Quark Gluon Plasma thermodynamics

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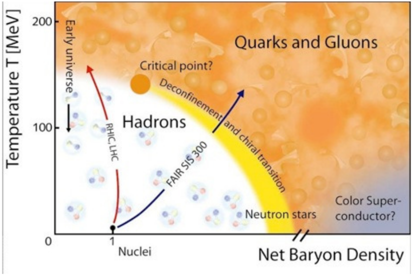
Heavy-ion collisions: exploring the QCD phase-diagram



QCD phases identified through the *order parameters*

- Polyakov loop $\langle L \rangle \sim e^{-\beta \Delta F_Q}$: energy cost to add an isolated color charge
- Chiral condensate $\langle \overline{q} q \rangle \sim$ effective mass of a "dressed" quark in a hadron

Region explored at LHC: *high-T/low-density* (early universe, $n_B/n_\gamma \sim 10^{-9}$)

- From QGP (color deconfinement, chiral symmetry restored)
- to hadronic phase (confined, chiral symmetry breaking[1])

NB $\langle \overline{q} q \rangle \neq 0$ responsible for most of the baryonic mass of the universe: *only* $\sim$*35 MeV of the proton mass from* $m_{u/d} \neq 0$

[1] V. Koch, *Aspects of chiral symmetry*, Int.J.Mod.Phys. E6 (1997)

Virtual experiments: lattice-QCD simulations

- The best (unique?) tool to study QCD in the non-perturbative regime
- Limited to the study of equilibrium quantities

QCD on the lattice

Expectation values of operators are evaluated on a discretized euclidean lattice ($1/T = N_\tau a$) starting from the QCD partition function

$$\mathcal{Z} = \int [dU] \exp\left[-\beta S_g(U)\right] \prod_q \det\left[M(U, m_q)\right]$$

through a MC sampling of the field configurations, where[2]

- $\beta = 6/g^2$
- S_g is the gauge action, weighting the different field configurations;
- $U \in SU(3)$ is the link variable connecting two lattice sites;
- $M \equiv \gamma_\mu D_\mu + m_q$ is the Dirac operator

[2] See M. Panero lattice-QCD lectures

QCD at high-temperature: expectations

Based on *asymptotic freedom*, for $T \gg \Lambda_{QCD}$ hot-QCD matter should behave like a non-interacting plasma of massless quarks (the ones for which $m_q \ll T$) and gluons. In such a regime T is the only scale μ at which evaluating the gauge coupling, for which one has

$$\lim_{T/\Lambda_{QCD}\to\infty} g(\mu \sim T) = 0$$

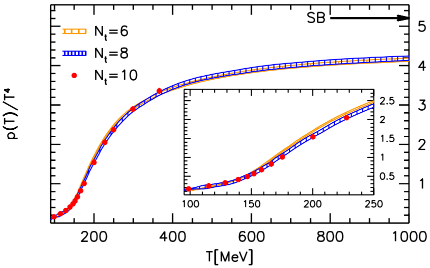
Hence one expects the asymptotic *Stefan-Boltzmann* behaviour

$$\epsilon = \frac{\pi^2}{30}\left[g_{\rm gluon} + \frac{7}{8} g_{\rm quark}\right] T^4,$$

where

$$g_{\rm gluon} = \underbrace{2 \times (N_c^2 - 1)}_{\text{pol.} \times \text{col.}} \quad \text{and} \quad g_{\rm quark} = \underbrace{2 \times 2 \times N_c \times N_f}_{q/\overline{q} \times \text{spin} \times \text{col.} \times \text{flav.}}$$

From the partition function on gets all the thermodynamical quantities:



Data by the W.B. Collaboration [JHEP 1011 (2010) 077]

- Pressure: $P = (T/V)\ \ln \mathcal{Z}$;

QCD on the lattice: results

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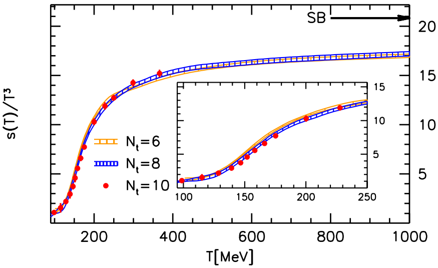

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- Pressure: $P = (T/V) \ln \mathcal{Z}$;
- Entropy density: $s = \partial P / \partial T$;

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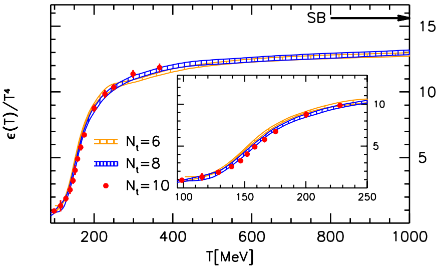

