\ &= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \end{aligned}$$

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Employing the Euler-Lagrange equations one gets then the conserved current (at the classical level! *quantum anomalies* may appear)

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \implies Q \equiv \int d^3x\, j^0(x), \quad \text{with} \quad \frac{dQ}{dt} = 0$$

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In case of invariance under a continuous group $\vec{\phi} \longrightarrow e^{-i\theta^a t^a}\vec{\phi}$ one has *a conseved current for each generator of the group*, i.e.

$$\phi_i \longrightarrow \phi_i - i\,\theta^a t^a_{ij}\,\phi_j \quad \implies \quad j^{\mu,a} = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, t^a_{ij}\,\phi_j$$

Partially conserved currents

It may happen that the Lagrangian contains a term $\mathcal{L}_1$ which breaks *explicitly* the symmetry, but is very small compared to the others:

$$\mathcal{L} \equiv \mathcal{L}_0 + \mathcal{L}_1, \quad \text{with} \quad \mathcal{L}_1 \ll \mathcal{L}_0$$

An example is represented by the mass term for the light quarks in the QCD Lagrangian, for which $m_{\rm light} \ll \Lambda_{\rm QCD}$.

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In this case the concept of a *partially conserved current* (PCC) is still very useful to understand qualitative features of the spectrum of the theory.
Under the field transformation $\phi \longrightarrow \phi + \delta\phi$ one has $\mathcal{L} \longrightarrow \mathcal{L} + \delta\mathcal{L}_1$, hence

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}_{PCC} = \delta\mathcal{L}_1$$

The massless QCD lagrangian: global symmetries

The matter part of the *massless* QCD Lagrangian can be written as

$$\mathcal{L}_q^{QCD} = \overline{q}\,[i\not{D}]q = \overline{q}_R\,[i\not{D}]q_R + \overline{q}_L\,[i\not{D}]q_L.$$

In the above $q \equiv [u, d, (s)]^T$ and $q_{R/L} \equiv \dfrac{1 \pm \gamma^5}{2} q$.

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The above Lagrangian is clearly invariant under the *global* symmetry

$$U_R(N_f) \times U_L(N_f) = U_R(1) \times SU_R(N_f) \times U_L(1) \times SU_L(N_f)$$

This corresponds to the following rotations *in flavour space*

$$q_R \longrightarrow e^{-i\alpha_R} e^{-i\theta_R^a t^a} q_R, \qquad q_L \longrightarrow e^{-i\alpha_L} e^{-i\theta_L^a t^a} q_L$$
$$\overline{q}_R \longrightarrow \overline{q}_R e^{i\alpha_R} e^{i\theta_R^a t^a}, \qquad q_L \longrightarrow \overline{q}_L e^{i\alpha_L} e^{i\theta_L^a t^a}$$

In the above t^a's $(a = 1, ... N_f^2 - 1)$ are the *generators* of the $SU(N_f)$ group and all the parameters $\alpha_R, \alpha_L, \theta_R^a, \theta_L^a$ are *independent*.

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In the above t^a's $(a = 1, \ldots N_f^2 - 1)$ are the *generators* of the $SU(N_f)$ group and all the parameters $\alpha_R, \alpha_L, \theta_R^a, \theta_L^a$ are *independent*.
One gets the *classical* conserved currents

$$j^\mu_{R/L} = \overline{q}_{R/L}\gamma^\mu q_{R/L}, \quad j^{\mu,a}_{R/L} = \overline{q}_{R/L}\gamma^\mu t^a q_{R/L}$$

For $N_f = 2$, $t^a = \tau^a/2$ (Pauli matrices) and $j^{\mu,a}_{R/L} = \overline{q}_{R/L}\gamma^\mu(\tau^a/2)q_{R/L}$

The massless QCD lagrangian: global symmetries (II)

It is convenient to combine the L/R currents into *vector* and *axial* ones

$$
\begin{aligned}
V^\mu &\equiv j_R^\mu + j_L^\mu = \overline{q}\gamma^\mu q,\\
A^\mu &\equiv j_R^\mu - j_L^\mu = \overline{q}\gamma^\mu \gamma^5 q,\\
V^{\mu,a} &\equiv j_R^{\mu,a} + j_L^{\mu,a} = \overline{q}\gamma^\mu (\tau^a/2) q,\\
A^{\mu,a} &\equiv j_R^{\mu,a} - j_L^{\mu,a} = \overline{q}\gamma^\mu \gamma^5 (\tau^a/2) q
\end{aligned}
$$

Going from $N_f = 2$ to $N_f = 3$ one simply replaces $(\tau^a/2) \longrightarrow (\lambda^a/2)$ (Gell-Man matrices in flavour space).
They are associated to the symmetry group

$$U_V(1) \times SU_V(N_f) \times U_A(1) \times SU_A(N_f)$$

We will comment on the role of each of these *classical* symmetries.

$U_V(1)$ symmetry

It corresponds to the invariance for

$$q \longrightarrow e^{-i\alpha} q, \quad \overline{q} \longrightarrow \overline{q} e^{i\alpha}$$

rotating *by the same angle* R and L components.
It is associated to the conservation of the *baryon number*

$$\begin{aligned} B \equiv (1/3) Q^V &= (1/3) \int d^3x \, q^\dagger(x) q(x) \\ &= (1/3) \int d^3x \, [q_R^\dagger(x) q_R(x) + q_L^\dagger(x) q_L(x)], \end{aligned}$$

i.e. the *net* number of quarks (right-handed *plus* left-handed).

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i.e. the *net* number of quarks (right-handed *plus* left-handed).
Baryon number is *exactly* conserved in QCD. This does not mean that it is exactly conserved in nature, as suggested by the matter-antimatter asymmetry in our universe

$U_A(1)$ symmetry

It corresponds to the invariance (of the massless *Lagrangian*) for

$$q \longrightarrow e^{-i\alpha\gamma^5} q, \quad \overline{q} \longrightarrow \overline{q} e^{-i\alpha\gamma^5} \quad (\text{since } \{\gamma^\mu, \gamma^5\} = 0)$$

rotating *by opposite angles* R and L components ($\gamma^5 q_{R/L} = \pm q_{R/L}$).
It *would be* associated to the conservation of the axial charge

$$Q_A = \int d^3x \, q^\dagger(x)\gamma^5 q(x) = \int d^3x \, [q_R^\dagger(x) q_R(x) - q_L^\dagger(x) q_L(x)],$$

i.e. to the number of right-handed minus left-handed quarks.

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i.e. to the number of right-handed minus left-handed quarks.
However, although being a symmetry of the *classical* QCD action, $U_A(1)$ is not a symmetry of the theory, being broken by quantum fluctuations:

$$\partial_\mu A^\mu = -N_f \frac{g^2}{16\pi^2} \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F^a_{\mu\nu} F^a_{\alpha\beta}$$
$$\equiv -N_f \frac{g^2}{16\pi^2} \widetilde{F}^{\alpha\beta,a} F^a_{\alpha\beta}$$

Quantum anomalies of axial currents at the basis of the $\pi^0 \to \gamma\gamma$ decay and of the η' being much heavier than the other pseudoscalar mesons

$SU_V(N_f)$ symmetry

It corresponds to the invariance for

$$q \longrightarrow e^{-i\theta^a t^a} q, \quad \overline{q} \longrightarrow \overline{q} e^{i\theta^a t^a}, \quad (t^a = \tau^a/2 \text{ or } \lambda^a/2)$$

rotating *by the same angles* θ^a's R and L components.
It leads to the conservation (we focus on $N_f = 2$) of the Isospin charges

$$Q_V^a \equiv \int d^3x \, q^\dagger(x) \frac{\tau^a}{2} q(x),$$

which play the role of *generators* of the Isospin rotations.
It is a symmetry of the Lagrangian *and* of the theory: QCD vacuum (and spectrum) invariant under Isospin transformations (more in the following)

$$e^{-i\theta^a Q_V^a} |0\rangle = |0\rangle \quad \Leftrightarrow \quad Q_V^a |0\rangle = 0$$

Isospin charges annihilate the vacuum!

$SU_A(N_f)$ symmetry

It corresponds to the invariance for

$$q \longrightarrow e^{-i\theta^a t^a \gamma^5} q, \quad \overline{q} \longrightarrow \overline{q} e^{-i\theta^a t^a \gamma^5}, \quad (t^a = \tau^a/2 \text{ or } \lambda^a/2)$$

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$$Q_A^a \equiv \int d^3x \, q^\dagger(x) \frac{\tau^a}{2} \gamma^5 q(x),$$

playing the role of *generators* of the chiral (flavour-changing) rotations. Although being a symmetry of the Lagrangian, it is *not* a symmetry of the theory: QCD vacuum (and spectrum) not invariant under chiral rotations (*spontaneous* symmetry breaking!)

$$e^{-i\theta^a Q_A^a}|0\rangle \neq |0\rangle \quad \Leftrightarrow \quad Q_A^a|0\rangle \neq 0 \quad i.e. \quad Q_A^a|0\rangle \equiv |\Phi^a\rangle$$

Chiral charges create physical states: pseudoscalar mesons

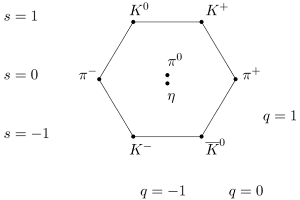
$$H_{QCD}|0\rangle = 0 \text{ and } [H_{\rm QCD}, Q_A^a] = 0 \quad \Longrightarrow \quad H|\Phi^a\rangle = 0 \quad \text{(Goldstone theorem)}$$

Pions (kaons and η) can be considered the Goldstone bosons associated to chiral-symmetry breaking: much lighter than other hadrons!

$SU_A(N_f)$ symmetry and the QCD spectrum

According to the *Goldstone theorem* there is one massless boson for each generator of a broken continuous symmetry.

In the case $N_f = 3$, with generators $t^a \equiv \frac{\lambda^a}{2}$ $(a = 1, ...8)$, they can be identified with the octet of pseudoscalar mesons



- $m_\pi \approx 138$ MeV
- $m_K \approx 495$ MeV
- $m_\eta \approx 548$ MeV

NB due to the *axial anomaly* the original symmetry group is reduced $U_A(N_f) \longrightarrow SU_A(N_f)$, hence the number of *spontaneously* broken generator is 8 instead of 9: in fact $m_{\eta'} = 958$ MeV, much heavier than all the other pseudoscala mesons

Quark masses and explicit symmetry-breaking

Consider the transformation of the mass term in the QCD Lagrangian

$$\mathcal{L}_1 = -\overline{q}mq = -\overline{q}_R m q_L - \overline{q}_L m q_R$$

One has, being $m \equiv \mathrm{diag}[m_u, m_d, (m_s)]$,

$$\overline{q}mq \underset{SU_V(N_f)}{\longrightarrow} \overline{q}mq - i\theta^a \overline{q} m t^a q + i\theta^a \overline{q} t^a m q + \ldots$$

$$\overline{q}mq \underset{SU_A(N_f)}{\longrightarrow} \overline{q}mq - i\theta^a \overline{q} m t^a \gamma^5 q - i\theta^a \overline{q}\gamma^5 t^a m q + \ldots$$

From $\partial_\mu j^\mu = \delta\mathcal{L}_1$ one gets:

$$\partial_\mu V^{\mu,a} = i\,\overline{q}\,[m, t^a]q \quad \text{and} \quad \partial_\mu A^{\mu,a} = i\,\overline{q}\,\{m, t^a\}\gamma^5 q$$

Focus on the $N_f = 2$ case:

- as long as $m_u = m_d \neq 0$, the isospin current is *exactly* conserved. Since $m_u \approx m_d$, isospin is an *almost exact symmetry* of QCD
- $m \neq 0$ *explicitly* breaks chiral symmetry. However, since $m \ll \Lambda_{\mathrm{QCD}}$, one speaks of Partially Conserved Axial Current (PCAC)

Transformations of mesonic currents

We now understand better the physical meaning of isospin and chiral transformation by considering their action on current operators having the quantum numbers to create/destroy different mesons:

pion-like state: $\vec{\pi} \equiv i\,\overline{q}\,\vec{\tau}\gamma^5 q$; sigma-like state: $\sigma \equiv \overline{q}q$

rho-like state: $\vec{\rho}^{\,\mu} \equiv \overline{q}\,\vec{\tau}\gamma^\mu q$; a_1-like state: $\vec{a_1}^{\,\mu} \equiv \overline{q}\,\vec{\tau}\gamma^\mu\gamma^5 q$

Channel	PS	S	V	PV
Particle	π	σ	ρ	a_1
Mass (MeV)	138	500	770	1260

- Pions are responsible for the long-range NN attractive interaction;
- Due to their large width, scalar mesons are difficult to identify. Usually one identify the sigma with the $f_0(500)$ state of the PDG, a broad 2-pion resonance, playing a major role in the nuclear binding;
- Vector mesons are responsible for the short-range repulsive NN interaction. In symmetric nuclear matter major role played by isoscalar $\omega(782)$ meson: $\omega^\mu \sim \overline{q}\,\gamma^\mu q$.

