Viscous Relativistic Hydrodynamics for heavy-ion collisions

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Ph.D. Lectures

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Hydrodynamics and heavy-ion collisions

The success of hydrodynamics in describing particle spectra in heavy-ion collisions measured *at RHIC came as a surprise*!

- The general setup and its implications
- The main predictions and what we can learn on the medium
  - Radial flow
  - Elliptic flow
- Recent developments (fluctuating initial conditions)
  - Flow in central collisions
  - Higher flow harmonics
  - Event-by-event flow measurements
- Devolping a *causal* relativistic *dissipative* hydrodynamic theory
  - from the entropy principle (Israel-Stewart theory)
  - from kinetic theory

Hydrodynamics: the general setup

- Hydrodynamics is applicable in a situation in which $\lambda_{\rm mfp} \ll L$
- In this limit the behavior of the system is entirely governed by the *conservation laws* (5 eqs. for 6 unknowns: P, ϵ, n_B, v^i)

$$\underbrace{\partial_\mu T^{\mu\nu} = 0}_{\rm four-momentum}\;, \qquad \underbrace{\partial_\mu j_B^\mu = 0}_{\rm baryon\ number}\;,$$

  where

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}, \quad j_B^\mu = n_B u^\mu \quad \text{and} \quad u^\mu = \gamma(1, \vec{v})$$

  NB: at rest $u^\mu = (1, \vec{0})$ and $T^{\mu\nu} = {\rm diag}(\epsilon, P, P, P)$.

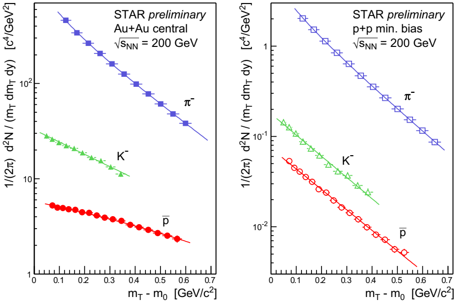
- Information on the medium is *entirely encoded* into the EOS

$$P = P(\epsilon, n_B) \quad (6^{\rm th}\ eq.)$$

- The transition from fluid to particles occurs at the *freeze-out hypersuface* $\Sigma^{\rm fo}$ (e.g. at $T = T_{\rm fo}$)

$$E(dN/d\vec{p}) = \frac{g}{(2\pi)^3} \int_{\Sigma^{\rm fo}} p^\mu d\Sigma_\mu \; \exp[-(p\cdot u)/T]$$

Hydro predictions: radial flow (I)



$$\frac{dN}{m_T dm_T} \sim e^{-m_T/\mathrm{T}_{\mathrm{slope}}} \equiv e^{-\sqrt{p_T^2+m^2}/\mathrm{T}_{\mathrm{slope}}}$$

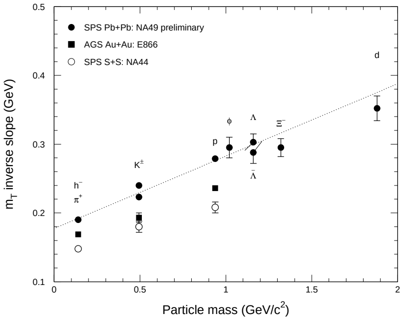
where $m_T \equiv \sqrt{p_T^2 + m^2}$

- $T_{\mathrm{slope}}(\sim 167\,\mathrm{MeV})$ *universal* in pp collisions;
- T_{slope} *growing with m* in AA collisions: spectrum gets harder!

Hydro predictions: radial flow (II)

Physical interpretation:

Thermal emission on top of a collective flow



$$\frac{1}{2}m\langle \boldsymbol{v}_\perp^2\rangle = \frac{1}{2}m\left\langle \left(\boldsymbol{v}_{\perp th} + \boldsymbol{v}_{\perp flow}\right)^2\right\rangle$$
$$= \frac{1}{2}m\langle \boldsymbol{v}_{\perp th}^2\rangle + \frac{1}{2}m\boldsymbol{v}_{\perp flow}^2$$

$$\Longrightarrow \quad T_{\rm slope} = T_{\rm fo} + \frac{1}{2}m\boldsymbol{v}_{\perp flow}^2$$

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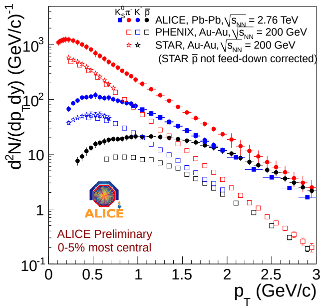

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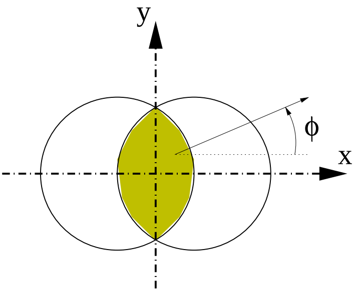
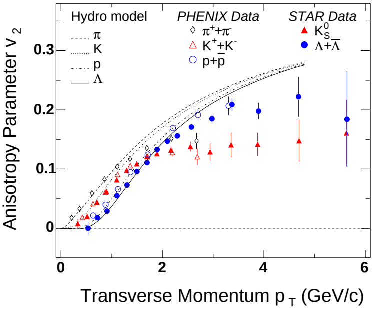
Radial flow gets larger going from RHIC to LHC!

Hydro predictions: elliptic flow



- In *non-central collisions* particle emission is not azimuthally-symmetric!

Hydro predictions: elliptic flow



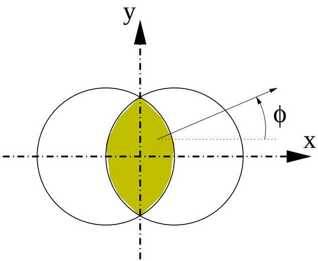
- In *non-central collisions* particle emission is not azimuthally-symmetric!
- The effect can be quantified through the *Fourier coefficient* v_2

$$\frac{dN}{d\phi} = \frac{N_0}{2\pi}\left(1 + 2v_2 \cos[2(\phi - \psi_{RP})] + \dots\right)$$

$$v_2 \equiv \langle \cos[2(\phi - \psi_{RP})] \rangle$$

- $v_2(p_T) \sim 0.2$ gives a modulation 1.4 vs 0.6 for in-plane vs out-of-plane particle emission!

Elliptic flow: physical interpretation



- Matter behaves like a fluid whose *expansion* is *driven by pressure gradients*

$$(\epsilon + P)\frac{dv^i}{dt} \underset{v \ll c}{=} -\frac{\partial P}{\partial x^i} \quad \text{(Euler equation)}$$

- Spatial anisotropy is converted into momentum anisotropy;
- At freeze-out *particles* are *mostly emitted along the reaction-plane*.

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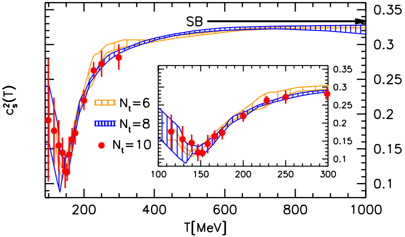

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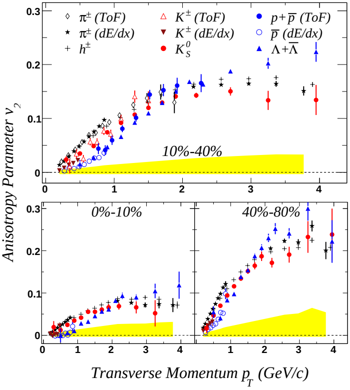
- Spatial anisotropy is converted into momentum anisotropy;
- At freeze-out *particles* are *mostly emitted along the reaction-plane*.
- It provides information on the EOS of the produced matte (Hadron Gas vs QGP) through the *speed of sound*: $\vec{\nabla} P = c_s^2 \vec{\nabla} \epsilon$

Elliptic flow: mass ordering

The mass ordering of v_2 is a direct consequence of the hydro expansion



- Particles emitted according to a thermal distribution $\sim\exp[-p\cdot u(x)/T_{\rm fo}]$ in the local rest-frame of the fluid-cell;
- Parametrizing the fluid velocity and the particle momentum as

$$u^\mu \equiv \gamma_\perp(\cosh Y, \boldsymbol{u}_\perp, \sinh Y),$$
$$p^\mu = (m_\perp \cosh y, \boldsymbol{p}_\perp, m_\perp \sinh y)$$

one gets $(v_z \equiv \tanh Y)$

$$p\cdot u = \gamma_\perp[m_\perp \cosh(y-Y) - \boldsymbol{p}_\perp\cdot\boldsymbol{u}_\perp]$$

- Dependence on m_T at the basis of *mass ordering* at fixed p_T

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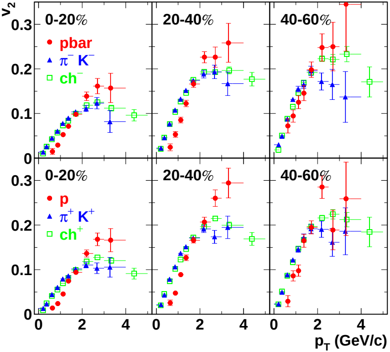

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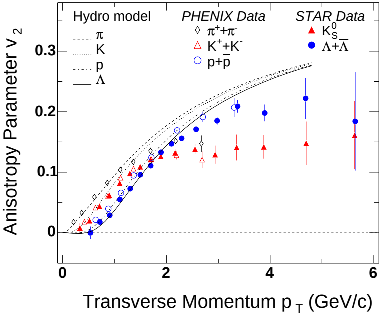

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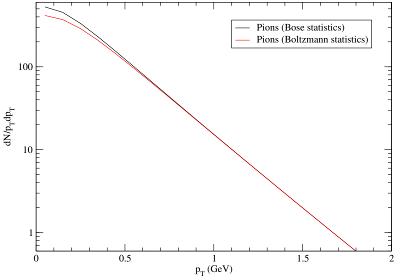
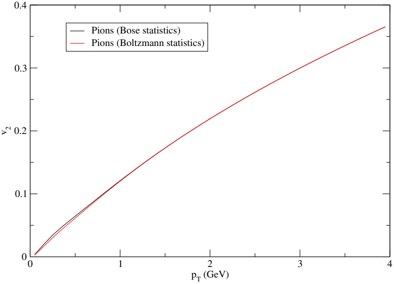
Classical vs quantum statistics





Results obtained with the ECHO-QGP code for an ideal hydrodynamic evolution for Au-Au collisions at $b = 7$ fm, $\tau_0 = 0.5$ fm/c and $e_0 = 30$ GeV/fm^3 and $T_{\rm fo} = 140$ MeV.

As one can see, also for pions, the effect of quantum corrections to the particle distributions is negligible, except at very low p_T.

The need for viscous corrections

- First data on elliptic flow at low p_T were very well reproduced by hydrodynamic calculations, so that one could conclude that "*the bulk of the fireball matter formed in Au-Au collisions at RHIC behaves very much like a perfect fluid*" (U. Heinz, nucl-th/0512051)

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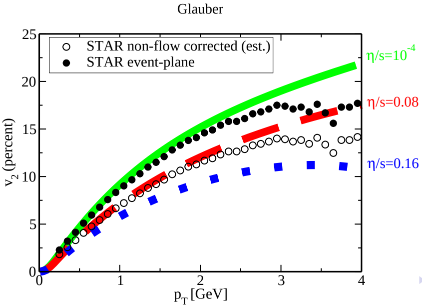
- However v_2 can get contribution from *non-flow* effects (resonance decays, jets, HBT correlations...) which must be subtracted, in order to leave simply the correlation of the particles with the reaction plane. This leads to a *decrease* of v_2, which is better reproduced introducing *viscous corrections*



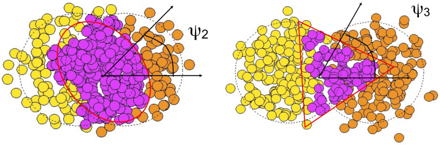
The QGP viscosity

From the comparison with the data one gets values for the *shear viscosity* close to the *universal lower bound* $\eta/s \approx 1/4\pi$ predicted by the AdS/CFT correspondence.
One can compare this with the values found for all the other known fluids:

fluid	P [Pa]	T [K]	η [Pa·s]	η/n [$\hbar$]	η/s [$\hbar/k_B$]
H_2O	$0.1{\cdot}10^6$	370	$2.9 \cdot 10^{-4}$	85	8.2
^{4}He	$0.1{\cdot}10^6$	2.0	$1.2 \cdot 10^{-6}$	0.5	1.9
H_2O	$22.6{\cdot}10^6$	650	$6.0 \cdot 10^{-5}$	32	2.0
^{4}He	$0.22{\cdot}10^6$	5.1	$1.7 \cdot 10^{-6}$	1.7	0.7
^{6}Li ($a = \infty$)	$12{\cdot}10^{-9}$	$23{\cdot}10^{-6}$	$\leq 1.7 \cdot 10^{-15}$	≤ 1	≤ 0.5
QGP	$88{\cdot}10^{33}$	$2{\cdot}10^{12}$	$\leq 5 \cdot 10^{11}$		≤ 0.4

leading to the conclusion that the QGP looks like the most *ideal fluid* ever observed

Event by event fluctuations



- Due to event-by-event fluctuations (e.g. of the nucleon positions) the initial density distribution is not smooth and can display *higher deformations*, each one with a *different azimuthal orientation*.
- Higher harmonics ($m > 2$) contribute to the angular distribution

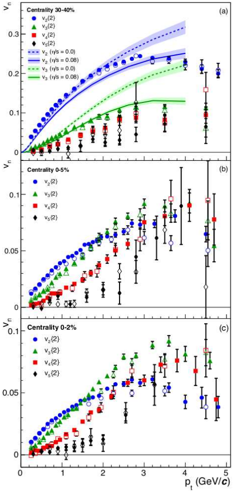
$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + 2\sum_m v_m \cos[m(\phi - \psi_m)]\right)$$

of the final hadrons, where *for each event* (verify!),

$$v_m = \langle \cos[m(\phi - \psi_m)] \rangle \quad \text{and} \quad \psi_m = \frac{1}{m} \arctan \frac{\sum_i w_i \sin(m\phi_i)}{\sum_i w_i \cos(m\phi_i)}$$

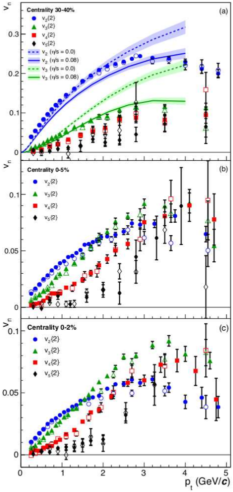
The choice $w_i = p_T^i$ for the weights increase the resolution on ψ_m (one deals with a *finite number* of hadrons!)