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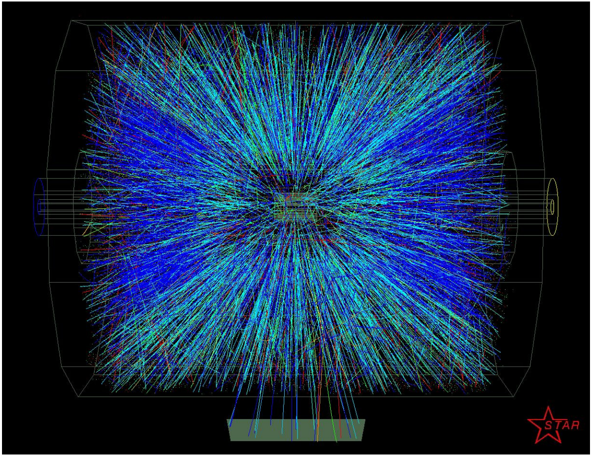
- Pressure: $P = (T/V)\ln \mathcal{Z}$;
- Entropy density: $s = \partial P/\partial T$;
- Energy density: $\epsilon = Ts - P$;

- Rapid rise in thermodynamical quantities suggesting a change in the *number of active degrees of freedom* (hadrons → partons): the most dramatic drop experienced by the early universe in which

$$H^2 = \frac{8\pi G}{3}\epsilon_{\rm rel} = \frac{8\pi G}{3}\frac{\pi^2}{30}g_* T^4$$

- One observes a systematic $\sim$20% deviation from the Stephan-Boltzmann limit even at large T: how to interpret it? In the last part of this lecture we will attempt an explanation

From final hadrons back to initial conditions



Au-Au collision at $\sqrt{s_{\rm NN}}=200$ GeV: tracks left by *charged particles* in the Time Projection Chamber of the STAR detector.

From final hadrons back to initial conditions

Invariant single-particle spectra

$$E\frac{d^3N}{d\vec{p}} = \frac{d^3N}{d^2p_T dy}$$

are generally expressed via the *transverse momentum* and the *rapidity*

$$p_T \equiv \sqrt{p_x^2 + p_y^2} \quad \text{and} \quad y \equiv \frac{1}{2}\ln\frac{E+p_z}{E-p_z}$$

Since y requires the knowledge of m (hence Particle IDentification, involving a longer analysis), one more often uses the *pseudorapidity*

$$\eta \equiv \frac{1}{2}\ln\frac{p+p_z}{p-p_z} = \frac{1}{2}\ln\frac{1+p_z/p}{1-p_z/p} = \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta}$$

Of course for relativistic particles $y \approx \eta$! In such a way one can simply perform two *geometrical* measurements from the tracks: p_T from the curvature in the B-field ($p_T = |q|B\,r$) and η from the polar angle. One of the first observables one can get is then the *rapidity density* of *charged particles* $dN_{\rm ch}/d\eta$

Bjorken coordinates in heavy-ion collisions

While a possible transverse expansion takes time to develop, the fluid produced in HICs is characterized by an initial Hubble-like *longitudinal expansion* $v_z^{\rm fluid} = z/t$: since the collision take place at $z = 0$, a fluid-cell found at z at a later time t is there because its constituens share a common average velocity v_z. The strong longitudinal expansion makes convenient to define the *longitudinal proper-time* and the *spatial rapidity*

$$\tau \equiv \sqrt{t^2 - z^2} \quad \text{and} \quad \eta_s \equiv \frac{1}{2} \ln \frac{t+z}{t-z} = \frac{1}{2} \ln \frac{1+z/t}{1-z/t}$$

and, accordingly,

$$t = \tau \cosh \eta_s \quad \text{and} \quad z = \tau \sinh \eta_s$$

Microscopic dynamics within a fluid cell depends on the time measured in its local rest frame. At large z, this "proper"-time would be boosted by a large γ-factor, requiring to follow the evolution of the system up to very large values of time $t = \sqrt{\tau^2 + z^2}$ measured in the lab-frame.
It is also convenient to introduce the *fluid-rapidity*

$$Y \equiv \frac{1}{2} \ln \frac{1+v_z^{\rm fluid}}{1-v_z^{\rm fluid}} \quad \text{so that} \quad u^\mu = \gamma_\perp(\cosh Y, \mathbf{u}_\perp, \sinh Y) \ \text{with} \ \gamma_\perp \equiv (1-\mathbf{u}_\perp^2)^{-1/2}$$

If the longitudinal dynamics follows a Bjorken expansion with $v_z^{\rm fluid} = z/t$ fluid-rapidity and spatial rapidity coincide: $\eta_s = Y$