Transformations of mesonic currents: $SU_V(N_f)$

The transformation of the quark fields is given by

$$q \longrightarrow e^{-i\theta^a(\tau^a/2)}q, \quad \overline{q} \longrightarrow \overline{q}e^{i\theta^a(\tau^a/2)}$$

From $\{\tau^a, \tau^b\} = 2\delta^{ab}$ and $[\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c$, one gets:

- Isoscalar mesons: $\overline{q}q \longrightarrow \overline{q}q$, hence $\sigma \longrightarrow \sigma$
- Isovector mesons

$$\begin{aligned}\pi^a: \quad i\,\overline{q}\,\tau^a\gamma^5 q \quad &\longrightarrow \quad i\,\overline{q}\,\tau^a\gamma^5 q + \theta^b\left(\overline{q}\,\tau^a\gamma^5\frac{\tau^b}{2}q - \overline{q}\,\frac{\tau^b}{2}\tau^a\gamma^5 q\right)\\ &= \quad i\,\overline{q}\,\tau^a\gamma^5 q + \epsilon^{abc}\theta^b\left(i\,\overline{q}\,\tau^c\gamma^5 q\right)\end{aligned}$$

Hence, they transform as vectors

$$\begin{aligned}\vec{\pi} \quad &\longrightarrow \quad \vec{\pi} + \vec{\theta}\times\vec{\pi}\\ \vec{\rho}^{\,\mu} \quad &\longrightarrow \quad \vec{\rho}^{\,\mu} + \vec{\theta}\times\vec{\rho}^{\,\mu}\\ \vec{a_1}^{\,\mu} \quad &\longrightarrow \quad \vec{a_1}^{\,\mu} + \vec{\theta}\times\vec{a_1}^{\,\mu}\end{aligned}$$

Isospin rotations mix mesons belonging to the same multiplet, having the same mass. It is a symmetry of the QCD spectrum!

Transformations of mesonic currents: $SU_A(N_f)$

The transformation of the quark fields is given by

$$q \longrightarrow e^{-i\theta^a(\tau^a/2)\gamma^5} q, \quad \overline{q} \longrightarrow \overline{q} e^{-i\theta^a(\tau^a/2)\gamma^5}$$

From $\{\tau^a, \tau^b\} = 2\delta^{ab}$ and $[\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c$, one gets for instance

$$\begin{aligned}\pi^a: \; i\,\overline{q}\,\tau^a\gamma^5 q \quad &\longrightarrow \quad i\,\overline{q}\,\tau^a\gamma^5 q + \theta^b \left(\overline{q}\,\tau^a \frac{\tau^b}{2} q + \overline{q}\,\frac{\tau^b}{2}\tau^a q\right) \\ &= \quad i\,\overline{q}\,\tau^a\gamma^5 q + \theta^a\,\overline{q}\,q\end{aligned}$$

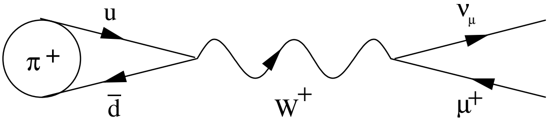
The complete set of chiral transformations is given by (verify!):

$$\begin{aligned}\vec{\pi} \longrightarrow \vec{\pi} + \vec{\theta}\,\sigma, \qquad & \sigma \longrightarrow \sigma - \vec{\theta}\cdot\vec{\pi} \\ \vec{\rho}^{\,\mu} \longrightarrow \vec{\rho}^{\,\mu} + \vec{\theta} \times \vec{a_1}^{\mu}, \qquad & \vec{a_1}^{\mu} \longrightarrow \vec{a_1}^{\mu} + \vec{\theta} \times \vec{\rho}^{\,\mu}\end{aligned}$$

Scalar/pseudoscalar and vector/pseudovector mesons are mapped one into the other by chiral rotations. However, such a symmetry is not found in the spectrum ($m_\pi \neq m_\sigma$, $m_\rho \neq m_{a_1}$), it is *spontaneously* broken!

Pion decay and PCAC

Pion decay is an electro-weak process



In the Fermi theory it can be described as a point-like current-current interaction, with both vector and axial components ($V - A$).

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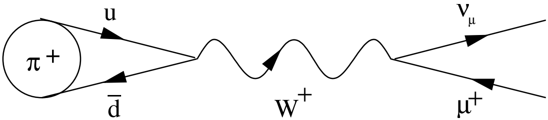

In the Fermi theory it can be described as a point-like current-current interaction, with both vector and axial components ($V - A$).
Only the axial current has the quantum number to annihilate a pion and the matrix element connecting the pion with the (QCD) vacuum is

$$\langle 0|A_\mu^a(x)|\pi^b(q)\rangle = -i\, f_\pi q_\mu \delta^{ab} e^{-iq\cdot x}$$

with $f_\pi = 93$ MeV taken from $\tau_\pi^{\rm exp}$.

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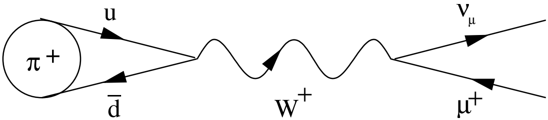

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$$\langle 0|A_\mu^a(x)|\pi^b(q)\rangle = -i\, f_\pi q_\mu \delta^{ab} e^{-iq\cdot x}$$

with $f_\pi = 93$ MeV taken from $\tau_\pi^{\rm exp}$. Taking the divergence one gets:

$$\langle 0|\partial^\mu A_\mu^a(x)|\pi^b(q)\rangle = -f_\pi q^2 \delta^{ab} e^{-iq\cdot x} = -f_\pi m_\pi^2 \delta^{ab} e^{-iq\cdot x}$$

In case axial current were exactly conserved, $\partial^\mu A_\mu^a = 0$ and hence $m_\pi = 0$.
Small pion mass comes from *explicit* symmetry breaking ($m_{u,d} \neq 0$)

Spontaneous chiral-symmetry breaking: the linear σ-model

The linear σ-model is the simplest hadronic lagrangian consistent with the isospin and chiral symmetry and including the possibility of *spontaneous breaking of chiral symmetry*. Here we illustrate its construction.

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\begin{aligned}
SU_V(2): \quad & \vec{\pi} \longrightarrow \vec{\pi} + \vec{\theta} \times \vec{\pi}, \quad & \sigma \longrightarrow \sigma \\
SU_A(2): \quad & \vec{\pi} \longrightarrow \vec{\pi} + \vec{\theta}\sigma, \quad & \sigma \longrightarrow \sigma - \vec{\theta} \cdot \vec{\pi}
\end{aligned}
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Hence, under infinitesimal transformations, for their squares one has:

$$\begin{array}{lll} SU_V(2): & \vec{\pi}^2 \longrightarrow \vec{\pi}^2, & \sigma^2 \longrightarrow \sigma^2 \\ SU_A(2): & \vec{\pi}^2 \longrightarrow \vec{\pi}^2 + 2\,\vec{\theta}\cdot\vec{\pi}\,\sigma, & \sigma^2 \longrightarrow \sigma^2 - 2\,\vec{\theta}\cdot\vec{\pi}\,\sigma \end{array}$$

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The combination $(\vec{\pi}^2 + \sigma^2)$ is invariant under $SU_{V/A}(2)$ transformations: pion and sigma fields must enter in the Lagrangian in such a form:

$$\mathcal{L}^{\phi}_{\sigma-\mathrm{mod}} = \frac{1}{2}(\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi} + \partial_\mu \sigma\, \partial^\mu \sigma) - V(\vec{\pi}^2 + \sigma^2)$$

where, with compact notation, $\phi^i \equiv (\vec{\pi}, \sigma)$ ($i = 1, \ldots 4$). The linear σ-model is also known, due to its symmetry, as $O(4)$ model

Let us now introduced also the nucleons. In order not to break chiral symmetry they must enter in the Lagrangian as *massless* particles:

$$\mathcal{L}^{\psi,\mathrm{kin}}_{\sigma-\mathrm{mod}} = i\overline{\psi}\partial\!\!\!/\psi, \quad \text{where} \quad \psi \equiv (p,n)^T$$

Under $SU_{V/A}(2)$ transformations the nucleon behaves as the quark field

$$SU_V(2): \quad \psi \longrightarrow e^{-i\theta^a(\tau^a/2)}\psi, \quad \overline{\psi} \longrightarrow \overline{\psi}e^{i\theta^a(\tau^a/2)}$$
$$SU_A(2): \quad \psi \longrightarrow e^{-i\theta^a(\tau^a/2)\gamma^5}\psi, \quad \overline{\psi} \longrightarrow \overline{\psi}e^{-i\theta^a(\tau^a/2)\gamma^5}$$

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$SU_V(2)\times SU_A(2)$ symmetry constraints its coupling with the mesons

$$\mathcal{L}^{\psi,\phi}_{\sigma-\text{mod}} = -g_\pi\,\overline{\psi}\,\left[i\gamma^5\vec{\pi}\cdot\vec{\tau}+\sigma\right]\psi$$

Let us verify it invariance under $SU_V(2)$ transformations:

$$\begin{aligned}\overline{\psi}\sigma\psi &\longrightarrow \overline{\psi}\sigma\psi\\ \overline{\psi}[i\gamma^5\vec{\pi}\cdot\vec{\tau}]\psi &\longrightarrow \overline{\psi}[i\gamma^5\vec{\pi}\cdot\vec{\tau}]\psi+i\frac{\theta^a}{2}\overline{\psi}[i\gamma^5\tau^a\tau^b\,\pi^b]\psi\\ &\quad -i\frac{\theta^a}{2}\overline{\psi}[i\gamma^5\tau^b\tau^a\,\pi^b]\psi+\overline{\psi}\left[i\gamma^5(\theta\times\pi)\cdot\vec{\tau}\right]\psi\end{aligned}$$

From $[\tau^a,\tau^b]=2i\epsilon^{abc}\tau^c$ one gets $\overline{\psi}[i\gamma^5\vec{\pi}\cdot\vec{\tau}]\psi \longrightarrow \overline{\psi}[i\gamma^5\vec{\pi}\cdot\vec{\tau}]\psi$

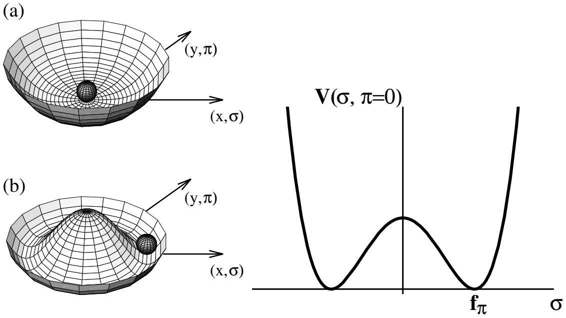
Repeat for $SU_A(2)$ transformations...

$$\overline{\psi}\sigma\psi \longrightarrow \overline{\psi}\sigma\psi - i\frac{\theta^a}{2}\overline{\psi}\tau^a\gamma^5\sigma\psi$$
$$-i\frac{\theta^a}{2}\overline{\psi}\sigma\tau^a\gamma^5\psi - \overline{\psi}(\vec{\theta}\cdot\vec{\pi})\psi$$
$$= \overline{\psi}\sigma\psi - \overline{\psi}[\, i\gamma^5\vec{\theta}\cdot\vec{\tau}\,\sigma\,]\psi - \overline{\psi}(\vec{\theta}\cdot\vec{\pi})\psi$$

$$\overline{\psi}[\, i\gamma^5\vec{\pi}\cdot\vec{\tau}\,]\psi \longrightarrow \overline{\psi}[\, i\gamma^5\vec{\pi}\cdot\vec{\tau}\,]\psi - i\frac{\theta^a}{2}\overline{\psi}[i\gamma^5\gamma^5\tau^a\tau^b\,\pi^b]\psi$$
$$-i\frac{\theta^a}{2}\overline{\psi}[i\gamma^5\tau^b\tau^a\,\pi^b\gamma^5]\psi + \overline{\psi}[\, i\gamma^5\vec{\theta}\cdot\vec{\tau}\,\sigma\,]\psi$$
$$= \overline{\psi}[\, i\gamma^5\vec{\pi}\cdot\vec{\tau}\,]\psi + \overline{\psi}(\vec{\theta}\cdot\vec{\pi})\psi + \overline{\psi}[\, i\gamma^5\vec{\theta}\cdot\vec{\tau}\,\sigma\,]\psi$$

Hence, while the scalar and pseudoscalar interactions are sepately invariant under isospin rotations, *only* the combination $\overline{\psi}[\sigma + i\gamma^5\vec{\pi}\cdot\vec{\tau}\,]\psi$ remains invariant under chiral transformations. This entails that a unique coupling constant governs both the scalar and the pseudoscalar interactions.