Event-by-event fluctuations: experimental consequences



Fluctuating initial conditions give rise to[a]:

- Non-vanishing v_2 in central collisions;
- Odd harmonics (v_3 and v_5)



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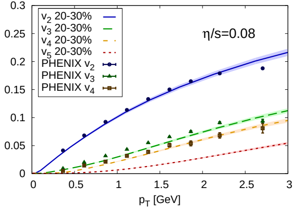
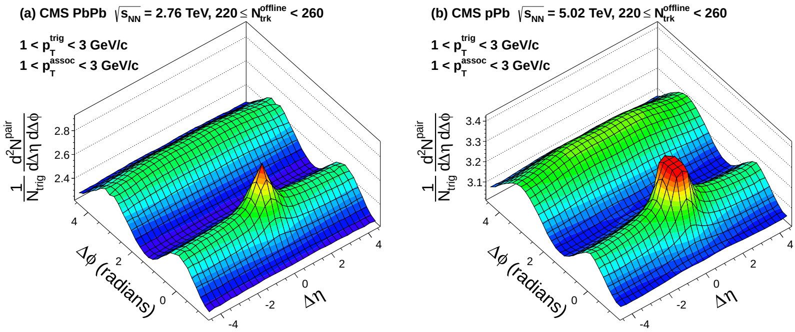
Hydro can reproduce also higher harmonics[b]



[a] ALICE, Phys.Rev.Lett. 107 (2011) 032301
[b] B: Schenke *et al.*, PRC 85, 024901 (2012)

Hydrodynamic behavior in small systems?



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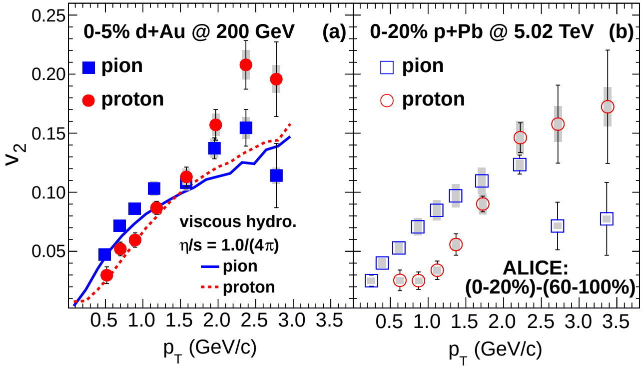

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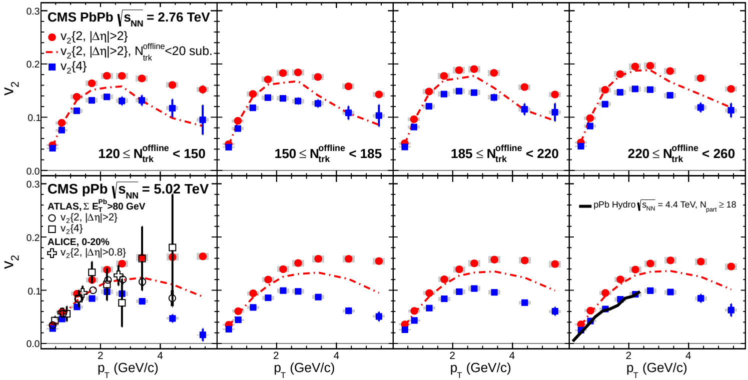

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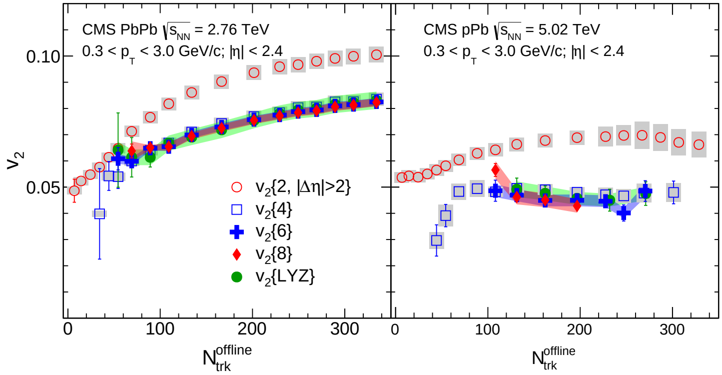

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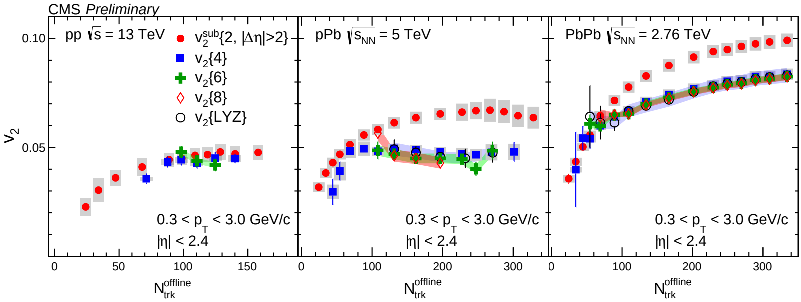

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Relativistic hydrodynamics

- Macroscopic approach: energy-momentum conservation
  - Ideal Fluid
  - Viscous fluid
    - First order Navier-Stokes theory (*causality problem*)
    - Second order (Israel-Stewart) theory
- Microscopic approach: kinetic theory

Relativistic hydrodynamics: the ideal case

In the absence of non-vanishing conserved charges ($n_B = 0$), the evolution of an *ideal fluid* is completely described by the *conservation of the ideal energy-momentum tensor*:

$$\partial_\mu T^{\mu\nu} = 0, \quad \text{where} \quad T^{\mu\nu} = T^{\mu\nu}_{\rm eq} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu}$$

It is convenient to project the above equations

- along the fluid velocity ($u_\nu \partial_\mu T^{\mu\nu} = 0$)

$$D\epsilon = -(\epsilon + P)\Theta, \quad (\text{with} \quad \underbrace{D \equiv u^\mu \partial_\mu}_{\text{comov. derivative}} \quad \text{and} \quad \underbrace{\Theta \equiv \partial_\mu u^\mu}_{\text{expansion rate}} \,)$$

- and perpendicularly to it ($\Delta_{\alpha\nu}\partial_\mu T^{\mu\nu} = 0$, with $\underbrace{\Delta_{\alpha\nu} \equiv g_{\alpha\nu} - u_\alpha u_\nu}_{\text{transv. project.}}$)

$$(\epsilon + P) D u^\alpha = \nabla^\alpha P \quad (\text{with } \nabla^\alpha \equiv \Delta^{\alpha\mu}\partial_\mu),$$

  which is the *relativistic* version of the *Euler equation* (fluid acceleration driven by pressure gradients)

$$\text{non relativistic limit :} \quad \underbrace{(\epsilon + P)}_{\approx \rho}\underbrace{(\partial_t + v^k \partial_k)}_{\equiv d/dt}\vec{v} = -\vec{\nabla} P$$

Ideal fluid: energy vs charge evolution

Let us now consider the more general case in which the fluid is characterized also by some non-vanishing *conserved charge*, e.g. the baryon number. In this case

$$\partial_\mu j_B^\mu = 0, \quad \text{where} \quad j_B^\mu = n_B u^\mu .$$

The conservation equation can be rewritten as:

$$D n_B = -n_B \Theta$$

to be compared with the equation for the energy

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The two quantities follow *different evolutions*:

- the baryon density gets simply diluted due to the expansion of the system;
- the energy density drops more rapidly, since during the expansion the system does work, i.e. disordered thermal energy is converted into ordered collective motion driven by pressure gradients

Isentropic evolution

In the absence of dissipative effects entropy conservation during the fluid evolution follows naturally. One introduces the entropy current $s^\mu \equiv s u^\mu$, where s in the entropy density in the LRF of the fluid, which obeys the continuity equation

$$\partial_\mu s^\mu = 0 \quad \longrightarrow \quad Ds + s\Theta = 0$$

The latter is identically satisfied, since

$$Ts = \epsilon + P - \mu_B n_B \quad \text{and} \quad d\epsilon = Tds + \mu_B dn_B$$

Hence

$$\frac{1}{T}D\epsilon - \frac{\mu_B}{T}dn_B + \frac{1}{T}[\epsilon + P - \mu_B n_B]\Theta = \frac{1}{T}[D\epsilon + (\epsilon + P)\Theta] - \frac{\mu_B}{T}[Dn_B + n_B\Theta] = 0,$$

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due to energy and baryon number conservation. This entails that, *for each fluid cell*, the evolution occurs at constant entropy per baryon s/n_B, i.e. its comoving derivative vanishes $D(s/n_B) = 0$. In fact:

$$D\left(\frac{s}{n_B}\right) = \frac{Ds}{n_B} - \frac{s}{n_B^2}Dn_B = -\frac{s\Theta}{n_B} + \frac{s}{n_B^2}n_B\Theta = 0$$

Linear perturbations: sound wave

Consider a linear perturbation propagating around a homogenous background along the x direction

$$\epsilon = \epsilon_0 + \delta\epsilon(t,x), \quad P = P_0 + \delta P(t,x), \quad u^\mu = (1,\vec{0}) + \delta u^\mu(t,x)$$

where $\delta P = (\partial P/\partial \epsilon)_\sigma \delta\epsilon \equiv c_S^2 \delta\epsilon$, since fluctuations occurs at constant entropy per baryon σ.

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$$\begin{aligned} \partial_t \delta\epsilon &= -(\epsilon_0 + P_0)\,\partial_x v^x \\ (\epsilon_0 + P_0)\,\partial_t v^x &= -c_s^2\,\partial_x \delta\epsilon \\ (\epsilon_0 + P_0)\,\partial_t v^{y/z} &= 0 \end{aligned}$$

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Taking a partial time-derivative of the second equation one gets

$$(\epsilon_0 + P_0)\,\partial_t^2 v^x + c_s^2 \partial_x(\partial_t \delta\epsilon) = 0$$

Exploiting the equation for the energy one obtains

$$\boxed{\left(\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) v^x(t,x) = 0}$$

which shows the longitudinal nature of the wave.

An exact model: pure Bjorken expansion

RHD equations can be solved analytically in the case of a pure longitudinal *Bjorken expansion*, namely

$$v^x = v^y = 0, \quad \text{and} \quad v^z = \frac{z}{t} \; (\text{Hubble} - \text{law})$$

The model, referring to an omogeneous infinite system in the transverse plane (no transverse pressure gradients!), is clearly an oversimplification, however it can be a good guidance to descrive the first instants of the evolution of a nuclear collisions, since *transverse flow* ($\vec{v}_\perp \neq 0$) takes time to develop, while from the very beginning $v^z \neq 0$.

An exact model: pure Bjorken expansion

RHD equations can be solved analytically in the case of a pure longitudinal *Bjorken expansion*, namely

$$v^x = v^y = 0, \quad \text{and} \quad v^z = \frac{z}{t} \;(\text{Hubble} - \text{law})$$

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The model is easily solved introducing the *longitudinal proper-time* $\tau \equiv \sqrt{t^2 - z^2}$ and one finds (verify!)

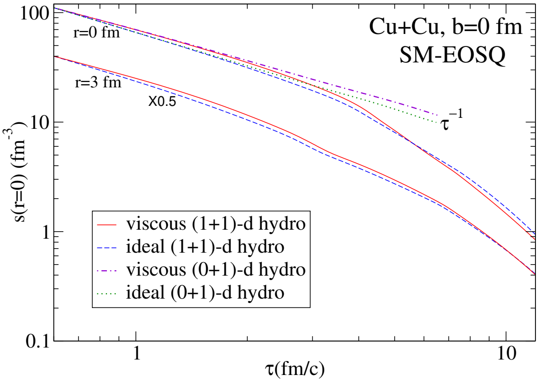
$$\theta \equiv \partial_\mu u^\mu = \frac{1}{\tau} \longrightarrow \frac{d\epsilon}{d\tau} = -\frac{\epsilon + P}{\tau}$$

For a fluid with $P = c_s^2 \epsilon$ EOS one gets then:

$$\frac{d\epsilon}{\epsilon} = -(1 + c_s^2)\frac{d\tau}{\tau} \longrightarrow \epsilon = \epsilon_0 \left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}$$

For a $P = \epsilon/3$ EOS one gets the temporal evolution $\epsilon \sim \tau^{-(4/3)}$, to compare with the one of the particle density $n \sim \tau^{-1}$

Not a bad approximation



For the first few fm/c, before the transverse expansion sets in (perturbations from the border propagate at finite velocity), the Bjorken model nicely describes the system evolution at the center of the fireball[1].

[1] H. Song and U. Heinz, Phys.Rev. C77 (2008) 064901

The Riemann problem: rarefaction wave

Another case one can treat exactly is the so-called Riemann problem, i.e. a flow which starts from an initial condition of the kind ϵ_L, P_L, u_L^μ for $x < 0$ and ϵ_R, P_R, u_R^μ for $x > 0$. It is also very important for numerical implementations of hydrodynamic equations, which should be able to capture shocks.