Initial conditions: "Bjorken" estimate

- It is useful to describe the evolution in term of the variables

$$\tau \equiv \sqrt{t^2 - z^2} \quad \text{and} \quad \eta_s \equiv \frac{1}{2} \ln \frac{t+z}{t-z}$$

  Independence of the initial conditions on η_s entails $v_z^{\rm fluid} = z/t$ throughout the fluid evolution;

- For a *purely longitudinal* Hubble-like expansion entropy conservation implies:

$$s\,\tau = s_0\,\tau_0 \quad \longrightarrow \quad s_0 = (s\,\tau)/\tau_0$$

- Entropy density is defined in the local fluid rest-frame:

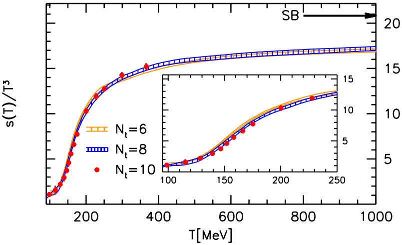
$$s \equiv \left.\frac{dS}{d\mathbf{x}_\perp dz}\right|_{z=0} = \frac{1}{\tau}\frac{dS}{d\mathbf{x}_\perp d\eta_s} \quad \longrightarrow \quad s\tau = \frac{dS}{d\mathbf{x}_\perp d\eta_s}$$

- Entropy is related to the *final multiplicity of charged particles* ($S \sim 3.6\,N$ for pions), so that (at decoupling $\eta \approx \eta_s$):

$$s_0 \approx \frac{1}{\tau_0}\frac{3.6}{\pi R_A^2}\frac{dN_{\rm ch}}{d\eta}\frac{3}{2}$$

"Bjorken" estimate: results

$$s_0 \approx \frac{1}{\tau_0} \frac{3.6}{\pi R_A^2} \frac{dN_{\rm ch}}{d\eta} \frac{3}{2}$$



- From $dN_{\rm ch}/d\eta \approx 1600$ measured by ALICE at LHC and $R_{\rm Pb} \approx 6$ fm one gets:

$$s_0 \approx (80\,{\rm fm}^{-2})/\tau_0$$

- τ_0 is found to be quite small (v_2 must develop early!):

$$0.1 \lesssim \tau_0 \lesssim 1\ {\rm fm} \ \longrightarrow\ 80 \lesssim s_0 \lesssim 800\ {\rm fm}^{-3}$$

- This should be compared with l-QCD

$$s(T = 200\,{\rm MeV}) \approx 10\,{\rm fm}^{-3}$$

Quantum harmonic oscillator at finite temperature

Its is possible to study the quantum statistical mechanics of a single harmonic oscillator coupled to a thermal bath. The generalization to finite temperature QFT will be straightforward.

$$H = \frac{p^2}{2} + \frac{1}{2}\omega_k^2 q^2$$

Introducing the raising/lowering operators $a^\dagger/a$ one gets:

$$H = \omega_k \left(a^\dagger a + \frac{1}{2} \right), \quad \text{with} \quad q = \frac{1}{\sqrt{2\omega_k}} \left(a + a^\dagger \right) \quad \text{and} \quad [a, a^\dagger] = 1$$

The partition function is easily evaluated:

$$Z = \mathrm{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta\omega_k(n+1/2)} = e^{-\beta\omega_k/2} \sum_{n=0}^{\infty} e^{-\beta\omega_k n} = e^{-\beta\omega_k/2} \frac{1}{1 - e^{-\beta\omega_k}}$$

One gets then for the *thermodynamical potential* Ω

$$Z = e^{-\beta\Omega} \quad \longrightarrow \quad \Omega = \Omega_{\text{vac}} + \Omega_T = \frac{\omega_k}{2} + \ln\left(1 - e^{-\beta\omega_k}\right),$$

where we have separated the contribution of *vacuum* (responsible for UV divergences in QFT) and *thermal* fluctuations.

The vacuum and thermal fluctuations of the displacement are given by

$$\langle q^2 \rangle = \frac{1}{2\omega_k} \langle \underbrace{aa^\dagger + a^\dagger a}_{[a,a^\dagger]+2a^\dagger a} \rangle = \frac{1 + 2N_k}{2\omega_k},$$

involving the Bose-Einstein distribution $\langle a^\dagger a \rangle = N_k \equiv 1/(e^{\beta\omega_k} - 1)$.
From the h.o. algebra

$$[H, a] = -\omega_k a \qquad [H, a^\dagger] = \omega_k a^\dagger$$

one gets the time-evolution of the raising/lowering operators

$$a(t) = a\, e^{-i\omega_k t} \qquad a^\dagger(t) = a^\dagger e^{i\omega_k t}$$

Dim:

$$a(t) = e^{iHt}\, a\, e^{-iHt} \;\longrightarrow\; \frac{da}{dt} = i\, e^{iHt}\, [H, a]\, e^{-iHt} = -i\omega_k a(t)$$