We have seen that, although the QCD (and the effective σ-model) Lagrangian is invariant under $SU_A(N_f)$, the *spectrum is not*. The potential must induce a *spontaneous breaking* of chiral symmetry



$$V = V(\vec{\pi}^2+\sigma^2) = \frac{\lambda}{4}\left[(\vec{\pi}^2+\sigma^2) - f_\pi^2\right]^2$$

Fields acquire a vacuum expectation value (VEV) minimizing the potential $\langle\vec{\pi}\rangle = 0$ and $\langle\sigma\rangle = \sigma_0 = f_\pi$.

NB Identification of f_π with the minimum of the potential to be proven!

We can perform fluctuations around the VEV and rewrite the full Lagrangian in terms of the new field variables $\vec{\pi}(x)$ and $\sigma(x) = f_\pi + s(x)$ It is convenient to employ the field $\phi^i \equiv (\vec{\pi}, \sigma) = (\vec{\pi}, f_\pi + s) = \phi_0^i + \eta^i$, with the VEV $\phi_0^i \equiv \delta^{i4} f_\pi$. The potential reads then

$$V = \frac{\lambda}{4}\left[(\phi^i)^2 - f_\pi^2\right]^2 = \frac{\lambda}{4}\left[(\phi_0^i + \eta^i)^2(\phi_0^j + \eta^j)^2 - 2(\phi_0^i + \eta^i)^2 f_\pi^2 + f_\pi^4\right]$$

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Constant and linear terms in the fluctuations η^i vanish and one gets

$$V = \frac{1}{2}(2\lambda f_\pi^2)\, s^2 + \lambda f_\pi\, s\,(\pi^2 + s^2) + \frac{\lambda}{4}(\pi^2 + s^2)^2$$

Notice that:

- Pions are massless: $m_\pi = 0$ (Goldstone theorem)
- The σ-meson gets a mass: $m_\sigma^2 = 2\lambda f_\pi^2$
- Cubic self-interaction terms among the mesons appear

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Notice that:

- Pions are massless: $m_\pi = 0$ (Goldstone theorem)
- The σ-meson gets a mass: $m_\sigma^2 = 2\lambda f_\pi^2$
- Cubic self-interaction terms among the mesons appear

From the coupling with the nucleon field one gets:

$$\mathcal{L}^{\psi,\mathrm{kin}}_{\sigma-\mathrm{mod}} + \mathcal{L}^{\psi,\phi}_{\sigma-\mathrm{mod}} = \overline{\psi}\,[i\partial\!\!\!/ - g_\pi f_\pi]\,\psi - g_\pi\,\overline{\psi}\,\left[i\gamma^5 \vec{\pi}\cdot\vec{\tau} + s\right]\psi$$

The nucleon gets its mass from chiral-symmetry breaking!

Nucleon mass: some comments

$$
\begin{aligned}
\mathcal{L}^{\psi,\mathrm{kin}}_{\sigma-\mathrm{mod}} + \mathcal{L}^{\psi,\phi}_{\sigma-\mathrm{mod}} &= i\overline{\psi}\partial\!\!\!/\psi - g_\pi \overline{\psi}\left[i\gamma^5\vec{\pi}\cdot\vec{\tau} + \sigma\right]\psi \\
&= \overline{\psi}\left[i\partial\!\!\!/ - g_\pi f_\pi\right]\psi - g_\pi \overline{\psi}\left[i\gamma^5\vec{\pi}\cdot\vec{\tau} + s\right]\psi \\
&= \overline{\psi}\left[i\partial\!\!\!/ - M_N\right]\psi - g_\pi \overline{\psi}\left[i\gamma^5\vec{\pi}\cdot\vec{\tau} + s\right]\psi
\end{aligned}
$$

- The model allows the nucleon to get a mass in a way consistent with the chiral-symmetry of the Lagrangian: no explicit symmetry breaking!
- Most of the present baryonic mass in the universe got its origin at the chiral transition, when $\langle\sigma\rangle = 0 \longrightarrow \langle\sigma\rangle \neq 0$
- One gets the *Goldberger-Treiman* relation $g_\pi = M_N/f_\pi$, entailing $g_\pi \approx 10$ (it will receive positive corrections): chiral symmetry allows one to get an estimate of the (QCD!) pion-nucleon coupling, just from the nucleon mass and the pion decay-constant (e.w. process!). The pion-nucleon coupling is very large!

Linear σ-model: the full Lagrangian

Let us collect all the terms and write the full Lagrangian:

$$\mathcal{L}_{\sigma-\mathrm{mod}} = \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi} + \frac{1}{2}\partial_\mu s\,\partial^\mu s - \frac{1}{2} m_\sigma^2 s^2$$

$$-\lambda f_\pi\, s\,(\pi^2 + s^2) - \frac{\lambda}{4}(\pi^2+s^2)^2 + \overline{\psi}\,[i\partial\!\!\!/ - M_N]\,\psi - g_\pi\,\overline{\psi}\,\left[i\gamma^5 \vec{\pi}\cdot\vec{\tau} + s\right]\psi$$

with $m_\sigma^2 = 2\lambda f_\pi^2$ and $M_N = g_\pi f_\pi$

Linear σ-model: the axial current

Starting from the Lagrangian written in terms of the original fields

$$\mathcal{L}_{\sigma-\mathrm{mod}} = \frac{1}{2}\left(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \partial_\mu\sigma\,\partial^\mu\sigma\right) - \frac{\lambda}{4}\left[(\vec{\pi}^2+\sigma^2) - f_\pi^2\right]^2 + i\overline{\psi}\partial\!\!\!/\psi - g_\pi\,\overline{\psi}\,\left[i\gamma^5\vec{\pi}\cdot\vec{\tau}+\sigma\right]\psi$$

and from the transformation law of the fields under $SU_A(2)$

$$\psi \longrightarrow \psi - i\gamma^5\frac{\tau^a}{2}\theta^a\psi \qquad \vec{\pi}\longrightarrow\vec{\pi}+\vec{\theta}\sigma \qquad \sigma\longrightarrow\sigma-\vec{\theta}\cdot\vec{\pi}$$

one gets:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^i)}\delta\Phi^i \quad\longrightarrow\quad A^a_\mu = \bar{\psi}\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi - \pi^a\partial_\mu\sigma + \sigma\partial_\mu\pi^a$$

Linear σ-model: the axial current

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$$\mathcal{L}_{\sigma-\mathrm{mod}} = \frac{1}{2}\left(\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi} + \partial_\mu \sigma\, \partial^\mu \sigma\right) - \frac{\lambda}{4}\left[(\vec{\pi}^2 + \sigma^2) - f_\pi^2\right]^2 + i\overline{\psi}\partial\!\!\!/\psi - g_\pi\, \overline{\psi}\,\left[i\gamma^5 \vec{\pi}\cdot\vec{\tau} + \sigma\right]\psi$$

and from the transformation law of the fields under $SU_A(2)$

$$\psi \longrightarrow \psi - i\gamma^5 \frac{\tau^a}{2}\theta^a \psi \qquad \vec{\pi} \longrightarrow \vec{\pi} + \vec{\theta}\sigma \qquad \sigma \longrightarrow \sigma - \vec{\theta}\cdot\vec{\pi}$$

one gets:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\delta\Phi^i \quad \longrightarrow \quad A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5 \frac{\tau^a}{2}\psi - \pi^a \partial_\mu \sigma + \sigma \partial_\mu \pi^a$$

In terms of the shifted fields one has

$$A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5 \frac{\tau^a}{2}\psi - \pi^a \partial_\mu s + f_\pi \partial_\mu \pi^a + s\,\partial_\mu \pi^a,$$

consistently with the pion-decay matrix element

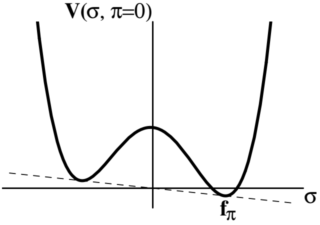
$$\langle 0|A_\mu^a(x)|\pi^b(q)\rangle = -i\, f_\pi q_\mu \delta^{ab} e^{-iq\cdot x}$$

Explicit symmetry-breaking

$\mathcal{L}_{\sigma-\mathrm{mod}}$ must be supplemented by a small *explicit* symmetry breaking term arising from the non-zero quark masses:

$$\mathcal{L}_1^{\mathrm{QCD}} = -\overline{q}mq \quad \longrightarrow \quad \mathcal{L}_1^{\sigma-\mathrm{mod}} = \epsilon\sigma$$

This corresponds to a modification of the potential



$$V = \frac{\lambda}{4}\left[(\vec{\pi}^2 + \sigma^2) - v_0^2\right]^2 - \epsilon\sigma$$

We require $\langle\sigma\rangle = f_\pi$, so to satisfy $M_N = g_\pi f_\pi$

$$\left.\frac{\partial V}{\partial \sigma}\right|_{f_\pi} = 0 \;\longrightarrow\; v_0^2 = f_\pi^2 - \frac{\epsilon}{\lambda f_\pi}$$

The σ-meson and pions receive a positive mass correction

$$m_\sigma^2 = \left.\frac{\partial^2 V}{\partial \sigma^2}\right|_{(0,f_\pi)} = 2\lambda f_\pi^2 + \frac{\epsilon}{f_\pi}, \quad m_\pi^2 = \left.\frac{\partial^2 V}{\partial \pi^2}\right|_{f_\pi} = \frac{\epsilon}{f_\pi} \neq 0$$

The nucleon mass

We have imposed the Golberger-Treiman relation $M_N = g_\pi f_\pi$ even in the presence of explicit symmetry breaking. However, we may ask which fraction of the nucleon mass comes from the spontaneous (v_0) and from the explicit (ϵ) symmetry-breaking. To linear order in ϵ we have:

$$v_0 \approx f_\pi - \frac{1}{2}\frac{\epsilon}{\lambda f_\pi^2} \longrightarrow M_N = g_\pi f_\pi = g_\pi \left(v_0 + \frac{\epsilon}{2\lambda f_\pi^2} \right)$$

Hence

$$\delta M_N = g_\pi \frac{\epsilon}{2\lambda f_\pi^2} = g_\pi \frac{f_\pi m_\pi^2}{2\lambda f_\pi^2} = M_N \frac{m_\pi^2}{m_\sigma^2 - m_\pi^2} \approx M_N \frac{m_\pi^2}{m_\sigma^2} \ll M_N$$

Experimentally one gets $\delta M_N \approx 50$ MeV

- only a small fraction of the baryonic mass of our universe is due to the Higgs mechanism (via $m_{u,d} \neq 0$) at the electro-weak phase transistion!
- most of the baryonic mass of our universe got its origin at the chiral QCD transition at $T \approx 150$ MeV

The width of the σ meson

From the $\sigma\pi\pi$ coupling one can calculate the $\Gamma_{\sigma\to\pi\pi}$ decay width:

$$\Gamma_{\sigma\to\pi\pi} = \frac{3}{32\pi m_\sigma}\sqrt{1-\left(\frac{2m_\pi}{m_\sigma}\right)^2}\left(\frac{m_\sigma^2-m_\pi^2}{f_\pi}\right)^2$$

getting $\Gamma_{\sigma\to\pi\pi} \approx 0.3 - 0.6$ GeV, depending on the value of the σ mass

Linear σ-model and PCAC

From the $SU_A(2)$ transformation of the symmetry-breaking term

$$\mathcal{L}_1 = \epsilon\sigma \quad \longrightarrow \quad \mathcal{L}_1 + \delta\mathcal{L}_1 = \epsilon\sigma - \epsilon\,\theta^a\pi^a$$

and from $\partial_\mu j^\mu = \delta\mathcal{L}_1$ one gets:

$$\partial^\mu A^a_\mu = -\epsilon\pi^a = -f_\pi m_\pi^2 \pi^a,$$

consistently with the pion-decay matrix element

$$\langle 0|\partial^\mu A^a_\mu(x)|\pi^b(q)\rangle = -f_\pi q^2 \delta^{ab} e^{-iq\cdot x} = -f_\pi m_\pi^2 \delta^{ab} e^{-iq\cdot x}$$

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Finally, from the link with the QCD Lagrangian

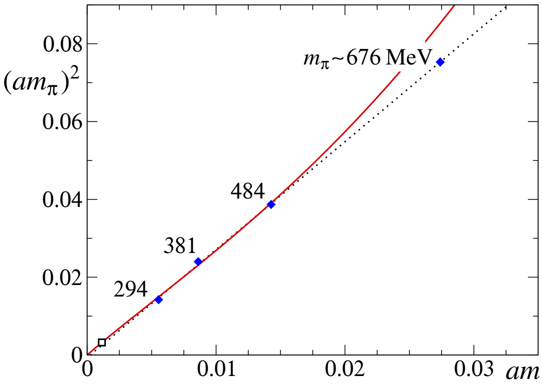
$$\mathcal{L}_1^{\sigma-\mathrm{mod}} = \epsilon\sigma \quad \mathcal{L}_1^{QCD} = -\overline{q}mq$$

one gets ($m_u \approx m_d$) the Gell-Mann-Oakes-Renner (GOR) relation:

$$\epsilon\langle\sigma\rangle = -m\langle\overline{q}q\rangle \quad \longrightarrow \quad f_\pi^2 m_\pi^2 = -\frac{m_u+m_d}{2}\langle\,\overline{u}u+\overline{d}d\,\rangle$$

establishing a link between hadronic and partonic quantities.