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$$D\epsilon = -(\epsilon + P)\Theta \quad \text{and} \quad (\epsilon + P)Du^\alpha = \nabla^\alpha P$$

depends now only on t and x, with $u^\mu = \gamma(1, \beta, 0, 0)$:

$$(\partial_t + \beta\partial_x)\epsilon + \gamma^2(\epsilon + P)(\beta\partial_t + \partial_x)\beta = 0$$
$$(\beta\partial_t + \partial_x)P + \gamma^2(\epsilon + P)(\partial_t + \beta\partial_x)\beta = 0.$$

Actually the solution depends only on the *self-similar* variable $\xi \equiv x/t$

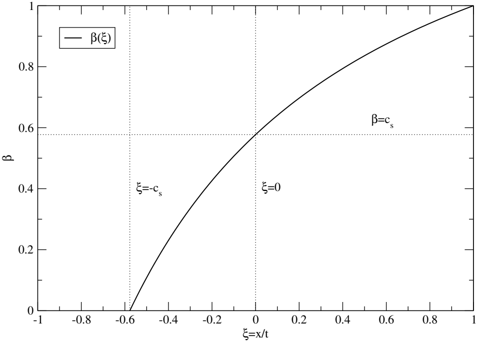
$$(-\xi + \beta)\epsilon' + \gamma^2(\epsilon + P)(-\beta\xi + 1)\beta' = 0$$
$$(-\beta\xi + 1)P' + \gamma^2(\epsilon + P)(-\xi + \beta)\beta' = 0,$$

with $P' = c_s^2 \epsilon'$

The Riemann problem: rarefaction wave

The above system has non-trivial solutions only if its determinant vanishes. One has then (with our initial condition):

$$\frac{(\beta-\xi)^2}{(1-\beta\xi)^2} = c_s^2 \quad \longrightarrow \quad \beta(\xi) = \frac{c_s+\xi}{1+\xi c_s}$$



The head of the rarefaction wave propagates backwards with velocity $\xi_{\rm rw} = -c_s$. In the region $\xi < -c_s$ the fluid is still unperturbed ($\epsilon = \epsilon_0$ and $\beta = 0$), while at the origin $\beta(0) = c_s$!

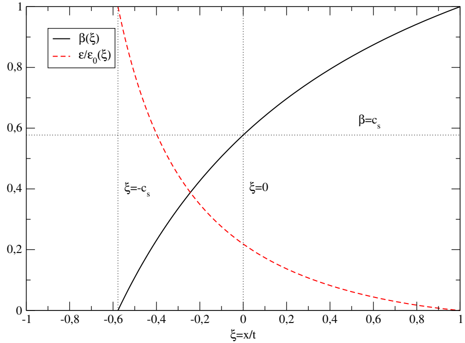
The Riemann problem: rarefaction wave

Substituting back the result into the system one gets:

$$d\epsilon + \gamma^2(1+c_s^2)\epsilon\,\frac{1-\beta\xi}{\beta-\xi}\,d\beta = 0 \quad\longrightarrow\quad \frac{d\epsilon}{\epsilon} + \frac{1+c_s^2}{c_s}\,\frac{d\beta}{1-\beta^2} = 0$$

With the boundary condition $\epsilon = \epsilon_0$ when $\beta = 0$ one gets

$$\epsilon = \epsilon_0 \exp\left[-\frac{1+c_s^2}{c_s}\,\text{atanh}\,\beta\right] = \epsilon_0 \exp\left[-\frac{1+c_s^2}{c_s}\,\text{atanh}\left(\frac{c_s+\xi}{1+\xi c_s}\right)\right]$$



- Black curve: fluid velocity
- Red curve: energy density

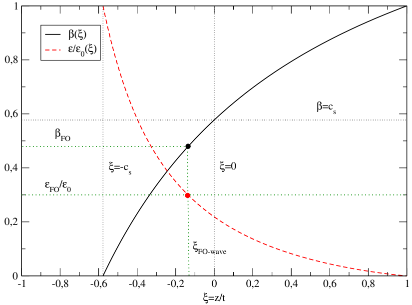
Freeze-out hypersurface and particle decoupling

In the case of the above Riemann problem the particle spectrum from a Cooper-Frye decoupling can be computed exactly. The freeze-out (FO) hypersurface is defined by the condition:

$$\epsilon_{\rm FO} = \epsilon_0 \exp\left[-\frac{1+c_s^2}{c_s}\,\text{atanh}\left(\frac{c_s+\xi}{1+\xi c_s}\right)\right],$$

which fixes also the value of the fluid-velocity at freeze-out

$$\epsilon_{\rm FO} = \epsilon_0 \exp\left[-\frac{1+c_s^2}{c_s}\,\text{atanh}\,\beta_{\rm FO}\right] \quad \longrightarrow \quad \beta_{\rm FO} = \tanh\left[\ln\left(\frac{\epsilon_0}{\epsilon_{\rm FO}}\right)^{c_s^2/(1+c_s^2)}\right]$$



From

$$\beta = \frac{c_s+\xi}{1+\xi c_s}$$

one gets a left/right-moving "FO-wave"

$$\xi_{\rm FO} = \frac{\beta_{\rm FO}-c_s}{1-\beta_{\rm FO}\,c_s}$$

depending whether $\beta_{\rm FO} < c_s$ or $\beta_{\rm FO} > c_s$

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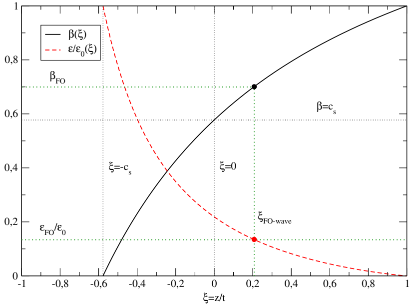

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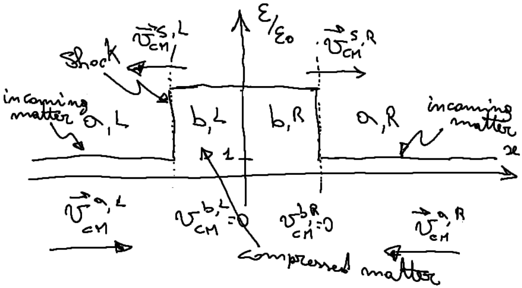
Compressed nuclear matter: heavy-ion collisions

Make two infinite slabs of nuclear matter collide, with initial condition

$$
\begin{aligned}
\epsilon(0,x) &= \epsilon_0 \qquad && -\infty < x < \infty \\
n(0,x) &= n_0 \qquad && -\infty < x < \infty \\
v(0,x<0) &= v_{\rm CM} \qquad && v(0,x>0) = -v_{\rm CM}
\end{aligned}
$$



Assuming complete stopping one has the formation of a compressed region *at rest* (b) and of a shock wave *propagating backward* with respect to the unperturbed incoming matter (a)

Shocks in hydrodynamics: general treatment

Hydrodynamic equations express in a differential way *global conservation laws*. Their general form is

$$\partial_t q(t, x) + \partial_x f(t, x) = 0$$

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In the presence of a shock front $x_s(t)$ quantities display a discontinuity, with different values on the left (1) and on the right (2) of the shock. The differential version of the conservation law is not well defined, but its integral form it is:

$$\int_{x_s(t)-\epsilon}^{x_s(t)+\epsilon} \partial_t q(t,x)\, dx + \int_{x_s(t)-\epsilon}^{x_s(t)+\epsilon} \partial_x f(t,x)\, dx = 0$$

Since the integration domain vanishes one has:

$$0 = \frac{d}{dt}\int_{x_s(t)-\epsilon}^{x_s(t)+\epsilon} q(t,x)dx = q_2\dot{x}_s - q_1\dot{x}_s + \int_{x_s(t)-\epsilon}^{x_s(t)+\epsilon} \partial_t q(t,x)dx$$

Hence one gets

$$(q_1 - q_2)\dot{x}_s + (f_2 - f_1) = 0 \quad \longrightarrow \quad \boxed{\dot{x}_s = \frac{f_2 - f_1}{x_2 - x_1}}$$

called Rankine-Hugoniot jump condition

Shocks in relativistic hydrodynamics

To study shocks in a 1-dimensional flow one starts from the equations

$$\begin{aligned} \partial_0 j^0 + \partial_x j^x &= 0 \\ \partial_0 T^{00} + \partial_x T^{x0} &= 0 \\ \partial_0 T^{00} + \partial_x T^{xx} &= 0 \end{aligned}$$

where

$$j^\mu = n u^\mu \ , \quad T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - P g^{\mu\nu} \quad \text{and} \quad u^\mu = \gamma(1, \vec{v})$$

The Rankine-Hugoniot conditions read then ($h \equiv \epsilon + P$)

$$\begin{aligned} n_a \gamma_a v_a &= n_b \gamma_b v_b \equiv J \\ h_a \gamma_a^2 v_a &= h_b \gamma_b^2 v_b \\ h_a \gamma_a^2 v_a^2 + P_a &= h_b \gamma_b^2 v_b^2 + P_b \end{aligned}$$

where the quantities are evaluated in the rest-frame of the shock

The Taub adiabatic

Combining the first and third equations one has

$$P_a - P_b = -J^2\left(\frac{h_a}{n_a^2} - \frac{h_b}{n_b^2}\right) \quad \longrightarrow \quad \left(\frac{h_a}{n_a^2} + \frac{h_b}{n_b^2}\right)(P_a - P_b) = -\gamma_a^2 v_a^2 \frac{h_a^2}{n_a^2} + \gamma_b^2 v_b^2 \frac{h_b^2}{n_b^2}$$

Combining the first and second equations one has

$$\frac{h_a \gamma_a}{n_a} = \frac{h_b \gamma_b}{n_b} \quad \longrightarrow \quad \frac{h_a^2 \gamma_a^2}{n_a^2} = \frac{h_b^2 \gamma_b^2}{n_b^2}$$

Hence

$$\left(\frac{h_a}{n_a^2} + \frac{h_b}{n_b^2}\right)(P_a - P_b) = -\gamma_a^2 v_a^2 \frac{h_a^2}{n_a^2} + \frac{h_a^2 \gamma_a^2}{n_a^2} - \frac{h_b^2 \gamma_b^2}{n_b^2} + \gamma_b^2 v_b^2 \frac{h_b^2}{n_b^2}$$

From which one gets

$$\boxed{\left(\frac{h_a}{n_a^2} + \frac{h_b}{n_b^2}\right)(P_a - P_b) = \frac{h_a^2}{n_a^2} - \frac{h_b^2}{n_b^2}}$$

which is know as Taub adiabatic.

Reference frames

Let us focus on the left-moving shock

Shock frame (shock at rest)

$$u^a_{\mathrm{S}} = \gamma_a(1, v_a) = (\cosh Y_a, \sinh Y_a)$$
$$u^b_{\mathrm{S}} = \gamma_b(1, v_b) = (\cosh Y_b, \sinh Y_b)$$
$$u^s_{\mathrm{S}} = (1, 0)$$

CM frame (compressed matter at rest)

$$u^a_{\mathrm{CM}} = \gamma_{\mathrm{CM}}(1, v_{\mathrm{CM}})$$
$$u^b_{\mathrm{CM}} = (1, 0)$$
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$$u^s_{\rm CM} = \gamma^s_{\rm CM}(1, v^s_{\rm CM})$$

The two reference frames are linked by the boost

$$u^\mu_{\rm S} = \begin{pmatrix} \cosh Y_b & \sinh Y_b \\ \sinh Y_b & \cosh Y_b \end{pmatrix}^\mu_{\ \nu} u^\nu_{\rm CM} \qquad u^\mu_{\rm CM} = \begin{pmatrix} \cosh Y_b & -\sinh Y_b \\ -\sinh Y_b & \cosh Y_b \end{pmatrix}^\mu_{\ \nu} u^\nu_{\rm S}$$

Hence one has

$$u^a_{\rm CM} = (\cosh(Y_a - Y_b), \sinh(Y_a - Y_b)) \ , \qquad u^s_{\rm CM} = (\cosh Y_b, -\sinh Y_b)$$

From which

$$\boxed{v_{\rm CM} \equiv \tanh Y_{\rm CM} = \frac{v_a - v_b}{1 - v_a v_b}}$$

Compression of nuclear matter (I)

We are left with the problem of evaluating the discontinuity of the thermodynamic quantities at the shock front knowing the beam velocity v_{CM} and the equation of state of the matter $P = P(\epsilon, n)$.

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We start evaluating the velocities on the left/right of the shock.

$$h_a \gamma_a^2 v_a = h_b \gamma_b^2 v_b$$
$$h_a \gamma_a^2 v_a^2 + P_a = h_b \gamma_b^2 v_b^2 + P_b$$

It is convenient to write $\gamma_i = \cosh Y_i$ and $v_i = \tanh Y_i$

$$h_a^2 \sinh^2 Y_a \cosh^2 Y_a = h_b^2 \sinh^2 Y_b \cosh^2 Y_b$$
$$h_a \sinh^2 Y_a - h_b \sinh^2 Y_b = P_b - P_a$$

Exploiting $\cosh^2 x = 1 + \sinh^2 x$ one gets

$$\boxed{v_a^2 \equiv \tanh^2 Y_a = \frac{(P_a - P_b)(P_a + \epsilon_b)}{(\epsilon_a - \epsilon_b)(\epsilon_a + P_b)}} \quad \boxed{v_b^2 \equiv \tanh^2 Y_b = \frac{(P_a - P_b)(P_b + \epsilon_a)}{(\epsilon_a - \epsilon_b)(\epsilon_b + P_a)}}$$

Compression of nuclear matter (II)

For the velocity of the incoming matter in the CM frame one has

$$v_{\rm CM} = \frac{v_a - v_b}{1 - v_a v_b} \quad \longrightarrow \quad v_{\rm CM}^2 = \frac{(P_b - P_a)(\epsilon_b - \epsilon_a)}{(\epsilon_a + P_b)(\epsilon_b + P_a)}$$

Hence

$$\gamma_{\rm CM}^2 = \frac{(\epsilon_a + P_b)(\epsilon_b + P_a)}{h_a \cdot h_b}$$

which can be rewritten as

$$\gamma_{\rm CM}^2 = \frac{(\epsilon_b + P_a)^2}{h_a^2} \frac{(\epsilon_a + P_b) h_a}{(\epsilon_b + P_a) h_b} = \frac{(\epsilon_b + P_a)^2}{h_a^2} \frac{[(\epsilon_a + P_a) + (P_b - P_a)] h_a}{[(\epsilon_b + P_b) - (P_b - P_a)] h_b}$$