In *thermal field theory* one often works with imaginary times $t = -i\tau$

$$a(\tau) = a\, e^{-\omega_k \tau} \qquad a^\dagger(\tau) = a^\dagger e^{\omega_k \tau} \quad \text{with} \quad \tau \in [0, \beta]$$

Temporal propagators

The propagator along the real-time axis is defined as:

$$D(t_1-t_2)=\theta(t_1-t_2)D^>(t_1-t_2)+\theta(t_2-t_1)D^<(t_1-t_2),$$

where (from the algebra of the raising/lowering oprators):

$$D^>(t_1-t_2)\equiv\langle q(t_1)q(t_2)\rangle=\frac{1}{2\omega_k}\left[(1+N_k)e^{-i\omega_k(t_1-t_2)}+N_k e^{i\omega_k(t_1-t_2)}\right]$$
$$D^<(t_1-t_2)\equiv\langle q(t_2)q(t_1)\rangle=\frac{1}{2\omega_k}\left[(1+N_k)e^{i\omega_k(t_1-t_2)}+N_k e^{-i\omega_k(t_1-t_2)}\right]$$

For the *imaginary-time* propagator $\mathcal{D}(\tau)\equiv D(t=-i\tau)$ one has:

$$\mathcal{D}(\tau)=\frac{1}{2\omega_k}\left[(1+N_k)e^{-\omega_k|\tau|}+N_k e^{\omega_k|\tau|}\right],\quad \text{with}\quad \tau\in[-\beta,\beta]$$

for which the symmetry property $\mathcal{D}(\tau-\beta)=\mathcal{D}(\tau)$ (with $0\le\tau\le\beta$) holds. The propagator can then be expanded in *Matsubara frequencies*

$$\mathcal{D}(\tau)=\frac{1}{\beta}\sum_{n=-\infty}^{\infty}e^{-i\omega_n\tau}\Delta(i\omega_n)\quad\text{with}\quad \omega_n\equiv(2\pi/\beta)n$$

Path integrals in statistical QM

We wish to provide a *path-integral representation* for the partition function of a quantum system in equilibrium with a thermal bath

$$Z \equiv \mathrm{Tr} e^{-\beta H} = \int dx \, \langle x | e^{-\beta H} | x \rangle$$

here written in the basis of *position-eigenstates* $\{|x\rangle\}$.
The starting point is the transition amplitude ${}_H\langle x_f, t_f | x_i, t_i \rangle_H$

$$\langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle = \int_{x(t_i)=x_i, x(t_f)=x_f} [\mathcal{D}x(t)] \; e^{i \int_{t_i}^{t_f} dt [1/2 (dx/dt)^2 - V(x)]}$$

Moving to imaginary times $t = -i\tau$ one has

$$\langle x_f | e^{-H(\tau_f - \tau_i)} | x_i \rangle = \int_{x(\tau_i)=x_i, x(\tau_f)=x_f} [\mathcal{D}x(\tau)] \; e^{- \int_{\tau_i}^{\tau_f} d\tau [1/2 (dx/d\tau)^2 + V(x)]}$$

Hence, setting $\tau_i = 0$ and $\tau_f = \beta$ and periodic boundary conditions,

$$Z = \int_{x(0)=x(\beta)} [\mathcal{D}x(\tau)] \; e^{- \int_0^{\beta} d\tau [1/2 (dx/d\tau)^2 + V(x)]}$$

Harmonic Oscillator partition function: path-integral

The h.o. partition function reads then (after integration by parts)

$$Z = \int_{q(0)=q(\beta)} [\mathcal{D}q(\tau)]\ e^{-\int_0^\beta d\tau (1/2) q(\tau)[-\partial_\tau^2 + \omega_k^2] q(\tau)}$$

Such a representation turns out to be convenient

- for the evaluation of *correlation functions*
- as a starting point for a *perturbative expansion* in the presence of interactions (i.e. non-gaussian terms in the action)

It is convenient to introduce a source term $j(\tau)$

$$Z[j] = \int_{q(0)=q(\beta)} [\mathcal{D}q(\tau)]\ e^{-\int_0^\beta d\tau [(1/2) q(\tau)(-\partial_\tau^2 + \omega_k^2) q(\tau) - j(\tau) q(\tau)]} \qquad (1)$$

Let us now introduce the *Green function* of the differential operator

$$(-\partial_\tau^2 + \omega_k^2)\mathcal{D}(\tau - \tau') = \delta(\tau - \tau')$$

We will show that it actually coincides with the previously introduced *thermal propagator*