GOR relation in lattice-QCD



(M. Lüscher, hep-lat/0509152)

The pion-exchange potential

The one-pion exchange potential (OPE) plays the major role in describing the "long-range" nucleon-nucleon interaction (although not necessarily in ensuring the binding of large nuclei). It corretly accounts for many properties of the deuteron, the lighest nuclear bound state. Its explicit expression can be obtained from the non-relativistic reduction of the NN scattering amplitude obtained from the pseudoscalar interaction of the linear sigma-model (the t-channel is considered):

$$i\mathcal{M} = \overline{u}(1')(g_\pi \gamma^5 \tau^a) u(1) \, \frac{i}{q^2 - m_\pi^2} \, \overline{u}(2')(g_\pi \gamma^5 \tau^a) u(2)$$

In order to take the non-relativistic limit it is convenient to work in the Dirac representation in which

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

The non-relativistic one-pion exchange potential in momentum space will be obtained considering for the nucleons the limit $p \ll M$ and setting the energy transfer $q^0 \to 0$

The pion-exchange potential

The non-relativistic nucleon spinors read

$$u(\vec{p}) = \sqrt{\frac{E+M}{2M}} \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M}\chi \end{pmatrix} \quad \overline{u}(\vec{p}) = \sqrt{\frac{E+M}{2M}} \left(\chi^\dagger, -\chi^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E+M} \right),$$

where $\chi = (1,0)^T$ or $(0,1)^T$ are two-dimensional Pauli spinors. One gets then, for $p \ll M$,

$$g_\pi \overline{u}(\vec{p}_1')\gamma^5 \tau^a u(\vec{p}_1) \approx \frac{g_\pi}{2M}\,\tau^a\,\vec{\sigma}\cdot(\vec{p}_1 - \vec{p}_1') = \frac{g_\pi}{2M}\,\tau^a\,\vec{\sigma}\cdot\vec{q},$$

where the matrix elements of the spin and isospin matrices are assumed to be taken between the final and initial nucleon state. In the static $q^0 \to 0$ limit the pion propagator reduces to

$$\frac{1}{q^2 - m_\pi^2} \underset{q^0\to 0}{\sim} \frac{-1}{\vec{q}^2 + m_\pi^2}$$

One gets then ($V_{1\pi} \equiv -\mathcal{M}_{\mathrm{nr}}$)

$$V_{1\pi}(\vec{q}) = -\frac{g_\pi^2}{4M^2}\,\vec{\tau}_1\cdot\vec{\tau}_2\,\frac{(\vec{\sigma}_1\cdot\vec{q})(\vec{\sigma}_2\cdot\vec{q})}{\vec{q}^2 + m_\pi^2}$$

The pion-exchange potential

It is convenient to decompose the spin structure of the potential into a central and a tensor part according to

$$\frac{(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})}{\vec{q}^2 + m_\pi^2} = \frac{1}{3}\frac{\vec{q}^2}{\vec{q}^2 + m_\pi^2}[\underbrace{3(\vec{\sigma}_1 \cdot \hat{q})(\vec{\sigma}_2 \cdot \hat{q}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2}_{S_{12}(\hat{q})} + \vec{\sigma}_1 \cdot \vec{\sigma}_2].$$

One obtains

$$V_{1\pi}(\vec{q}) = -\frac{1}{3}\frac{g_\pi^2}{4M^2}\left[\left(1 - \frac{m_\pi^2}{\vec{q}^2 + m_\pi^2}\right)\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{\vec{q}^2}{\vec{q}^2 + m_\pi^2} S_{12}(\hat{q})\right]\tau_1 \cdot \vec{\tau}_2$$

In going to coordinate space, for the tensor piece one needs to evaluate a Fourier transform of the kind

$$\int \frac{d\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} f(\vec{q}^2)[3\hat{q}^i\hat{q}^j - \delta^{ij}] = (3\hat{r}^i\hat{r}^j - \delta^{ij})A(r)$$

Contracting the above expression with $\hat{r}^i\hat{r}^j$ one gets

$$A(r) = \frac{1}{2}\int \frac{d\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} f(\vec{q}^2)[3(\hat{q}\cdot\hat{r})(\hat{q}\cdot\hat{r}) - 1]$$

The pion-exchange potential

It is then possible to write the OPE potential in coordinate space

$$V_{1\pi}(\vec{r}) = V_{1\pi}^C(\vec{r}) + V_{1\pi}^T(\vec{r}),$$

where the central piece reads

$$V_{1\pi}^C(\vec{r}) = \frac{g_\pi^2}{4M^2}\frac{1}{3}\left[\frac{m_\pi^2}{4\pi r}e^{-m_\pi r} - \delta(\vec{r})\right](\vec{\sigma}_1\cdot\vec{\sigma}_2)(\vec{\tau}_1\cdot\vec{\tau}_2)$$

and, being $S_{12}(\hat{r}) \equiv [3(\vec{\sigma}_1\cdot\hat{r})(\vec{\sigma}_2\cdot\hat{r}) - \vec{\sigma}_1\cdot\vec{\sigma}_2]$, the tensor one

$$V_{1\pi}^T(\vec{r}) = \frac{g_\pi^2}{4M^2}\frac{1}{3}\left[\frac{m_\pi^2}{4\pi r} + 3\frac{m_\pi}{4\pi r^2} + 3\frac{1}{4\pi r^3}\right]e^{-m_\pi r}S_{12}(\hat{r})(\vec{\tau}_1\cdot\vec{\tau}_2)$$

Whether the potential is attractive or repulsive depends on the spin-isospin state of the nucleon-nucleon pair

- $S=I=0$: $(\vec{\sigma}_1\cdot\vec{\sigma}_2)(\vec{\tau}_1\cdot\vec{\tau}_2) = (-3)(-3) = 9$
- $S=I=1$: $(\vec{\sigma}_1\cdot\vec{\sigma}_2)(\vec{\tau}_1\cdot\vec{\tau}_2) = (1)(1) = 1$
- $S=1$ and $I=0$ (or viceversa): $(\vec{\sigma}_1\cdot\vec{\sigma}_2)(\vec{\tau}_1\cdot\vec{\tau}_2) = (1)(-3) = -3$

The OPE potential and the deuteron

The deuteron is *the only NN bound state*. It occurs in the $S=1$ and $I=0$ channel, in which both the central and the tensor potential provide an attractive contribution. Notice that $[\vec{L}, V^T_{1\pi}] \neq 0$, so that the deuteron wavefunction cannot be an eigenstate of the orbital momentum. The latter is a superposition of a s-wave and a d-wave:

$$|D\rangle = \alpha|L=0\rangle + \beta|L=2\rangle \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1.$$

One finds $|\alpha|^2 \approx 0.96$ and $|\beta|^2 \approx 0.04$, which explain both the *magnetic moment* $\langle\vec{\mu}\rangle \approx 0.857\mu_0$ ($\mu_0 \equiv e\hbar/2Mc$) and the positive *electric quadrupole moment* $\langle Q\rangle \equiv e\langle 3z^2 - r^2\rangle/4 \approx 0.286\, e{\cdot}\mathrm{fm}^2$ of the deuteron. Notice that for a pure s-wave, due to spherical symmetry, one would have $\langle Q\rangle_s = 0$. The tensor interaction favours a charge distribution aligned along the spin axis and explains the *positive* quadrupole moment, i.e. $\langle z^2\rangle > \langle x^2\rangle, \langle y^2\rangle$. Take the spin configuration $|+,+\rangle$; one has:

$$\langle V^T_{1\pi}(r\hat{u}_z)\rangle \sim -3\langle +,+|3\sigma^z_1\sigma^z_2 - \vec{\sigma}_1\cdot\vec{\sigma}_2|+,+\rangle = -3(3-1) = -6$$
$$\langle V^T_{1\pi}(r\hat{u}_x)\rangle \sim -3\langle +,+|3\sigma^x_1\sigma^x_2 - \vec{\sigma}_1\cdot\vec{\sigma}_2|+,+\rangle = -3(-1) = 3$$

The OPE potential and the virtual NN levels

The OPE potential is attractive also in the $S=0$ and $I=1$ channel. However, in this case, the attraction is weaker since the tensor potential vanishes acting on a spin-singlet state $|S=0\rangle = 1/\sqrt{2}(|+,-\rangle - |-,+\rangle)$. Taking $\hat{r}$ along the z-axis one has

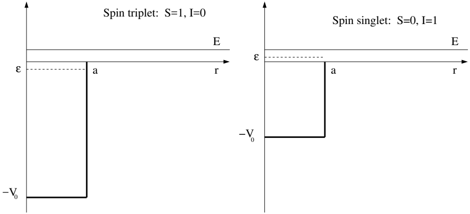
$$(3\sigma_1^z\sigma_2^z - \vec{\sigma}_1 \cdot \vec{\sigma}_2)|S=0\rangle = -3-(-3) = 0.$$

The attractive interaction is not sufficient to support a bound state (i.e. with $E<0$). However, low-energy experiments display the existence of a virtual positive-energy level: the wave-function of $E \to 0^+$ scattering states is large close to the origin. Due to *isospin symmetry* such a state should be present also in the p-p case. Such a resonant "diproton" state is of strong importance in the fusion reactions occurring in the stars. Two protons can cross the Coulomb repulsive barrier through tunneling and come close to each other so that the attractive nuclear interaction can lead to the formation of a resonant diproton state, which would decay back very soon into a p-p pair. However, in a *very* small faction of cases, the *resonant* "diproton" can decay *weakly* into a *bound* deuteron while the two protons are close to each other

$$\mathrm{p}+\mathrm{p} \to {}^2\mathrm{He}^* \to {}^2\mathrm{H} + e^+ + \nu_e$$

NB: *if the diproton were bound*, the weak decay would occur each time the protons cross the Coulomb barrier and the stellar evolution would be different!

NN scattering: triplet vs singlet



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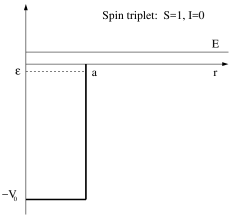

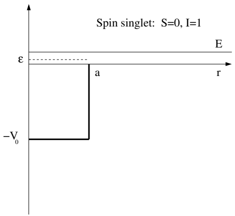

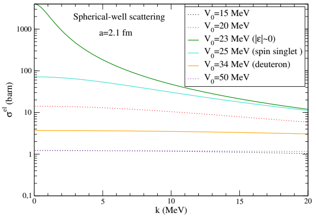

For $k \equiv \sqrt{2\mu E/\hbar^2} \to 0^+$ (i.e. $ka \ll 1$) only the $l=0$ term of the partial-wave expansion of the scattering amplitude matters

$$\sigma = 4\pi \sum_{l=0}^{\infty} (2l+1)|a_l(k)|^2 \quad \text{with} \quad a_l(k) = \frac{1}{k\cot\delta_l(k) - ik},$$

parametrized in term of a *scattering length* α and an *effective range* r_0

$$k\cot\delta_0(k) \approx -\frac{1}{\alpha} + \frac{1}{2} r_0 k^2 + \ldots \quad \longrightarrow \quad \sigma \underset{k\to 0}{\sim} 4\pi\alpha^2$$

One gets $\alpha_{S=1} \approx 5.35$ fm and $\alpha_{s=0} \approx -23.55$ fm, much larger than the range of the potential due to $|\epsilon| \ll V_0$ ($\alpha^{-1} \underset{\epsilon\to 0}{\sim} \sqrt{2\mu|\epsilon|/\hbar^2}$), leading to the spin-averaged cross-section $\sigma^{\rm el} = 0.25\sigma^{\rm el}_{S=0} + 0.75\sigma^{\rm el}_{S=1} \approx 20$ barn

If the di-proton were bound...