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Using the Taub equation

$$\left(\frac{h_a}{n_a^2} + \frac{h_b}{n_b^2}\right)(P_a - P_b) = \frac{h_a^2}{n_a^2} - \frac{h_b^2}{n_b^2}$$

one gets for $P_a = 0$, $\epsilon_a = \epsilon_0$ and $n_a = n_0$ (nuclear matter at saturation)

$$\boxed{\gamma_{\rm CM} = \frac{\epsilon_b + P_a}{h_a}\frac{n_a}{n_b} = \frac{(\epsilon_b/n_b)}{(\epsilon_0/n_0)}}$$

Compression of nuclear matter (III)

Exploiting the Taub equation one gets for the pressure of the compressed matter

$$\boxed{P(\epsilon, n) = \frac{\dfrac{(\epsilon + P(\epsilon, n))^2}{n^2} - \dfrac{\epsilon_0^2}{n_0^2}}{\dfrac{\epsilon + P(\epsilon, n)}{n^2} + \dfrac{\epsilon_0}{n_0^2}}}$$

Viscous hydrodynamics

Better flow measurements required the introduction of *dissipative corrections* (viscosity, heat-flow and charge diffusion):

$$T^{\mu\nu} = T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu} \quad (\Pi^{\mu\nu} \equiv \pi^{\mu\nu} - \Pi\Delta^{\mu\nu} \text{ with } \pi^{\mu}_{\mu} = 0)$$

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One must first of all establish the link with the ordinary thermodynamic quantities. This can be done imposing the Landau matching condition

$$u_\mu \pi^{\mu\nu} = u_\mu q^\mu = 0 \longrightarrow u_\mu u_\nu T^{\mu\nu} = u_\mu u_\nu T^{\mu\nu}_{\rm eq} = \epsilon \;\; (\overline{T}^{00} = \overline{T}^{00}_{\rm eq} = \epsilon \text{ in the LRF})$$

$$u_\mu V^\mu_B = 0 \longrightarrow u_\mu j^\mu_B = u_\mu j^\mu_{B(\rm eq)} = n_B \;\; (\bar{j}^0_B = \bar{j}^0_{B(\rm eq)} = n_B \text{ in the LRF})$$

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One must define what one means with *fluid velocity* : whether it refers to the transport of the *conserved charge* or of the *energy*

- Eckart frame: $V^\mu_B = 0$. In the LRF no charge diffusion ($j^i_B = 0$), but one can have energy flowing in/out-of the cell (heat conduction)
- Landau frame: $q^\mu = 0 \longrightarrow u_\mu T^{\mu\nu} = \epsilon u^\nu$. In the LRF no heat-flow throughout the cell ($T^{0i} = 0$), but it is possible to have charge diffusion. The Landau frame is the natural choice in nuclear collisions, in which $n_B \approx 0$.

Viscous hydrodynamics

In the following we will work in the Landau frame ($q^\mu = 0$), in which

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- Projecting along u_ν:

$$\boxed{D\epsilon + (\epsilon + P + \Pi)\Theta - \pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} = 0},$$

after replacing $\nabla_\mu u_\nu \longrightarrow \nabla_{<\mu}u_{\nu>} \equiv \frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) - \frac{1}{3}\Delta_{\mu\nu}\Theta$

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$$\boxed{(\epsilon + P + \Pi)Du^\alpha = \nabla^\alpha(P + \Pi) - \Delta^\alpha_\nu\partial_\mu\pi^{\mu\nu}}$$

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From baryon-number conservation $\partial_\mu j^\mu_B = 0$ one has

$$\boxed{Dn_B + n_B\Theta + \partial_\mu V^\mu_B = 0}$$

Fixing the viscous tensor: first order formalism ($n_B = 0$)

A way to fix the viscous tensor is through the 2^{nd} law of thermodynamics, imposing $\partial_\mu s^\mu \geq 0$.

Fixing the viscous tensor: first order formalism ($n_B = 0$)

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Employing

$$D\epsilon = -(\epsilon + P + \Pi)\Theta + \pi^{\mu\nu}\nabla_{<\mu}u_{\nu>},$$

one gets

$$\partial_\mu s^\mu = \frac{1}{T}\left[-\Pi\Theta + \pi^{\mu\nu}\nabla_{<\mu}u_{\nu>}\right] \geq 0$$

which is identically satisfied if (relativistic Navier Stokes result)

$$\Pi = -\zeta\Theta \quad \text{and} \quad \pi^{\mu\nu} = 2\eta\nabla^{<\mu}u^{\nu>},$$

where ζ and η are the bulk and shear *viscosity coefficients*.

The finite baryon-density case

In the finite-density case one employ for the entropy current the ansatz $s^\mu = su^\mu - \alpha V^\mu$ together with the thermodynamic relations

$$Ts = \epsilon + P - \mu n \quad \text{and} \quad d\epsilon = T\,ds + \mu dn$$

The second law of thermodynamics $T\partial_\mu s^\mu \geq 0$ leads to:

$$D\epsilon + (\epsilon + P)\Theta - \mu(Dn + n\Theta) - TV^\mu \partial_\mu \alpha - T\alpha(\partial_\mu V^\mu) \geq 0$$

Employing the equations for the energy and charge conservation one gets

$$-\Pi\Theta + \pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} + (\mu - T\alpha)\partial_\mu V^\mu - TV^\mu \partial_\mu \alpha \geq 0,$$

which entails also $\alpha = \mu/T$ (the inequality should hold always, independently of the sign of $\partial_\mu V^\mu$). Hence (V^ν being *spacelike*)

$$-TV^\nu \nabla_\nu \left(\frac{\mu}{T}\right) \geq 0 \;\longrightarrow\; \boxed{V^\nu = \lambda T^2 \nabla^\nu \left(\frac{\mu}{T}\right)},$$

where λ is the baryon-number *diffusion constant*. In the following, in any case, we will focus on the $n_B = 0$ case.

Non-relativistic limit

In the non-relativistic limit one recovers the usual Navier-Stokes equation. The viscous contribution to the momentum-flux tensor $T_{ij} = T_{ij}^{\rm id} - \sigma'_{ij}$ reads (an euclidean metric is assumed in raising/lowering the indices)

$$\sigma'_{ij} = \eta(\partial_i v_j + \partial_j v_i - 2/3\delta_{ij}\Theta) + \zeta\delta_{ij}\Theta \quad \text{with} \quad \Theta \equiv \partial_k v_k.$$

The combination of velocity-gradients ensures that $\sigma'_{ij} = 0$ for a uniform rotation $\boldsymbol{v} = \Omega \times \boldsymbol{r}$. This leads to a correction to the Euler equation

$$\rho(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})v_i = -\partial_i P + \partial_k \sigma'_{ik}.$$

For constant shear and bulk viscosity one gets

$$\rho(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P + \eta\Delta\boldsymbol{v} + \left(\zeta + \frac{1}{3}\eta\right)\boldsymbol{\nabla}\Theta.$$

For an incompressible fluid the expansion rate vanishes, $\Theta = 0$, and one has the Navier-Stokes equation

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \frac{\eta}{\rho}\Delta\boldsymbol{v},$$

where the dissipative corrections depends on the *kinematic viscosity* η/ρ.
In the relativistic case for the inertial term $\rho \to \epsilon + P \equiv Ts$

Linear perturbations

We consider linear perturbations around equilibrium depending on (t, x)

$$\epsilon = \epsilon_0 + \delta\epsilon(t, x), \quad P = P_0 + \delta P(t, x), \quad u^\mu = (1, \vec{0}) + \delta u^\mu(t, x)$$

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For the viscous tensor one has $(\theta \equiv \partial_\mu u^\mu = \partial_x \delta u^x)$

$$\pi^{xx} = 2\eta_0 \nabla^{<x} u^{x>} = 2\,\eta_0 \left[\frac{1}{2}(-2\partial_x \delta u^x) - \frac{1}{3}\Delta^{xx}\theta\right] = -\frac{4}{3}\eta_0 \partial_x \delta u^x + \mathcal{O}(\delta^2)$$

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$$\Pi = -\zeta_0 \theta = -\zeta_0 \partial_x \delta u^x,$$

which are the only non-vanishing components one has to consider. In fact, from $u_\mu \pi^{\mu\nu} = 0$ and $u_0 = 1 + \mathcal{O}(\delta^2)$ one has

$$u_0 \pi^{0\nu} + u_i \pi^{i\nu} = 0 \quad \longrightarrow \quad \pi^{0\nu} \approx -\delta u_i \pi^{i\nu} = \mathcal{O}(\delta^2)$$

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The perturbations will be expanded in Fourier modes, e.g.

$$\delta\epsilon(t,x) = \int_k e^{-i\omega t + ikx}\delta\epsilon_{\omega,k}$$

finding the *dispersion relation* $\omega = \omega(k)$ solution of the eqs.

Linear perturbations: sound mode

We must then linearize the equations

$$D\epsilon = -(\epsilon + P + \Pi)\Theta + \pi^{\mu\nu}\nabla_{<\mu}u_{\nu>},$$
$$(\epsilon + P + \Pi)Du^{\alpha} = \nabla^{\alpha}(P + \Pi) - \Delta^{\alpha}_{\nu}\partial_{\mu}\pi^{\mu\nu}$$

For the longitudinal mode we get

$$\partial_t \delta\epsilon = -(\epsilon_0 + P_0)\partial_x \delta u^x$$
$$(\epsilon_0 + P_0)\partial_t \delta u^x = -\partial_x \delta P - \partial_x(\Pi + \pi^{xx})$$

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In Fourier space one has ($\partial P/\partial\epsilon \equiv c_s^2$):

$$-i\omega\delta\epsilon_{\omega,k} + ik(\epsilon_0 + P_0)\delta u^x_{\omega,k} = 0$$
$$(\epsilon_0 + P_0)(-i\omega)\delta u^x_{\omega,k} + ik(\partial P/\partial\epsilon)\delta\epsilon_{\omega,k} + ik(\delta\Pi_{\omega,k} + \delta\pi^{xx}_{\omega,k}) = 0$$

with $\delta\pi^{xx}_{\omega,k} = (-ik)(4/3)\eta_0\delta u^x_{\omega,k}$ and $\delta\Pi_{\omega,k} = (-ik)\zeta_0\delta u^x_{\omega,k}$

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Solving for $\delta u^x_{\omega,k}$ or $\delta\epsilon_{\omega,k}$ one gets

$$\omega^2 - c_s^2k^2 + i\omega\left[\frac{(4/3)\eta_0 + \zeta_0}{\epsilon_0 + P_0}\right]k^2 = 0$$

Linear perturbations: sound mode

In order to find the dispersion relation of the mode one has to solve

$$\omega^2 + i\left[\frac{(4/3)\eta_0 + \zeta_0}{\epsilon_0 + P_0}\right] k^2\omega - c_s^2 k^2 = 0$$

which has *two complex solutions*, whose long wavelength expansion is

$$\omega \underset{k\to 0}{\sim} \pm c_s k - i \underbrace{\left[\frac{(4/3)\eta_0 + \zeta_0}{2(\epsilon_0 + P_0)}\right]}_{\equiv\alpha} k^2 + \mathcal{O}(k^3)$$

Leading to

$$\delta u^x(t,x) = \delta u^x_{\omega_k,k}\, e^{ik(x\mp c_s t)} e^{-\alpha k^2 t}$$

The viscosity is responsible for the damping of the sound! *The damping is stronger for short-wavelength modes*.

For the group velocity one has:

$$v_g \equiv (d\omega/dk) \underset{k\to 0}{\sim} c_s, \quad \text{with at most} \quad c_s = 1/\sqrt{3} < c$$

for an ideal conformal fluid of massless particles.

Linear perturbations: shear-mode

We now consider the propagation of the perturbation

$$\delta u^y(t,x) \quad \text{with} \quad \delta u^y(t=0,x) \equiv \delta u_0^y(x) = \delta u_0\, \delta(x)$$

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The relevant linearized equations read

$$(\epsilon_0 + P_0)\partial_t \delta u^y + \partial_x \pi^{xy} = 0 \quad \text{with} \quad \pi^{xy} = -\eta_0 \partial_x \delta u^y$$

leading to the *diffusion equation*

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leading to

$$\delta u^y(t,x) = \int \frac{dk}{2\pi} e^{-\frac{\eta_0}{\epsilon_0+P_0}k^2 t} e^{ikx} \delta u^y_{\omega_k,k}$$

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$$\delta u^y(t,x) = \int \frac{dk}{2\pi} e^{-\frac{\eta_0}{\epsilon_0+P_0}k^2 t} e^{ikx} \delta u^y_{\omega_k,k}$$

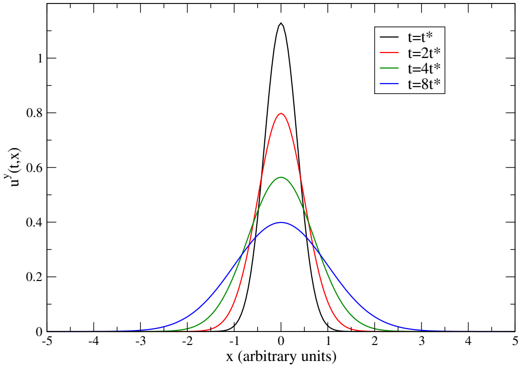
In order to satisfy the initial condition one must have $\delta u^y_{\omega_k,k} = \delta u_0$ so that

$$\delta u^y(t,x) = \int \frac{dk}{2\pi} e^{-\frac{\eta_0}{\epsilon_0+P_0}k^2 t} e^{ikx} \delta u_0$$

Diffusion of shear perturbations: causality problems

The integral in Eq. (1) can be evaluated completing the square. One gets:

$$\delta u^y(t,x) = \sqrt{\frac{1}{4\pi[\eta_0/(\epsilon_0+P_0)]t}}\exp\left[-\frac{x^2}{4[\eta_0/(\epsilon_0+P_0)]t}\right]$$



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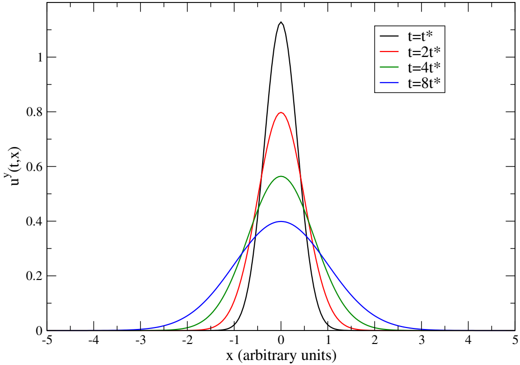

In response to the initial perturbation one gets a *non-vanishing* fluid velocity even in *causally disconnected* regions (i.e. $x > ct$)!