Performing the change of variables

$$q(\tau) = q'(\tau) + \int d\tau' \mathcal{D}(\tau - \tau') j(\tau')$$

the generating functional $Z[j]$ can be written as

$$Z[j] = Z[0] \times \exp\left[\frac{1}{2}\int d\tau\, d\tau' j(\tau)\mathcal{D}(\tau - \tau') j(\tau')\right] \qquad (2)$$

Combining Eqs. (1) and (2) one gets

$$\left.\frac{\delta^2 \ln Z[j]}{\delta j(\tau_1)\delta j(\tau_2)}\right|_{j=0} = \mathcal{D}(\tau_1 - \tau_2) = \langle T_\tau\, q(\tau_1) q(\tau_2)\rangle$$

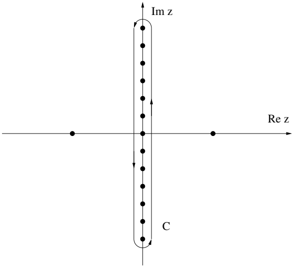
The equivalence can be verified also decomposing the Green function $\mathcal{D}(\tau - \tau')$ in its Matsubara modes $\Delta(i\omega_n)$

$$(-\partial_\tau^2 + \omega_k^2)\frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-i\omega_n(\tau-\tau')}\Delta(i\omega_n) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{-i\omega_n(\tau-\tau')}$$

One gets then

$$\Delta(i\omega_n) = \frac{1}{\omega_n^2 + \omega_k^2}$$

The sum over Matsubara frequencies



The function to integrate has simple poles

- at $z = i\omega_n$ $(n = 0, \pm 1, \pm 2 \ldots)$
- at $z = \pm\omega_k$

The Matsubara sum can be expressed as (for $0 \leq \tau \leq \beta$)

$$\frac{1}{\beta}\sum_n e^{-i\omega_n \tau}\Delta(i\omega_n) = \oint \frac{dz}{2\pi i}\frac{e^{-z\tau}}{1-e^{-\beta z}}\frac{-1}{(z-\omega_k)(z+\omega_k)}$$

as can be verified expanding $(1 - e^{-\beta z})$ around $z = i\omega_n$.
The integral can be also evaluated taking the residues of the poles *outside* the C contour at $z = \pm\omega_k$. One gets:

$$\mathcal{D}(\tau > 0) = \frac{1}{2\omega_k}\left[(1 + N_k)e^{-\omega_k \tau} + N_k e^{\omega_k \tau}\right]$$

which coincides with what previously found!

Thermodynamics of the free scalar field

Let us now promote the displacement operator to a *scalar field* function of space and time:

$$q(t) \quad \longrightarrow \quad \phi(t,\vec{x})$$

The free Lagrangian/Hamiltonian involves also spatial gradients:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \quad \mathcal{H}_0 = \frac{\pi^2}{2} + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

The raising/lowering operators now create/destroy particles with momentum $\vec{k}$ and energy $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$ and one has

$$H_0 = \sum_{\vec{k}} \omega_k \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \right) \quad \longrightarrow \quad \Omega_0 = \sum_{\vec{k}} \frac{\omega_k}{2} + \sum_{\vec{k}} \ln\left(1 - e^{-\beta\omega_k}\right)$$

The (UV divergent!) contribution of *vacuum-fluctuations* does not depend on T and can be dropped (if we are not interested in gravity, cosmological constant....). In the infinite-volume limit one gets:

$$\sum_{\vec{k}} \to V\!\int \frac{d\vec{k}}{(2\pi)^3} \quad \Longrightarrow \quad P_0 = -\frac{\Omega_0}{V} = -\frac{1}{\beta}\int \frac{d\vec{k}}{(2\pi)^3} \ln\left(1 - e^{-\beta\omega_k}\right) = \frac{\pi^2}{90}T^4$$

From the "box-quantization" of the field

$$\phi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right] \quad \text{with} \quad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'}$$

one can estimate its fluctuations ($\langle a^\dagger_{\vec{k}} a_{\vec{k}'} \rangle = \delta_{\vec{k},\vec{k}'} N_k$)

$$\langle \phi^2(\vec{x}) \rangle = \frac{1}{V} \sum_{\vec{k}} \frac{1 + 2N_k}{2\omega_k} \underset{\text{inf. volume}}{\longrightarrow} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k}$$

where one can isolate the (UV divergent!) vacuum fluctuations

$$\langle \phi^2 \rangle_{\text{vac.}} \sim \int^\Lambda \frac{k^2 dk}{k} \sim \Lambda^2$$

From the commutation relations

$$[H, a_{\vec{k}}] = -\omega_k a_{\vec{k}} \qquad [H, a^\dagger_{\vec{k}}] = \omega_k a^\dagger_{\vec{k}}$$

one gets that the evolution of the $a_{\vec{k}}/a^\dagger_{\vec{k}}$ operators is

$$a_{\vec{k}}(\tau) = a_{\vec{k}}\, e^{-\omega_k \tau} \qquad a^\dagger_{\vec{k}}(\tau) = a^\dagger_{\vec{k}} e^{\omega_k \tau} \quad \text{with} \quad \tau \in [0, \beta]$$