What would happen if the di-proton were bound? It would be sufficient that the potential well were just a bit deeper (i.e. stronger coupling) or larger (i.e. lower pion mass). The consequences were analyzed in a wonderful paper by Dyson

ENERGY IN THE UNIVERSE

The energy flows on the earth are embedded in the energy flows in the universe. A delicate balance among gravitation, nuclear reactions and radiation keep the energy from flowing too fast

by Freeman J. Dyson

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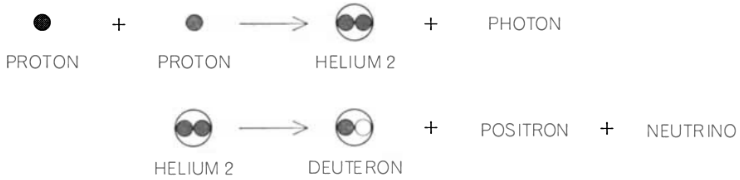


If the di-proton were bound each time two protons cross the Coulomb barrier they would form a $^2\mathrm{He}$ nucleus, emitting a photon. The $^2\mathrm{He}$ would then decay weakly into deuteron. There would be no deuteron bottleneck in Big Bang and stellar nucleosynthesis. Stars like the Sun would end very soon their fuel, because nuclear reactions would occur very fast.

$O(4)$ model at finite temperature: χ-symmetry restoration

We now wish to study the temperature evolution of the *order parameter* associated to chiral-symmetry breaking, i.e.

$$\langle\sigma\rangle_T \text{ (hadron level)} \quad \text{or} \quad \langle\overline{q}q\rangle_T \text{ (quark level)}$$

It is convenient to consider $\mathcal{L}_{\sigma-\mathrm{mod}}$ as the $N\!=\!4$ case of the $O(N)$ model, performing the calculations in the $N \to \infty$ limit. Hence:

$$V = \frac{\lambda}{4}\left[(\phi^i)^2 - f_\pi^2\right]^2, \quad \text{with} \quad \phi^i \equiv (\vec{\pi}, \sigma)$$

At finite temperature we can consider the thermal expectation value

$$\left\langle \frac{\partial V}{\partial \phi^j} \right\rangle_T = 0 \quad \longrightarrow \quad \langle(\phi^i)^2\phi^j\rangle_T - f_\pi^2\langle\phi^j\rangle_T = 0$$

We can split the fields as follows:

$$\phi^i \equiv \langle\phi^i\rangle_T + \eta^i, \quad \text{where} \quad \langle\vec{\pi}\rangle_T = 0,\ \langle\sigma\rangle_T \equiv \sigma_T$$

One gets ($\langle\eta^i\rangle_T = \langle\eta^i\eta^j\eta^k\rangle_T = 0$):

$$\left(\langle\phi^i\rangle_T\right)^2\langle\phi^j\rangle_T + \underbrace{\langle\eta^i\eta^i\rangle_T}_{\mathcal{O}(N)}\langle\phi^j\rangle_T + 2\underbrace{\langle\eta^i\eta^j\rangle_T}_{\mathcal{O}(1)}\langle\phi^i\rangle_T - f_\pi^2\langle\phi^j\rangle_T = 0$$

It is now convenient to perform the large-N approximation ($N=4$ corresponding to the physical case, with 3 pions and one σ-mesons)

$$\left(\langle\phi^i\rangle_T\right)^2\langle\phi^j\rangle_T + \langle\eta^i\eta^i\rangle_T\langle\phi^j\rangle_T - f_\pi^2\langle\phi^j\rangle_T = 0$$

We focus on the equation for the $\langle\phi^N\rangle_T \equiv \langle\sigma\rangle_T \equiv \sigma_T$. One has:

$$\sigma_T^2 = f_\pi^2 - \langle\eta^i\eta^i\rangle_T \quad \text{with} \quad \langle\eta^i\eta^i\rangle_T = \sum_i \int \frac{d\vec{k}}{(2\pi)^3}\frac{N_k}{\epsilon_k^i} \underset{m_i\to 0}{\sim} \sum_i \frac{T^2}{12}$$

In the above N_k is a Bose distribution and $\epsilon_k^i = \sqrt{k^2+m_i^2}$.
We can study two different limits:

- $T\to 0$: $N-1$ massless pions; σ-meson ($m_\sigma \gg T$) can be neglected

$$\sigma_T^2 = f_\pi^2\left(1-(N-1)\frac{T^2}{12f_\pi^2}\right) \underset{N\to 4}{\sim} f_\pi^2\left(1-\frac{T^2}{4f_\pi^2}\right)$$

- $T\to T_c$: pions and σ-meson ($m_\sigma \gg T$) become degenerate and massless

$$\sigma_T^2 = f_\pi^2\left(1-N\frac{T^2}{12f_\pi^2}\right) \underset{N\to 4}{\sim} f_\pi^2\left(1-\frac{T^2}{3f_\pi^2}\right),$$

from which one gets the estimate $T_c \approx \sqrt{3}f_\pi \approx 160$ MeV.

The Gross-Neveu model

A nice model of dynamical breaking of chiral symmetry is given by the Gross-Neveu Lagrangian

$$\mathcal{L}_{\rm GN} = \overline{\psi} i\partial\!\!\!/ \psi + \frac{g_0}{2N}(\overline{\psi}_a \psi_a)^2 \qquad a = 1...N,$$

where a label some internal degree of freedom. The model is well suited to perform a $1/N$ expansion and is usually studied in 2 dimensions, where it is renormalizable ($[\psi] = L^{-1/2}$) and analytic results can be obtained. In the 2-dimensional case the Lagrangian has the discrete chiral symmetry

$$\psi \to \gamma^5 \psi, \quad \overline{\psi} \to -\overline{\psi}\gamma^5 \quad \text{with} \quad \gamma^5 \equiv \gamma^0\gamma^1,$$

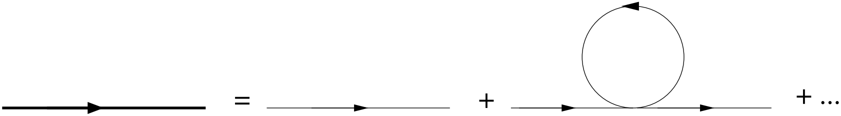
which follows from the usual properties of γ^5

$$(\gamma^5)^\dagger = \gamma^5, \quad (\gamma^5)^2 = 1, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

Under this discrete chiral symmetry $\overline{\psi}\psi \to -\overline{\psi}\psi$, so that the *chiral condensate* $\langle\overline{\psi}\psi\rangle$ can be used as an order parameter for the phase transition.

The quark self-energy and propagator

The perturbative expansion of the quark propagator $iS(p)^{-1} \equiv \not{p} - \Sigma$ or, in coordinate space,
$S(x-y) \equiv S_F(x-y) + \int_{zz'} S_F(x-z)(-i\Sigma(z,z'))S(x-y)$



$$S^{ij}(x-y) = S_F^{ij}(x-y) + i\frac{g_0}{2N}\int_z \langle T\psi^i(x)\overline{\psi}^j(y)\overline{\psi}^a(z)\psi^a(z)\overline{\psi}^b(z)\psi^b(z)\rangle + \ldots$$

allows one to identify the lowest order contribution to the self-energy

$$\Sigma^{\rm pert} = \Sigma^{\rm dir} + \Sigma^{\rm exch} = \frac{g_0}{2N} 2\left[{\rm Tr} S_F(z,z) - S_F(z,z)\right],$$

where the exchange term ($\sim \delta^{ij}$) is subleading by a factor $1/N$ wrt to the direct one ($\sim \delta^{ij}\delta^{bb}$) and can be dropped in the large-N limit. Notice that $\Sigma^{\rm pert}$ turns out to be momentum-independent (contact interaction!) and that, for massless quark, it vanishes (${\rm Tr}\not{p} = 0$).

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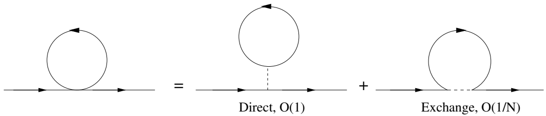

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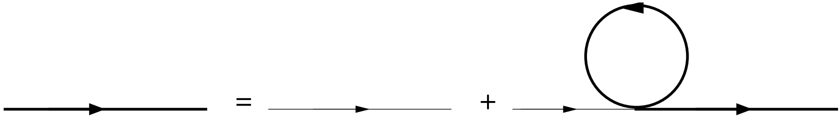
The self-consistent Hartree approximation

Even starting from massless quarks one can obtain a non-trivial solution by adopting a self-consistent approach (Hartree approximation)



$$\Sigma_H \equiv \frac{g_0}{2N} 2\,\mathrm{Tr} S_H(z,z) \quad \longrightarrow \quad m^* = \frac{g_0}{2N} 2(2N) \int \frac{d^2p}{(2\pi)^2} \frac{i\, m^*}{p^2 - m^{*2} + i\epsilon}$$

A self-consistent solution for the dynamically-generated quark mass arises from the *gap-equation* (after Wick-rotating $p^0 \equiv i p_E^0$)

$$1 = 2g_0 \int \frac{d^2 p_E}{(2\pi)^2} \frac{1}{p_E^2 + m^{*2}}$$

The latter looks similar to the gap-equation of the BCS theory of superconductivity (energies ξ measured wrt to the Fermi surface):

$$1 = \frac{V_0}{2} \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \mathcal{N}(\xi) \frac{1}{\sqrt{\xi^2 + \Delta^2}}$$

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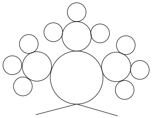

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$$1 = 2g_0 \int \frac{d^2p_E}{(2\pi)^2} \frac{1}{p_E^2 + m^{*2}}$$

The latter looks similar to the gap-equation of the BCS theory of superconductivity (energies ξ measured wrt to the Fermi surface):

$$1 = \frac{V_0}{2} \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \mathcal{N}(\xi) \frac{1}{\sqrt{\xi^2 + \Delta^2}}$$

The self-consistent Hartree approximation

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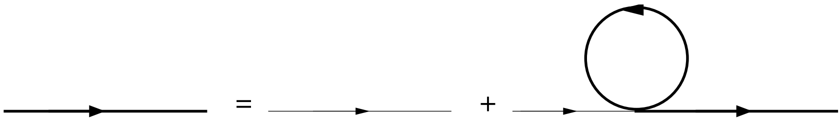

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Notice that the dynamical generation an effective quark mass is associated to the spontaneous breaking of chiral symmetry:

$$\langle \overline{\psi}\psi \rangle = -\mathrm{Tr} S(z, z^+) = -\frac{N}{g_0} m^*$$

Renormalization and running coupling

The integral involved in the gap-equation is UV divergent and needs to be *regularized* by a cutoff Λ

$$\frac{1}{2g_0} = \int^{\Lambda} \frac{d^2 p_E}{(2\pi)^2} \frac{1}{p_E^2 + m^{*2}} \quad \longrightarrow \quad \frac{1}{g_0} = \frac{1}{2\pi} \ln\left(1 + \frac{\Lambda^2}{m^{*2}}\right) \approx \frac{1}{2\pi} \ln \frac{\Lambda^2}{m^{*2}}$$

Let us now introduced the renormalized coupling at the scale M

$$\frac{1}{g(M)} = \frac{1}{g_0} - \frac{1}{2\pi} \ln \frac{\Lambda^2}{M^2} + \frac{1}{\pi}$$

Substituting the result for g_0 the dependence on Λ cancels and one gets

$$\frac{1}{g(M)} = \frac{1}{2\pi} \ln \frac{M^2}{m^{*2}} + \frac{1}{\pi} \quad \longrightarrow \quad m^{*2} = M^2 \exp\left(2 - \frac{2\pi}{g(M)}\right)$$

Notice the non-analytic dependence on the strength of the interaction, similar to the one in the BCS theory of superconductivity

$$\Delta \approx 2\hbar\omega_c \exp\left(-\frac{1}{\mathcal{N} V_0}\right)$$

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The coupling at the scale M' is related to the one at the scale M by

$$\frac{1}{g(M')} = \frac{1}{g(M)} + \frac{1}{2\pi} \ln \frac{M'^2}{M^2}$$

The theory is *asymptotically free*!