Relativistic causal theory: second order formalism

The naive relativistic generalization of the Navier Stokes equations violates causality! This pathology can be cured *including viscous corrections into the entropy current*, of second order in the gradients (Israel-Stewart theory):

$$s^\mu = s^\mu_{\rm eq} + Q^\mu = s u^\mu - \left(\beta_0 \Pi^2 + \beta_2 \pi_{\alpha\beta}\pi^{\alpha\beta}\right) \frac{u^\mu}{2T}$$

One gets then ($Df \equiv \dot{f}$):

$$T\partial_\mu s^\mu = \Pi\left[-\Theta - \beta_0\dot{\Pi} - T\Pi\,\partial_\mu(\beta_0 u^\mu/2T)\right] + \pi^{\alpha\beta}\left[\nabla_{<\alpha}u_{\beta>} - \beta_2\dot{\pi}_{\alpha\beta} - T\pi_{\alpha\beta}\,\partial_\mu(\beta_2 u^\mu/2T)\right] \geq 0,$$

which is satisfied if $\Pi \approx \zeta[-\Theta - \beta_0\dot{\Pi}]$ and $\pi_{\alpha\beta} \approx 2\eta[\nabla_{<\alpha}u_{\beta>} - \beta_2\dot{\pi}_{\alpha\beta}]$.

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which is satisfied if $\Pi \approx \zeta[-\Theta - \beta_0\dot{\Pi}]$ and $\pi_{\alpha\beta} \approx 2\eta[\nabla_{<\alpha}u_{\beta>} - \beta_2\dot{\pi}_{\alpha\beta}]$. One has then to evolve also the components of the viscous tensor (6 independent equations, due to $u_\mu\pi^{\mu\nu}=0$ and $\pi^\mu_\mu=0$)

$$\dot{\Pi} \approx -\frac{1}{\zeta\beta_0}[\Pi + \zeta\Theta] \quad \text{and} \quad \dot{\pi}_{\alpha\beta} \approx -\frac{1}{2\eta\beta_2}[\pi_{\alpha\beta} - 2\eta\nabla_{<\alpha}u_{\beta>}],$$

whera $\tau_\Pi \equiv \zeta\beta_0$ and $\tau_\pi \equiv 2\eta\beta_2$ play the role of *relaxation times*.

Israel Stewart equations for a conformal fluid

For a conformal fluid the only energy scale is the temperature, hence the evolution of all dimensionful quantities depends on the one of the temperature. Since $\epsilon = 3P \sim T^4$ and $\beta_2 \sim T^{-4}$ one has

$$D\epsilon + (\epsilon + P)\theta \approx 0 \quad \longrightarrow \quad \frac{DT}{T} + \frac{1}{3}\theta \approx 0$$

$$\frac{D\beta_2}{\beta_2} = -4\frac{DT}{T} \quad \longrightarrow \quad \frac{D\beta_2}{\beta_2} = \frac{4}{3}\theta$$

Substituing this into

$$\pi^{\alpha\beta} = 2\eta\,[\nabla_{<\alpha} u_{\beta>} - \beta_2\dot{\pi}_{\alpha\beta} - T\pi_{\alpha\beta}\,\partial_\mu(\beta_2 u^\mu/2T)]$$

and exploiting the definition $\tau_\pi \equiv 2\eta\beta_2$ one gets:

$$\boxed{\dot{\pi}_{\alpha\beta} = -\frac{1}{\tau_\pi}\left[\pi_{\alpha\beta} - 2\eta\,\sigma_{\alpha\beta} + \frac{4}{3}\tau_\pi\pi_{\alpha\beta}\theta\right]}$$

Linear perturbations: causality restoration

The equations to solve are now

$$(\epsilon_0 + P_0)\partial_t \delta u^y + \partial_x \pi^{xy} = 0 \quad \text{and} \quad \tau_\pi \partial_t \pi^{xy} + \pi^{xy} = -\eta_0 \partial_x \delta u^y$$

which in Fourier space read

$$(\epsilon_0 + P_0)(-i\omega)\delta u^y_{\omega,k} + (ik)\delta\pi^{xy}_{\omega,k} = 0$$
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$$\text{Solving for } \delta u^y_{\omega,k} \text{ one gets} \quad \omega = -i\frac{\eta_0}{\epsilon_0 + P_0}\frac{k^2}{1 - i\omega\tau_\pi}$$

The *dispersion relation* of the shear mode is then

$$\omega = \frac{-i \pm \sqrt{-1 + 4[\eta_0/(\epsilon_0 + P_0)]k^2\tau_\pi}}{2\tau_\pi}$$

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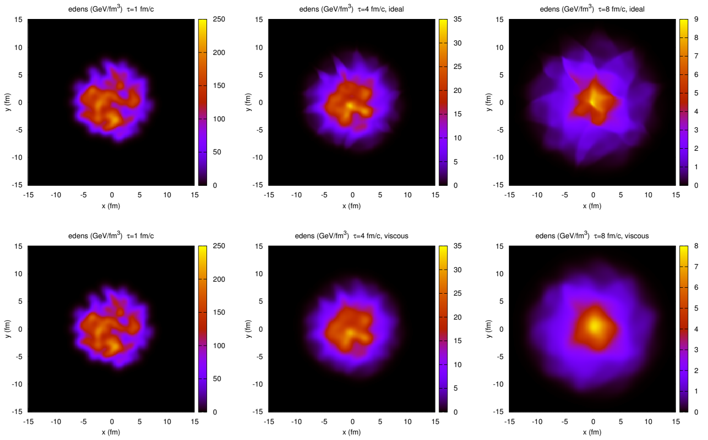
Its short-wavelength limit is given by:

$$\omega_k \underset{k\to\infty}{\sim} \sqrt{\frac{\eta_0}{\epsilon_0 + P_0}\frac{1}{\tau_\pi}}\,k \quad \Longrightarrow \quad v^T \equiv \frac{d\omega_k}{dk} \underset{k\to\infty}{\sim} \sqrt{\frac{\eta_0}{\epsilon_0 + P_0}\frac{1}{\tau_\pi}}$$

For a conformal fluid of massless particles the relaxation time is $\tau_\pi = 5\left(\frac{\eta_0}{s_0}\right)\frac{1}{T_0} = 5\frac{\eta_0}{\epsilon_0 + P_0}$, so that $v^T < c$!

Ideal vs viscous evolution

As an example, starting from the *same initial condition* (an ultra-central Au-Au collisions at $\sqrt{s_{\rm NN}} = 200\,GeV$), we display the different evolution of the energy density in the ideal (upper panels) and viscous (lower panels) case



Viscosity damps short-wavelength modes!

Microspopic approach to hydrodynamics: kinetic theory

The energy-momentum tensor can be written as the following integral over the *on-shell* ($p^0 = E_p$) single-particle distribution $f(x,p)$

$$T^{\mu\nu}(x) \equiv \int d\chi\, p^\mu p^\nu f(x,p), \quad \text{where} \quad d\chi \equiv \frac{d\vec{p}}{(2\pi)^3 E_p}.$$

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The evolution of $f(x,p)$ is described by the Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = -\mathcal{C}[f] \quad \text{where} \quad \mathcal{C}[f] \equiv \mathcal{C}^{\rm loss}[f] - \mathcal{C}^{\rm gain}[f]$$

Notice that, defining $\lambda_{\rm mfp} \equiv 1/(\sigma n)$, $\mathrm{RHS}=0$ in two opposite regimes

- in the *free-streaming limit* ($\lambda_{\rm mfp}/L \to \infty$, not so interesting)
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NB1: whether a hydro picture applies depends on the $\lambda_{\rm mfp}/L$ ratio: the air at room temperature and $P = 1$ atm can be considered either a perfect (almost) collisionless gas ($\lambda_{\rm mfp} \approx 68$ nm vs $d \approx 3$ nm) or as a fluid, if we are interested at its macroscopic behavior (e.g. weather forecast).

Microspopic approach to hydrodynamics: kinetic theory

The energy-momentum tensor can be written as the following integral over the *on-shell* ($p^0 = E_p$) single-particle distribution $f(x, p)$

$$T^{\mu\nu}(x) \equiv \int d\chi\, p^\mu p^\nu f(x,p), \quad \text{where} \quad d\chi \equiv \frac{d\vec{p}}{(2\pi)^3 E_p}.$$

The evolution of $f(x, p)$ is described by the Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = -\mathcal{C}[f] \quad \text{where} \quad \mathcal{C}[f] \equiv \mathcal{C}^{\text{loss}}[f] - \mathcal{C}^{\text{gain}}[f]$$

Notice that, defining $\lambda_{\text{mfp}} \equiv 1/(\sigma n)$, $\text{RHS} = 0$ in two opposite regimes

- in the *free-streaming limit* ($\lambda_{\text{mfp}}/L \to \infty$, not so interesting)
- in the *ideal fluid limit* ($\lambda_{\text{mfp}}/L \to 0$: strongly coupled regime!)

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NB2 Kinetic theory is more microscopic, but it relies on a (quasi-)particle description: excitations in a strongly-coupled many-body system can be very different!

Particle and energy-momentum conservation

Let us consider the particle-number current

$$j^\mu(x) \equiv \int d\chi\, p^\mu f(x,p) = \int \frac{d\vec{p}}{(2\pi)^3} \frac{p^\mu}{E_p} f(x,p)$$

From the $0^{\rm th}$ moment of the Boltzmann equation one gets its conservation law

$$\partial_\mu \int d\chi\, p^\mu f(x,p) = \int d\chi\, p^\mu \partial_\mu f(x,p) = -\int d\chi\, \mathcal{C}[f]$$

so that $\partial_\mu j^\mu = 0 \Leftrightarrow \int d\chi\, \mathcal{C}[f] = 0$: it is not necessary that the collision term vanishes, but just its momentum intergral, in order to conserve the number of particles.

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From the $1^{\rm st}$ moment of the Boltzmann equation one obtains the conservation of the energy-momentum tensor

$$\partial_\mu \int d\chi\, p^\mu p^\nu f(x,p) = \int d\chi\, p^\nu p^\mu \partial_\mu f(x,p) = -\int d\chi\, p^\nu \mathcal{C}[f]$$

implying $\partial_\mu T^{\mu\nu} = 0 \Leftrightarrow \int d\chi\, p^\nu \mathcal{C}[f] = 0$

Conserved currents in QFT

In ordinary kinetic theory particle number usually refers to atoms and its is always conserved: the previous continuity equation holds.
In a relativistic system described by Quantum Field Theory the situation is more complex:

- There are particles (e.g. photons, gluons) whose number is not conserved, since they can be radiated or absorbed without violating any conservation law;
- There are particles (e.g. quarks, electrons...) which obey global conservation law and for which only their *net number* is conserved. In this case the conserved current is given by

$$j^\mu \equiv \int d\chi\, p^\mu \left[f_q(x,p) - f_{\bar{q}}(x,p)\right]$$

NB In the following, even when not written explicitly, in the equations for current conservation what enters is always the difference between particle and antiparticle distributions!

Kinetic theory and ideal hydrodynamics

In the case of an ideal fluid one has $f(x,p) = f_{\rm eq}(p \cdot u(x))$ so that

$$T^{\mu\nu}_{\rm eq} = \int d\chi\, p^\mu p^\nu f_{\rm eq}(p\cdot u) = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$$

For *classical particles* with g internal degrees of freedom $f_{\rm eq} = g\, e^{-p\cdot u/T}$.

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Exploiting $u_\mu \Delta^{\mu\nu} = 0$ and $\Delta_{\mu\nu}\Delta^{\mu\nu} = 3$ one gets:

$$\epsilon = u_\mu u_\nu T^{\mu\nu}_{\rm eq} = \int d\chi \, (p\cdot u)^2 f_{\rm eq}(p\cdot u)$$

$$P = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}_{\rm eq} = -\frac{1}{3}\Delta_{\mu\nu}\int d\chi \, p^\mu p^\nu (g_{\mu\nu} - u_\mu u_\nu) f_{\rm eq}(p\cdot u) \underset{p^2\to 0}{\sim} \frac{\epsilon}{3}$$

where the last equality holds for a fluid of massless particles.

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where the last equality holds for a fluid of massless particles. In this case ϵ and P can be easily evaluated working in the LRF:

$$\epsilon = g\int \frac{d\vec{p}}{(2\pi)^3 E_p} E_p^2\, e^{-E/T} = g\frac{1}{2\pi^2}\int_0^\infty p^3\, dp\, e^{-p/T}$$

$$= g\frac{T^4}{2\pi^2}\int_0^\infty dx\, x^3\, e^{-x} = g\frac{T^4}{2\pi^2}\Gamma(4) = g\frac{3T^4}{\pi^2}$$

and $P = \epsilon/3 = gT^4/\pi^2$.

Partons in the QGP

The average density of partons (gluons, quarks and antiquarks) in a hot QGP is given by:

$$n = g \int \frac{d\vec{p}}{(2\pi)^3 E_p} E_p \, e^{-E/T} = g \frac{T^3}{2\pi^2} \int_0^\infty dx \, x^2 \, e^{-x} = g \frac{T^3}{2\pi^2} \Gamma(3) = g \frac{T^3}{\pi^2}$$

The average energy per parton is given by $\epsilon / n = 3T$.