The propagator is then a generalization of the harmonic oscillator result

$$\mathcal{D}_0(\tau, \vec{k}) = \frac{1}{2\omega_k} \left[(1 + N_k) e^{-\omega_k |\tau|} + N_k e^{\omega_k |\tau|} \right], \quad \text{with} \quad \tau \in [-\beta, \beta]$$

Free scalar field: path-integral

The partition function of the free scalar field is given by

$$Z_0 = \int_{\phi(0,\mathbf{x})=\phi(\beta,\mathbf{x})} [\mathcal{D}\phi(\tau,\mathbf{x})]\; e^{-\int_0^\beta d\tau d\mathbf{x}\,(1/2)[(\partial_\tau\phi)^2+(\boldsymbol{\nabla}\phi)^2+m^2\phi^2]}$$

After partial integration...

$$Z_0 = \int_{\phi(0,\mathbf{x})=\phi(\beta,\mathbf{x})} [\mathcal{D}\phi(\tau,\mathbf{x})]\; \underbrace{e^{-\int_0^\beta d\tau d\mathbf{x}\,(1/2)\phi[-\partial_\tau^2-\boldsymbol{\nabla}^2+m^2]\phi}}_{\equiv e^{-S_{\mathrm{E}}^0[\phi]}}$$

The euclidean propagator can then be view as the Green function

$$[-\partial_\tau^2 - \boldsymbol{\nabla}^2 + m^2]\mathcal{D}_0(\tau-\tau') = \delta(\tau-\tau')\delta(\mathbf{x}-\mathbf{x}')$$

whose Matsubara-Fourier components are given by

$$\Delta_0(i\omega_n,\vec{k}) = \frac{1}{\omega_n^2+\vec{k}^2+m^2} \equiv \frac{1}{\omega_n^2+\omega_k^2},$$

as in the case of the harmonic oscillator.

The interacting scalar field: thermodynamics

In order to illustrate the effects of the interaction on the thermodynamics we will consider the usual case of $\lambda\phi^4$ theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm I} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

whose partition function reads

$$Z = \int_{\phi(0,\mathbf{x})=\phi(\beta,\mathbf{x})} [\mathcal{D}\phi(\tau,\mathbf{x})]\; e^{-\int_0^\beta d\tau d\mathbf{x}\,[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4]}$$

One can attempt a *perturbative expansion* around the free result

$$Z = \int_{\phi(0,\mathbf{x})=\phi(\beta,\mathbf{x})} [\mathcal{D}\phi(\tau,\mathbf{x})]\; e^{-S_{\rm E}^0[\phi]} \times \left[1 - (\lambda/4!)\int_0^\beta d\tau d\mathbf{x}\,\phi^4(\tau,\mathbf{x}) + \mathcal{O}(\lambda^2)\right]$$

Using the free action to evaluate the thermal averages one gets

$$Z = Z_0 - Z_0(\lambda/4!)\beta V\langle\phi^4\rangle_0 + \dots = Z_0\left[1 - (\lambda/4!)\beta V\langle\phi^4\rangle_0 + \dots\right]$$

One gets then for the pressure $P \equiv \ln(Z)/\beta V$

$$P \equiv P_0 + P_1 + \dots = P_0 + \ln\left[1 - (\lambda/4!)\beta V\langle\phi^4\rangle_0 + \dots\right]/\beta V$$
$$= P_0 - (\lambda/4!)\langle\phi^4\rangle_0 + \dots$$

The pressure: the first perturbative correction

Before addressing its exact evaluation we see that the first perturbative correction to the pressure $P_1 = -(\lambda/4!)\langle\phi^4\rangle_0$ is negative: we expect the interacting result to stay *below* the Stefan-Boltzmann limit.
The correction can be computed through *Wick's theorem*:

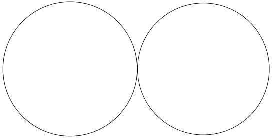
$$P_1 = -(\lambda/4!)\,3\,(\langle\phi^2\rangle_0)^2 = -(\lambda/4!)\,3\,[\mathcal{D}_0(\tau=0,\mathbf{x}=0)]^2$$
$$= -\frac{\lambda}{8}\left[\int\frac{d\vec{k}}{(2\pi)^3}\frac{1+2N_k}{2\omega_k}\right]^2$$