The effective potential (I)

A convenient method to study the Gross-Neveu model in the large-N limit is to introduce the *auxiliary field* σ

$$\mathcal{L} \to \mathcal{L}_\sigma \equiv \mathcal{L} - \frac{N}{2g_0}\left(\sigma - \frac{g_0}{N}\overline{\psi}^a\psi^a\right)^2$$

Actually σ is not a dynamical field (there is no kinetic term) and from its equation of motion $\sigma = (g_0/N)\overline{\psi}\psi$ one can see that the modified Lagrangian in equivalent to the original one. However $\mathcal{L}_\sigma$ make the study of the large-N limit easier:

$$\mathcal{L}_\sigma = \overline{\psi}^a i\partial\!\!\!/\psi^a + \sigma\overline{\psi}^a\psi^a - \frac{N}{2g_0}\sigma^2$$

In particular the associated partition function can be evaluated exactly:

$$Z_\sigma = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left\{i\int d^2x\left[\overline{\psi}(i\partial\!\!\!/ + \sigma)\psi - \frac{N}{2g_0}\sigma^2\right]\right\}$$

$$= e^{-i(N/2g_0)\int d^2x\sigma^2}\det(i\partial\!\!\!/ + \sigma) = Z_0 e^{-i(N/2g_0)\int d^2x\sigma^2}\det\left(1 + \frac{\sigma}{i\partial\!\!\!/}\right)^N$$

The effective potential (II)

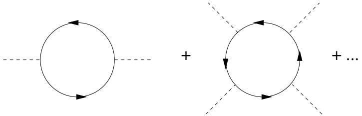
We now consider σ constant and look for its vacuum expectation-value from minimizing the effective potential $V_{\rm eff}(\sigma)$

$$Z_\sigma/Z_0 \equiv e^{-iV_{\rm eff}LT} = e^{-i(N/2g_0)\sigma^2 LT}\,\exp\left[N\,\mathrm{Tr}\,\ln\left(1+\frac{\sigma}{i\not{\partial}}\right)\right]$$

Expanding the log and taking into account that the trace of an odd number of Dirac matrices vanishes one gets:

$$V_{\rm eff}(\sigma) = \frac{N}{2g_0}\sigma^2 - iN\sum_{n=1}^{\infty}\frac{1}{2n}\mathrm{Tr}\int\frac{d^2p}{(2\pi)^2}\left(\frac{\not{p}\sigma}{p^2+i\epsilon}\right)^{2n}$$

which has the following interpretation in terms of Feynman diagrams



The effective potential (III)

From $\not{p}^2 = p^2$ and $\mathrm{Tr}(\mathbb{I})=2$ one can resum the series

$$V_{\rm eff}(\sigma) = N\left[\frac{\sigma^2}{2g_0} + i\int\frac{d^2p}{(2\pi)^2}\ln\left(1-\frac{\sigma^2}{p^2}\right)\right]$$

After Wick-rotating $p^0 \equiv ip_E^0$ one gets

$$V_{\rm eff}(\sigma) = N\left[\frac{\sigma^2}{2g_0} - \int\frac{d^2p_E}{(2\pi)^2}\ln\left(1+\frac{\sigma^2}{p_E^2}\right)\right]$$

Looking for the minimum of the effective potential one revovers the gap-equation

$$\left.\frac{dV_{\rm eff}}{d\sigma}\right|_{\sigma_0} = 0 \quad \longleftrightarrow \quad \frac{1}{2g_0} = \int\frac{d^2p_E}{(2\pi)^2}\frac{1}{p_E^2+\sigma_0^2}$$

previously obtained via a self-consistent solution of the Dyson equation (Hartree approximation) for the quark propagator.

The Nambu Jona-Lasinio model

A more realistic low-energy model of QCD, expressed in terms of quark degrees of freedom, is the Nambu Jona-Lasinio model. Its Lagrangian is very similar to the Gross-Neveu one, but it is written in 4 space-time dimensions. Hence the model is *non-renormalizable* and requires the introduction of an UV cutoff, which will be one of the few parameters of the model. The simplest version of its Lagrangian is

$$\mathcal{L}_{\mathrm{NJL}} = \overline{q}(i\partial\!\!\!/ - m_0)q + G[(\overline{q}q)^2 + (\overline{q}i\gamma^5\vec{\tau}q)^2],$$

where τ^i ($i = 1...N_f^2 - 1$) are Pauli/Gell-Mann matrices in flavor space.

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where τ^i $(i = 1...N_f^2 - 1)$ are Pauli/Gell-Mann matrices in flavor space. The above Lagrangian, in (Hartree) mean-field approximation, immediately leads to the development of an effective consituent mass of the quarks:

$$\overline{q}q = \langle\overline{q}q\rangle + \delta_{\overline{q}q} \quad \longrightarrow \quad (\overline{q}q)^2 = \langle\overline{q}q\rangle^2 + 2\langle\overline{q}q\rangle\delta_{\overline{q}q} + ... = 2\langle\overline{q}q\rangle\overline{q}q - \langle\overline{q}q\rangle^2 + ...$$

entailing

$$\mathcal{L}_{\rm NJL}^{\rm MF} = \overline{q}\,[i\partial\!\!\!/ - (m_0 - 2G\langle\overline{q}q\rangle)]\,q - G\langle\overline{q}q\rangle^2,$$

allowing one to identify the effective mass

$$M^\star = m_0 - 2G\langle\overline{q}q\rangle$$

The gap-equation

The mean-field approximation leads to a self-consistent gap-equation for the effective mass of the constituent quarks, which will be found to be around one third of the proton mass:

$$M^\star = m_0 - 2G\langle\overline{q}q\rangle = m_0 + 2G\,\mathrm{Tr}S(z,z^+)$$
$$= m_0 + 2G\int\frac{d^4p}{(2\pi)^4}e^{ip^0\eta}\,\mathrm{Tr}S(p)$$

One gets the self-consistent equation

$$M^\star - m_0 = 4N_cN_fG\int^\Lambda\frac{d\vec{p}}{(2\pi)^3}\frac{M^\star}{\sqrt{\vec{p}^2+M^{\star 2}}}$$

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The development of an effective mass of the consituent quarks corresponds to the appearance a non-vanishing expectation value of the chiral condensate

$$\langle\overline{q}q\rangle = -\frac{M^\star - m_0}{2G} \neq 0$$

and hence to the dynamical breaking of chiral symmetry, since

$$\langle\overline{q}q\rangle = \langle\overline{q}_Rq_L + \overline{q}_Lq_R\rangle$$

The quark propagator and self-energy

The perturbative expansion of the quark propagator $iS(p)^{-1} \equiv \not{p} - m_0 - \Sigma$ or, in coordinate space,

$$S(x-y) \equiv S^0(x-y) + \int_{zz'} S^0(x-z)(-i\Sigma(z,z'))S(x-y)$$

$$S_{ij}(x-y) = S^0_{ij}(x-y) + iG\int_z \langle T q_i(x)\overline{q}_j(y)\overline{q}_a(z)\Gamma^M_{ab}q_b(z)\overline{q}_c(z)\Gamma^M_{cd}q_d(z)\rangle + \ldots$$

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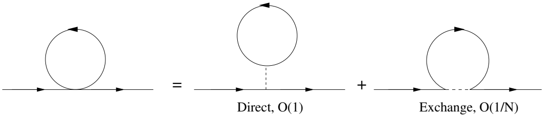

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allows one to identify the lowest order contribution to the self-energy

$$\Sigma^{\rm pert} = \Sigma^{\rm dir} + \Sigma^{\rm exch} = 2G\ [N_c N_f + 1/2]\ {\rm Tr_D} S^0(z, z^+),$$

$$= 2G[N_c N_f + 1/2] \int \frac{d^4p}{(2\pi)^4} e^{ip^0\eta} {\rm Tr_D} S^0(p)$$

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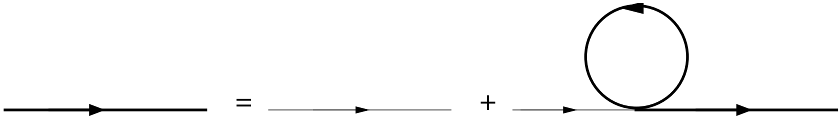
Apparently, starting from massless quarks, within a perturbative expansion one gets a vanishing self-energy correction to the propagator

The self-consistent Hartree approximation

Even starting from massless quarks one can obtain a non-trivial solution by adopting a self-consistent approach (Hartree approximation)



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which is the same result found performing the mean-field approximation. In the massless limit it looks similar to the gap-equation of the BCS theory of superconductivity (energies ξ measured wrt to the Fermi surface):

$$1 = \frac{V_0}{2} \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \mathcal{N}(\xi) \frac{1}{\sqrt{\xi^2 + \Delta^2}}$$

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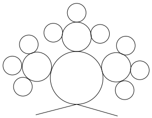

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The NJL model at finite temperature and density

From the mean-field NJL Lagrangian one derive the thermodynamic potential ($E_p^* \equiv \sqrt{\vec{p}^2 + M^{\star 2}}$)

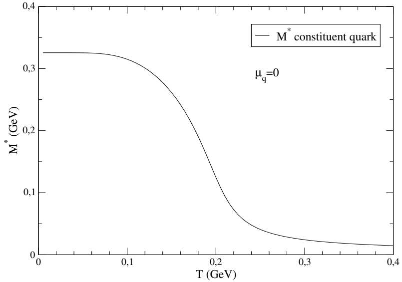
$$\Omega/V = \frac{(M^\star - m_0)^2}{4G} - 2N_c N_f \left\{ \int^\Lambda \frac{d\vec{p}}{(2\pi)^3} E_p^* + \frac{1}{\beta} \int^\Lambda \frac{d\vec{p}}{(2\pi)^3} \left[\ln(1 + e^{-\beta(E_p^* - \mu)}) + \ln(1 + e^{-\beta(E_p^* + \mu)}) \right] \right\}$$

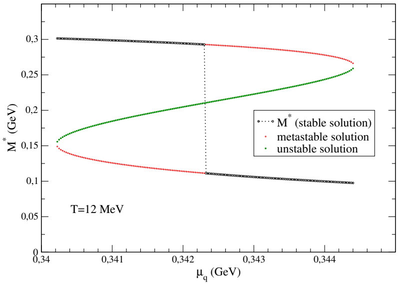
The in-medium gap-equation is obtained minimizing the thermodynamic potential, $\partial\Omega/\partial M^\star = 0$:

$$\boxed{M^\star = m_0 + 4N_c N_f G \int^\Lambda \frac{d\vec{p}}{(2\pi)^3} \frac{M^\star}{E_p^*} \left[1 - f(E_p^* - \mu) - f(E_p^* + \mu) \right]}$$

One can then explore the phase-diagram of the model

Numerical results: gap equation





- For high-T and low-μ_q the transition is a smooth crossover;
- For low-T and high-μ_q the transition becomes a first-order one, with a discontinuity in the effective mass (order parameter) and the appearance of *metastable* (local minimum of Ω) and *unstable* (maximum of Ω) solutions. Notice that at $T=0$ there is a minimum value for the chemical potential $E_F^{\min} = M^*_{\rm vac}$ to have a medium with non-zero quark density.