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The average energy per parton is given by $\epsilon/n = 3T$. The number of active d.o.f. to sum over when $m_u, m_d, m_s \ll T \ll m_c$ is given by $(N_c = N_f = 3)$

$$g_{\rm dof} = \underbrace{2 \times (N_c^2 - 1)}_{\rm gluons} + \underbrace{2 \times 2 \times N_c \times N_f}_{\rm quarks+antiquarks} = 16 + 36 = 52$$

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Partonic cross sections must be of order $\sigma \sim \alpha_s^2/T^2$, hence

$$\lambda_{\rm mfp} = \frac{1}{n\sigma} \sim \frac{\pi^2}{g_{\rm dof} \, T^3 \alpha_s^2 / T^2} = \frac{\pi^2}{g_{\rm dof} \alpha_s^2 T}$$

For $\alpha_s \sim 0.3$ and $T \sim 0.4$ GeV one has

$$\lambda_{\rm mfp} \sim \frac{10}{50 \cdot 0.1 \cdot 0.4 \, {\rm GeV}} \sim 1 \, {\rm fm} \ll L \sim 10 \, {\rm fm},$$

hence hydrodynamic conditions are marginally satisfied.

Kinetic theory and dissipative hydrodynamics

In a non-ideal fluid one gets *viscous corrections* to the particle distribution and hence to the energy-momentum tensor and conserved currents

$$T^{\mu\nu} = \int d\chi\, p^\mu p^\nu [f_{\rm eq}(p\cdot u) + \delta f(x,p)] = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \Pi^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu$$

$$j^\mu = \int d\chi\, p^\mu [f_{\rm eq}(p\cdot u) + \delta f(x,p)] = n u^\mu + V^\mu$$

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From the Landau matching conditions $u_\mu \Pi^{\mu\nu} = u_\mu q^\mu = u_\mu V^\mu = 0$, i.e.

$$\epsilon = u_\mu u_\nu T^{\mu\nu} = u_\mu u_\nu T^{\mu\nu}_{\rm eq} \quad \text{and} \quad n = u_\mu j^\mu = u_\mu j^\mu_{\rm eq}$$

allowing one to define an effective temperature and chemical potential even for an *off-equilibrium system* one gets

$$\boxed{\int d\chi\, (p\cdot u)^2\, \delta f(x,p) = 0} \quad \text{and} \quad \boxed{\int d\chi\, (p\cdot u)\, \delta f(x,p) = 0}$$

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Choosing the Landau frame $q^\mu = 0 \longleftrightarrow u_\mu T^{\mu\nu} = u_\mu T^{\mu\nu}_{\rm eq} = \epsilon u^\nu$ sets also

$$\boxed{\int d\chi\, (p\cdot u) p^\nu\, \delta f(x,p) = 0}$$

δf and dissipative corrections (I)

In the Landau frame, with no heat-conduction, we found

$$\Pi^{\mu\nu} \equiv \int d\chi p^\mu p^\nu \delta f,$$

which can be decomposed as

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \frac{1}{3}\Pi^\alpha_\alpha \Delta^{\mu\nu} \equiv \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi$$

One has

$$\Pi = -\frac{1}{3}\Delta_{\mu\nu}\Pi^{\mu\nu} \quad \text{and} \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\Pi^{\alpha\beta}$$

where $\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$.

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$$\boxed{\Pi = -\frac{1}{3}\Delta_{\mu\nu}\int d\chi p^\mu p^\nu \delta f} \quad \text{and} \quad \boxed{\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int d\chi p^\alpha p^\beta \delta f}$$

For the QGP one has

$$\Pi = -\frac{1}{3}\Delta_{\mu\nu}\int d\chi p^\mu p^\nu [\delta f_g + \delta f_q + \delta f_{\bar{q}}] \qquad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int d\chi p^\alpha p^\beta [\delta f_g + \delta f_q + \delta f_{\bar{q}}]$$

δf and dissipative corrections (II)

The diffusion current was found to be

$$V^\mu = \int d\chi p^\mu \delta f$$

The Landau matching conditions force it to be transverse to the fluid four-velocity, $u_\mu V^\mu = 0$, hence

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One is left with the problem of evaluating δf

The relaxation-time approximation

The Relaxation Time Approximation for the collision integral allows one to simplify the structure of the BE ($\delta f \equiv f - f_{\mathrm{eq}}$)

$$p^\mu \partial_\mu f = -\frac{(p \cdot u)}{\tau_R} \delta f$$

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- Particle conservation

$$\partial_\mu \int d\chi \, p^\mu f = 0 \quad \Longleftrightarrow \quad \boxed{\int d\chi \frac{(p \cdot u)}{\tau_R} \delta f = 0}$$

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- Energy-momentum conservation

$$\partial_\mu \int d\chi \, p^\mu p^\nu f = 0 \quad \Longleftrightarrow \quad \boxed{\int d\chi \frac{(p \cdot u)}{\tau_R} p^\nu \delta f = 0}$$

  If τ_R independent of p it is equivalent to setting the Landau frame, which is the only consistent choice in the RTA of the BE

RTA and Chapman-Enskog expansion (I)

Write explicitly the RTA form of the BE for a non-relativistic flow

$$E\left(\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial\vec{x}}\right)f(t,\vec{x};\vec{p})=-\frac{E}{\tau_R}\delta f(t,\vec{x};\vec{p})$$

Rewrite the equation with dimensionless variables, factorizing the typical macroscopic or intrincic scales at which we explore the medium:

$$t'\equiv t/\mathcal{T},\quad x'=x/\mathcal{L},\quad \mathcal{L}=U\mathcal{T},\quad v'\equiv v/c_s,$$

where U is the macroscopic collective velocity of the fluid and the speed of sound c_s is the typical particle velocity. One has

$$\left(\frac{\tau_R}{\mathcal{T}}\frac{\partial}{\partial t'}+\frac{\tau_R c_s}{\mathcal{L}}\,\vec{v}'\cdot\frac{\partial}{\partial\vec{x}'}\right)f=-\delta f$$

Since $\tau_R\approx\lambda_{\rm mfp}/c_s$ one gets

$$\left(\frac{\lambda_{\rm mfp}}{\mathcal{L}}\frac{U}{c_s}\frac{\partial}{\partial t'}+\frac{\lambda_{\rm mfp}}{\mathcal{L}}\,\vec{v}'\cdot\frac{\partial}{\partial\vec{x}'}\right)f=-\delta f$$

RTA and Chapman-Enskog expansion (II)

Introducing the Mach number $\mathrm{Ma} \equiv U/c_s$ and the Knudsen number $\mathrm{Kn} \equiv \lambda_{\mathrm{mfp}}/\mathcal{L}$ one has

$$\left(\mathrm{Ma} \cdot \mathrm{Kn} \cdot \frac{\partial}{\partial t'} + \mathrm{Kn} \cdot \vec{v}' \cdot \frac{\partial}{\partial \vec{x}'}\right) f = -\delta f$$

Expand now the particle distribution around the local thermal equilibrium solution $f_0 \equiv f_{\mathrm{eq}}$ using the Knudsen number $\varepsilon \equiv \mathrm{Kn}$ as a small parameter:

$$f = f_0 + \varepsilon \Delta f_1 + \varepsilon^2 \Delta f_2 + \ldots$$

Substituting the above expansion into the BE one gets

$$\left(\mathrm{Ma} \cdot \varepsilon \cdot \frac{\partial}{\partial t'} + \varepsilon \cdot \vec{v}' \cdot \frac{\partial}{\partial \vec{x}'}\right) (f_0 + \varepsilon \Delta f_1 + \varepsilon^2 \Delta f_2 + \ldots) = -\varepsilon \Delta f_1 - \varepsilon^2 \Delta f_2 + \ldots$$

It is evident that to get Δf_1 one substitutes f_0 in the LHS and, in general, the Δf_n correction comes from the Δf_{n-1} term in the LHS

RTA and Chapman-Enskog expansion (III)

The Chapman-Enskog expansion arises naturally from an iterative solution of the BE in RTA

$$f = f_0 - \frac{\tau_R}{(p\cdot u)}(p\cdot\partial)\, f$$

One gets

$$f_1 = f_0 - \frac{\tau_R}{(p\cdot u)}(p\cdot\partial)\, f_0 \equiv f_0 + \delta f_1$$
$$f_2 = f_0 - \frac{\tau_R}{(p\cdot u)}(p\cdot\partial)\, f_1 \equiv f_0 + \delta f_1 + \delta f_2$$
$$f_3 = \dots$$

In particular one has

$$\boxed{\delta f_1 = -\frac{\tau_R}{(p\cdot u)}(p\cdot\partial)\, f_0} \quad \text{and} \quad \boxed{\delta f_2 = \frac{\tau_R}{(p\cdot u)}(p\cdot\partial)\left[\frac{\tau_R}{(p\cdot u)}(p\cdot\partial)\, f_0\right]}$$

The Chapman-Enskog approach amounts then to a *gradient expansion* around the local thermal equilibrium solution

First-order results: shear tensor

The first-order result is obtained replacing $\delta f \to \delta f_1$. For the shear tensor one has for instance

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta \frac{(-\tau_R)}{(p\cdot u)} p^\gamma \partial_\gamma f_0$$

From $f_0 = g\, e^{-\beta(p\cdot u)}$ and the decomposition $\partial_\mu = u_\mu D + \nabla_\mu$ one gets

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta \frac{(-\tau_R)}{(p\cdot u)} \left[-\beta(p\cdot u) p^\delta D u_\delta - \beta p^\gamma p^\delta \nabla_\gamma u_\delta \right.$$
$$\left. -(p\cdot u)^2 D\beta - (p\cdot u) p^\gamma \nabla_\gamma \beta \right] f_0$$

Three different kind of terms can be identified

$$I = \Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta g(p\cdot u) = \Delta^{\mu\nu}_{\alpha\beta}[A_{2,0} u^\alpha u^\beta + A_{0,2}\Delta^{\alpha\beta}] = 0$$

$$II = \Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta p^\gamma g(pu) = \Delta^{\mu\nu}_{\alpha\beta}[A_{3,0} u^\alpha u^\beta u^\gamma + A_{1,2}(u^\alpha \Delta^{\beta\gamma} + \mathrm{perm})] = 0$$

$$III = \Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta p^\gamma p^\delta g(p\cdot u) \neq 0$$

First-order results: shear tensor

Consider in detail the only structure contributing

$$\Delta^{\mu\nu}_{\alpha\beta}\int d\chi p^\alpha p^\beta p^\gamma p^\delta g(p\cdot u) = \Delta^{\mu\nu}_{\alpha\beta}\left[A_{4,0}u^\alpha u^\beta u^\gamma u^\delta + A_{2,2}(u^\alpha u^\beta \Delta^{\gamma\delta} + \mathrm{perm}) + A_{0,4}(\Delta^{\alpha\beta}\Delta^{\gamma\delta} + \Delta^{\alpha\gamma}\Delta^{\beta\delta} + \Delta^{\alpha\delta}\Delta^{\beta\gamma})\right]$$

Only the last two terms, once contracted, contribute. One gets

$$\Delta^{\mu\nu}_{\alpha\beta}\int d\chi p^\alpha p^\beta p^\gamma p^\delta g(p\cdot u) = 2\,A_{0,4}\left[\frac{1}{2}(\Delta^{\mu\gamma}\Delta^{\nu\delta} + (\Delta^{\nu\gamma}\Delta^{\mu\delta}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\gamma\delta}\right]$$

where

$$A_{0,4} = \frac{1}{15}\int d\chi \Delta_{\alpha\gamma}\Delta_{\beta\delta}p^\alpha p^\beta p^\gamma p^\delta\, g(p\cdot u) \underset{m\to 0}{\sim} \frac{1}{15}\int d\chi (p\cdot u)^4\, g(p\cdot u)$$

and

$$g(p\cdot u) = \frac{\beta\tau_R}{(p\cdot u)}\, g_{\mathrm{dof}}\, e^{-\beta(p\cdot u)}$$

First-order results: shear tensor

One obtains then

$$\pi^{\mu\nu} = 2A_{0,4}\left[\frac{1}{2}(\Delta^{\mu\gamma}\Delta^{\nu\delta} + (\Delta^{\nu\gamma}\Delta^{\mu\delta}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\gamma\delta}\right]\nabla_\gamma u_\delta$$
$$= 2A_{0,4}\left[\frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\Delta^{\mu\nu}\Theta\right] \equiv 2\eta\sigma^{\mu\nu}$$

We have then identitified the *physical meaning* of $A_{0,4}$, which represents the shear viscosity η of the medium, which we can now calculate:

$$\eta = \frac{g_{\rm dof}}{15}\frac{\tau_R}{T}\int d\chi(p\cdot u)^3 e^{-p\cdot u/T} = \frac{g_{\rm dof}}{15}\frac{\tau_R}{T}\frac{1}{2\pi^2}\int_0^\infty dp\, p^{5-1}e^{-p/T}$$
$$= \frac{g_{\rm dof}}{15}\frac{\tau_R}{T}\frac{1}{2\pi^2}T^5\Gamma(5) = \frac{1}{5}\tau_R\, g_{\rm dof}\frac{4T^4}{\pi^2}$$

For a relativistic fluid of massless particles with $\epsilon = 3P$ one has then

$$\eta = \frac{1}{5}\tau_R(\epsilon + P) = \frac{4}{5}\tau_R P$$

On the η/s ratio

The QGP produced in HIC's was found to be the "most ideal" fluid in the universe, with a very small η/s ratio. Let us estimate this ratio in our approach. From $\epsilon + P = Ts$ one has

$$\eta = \frac{1}{5}\tau_R(\epsilon + P) = \frac{1}{5}\tau_R Ts \quad \longrightarrow \quad \frac{\eta}{s} = \frac{T}{5}\tau_R = \frac{T}{5}\frac{1}{n\sigma},$$

where in the relativistic limit $\tau_R \approx \lambda_{\rm mfp}$. One has

$$n = g_{\rm dof}\frac{T^3}{\pi^2}, \quad \sigma \sim \frac{\alpha_s^2}{T^2} \quad \longrightarrow \quad \frac{\eta}{s} \approx \frac{\pi^2}{5 g_{\rm dof}\alpha_s^2} \approx \frac{10}{5\cdot 50\cdot 0.1} \approx 0.4,$$

where we employed $g_{\rm dof} \approx 50$ and $\alpha_s \approx 0.3$.