The calculation is plagued by UV divergences due to *vacuum fluctuations*. However, the UV divergent term in the integral does not depend on T and, for the moment, one can simply subtract it.
The final result, *including the first perturbative correction* is

$$P_0 + P_1 = -\frac{1}{\beta}\int\frac{d\vec{k}}{(2\pi)^3}\ln\left(1-e^{-\beta\omega_k}\right) - \frac{\lambda}{8}\left[\int\frac{d\vec{k}}{(2\pi)^3}\frac{N_k}{\omega_k}\right]^2$$
$$= \left(\frac{\pi^2}{90} - \frac{\lambda}{1152}\right)T^4$$

Naively one expects higher order corrections to be $\mathcal{O}(\lambda^2)$

A closer look at the UV divergences

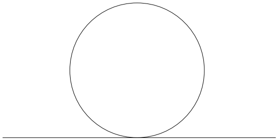


The perturbative correction P_1 to the pressure arises from the above diagram an can be recast as

$$P_1 = -\frac{\lambda}{8}\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}\right]^2 - \frac{\lambda}{8}\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{N_k}{\omega_k}\right]^2 - \frac{\lambda}{4}\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}\right]\left[\int \frac{d\vec{k}'}{(2\pi)^3}\frac{N_{k'}}{\omega_{k'}}\right]$$

If the first term can be simply dropped, being T-independent, the third term is in principle more subtle, since *vacuum (UV divergent!) and thermal fluctuations look apparently coupled*.

A closer look at the UV divergences



The source of the UV divergence is in the self-energy correction to the single-particle propagator

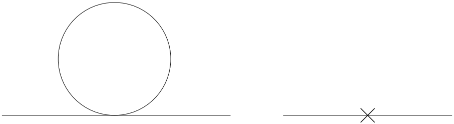
$$\Delta^{-1} = \Delta_0^{-1} + \Sigma = \omega_n^2 + \omega_k^2 + \frac{\lambda}{2}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1+2N_k}{2\omega_k},$$

where the statistical factor in the self-energy arises from the 12 different possible contractions from the Wick's theorem ($\int d^4z \equiv \int d\tau_z d\vec{z}$)

$$\int [\mathcal{D}\phi]\,\phi(x)\phi(y)\left(-\frac{\lambda}{4!}\right)\int_0^\beta d^4z\phi(z)\phi(z)\phi(z)\phi(z)\,e^{-S_{\mathrm{E}}^0[\phi]}$$

$$\sim -\frac{\lambda}{2}\int_0^\beta d^4z\mathcal{D}_0(x-z)\mathcal{D}_0(z-y)\mathcal{D}_0(\tau=0,\vec{x}=0)$$

A closer look at the UV divergences



The UV divergence can be cured adding a *mass counterterm* to $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} + \mathcal{L}_{\mathrm{ct}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{1}{2}\delta m^2\phi^2 + \dots$$

so that the self-energy reads now

$$\Sigma = \Sigma_{\mathrm{vac}} + \Sigma_T = \frac{\lambda}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1+2N_k}{2\omega_k} + \delta m^2$$

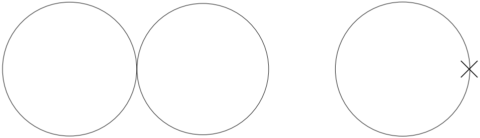
One can now impose the *renormalization condition*

$$\Sigma_{\mathrm{vac}} = \frac{\lambda}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k} + \delta m^2 = 0 \quad \longrightarrow \quad \delta m^2 = -\frac{\lambda}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}$$

The self-energy is now finite and can be identified with a *thermal mass*

$$\Sigma_T = \frac{\lambda}{2}\int\frac{d\vec{k}}{(2\pi)^3}\frac{N_k}{\omega_k} \underset{m\to 0}{=} \frac{\lambda T^2}{24} \equiv m_T^2$$

A closer look at the UV divergences



This suffices to cure also the UV divergence in the pressure. In fact now

$$Z = \int [\mathcal{D}\phi(\tau, \mathbf{x})] \; e^{-S_{\mathrm{E}}^{0}[\phi]} \times \left[1 - \int_0^\beta d\tau d\mathbf{x} \left(\frac{\lambda}{4!}\phi^4(\tau,\mathbf{x}) + \frac{1}{2}\delta m^2 \phi^2(\tau,\mathbf{x}) \right) + \dots \right]$$

so that

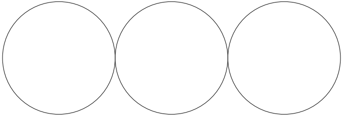
$$P_1 = -\frac{\lambda}{8}[\mathcal{D}_0(\tau = \mathbf{x} = 0)]^2 - \frac{1}{2}\delta m^2 \mathcal{D}_0(\tau = \mathbf{x} = 0)$$

$$= -\frac{\lambda}{8}\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{1+2N_k}{2\omega_k}\right]^2 - \frac{1}{2}\left[-\frac{\lambda}{2}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}\right]\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{1+2N_k}{2\omega_k}\right]$$

where the dangerous "vacuum $\times$ thermal" term cancels.

