Quark Gluon Plasma thermodynamics

Andrea Beraudo

INFN - Sezione di Torino

Ph.D. Lectures,
AA 2015-16 Torino
QCD phases identified through the order parameters

- Polyakov loop $\langle L \rangle \sim e^{-\beta \Delta F_Q}$: energy cost to add an isolated color charge
- Chiral condensate $\langle \bar{q}q \rangle \sim$ effective mass of a “dressed” quark in a hadron

Region explored at LHC: high-$T$/low-density (early universe, $n_B/n_\gamma \sim 10^{-9}$)

- From QGP (color deconfinement, chiral symmetry restored)
- to hadronic phase (confined, chiral symmetry breaking$^1$)

NB $\langle \bar{q}q \rangle \neq 0$ responsible for most of the baryonic mass of the universe: only $\sim 35$ MeV of the proton mass from $m_u/d \neq 0$

Virtual experiments: lattice-QCD simulations

- The best (unique?) tool to study QCD in the non-perturbative regime
- Limited to the study of equilibrium quantities
The QCD partition function

\[ Z = \int [dU] \exp[-\beta S_g(U)] \prod_q \det[M(U, m_q)] \]

is evaluated on the lattice through a MC sampling of the field configurations, where\(^2\)

- \( \beta = 6/g^2 \)
- \( S_g \) is the gauge action, weighting the different field configurations;
- \( U \in SU(3) \) is the link variable connecting two lattice sites;
- \( M \equiv \gamma_\mu D_\mu + m_q \) is the Dirac operator

\(^2\)See M. Panero lattice-QCD lectures
QCD at high-temperature: expectations

Based on *asymptotic freedom*, for \( T \gg \Lambda_{QCD} \) hot-QCD matter should behave like a non-interacting plasma of massless quarks (the ones for which \( m_q \ll T \)) and gluons. In such a regime \( T \) is the only scale \( \mu \) at which evaluating the gauge coupling, for which one has

\[
\lim_{T/\Lambda_{QCD} \to \infty} g(\mu \sim T) = 0
\]

Hence one expects the *asymptotic Stefan-Boltzmann* behaviour

\[
\epsilon = \frac{\pi^2}{30} \left[ g_{\text{gluon}} + \frac{7}{8} g_{\text{quark}} \right] T^4,
\]

where

\[
g_{\text{gluon}} = 2 \times (N_c^2 - 1) \quad \text{pol.} \times \text{col.}
\quad \text{and} \quad g_{\text{quark}} = 2 \times 2 \times N_c \times N_f \quad q/\bar{q} \times \text{spin} \times \text{col.} \times \text{flav.}
\]
QCD on the lattice: results

From the partition function on gets all the thermodynamical quantities:

\[
\begin{align*}
\text{Pressure: } & \quad P = \left( \frac{T}{V} \right) \ln Z; \\
\text{Entropy density: } & \quad s = \frac{\partial P}{\partial T}; \\
\text{Energy density: } & \quad \epsilon = Ts - P;
\end{align*}
\]

The rapid rise in thermodynamical quantities suggests a change in the number of active degrees of freedom (hadrons \(\rightarrow\) partons):

\[
H_2 = 8\pi G_3 \epsilon_{\text{rel}} = 8\pi G_3 \pi^2 30 g^* T^4
\]

One observes a systematic \(\sim 20\%\) deviation from the Stephan-Boltzmann limit even at large \(T\): how to interpret it?

In the last part of this lecture we will attempt an explanation.
From the partition function on gets all the thermodynamical quantities:

Data by the W.B. Collaboration
[JHEP 1011 (2010) 077]

- Pressure: \( P = \left( \frac{T}{V} \right) \ln Z \);
- Entropy density: \( s = \frac{\partial P}{\partial T} \);

Rapid rise in thermodynamical quantities suggesting a change in the number of active degrees of freedom (hadrons → partons):

The most dramatic drop experienced by the early universe in which

\[ H_\text{2} = 8 \pi G_3 \epsilon_{\text{rel}} \sim 8 \pi G_3 \pi^2 \frac{30}{g^*} T^4 \]

One observes a systematic \( \sim 20\% \) deviation from the Stephan-Boltzmann limit even at large \( T \): how to interpret it?