On the η/s ratio

The QGP produced in HIC's was found to be the "most ideal" fluid in the universe, with a very small η/s ratio. Let us estimate this ratio in our approach. From $\epsilon + P = Ts$ one has

$$\eta = \frac{1}{5}\tau_R(\epsilon + P) = \frac{1}{5}\tau_R Ts \quad \longrightarrow \quad \frac{\eta}{s} = \frac{T}{5}\tau_R = \frac{T}{5}\frac{1}{n\sigma},$$

where in the relativistic limit $\tau_R \approx \lambda_{\rm mfp}$. One has

$$n = g_{\rm dof}\frac{T^3}{\pi^2}, \quad \sigma \sim \frac{\alpha_s^2}{T^2} \quad \longrightarrow \quad \frac{\eta}{s} \approx \frac{\pi^2}{5 g_{\rm dof}\alpha_s^2} \approx \frac{10}{5 \cdot 50 \cdot 0.1} \approx 0.4,$$

where we employed $g_{\rm dof} \approx 50$ and $\alpha_s \approx 0.3$.

In order to reproduce the experimental data one needs a 2-4 times smaller value, but this naive estimate is already able to reproduce the correct order of magnitude. The vary small η/s ratio arises here from the huge number of relativistic degrees of freedom which are thermally excited

Comparison with the non-relativistic limit

The result of kinetic theory for a gas of non-relativistic particles is

$$\eta_{\rm nr} = n(K_B T)\tau_R$$

To be compared with the relativistic result ($\epsilon = 3nT = 3P$, $K_B = 1$)

$$\eta_{\rm rel} = \frac{4}{5}P\tau_R = \frac{4}{5}nT\tau_R$$

where n is the total density of partons of any kind.

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where n is the total density of partons of any kind.
Maxwell showed that in a non-relativistic gas the viscosity is independent of the density. Since $\tau_R \approx \lambda_{\rm mfp}/\bar{v}$ one has

$$\eta_{\rm nr} \approx n(K_B T)\frac{\lambda_{\rm mfp}}{\bar{v}} = n\frac{K_B T}{\bar{v}}\frac{1}{n\sigma} \sim \frac{\sqrt{M K_B T}}{\sigma}$$

On the contrary, in a ultrarelativistic fluid $s = 4n$, hence

$$\frac{\eta}{s} \approx \frac{\pi^2}{5 g_{\rm dof}\alpha_s^2} \quad \longrightarrow \quad \eta \approx \frac{4\pi^2}{5 g_{\rm dof}\alpha_s^2} n \,,$$

growing linearly with the particle density, since $n \sim T^3$ and $\sigma \sim T^{-2}$.

Evaluation of δf_1

The first-order (in the gradients) dissipative correction to the distribution funtion is given by

$$\delta f_1 = -\frac{\tau_R}{(p \cdot u)}(p \cdot \partial)\, f_0 = -\frac{\tau_R}{(p \cdot u)} p^\mu \left[-\beta p^\nu (u_\mu D u_\nu + \nabla_\mu u_\nu) - (p \cdot u)(u_\mu D\beta + \nabla_\mu \beta)\right] f_0$$

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$$\left. -(p\cdot u)(u_\mu D\beta + \nabla_\mu \beta)\right] f_0$$

Due to the contraction with $p^\mu p^\nu$ one can substitute

$$\nabla_\mu u_\nu \longrightarrow \sigma_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\Theta$$

At first order in the gradients one can express the derivatives of β through the ideal hydrodynamic equations for a conformal fluid:

$$D\beta = \frac{1}{3}\beta\Theta \qquad \nabla_\mu \beta = -\beta D u_\mu$$

Almost all terms cancel except one, leading to

$$\delta f_1 = \beta \frac{p^\mu p^\nu}{(p\cdot u)} \tau_R \sigma_{\mu\nu} f_0$$

If the system relaxes infinitely fast ($\tau_R \to 0$) the dissipative correction vanishes, as expected

Evaluation of δf_1

One can go one step further, exploiting the first-order result

$$\pi^{\mu\nu} = 2\frac{\epsilon + P}{5}\tau_R \sigma^{\mu\nu}$$

leading to

$$\boxed{\delta f_1 = \frac{5}{2T(p\cdot u)(\epsilon + P)} p^\mu p^\nu \pi_{\mu\nu} f_0}$$

Evaluation of δf_1

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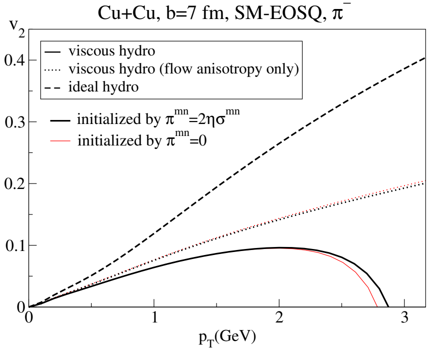


The correction δf affects the distributions of final hadrons decoupling from the fireball[a]

[a] H. Song and U. Heinz, Phys.Rev.C 77

Evolution equation for the shear tensor

In order to get a causal theory one must write an evolution equation for $\pi^{\mu\nu}$, whose comoving derivative contains two terms:

$$D\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int d\chi p^\alpha p^\beta D(\delta f) + (D\Delta^{\mu\nu}_{\alpha\beta})\int d\chi p^\alpha p^\beta \delta f$$

The last term involves the contraction

$$(D\Delta^{\mu\nu}_{\alpha\beta})\Pi^{\alpha\beta} = (D\Delta^{\mu\nu}_{\alpha\beta})\pi^{\alpha\beta} - (D\Delta^{\mu\nu}_{\alpha\beta})\Delta^{\alpha\beta}\Pi$$

All the terms contracted with $\Delta^{\alpha\beta}$ vanish. In fact

$$(D\Delta^{\mu\nu}_{\alpha\beta})\Delta^{\alpha\beta} = -\frac{1}{2}\cdot 2\cdot(u^\mu Du^\nu + u^\nu Du^\mu) + \frac{1}{3}\cdot 3\cdot(u^\mu Du^\nu + u^\nu Du^\mu) = 0$$

One gets a non-zero contribution from the contraction with $\pi^{\alpha\beta}$

$$(D\Delta^{\mu\nu}_{\alpha\beta})\pi^{\alpha\beta} = -(\pi^{\mu\alpha}u^\nu + \pi^{\alpha\nu}u^\mu)Du_\alpha$$

The latter vanishes if contracted with $\Delta^{\rho\sigma}_{\mu\nu}$. Hence, after defining

$\dot{\pi}^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\rho\sigma}\dot{\pi}^{\rho\sigma}$ one has $\boxed{\dot{\pi}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\int d\chi p^\alpha p^\beta \dot{\delta f}}$

Evolution equation for the shear tensor

From the BE in RTA

$$\delta f = -\frac{\tau_R}{(p\cdot u)} p^\mu \partial_\mu f \quad \text{and} \quad \partial_\mu = u_\mu D + \nabla_\mu$$

one gets

$$\frac{\delta f}{\tau_R} = -Df - \frac{(p\cdot\nabla)}{(p\cdot u)} f = -Df_0 - D\delta f - \frac{(p\cdot\nabla)}{(p\cdot u)} f$$

Hence one has

$$\dot{\delta f} = -\dot{f}_0 - \frac{(p\cdot\nabla)}{(p\cdot u)} f - \frac{\delta f}{\tau_R}$$

which can be substituted into the previous equation, getting

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta \left[\dot{f}_0 + \frac{(p\cdot\nabla)}{(p\cdot u)} f + \frac{\delta f}{\tau_R}\right]$$

leading to

$$\boxed{\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = -\Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta \left[\dot{f}_0 + \frac{(p\cdot\nabla)}{(p\cdot u)} f\right]}$$

Evolution equation for the shear tensor

In first approximation one can substitute in the RHS $f \to f_0$ getting

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} \approx -\Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta \left[\dot{f}_0 + \frac{(p\cdot\nabla)}{(p\cdot u)} f_0\right]$$

In this form the RHS was already calculated when considered the first-order theory, getting

$$\Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta (-\tau_R) \left[\dot{f}_0 + \frac{(p\cdot\nabla)}{(p\cdot u)} f_0\right] = 2\eta\sigma^{\mu\nu},$$

with $\eta = (\epsilon + P)\tau_R/5$. Hence one finds:

$$\boxed{\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = \frac{2\eta\sigma^{\mu\nu}}{\tau_R} + \ldots}$$

which allows one to restore causality. For $\tau_R \to 0$ the system immediately approach the Navier-Stokes result $\pi^{\mu\nu}_{\rm NS} = 2\eta\sigma^{\mu\nu}$

Evolution equation for $\pi^{\mu\nu}$: full 2$^{\rm nd}$-order result

The full 2$^{\rm nd}$-order result is obtained substituting $f \to f_0 + \delta f_1$ in the RHS, obtaining

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} \approx -\Delta^{\mu\nu}_{\alpha\beta} \int d\chi p^\alpha p^\beta \left[\dot{f}_0 + \frac{(p\cdot\nabla)}{(p\cdot u)} f_0 + \frac{(p\cdot\nabla)}{(p\cdot u)} \delta f_1\right],$$

where

$$\delta f_1 = \frac{5}{2T(p\cdot u)(\epsilon + P)} p^\mu p^\nu \pi_{\mu\nu}\, f_0$$

Exploiting the hydrodynamic equations to express the derivatives of the temperature in terms of derivatives of the four-velocity one gets the complete second-order result for a fluid of massless particles

$$\boxed{\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = 2\left(\frac{\epsilon + P}{5}\right)\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\Theta - \frac{10}{7}\pi_\rho^{\langle\mu}\sigma^{\nu\rangle\rho} + 2\pi_\rho^{\langle\mu}\omega^{\nu\rangle\rho}}$$

where we have introduced the vorticity $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ and we have defined $A^{\langle\mu}B^{\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^\alpha B^\beta$.

Why does hydro work so well?

Let us consider, for simplicity, the case of a pure Bjorken expansion

$$u^\mu = (t/\tau, 0, 0, z/\tau) = (\cosh\eta_s, 0, 0, \sinh\eta_s)$$

From $u_\mu \pi^{\mu\nu} = 0$, $\pi^\mu_\mu = 0$, $u_\perp = 0$ and $SO(2)$ invariance in the transverse plane one has, in the LRF:

$$T^{\mu\nu}_{\rm LRF} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P+\pi/2 & 0 & 0 \\ 0 & 0 & P+\pi/2 & 0 \\ 0 & 0 & 0 & P-\pi \end{pmatrix} \equiv \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_T & 0 & 0 \\ 0 & 0 & P_T & 0 \\ 0 & 0 & 0 & P_L \end{pmatrix}$$

The viscous hydrodynamic equations reduce to ($\tau \equiv \sqrt{t^2 - z^2}$)

$$\frac{d\epsilon}{d\tau} = -\frac{\epsilon + P - \pi}{\tau}$$
$$\frac{d\pi}{d\tau} = -\frac{1}{\tau_R}\left(\pi - \frac{4}{3}\frac{\eta}{\tau}\right) - \lambda\frac{\pi}{\tau},$$

where IS $\to \lambda = 4/3$, CE2 $\to \lambda = 38/21$ and $\pi \underset{\tau_R \to 0}{\sim} \pi_{\rm NS} \equiv 4\eta/3\tau$

Why does hydro work so well?

In the case of a pure Bjorken expansion it is possible to obtain an exact solution of the Boltzmann equation

$$\frac{df}{d\tau} = -\frac{f - f_{\rm eq}}{\tau_R}\,.$$

Introducing the dimensionless variable $w \equiv tp^z - zE$ one gets:

$$\boxed{f(\tau, p_T, w) = D(\tau, \tau_0) f_0(p_T, w) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_R(\tau')} D(\tau, \tau') f_{\rm eq}(\tau', p_T, w)},$$

where

$$\tau_R(\tau) = \frac{5(\eta/s)}{T(\tau)} \quad \text{and} \quad D(\tau_2, \tau_1) \equiv \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau'}{\tau_R(\tau')}\right]$$

is a sort of Sudakov factor representing the no-interaction probability from τ_1 to τ_2. The solution can then be inserted into

$$T^{\mu\nu} = \int d\chi p^\mu p^\nu f(x, p)$$

Why does hydro work so well?

As an initial condition we take a *distorted thermal distribution*

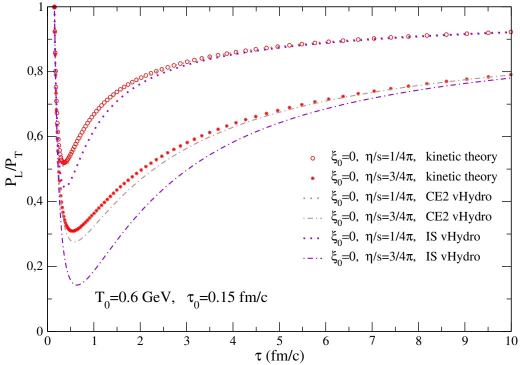
$$f_0(p_T, w) = g_{\rm dof} \exp\left[-\frac{\sqrt{(1+\xi_0)w^2 + p_T^2\tau_0^2}}{\Lambda_0\tau_0}\right],$$

where Λ_0 sets the typical transverse-momentum scale and ξ_0 quantifies the anisotropy:

$$\xi_0 = \frac{\langle p_T^2\rangle_0}{2\langle p_z^2\rangle_0} - 1\,.$$

Due to the initial huge expansion rate along the $z-$axis, $\theta = 1/\tau$, one must have $\langle p_z^2\rangle \ll \langle p_T^2\rangle$. Hence physical initial conditions must correspond to $\xi_0 > 0$ (*oblate* momentum-space distribution)

Why does hydro work so well?