Higher order corrections

Higher order corrections to the pressure arise from diagrams like



involving the evaluation of

$$\int [\mathcal{D}\phi(x)]\, e^{-S_E^0[\phi]} \int_0^\beta d^4z\, d^4z' \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \phi^4(z)\phi^4(z')$$

From $PV \equiv (1/\beta)\ln Z$ and the Wick's contractions one gets

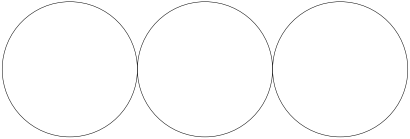
$$P_2^{\mathrm{ring}} = \frac{\lambda^2}{16}\left[\mathcal{D}_0(0,\mathbf{0})\right]^2 \frac{1}{\beta V}\int_0^\beta d^4z\, d^4z' [\mathcal{D}_0(\tau_z - \tau_z', \mathbf{z}-\mathbf{z}')]^2$$

Since

$$(1/\beta)\int_0^\beta d\tau e^{-i(\omega_n - \omega_{n'})\tau} = \delta_{n,n'} \quad \text{and} \quad \int d\mathbf{x} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

$$\text{one gets} \quad P_2^{\mathrm{ring}} = \frac{\lambda^2}{16}\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{N_k}{\omega_k}\right]^2 \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{(\omega_n^2+\omega_k^2)^2}$$

IR divergences

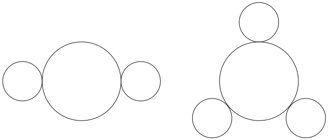


The expression we got for the “ring” diagram

$$P_2^{\text{ring}} = \frac{\lambda^2}{16}\left[\int \frac{d\vec{k}}{(2\pi)^3}\frac{N_k}{\omega_k}\right]^2 \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d\vec{k}}{(2\pi)^3}\frac{1}{(\omega_n^2+\omega_k^2)^2}$$

in the massless limit $\omega_k = k$ is plagued by an IR divergence $\sim \int dk/k^2$ arising from the $\omega_n = 0$ Matsubara mode.

Resummation of ring diagrams



The situation is even worse for higher-order similar diagrams, with a behaviour of the zero Matsubara mode $\sim \int dk/k^{2(n-1)}$. However *the whole set of ring diagrams can be resummed* (check!):

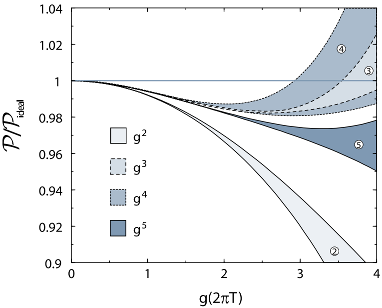
$$P^{\mathrm{ring}} = \frac{-1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d\vec{k}}{(2\pi)^3} \left[\ln(1 + \Sigma\Delta_0(i\omega_n, \vec{k})) - \Sigma\Delta_0(i\omega_n, \vec{k}) \right]$$

In the above $\Sigma = \lambda T^2/24 \equiv m_T^2$ (analogous of Debye screening mass in QED/QCD) and the dominant contribution come from the $n = 0$ mode

$$P_{n=0}^{\mathrm{ring}} = \frac{-1}{2\beta} \int \frac{d\vec{k}}{(2\pi)^3} \left[\ln\left(1 + \frac{m_T^2}{k^2}\right) - \frac{m_T^2}{k^2} \right] = \frac{m_T^3 T}{12\pi} \sim \lambda^{3/2} T^4$$

Notice the *non-analytic behaviour* in the coupling with an expansion $P \sim P_0 + P_1 + P_{3/2} + \ldots$arising from the *resummation of an infinite set of diagrams*

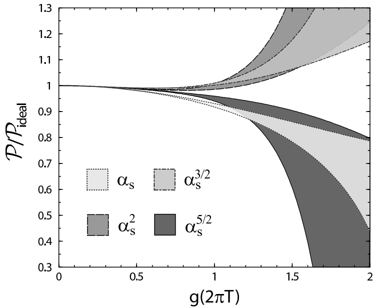
Numerical results: scalar field theory



With proper resummations the expansion can be pushed to higher orders[3]. Notice that, if the coupling is not sufficiently weak, the convergence is poor

[3] J.O. Andersen and M. Strickland, Annals Phys. 317 (2005) 281-353

Numerical results: scalar field theory



The situation looks even worse in QCD[4] and one has to develop more clever and powerful resummation schemes.

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[4] J.O. Andersen and M. Strickland, Annals Phys. 317 (2005) 281-353