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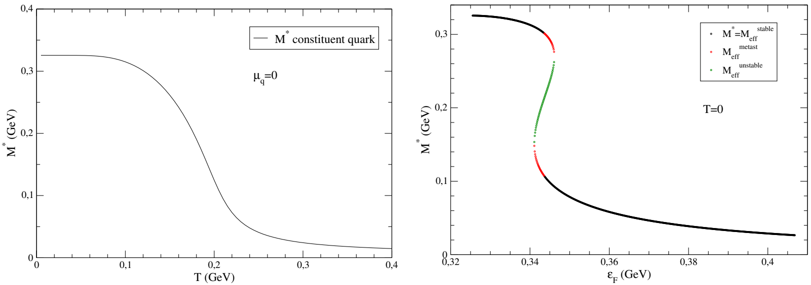

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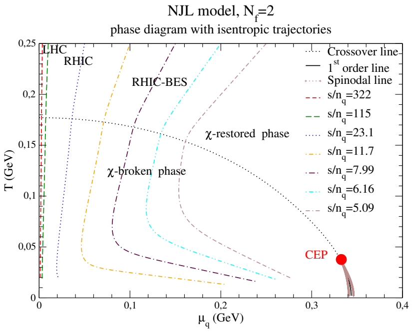
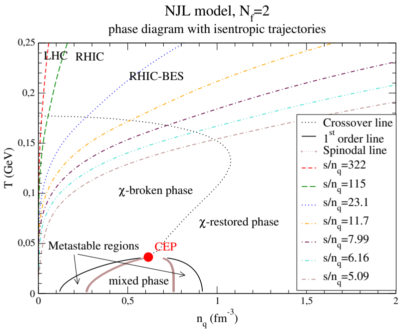
Numerical results: phase diagram

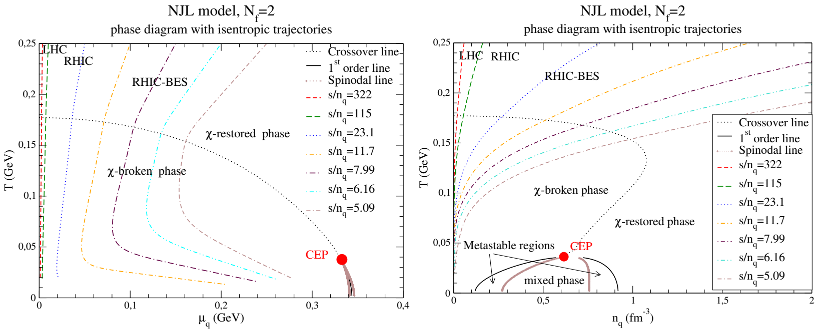




One can draw the phase diagram in the $\mu_q - T$ or $n_q - T$ plane, along with the isentropic trajectories s/n_q=const followed by the matter in heavy-ion collisions. Notice that when the transition becomes of first order

- There is the possibility of having metastability;

Numerical results: phase diagram



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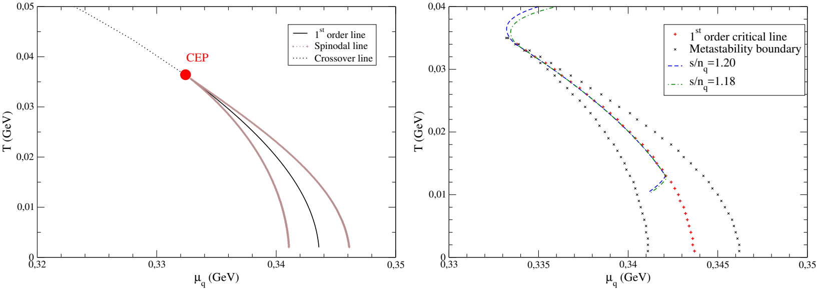

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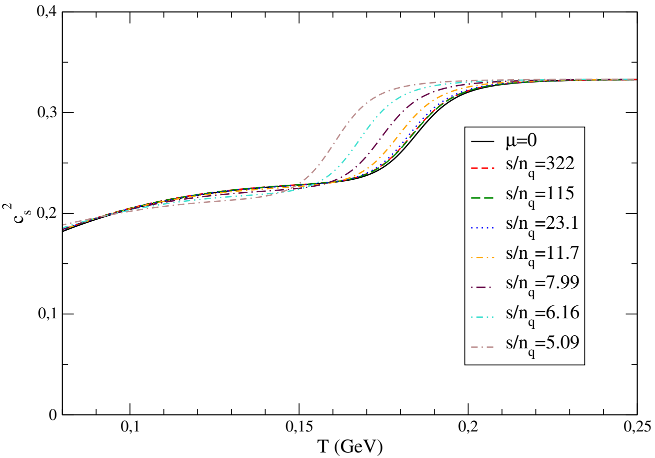
Numerical results: the speed of sound



- Approaching the relativistic limit $c_s^2 = 1/3$ at high temperature
- Rapid drop around the transition

Fluctuations of conserved charges: theory setup (I)

The moments $\langle m^n \rangle$ of a PDF $P(m)$ (with $\sum_m P(m) = 1$) are defined as

$$\langle m^n \rangle \equiv \sum_m m^n P(m)$$

and can be derived from the generating function

$$G(\theta) \equiv \sum_m e^{m\theta} P(m) \quad \longrightarrow \quad \langle m^n \rangle = \left. \frac{d^n}{d\theta^n} G(\theta) \right|_{\theta=0}$$

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It may be more convenient to characterize a PDF through its cumulants $\langle m^n \rangle_c$, derived from the generating function

$$K(\theta) = \ln G(\theta) \quad \longrightarrow \quad \langle m^n \rangle_c = \left. \frac{d^n}{d\theta^n} K(\theta) \right|_{\theta=0}$$

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Defining $\delta m \equiv m - \langle m \rangle$ one gets

$$\langle m \rangle_c = \langle m \rangle \,, \quad \langle m^2 \rangle_c = \langle \delta m^2 \rangle \,, \quad \langle m^3 \rangle_c = \langle \delta m^3 \rangle \,, \quad \langle m^4 \rangle_c = \langle \delta m^4 \rangle - 3 \langle \delta m^2 \rangle^2$$

One can get information on how broad, how asymmetric and how different from a Gaussian a PDF is

Fluctuations of conserved charges: theory setup (II)

In statistical mechanics the logarithm of the grand-canonical partition function

$$Z \equiv \mathrm{Tr} e^{-\beta(\hat{H}-\mu\hat{N})} \equiv e^{-\beta\Omega}$$

plays the role of cumulant generating function. One has ($\tilde{\mu} \equiv \mu/T$)

$$\frac{\partial \ln Z}{\partial \tilde{\mu}} = \frac{\mathrm{Tr}(\,\hat{\mathrm{N}}\,\mathrm{e}^{-\beta(\hat{\mathrm{H}}-\mu\hat{\mathrm{N}})})}{Z} = \langle \hat{N} \rangle = \langle \hat{N} \rangle_c$$

$$\frac{\partial^2 \ln Z}{\partial \tilde{\mu}^2} = \frac{\mathrm{Tr}(\,\hat{\mathrm{N}}^2\,\mathrm{e}^{-\beta(\hat{\mathrm{H}}-\mu\hat{\mathrm{N}})})}{Z} - \left(\frac{\mathrm{Tr}(\,\hat{\mathrm{N}}\,\mathrm{e}^{-\beta(\hat{\mathrm{H}}-\mu\hat{\mathrm{N}})})}{Z}\right)^2 = \langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 = \langle \hat{N}^2 \rangle_c$$

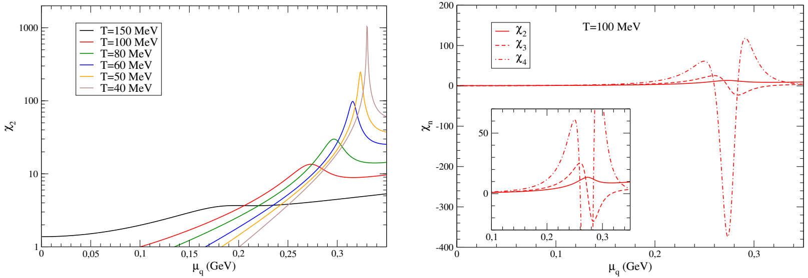
From $\Omega = -\ln Z/\beta = -PV$ one gets the general result

$$\langle \hat{N}^n \rangle_c = \frac{\partial^n (P/T)}{\partial (\mu/T)^n} \cdot V \equiv \tilde{\chi}_n \cdot V$$

It is useful to introduce the dimensionless generalized susceptibility

$$\boxed{\chi_n = \frac{\partial^n (P/T^4)}{\partial (\mu/T)^n}}$$

Fluctuations of conserved charges: numerical results



- All susceptibilities display peaks/oscillations around the critical temperature;
- Susceptibilities gets more and more peaked for higher-order susceptibilities and as one approaches the CEP.

In principle the susceptibilities χ_n are directly connected to the fluctuations of conserved charges (B, S, Q) measured by the detectors

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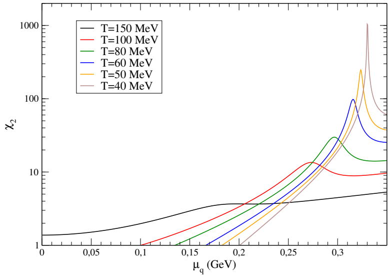

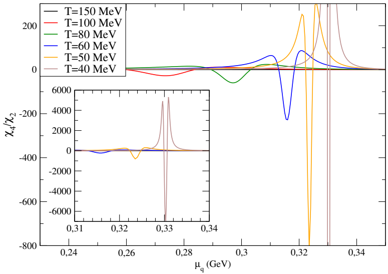

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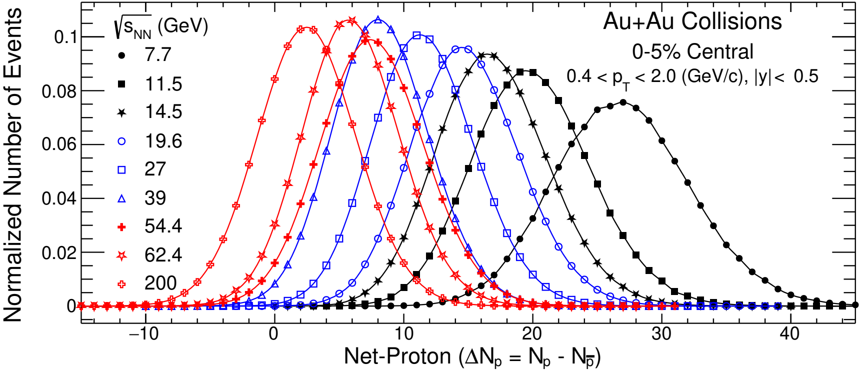

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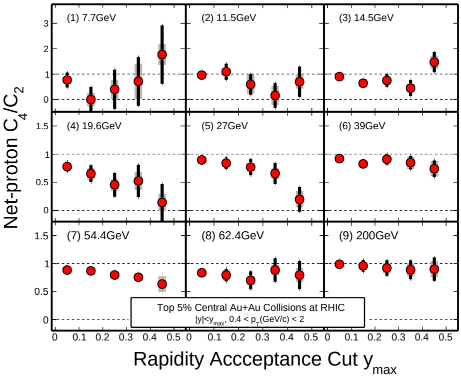
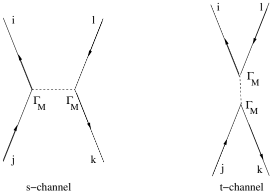


Resonant $q\overline{q}$ scattering and the mesons

Quark-antiquark scattering, at the tree-level described by the diagrams



close to the meson mass is actually dominated by the resonant scattering in the s-channel obtained resumming all the $q\overline{q}$ loops:

$$i\mathcal{T}_M(q) = 2iG\Gamma_M \frac{1}{1-2G\Pi_M(q)}\Gamma_M \equiv \Gamma_M \, it_M(q)\Gamma_M$$

with

$$t_M(q) = \frac{2G}{1-2G\Pi_M(q)}$$

from which one gets the meson mass (in the various channels):

$$\boxed{1 - 2G\mathrm{Re}\Pi_M(m_M, \vec{q}=0) = 0}$$

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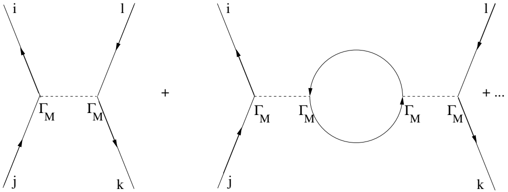

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In-medium pion spectral function

The in-medium pion spectral function can be obtained starting from the polarization propagator with two PS vertices (z is a complex energy):

$$\Pi_{PP}(z,\vec{q}=0) = -4N_c \int \frac{d\vec{k}}{(2\pi)^3}[1-2n_F(\epsilon_k)]\left[\frac{1}{z-2E_k^*} - \frac{1}{z+2E_k^*}\right] .$$

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$$\rho_\pi(\omega) \equiv 2\mathrm{Im}t_\pi(\omega+i\eta) = 2\pi\,\frac{1}{\pi}\frac{2G\mathrm{Im}\Pi^R_{PP}(\omega)}{(1-2G\mathrm{Re}\Pi^R_{PP}(\omega))^2+(2G\mathrm{Im}\Pi^R_{PP}(\omega))^2}$$

When does $\Pi^R(\omega)$ develop and imaginary part?

$$\mathrm{Im}\Pi^R_{PP}(\omega) = 4N_c \int \frac{d\vec{k}}{(2\pi)^3}[1-2n_F(\epsilon_k)]\left[\pi\delta(\omega-2E_k^*)-\pi\delta(\omega+2E_k^*)\right]$$

$$\mathrm{Im}\Pi^R(\omega)\neq 0 \quad \Longleftrightarrow \quad |\omega|>2M^*$$

In-medium pion spectral function

What about the behaviour around the pion mass previously defined? It depends whether