- 2^{nd}-order Chapmam-Enskog much better approximation of exact result than Israel-Stewart

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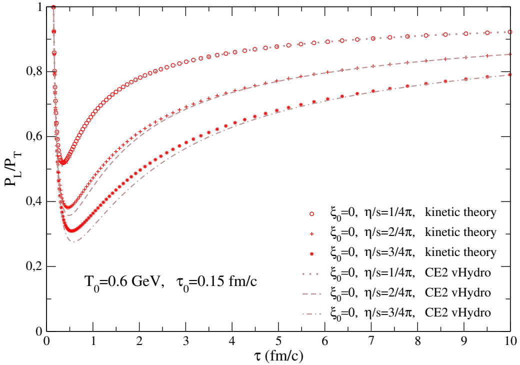

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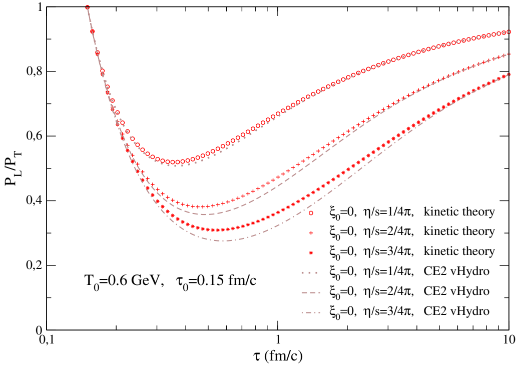

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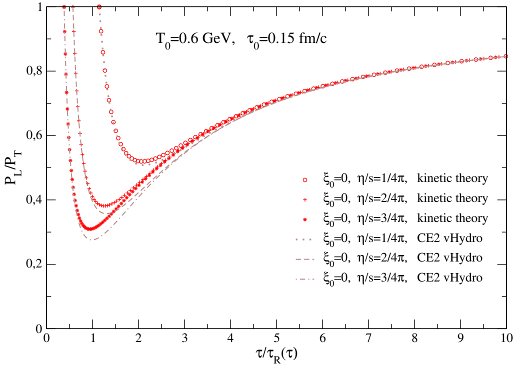

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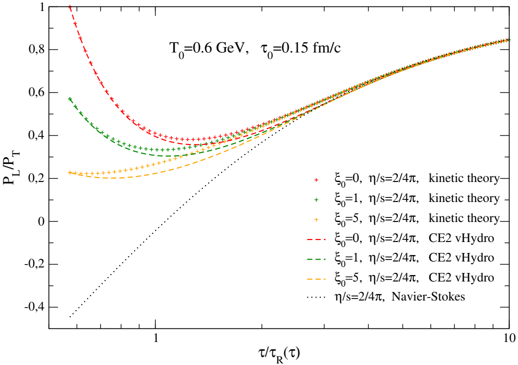

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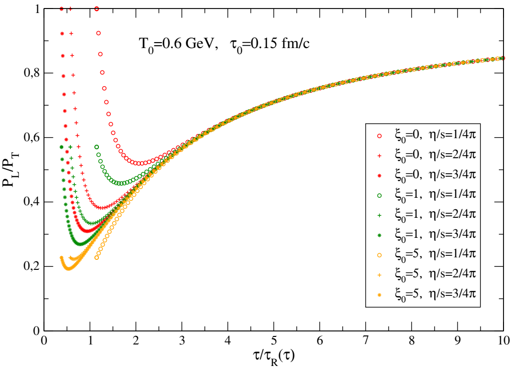
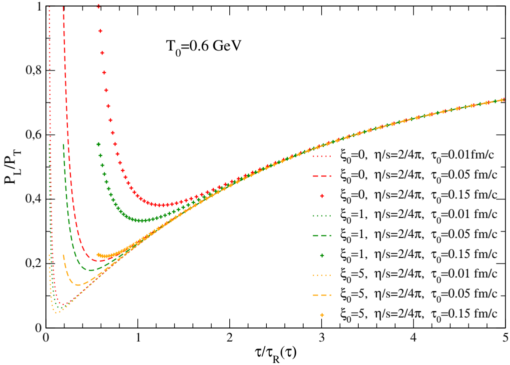
- Differences from kinetic theory increase with η/s: expansion in Kn
- Curves as functions of $\tau/\tau_R(\tau)$ approach a universal result *before* isotropization is reached (attractor)
- Independently from ξ_0, CE2 curves approach first Navier-Stokes and then kinetic-theory solution

Late and early-time attractor in kinetic theory



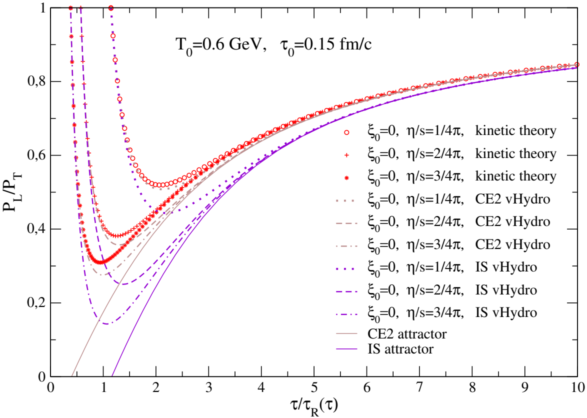
- Once initialized at τ_0 the system approaches a universal curve, indendently from initial anisotropy and specific viscosity (late-time attractor);

Late and early-time attractor in kinetic theory



- Once initialized at τ_0 the system approaches a universal curve, indendently from initial anisotropy and specific viscosity (late-time attractor);
- By taking τ_0 smaller and smaller $\rightarrow$ early-time attractor, characterized by infinite initial anisotropy at $\tau_0 = 0^+$

Attractors in viscous hydrodynamics



- Both IS and CE2 approximations to viscous hydrodynamics display an attractor, however CE2 solution in much better agreement with kinetic theory;
- vHydro attractor unphysical at very early time ($P_L < 0$)

Where does vHydro attractor come from?

It is useful to introduce the dimensionless variables

$$\bar{w} \equiv \frac{\tau}{\tau_R(\tau)}, \quad \phi \equiv \frac{\partial \ln \bar{w}}{\partial \ln \tau} = \frac{\tau \partial_\tau \bar{w}}{\bar{w}}$$

For a conformal fluid the only physical scale is the temperature, hence:

$$\bar{w} \sim \tau \cdot T \sim \tau \cdot \epsilon^{1/4}$$

so that

$$\phi = \frac{\tau \epsilon^{1/4} + \tau^2 (1/4) \epsilon^{-3/4} \partial_\tau \epsilon}{\tau \epsilon^{1/4}} \quad \longrightarrow \quad \boxed{\phi = 1 + \frac{1}{4} \frac{\partial \ln \epsilon}{\partial \ln \tau}}.$$

From the energy-conservation law for a conformal fluid ($P = \epsilon/3$) one has

$$\frac{\partial \epsilon}{\partial \tau} = -\frac{4}{3} \frac{\epsilon}{\tau} + \frac{\pi}{\tau} \quad \longrightarrow \quad \frac{\partial \ln \epsilon}{\partial \ln \tau} = -\frac{4}{3} + \frac{\pi}{\epsilon},$$

leading to

$$\boxed{\frac{\pi}{\epsilon} = 4 \left(\phi - \frac{2}{3} \right)} \quad \longrightarrow \lim_{\bar{w} \to \infty} \phi = \frac{2}{3}$$

Where does vHydro attractor come from?

One has then ($\phi' \equiv \partial\phi/\partial\bar{w}$):

$$\tau\partial_\tau\left(\frac{\pi}{\epsilon}\right) = 4\tau\partial_\tau\phi = 4\tau\phi'\partial_\tau\bar{w} = 4\bar{w}\phi'\frac{\partial\ln\bar{w}}{\partial\ln\tau} = 4\bar{w}\phi\phi'$$

The LHS of the above equation can be written as:

$$\tau\partial_\tau\left(\frac{\pi}{\epsilon}\right) = \tau\frac{\dot{\pi}}{\epsilon} - \frac{\pi}{\epsilon}\frac{\partial\ln\epsilon}{\partial\tau}$$

Hence one gets:

$$4\bar{w}\phi\phi' = \tau\frac{\dot{\pi}}{\epsilon} - 4\left(\phi - \frac{2}{3}\right)4(\phi - 1)\,.$$

One can now exploit the evolution equation for the viscous contribution:

$$\tau_R\frac{d\pi}{d\tau} + \pi + \lambda\frac{\tau_R}{\tau}\pi - \frac{4}{3}\frac{4}{15}\frac{\epsilon}{(\tau/\tau_R)} = 0 \quad\longrightarrow\quad \tau\frac{\dot{\pi}}{\epsilon} + \frac{\pi}{\epsilon}\bar{w} + \lambda\frac{\pi}{\epsilon} - \frac{16}{45} = 0\,,$$

Where does vHydro attractor come from?

One gets:

$$4\bar{w}\phi\phi' + 16\left(\phi - \frac{2}{3}\right)(\phi - 1) = \frac{16}{45} - 4\left(\phi - \frac{2}{3}\right)(\bar{w} + \lambda)$$

Hence the differential equation to solve is:

$$\bar{w}\phi\phi' = -4\phi^2 + \left(\frac{20}{3} - \lambda - \bar{w}\right)\phi - \left(\frac{116}{45} - \frac{2}{3}(\lambda + \bar{w})\right)$$

In order to have a regular solution for $\bar{w} \to 0$ one must have

$$\boxed{4\phi_0^2 - \left(\frac{20}{3} - \lambda\right)\phi_0 + \left(\frac{116}{45} - \frac{2}{3}\lambda\right) = 0},$$

whose stable solutions are:

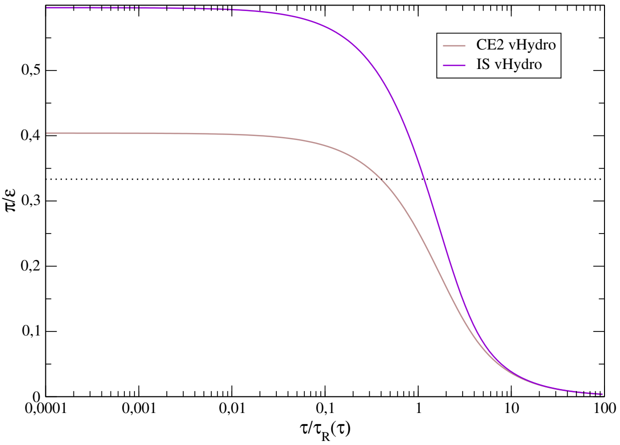
$$\phi_0^{\mathrm{IS}} = \frac{1}{15}(10 + \sqrt{5}), \qquad \phi_0^{\mathrm{CE2}} = \frac{1}{140}(85 + \sqrt{505}).$$

Attractor for the pressure anisotropy

$$\frac{P_L}{P_T} = \frac{P - \pi}{P + \pi/2} = \frac{1 - 3\frac{\pi}{\epsilon}}{1 + \frac{3}{2}\frac{\pi}{\epsilon}} = \frac{3 - 4\phi}{2\phi - 1}$$

Requiring both $P_L > 0$ and $P_T > 0$ one gets $\phi < 3/4$ and $\phi > 1/2$. Only when these conditions are satisfied the attractor solution is physically acceptable

vHydro attractor: numerical solutions



For $\bar{w} \lesssim 0.43$ the CE2 vHdydro attractor solution is unphysical, since it would correspond to negative longitudinal pressure

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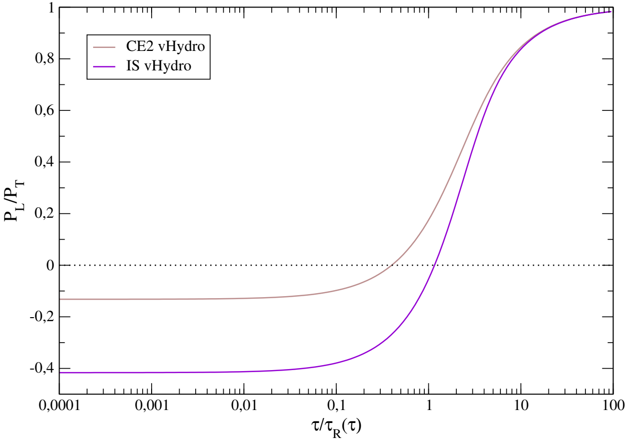

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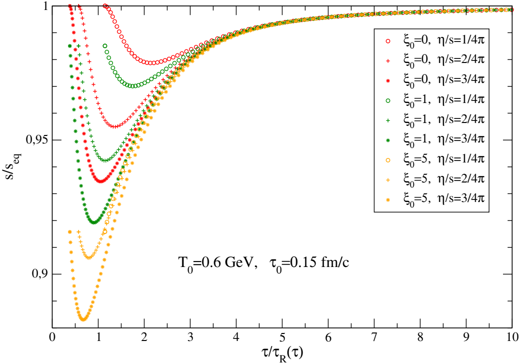
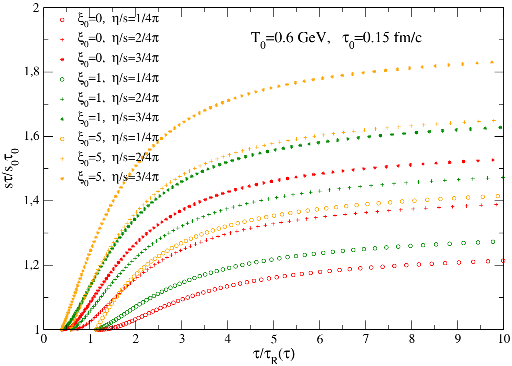
Entropy production



$$s(\tau) = -g_{\rm dof} \int d\chi (p \cdot u) f(\tau, p_T, w) [\ln(f(\tau, p_T, w)) - 1] \, .$$

- When large anisotropy $\longrightarrow$ entropy lower than $s_{\rm eq} \equiv (\epsilon + P)/T$

Entropy production



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- Most entropy produced before hydrodynamization, $\tau/\tau_R \lesssim 3$

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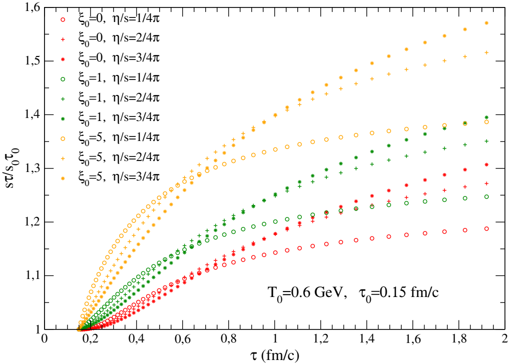

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- When large anisotropy $\longrightarrow$ entropy lower than $s_{\rm eq} \equiv (\epsilon + P)/T$
- Most entropy produced before hydrodynamization, $\tau/\tau_R \lesssim 3$
- *Initially*, lower entropy production for large η/s (free-streaming)

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