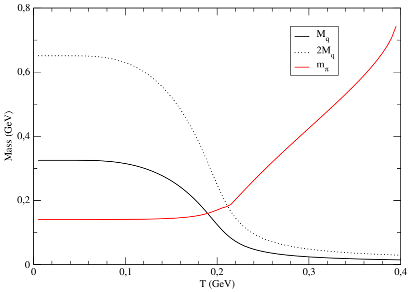
- $\omega = m_\pi < 2M^*$: in this case $\mathrm{Im}\Pi^R_{PP}(\omega) = 0$ and the spectral function reduces to a Dirac delta

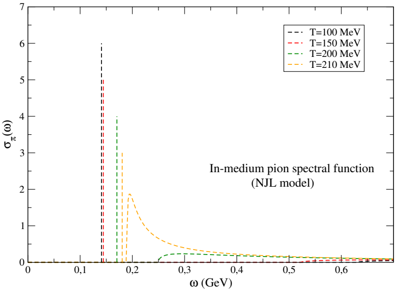
$$\rho_\pi(\omega) = 2\pi\delta[1 - 2G\mathrm{Re}\Pi^R_{PP}(\omega)] = 2\pi\delta\left[-2G\left.\frac{\partial\mathrm{Re}\Pi^R_{PP}}{\partial\omega}\right|_{m_\pi}(\omega - m_\pi)\right]$$

$$= \frac{2\pi}{2G\left|\frac{\partial\mathrm{Re}\Pi^R_{PP}}{\partial\omega}\right|}\delta(\omega - m_\pi)$$

- $\omega = m_\pi > 2M^*$: in this case $\mathrm{Im}\Pi^R_{PP}(\omega) \neq 0$, there is phase-space available for the $\pi \to q\bar{q}$ decay and the spectral function displays a broad peak embedded in the $q\bar{q}$ continuum around the pion mass

Numerical results: pion mass and spectral function





- The pion appears as a zero-width pole of the T-matrix $t_\pi(q)$ up to the Mott temperature T_M, defined as $m_\pi(T_M) = 2M_q^\star(T_M)$, where the decay channel into two dressed quarks opens up;
- For $T > T_M$ the pion remains a well defined collective excitation, identified by a sharp peark in the spectral function $2\mathrm{Im}t_\pi(\omega + i\eta, \vec{q} = 0)$ in the pseudoscalar channel, getting broader as the temperature increases.

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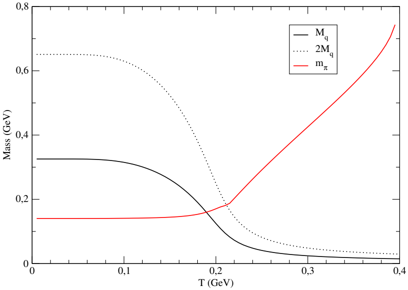

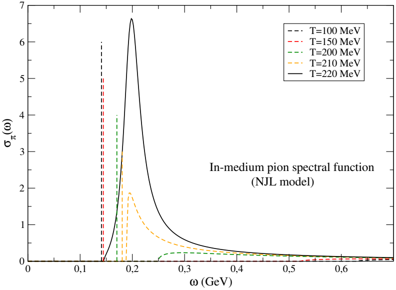

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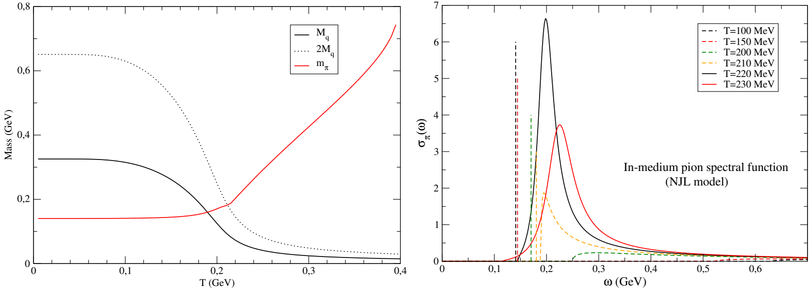

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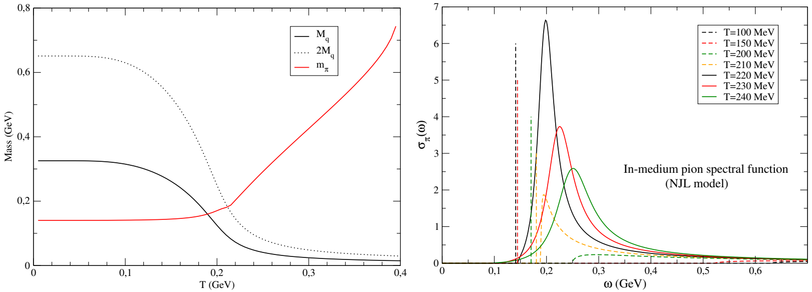

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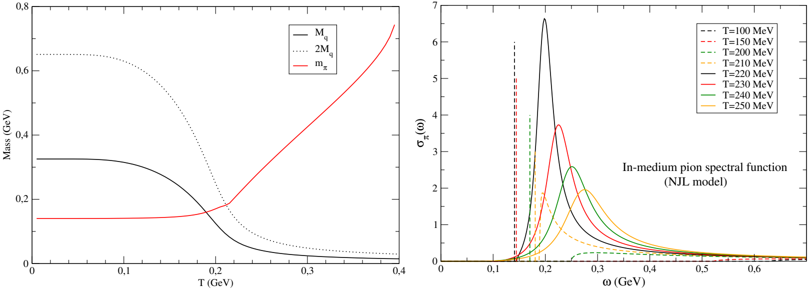

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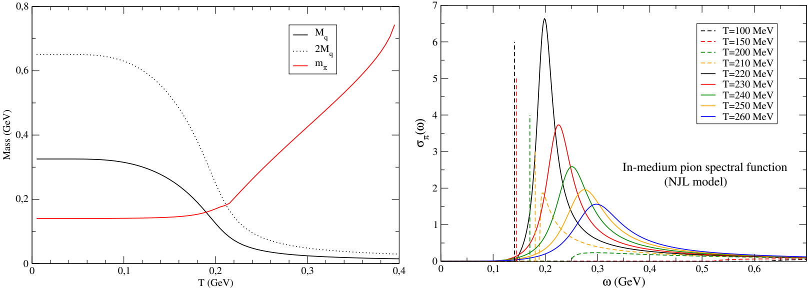

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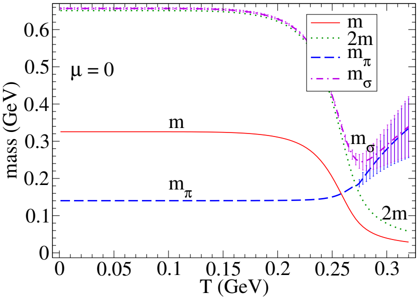
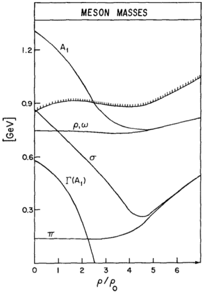
χ-symmetry restoration: NJL results





NJL-like models above a critical temperature/density leads to the degeneracy of chiral partners:

- Scalar and Pseudoscalar mesons (H. Hansen et al., Phys.Rev.D 75 (2007) 065004);
- Vector and Axial-Vector mesons, including in $\mathcal{L}_{\rm NJL}$ the vector coupling (V. Bernard and U.G. Meissner, NPA 489 (1988) 647)

$$\Delta\mathcal{L} = -G_V \left[(\bar{q}\gamma^\mu\vec{\tau}q)^2 + (\bar{q}\gamma^\mu\gamma^5\vec{\tau}q)^2 \right]$$

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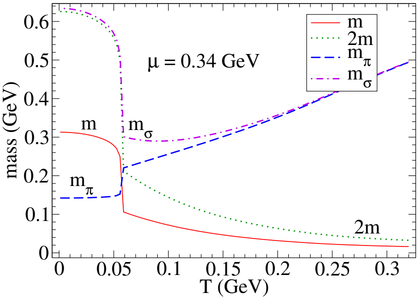

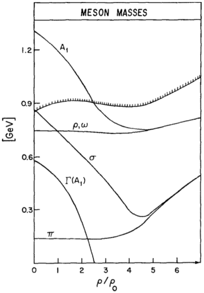

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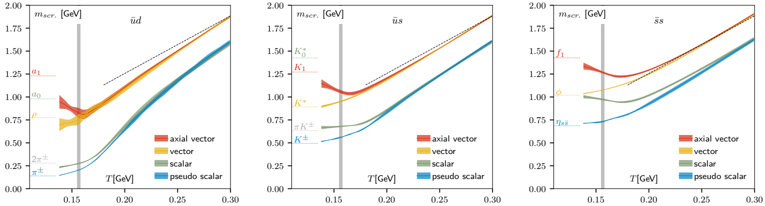
χ-symmetry restoration: lattice-QCD findings



Finite-temperature lattice-QCD calculations do not allow one to get the *pole* masses of the mesons. However they can provide the *screening masses* defined from the decay of meson correlators along the z-axis[1]

$$\int d\tau\, dx\, dy\, \langle J_M(\tau,x,y,z) J_M^\dagger(0,0,0,0)\rangle \sim e^{-m_{\rm scr} z}$$

- V-AV and S-PS mesons tend to merge;
- In the strange sector the degeneracy occurs later

[1] HotQCD Collaboration, PRD 100 (2019) 094510

χ-symmetry restoration: experimental studies

Signatures of chiral-symmetry restoration in the hot/dense medium produced in heavy-ion collisions studied looking at the ρ-meson spectral function. In fact

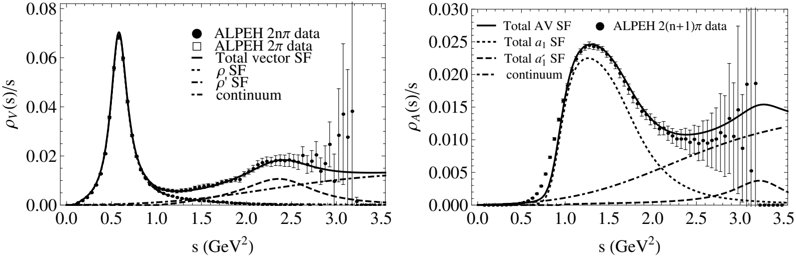
- $SU_A(2)$ transformations map ρ and a_1 meson into each other

$$\vec{\rho}^{\mu} \longrightarrow \vec{\rho}^{\mu} + \vec{\theta} \times \vec{a_1}^{\mu}, \quad \vec{a_1}^{\mu} \longrightarrow \vec{a_1}^{\mu} + \vec{\theta} \times \vec{\rho}^{\mu},$$

  so that, *if chiral-symmetry were restored*, their spectral functions should merge: $\rho_V(\omega, \vec{p}) = \rho_A(\omega, \vec{p})$

- Decay-width $\Gamma \approx 150$ MeV $\longrightarrow$ lifetime $\tau \approx 1.3$ fm/c $\ll \tau_{\rm fireball}$: it decays *inside* the medium and its spectral function is affected by medium-modifications
- In the $\rho \to e^+e^-$ and $\rho \to \mu^+\mu^-$ channels, decay products do not further interact with the medium (no colour-charge!) and carry direct information on in-medium spectral function. Furthermore, muons can be easily identified placing an absorber before the detector

χ-symmetry restoration: experimental studies



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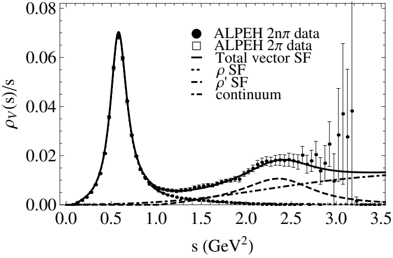

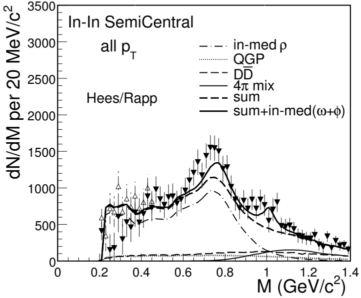

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- In the meddium (AA collisions) the ρ-meson spectral funtion is much broader and its strength above 1 GeV may suggest a *mixing* with the a_1 meson.

Bibliography

- Most of the material used for this lecture can be found in: V. Koch, *Aspects of Chiral Symmetry*, nucl-th/9706075;
- For a simplified treatment of the finite-temperature part see also: J.I. Kapusta and C. Gale, *Finite-Temperature Field Theory: Principles and Applications*, Cambridge University Press;
- For a comprehensive introduction to chiral effective theories see: S. Scherer and M.R. Schindler, *A Chiral Perturbation Theory Primer*, hep-ph/0505265;
- For a review of the NJL model see: S.P. Klevansky, *The Nambu-Jona-Lasinio model of quantum chromodynamics*, Rev.Mod.Phys.64 (1992) 649-708.