In the last part of this lecture we will attempt an explanation.
QCD on the lattice: results

From the partition function on gets all the thermodynamical quantities:

\[ P = \left( \frac{T}{V} \right) \ln Z; \]
\[ s = \partial P / \partial T; \]
\[ \epsilon = Ts - P; \]

- Rapid rise in thermodynamical quantities suggesting a change in the number of active degrees of freedom (hadrons → partons):
  - the most dramatic drop experienced by the early universe in which the Hubble parameter is
  \[ H^2 = \frac{8\pi G}{3} \frac{\epsilon_{\text{rel}}}{3} = \frac{8\pi G}{3} \frac{\pi^2}{30} g_\ast T^4 \]

- One observes a systematic \( \sim 20\% \) deviation from the Stephan-Boltzmann limit even at large \( T \): how to interpret it?

In the last part of this lecture we will attempt an explanation...
From final hadrons back to initial conditions

Au-Au collision at $\sqrt{s_{NN}} = 200$ GeV: tracks left by *charged particles* in the Time Projection Chamber of the STAR detector.
Invariant single-particle spectra

\[ E \frac{d^3 N}{d^3 \vec{p}} = E \frac{d^3 N}{d^2 p_T dy} \]

are generally expressed via the *transverse momentum* and the *rapidity*

\[ p_T \equiv \sqrt{p_x^2 + p_y^2} \quad \text{and} \quad y \equiv \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \]

Since \( y \) requires the knowledge of \( m \) (hence Particle IDentification, involving a longer analysis), one more often uses the *pseudorapidity*

\[ \eta \equiv \frac{1}{2} \ln \frac{p + p_z}{p - p_z} = \frac{1}{2} \ln \frac{1 + p_z/p}{1 - p_z/p} = \frac{1}{2} \ln \frac{1 + \cos \theta}{1 - \cos \theta} \]

Of course for relativistic particles \( y \approx \eta \)! In such a way one can simply perform two *geometrical* measurements from the tracks: \( p_T \) from the curvature in the B-field (\( p_T = |q| B r \)) and \( \eta \) from the polar angle. One of the first observables one can get is then the *rapidity density of charged particles* \( dN_{ch}/d\eta \)
Bjorken coordinates in heavy-ion collisions

While a possible transverse expansion takes time to develop, the fluid produced in HICs is characterized by an initial Hubble-like longitudinal expansion $v_{\text{fluid}}^z = z/t$: since the collision take place at $z = 0$, a fluid-cell found at $z$ at a later time $t$ is there because its constituents share a common average velocity $v_z$. The strong longitudinal expansion makes convenient to define the longitudinal proper-time and the spatial rapidity

$$\tau \equiv \sqrt{t^2 - z^2} \quad \text{and} \quad \eta_s \equiv \frac{1}{2} \ln \frac{t + z}{t - z} = \frac{1}{2} \ln \frac{1 + z/t}{1 - z/t}$$

and, accordingly,

$$t = \tau \cosh \eta_s \quad \text{and} \quad z = \tau \sinh \eta_s$$

Microscopic dynamics within a fluid cell depends on the time measured in its local rest frame. At large $z$, this “proper”-time would be boosted by a large $\gamma$-factor, requiring to follow the evolution of the system up to very large values of time $t = \sqrt{\tau^2 + z^2}$ measured in the lab-frame.

It is also convenient to introduce the fluid-rapidity

$$Y \equiv \frac{1}{2} \ln \frac{1 + v_{\text{fluid}}^z}{1 - v_{\text{fluid}}^z} \quad \text{so that} \quad u^\mu = \gamma_\perp (\cosh Y, u_\perp, \sinh Y) \quad \text{with} \quad \gamma_\perp \equiv (1 - u_\perp^2)^{-1/2}$$

If the longitudinal dynamics follows a Bjorken expansion with $v_{\text{fluid}}^z = z/t$

fluid-rapidity and spatial rapidity coincide: $\eta_s = Y$
Initial conditions: “Bjorken” estimate

- It is useful to describe the evolution in term of the variables
  \[ \tau \equiv \sqrt{t^2 - z^2} \quad \text{and} \quad \eta_s \equiv \frac{1}{2} \ln \frac{t + z}{t - z} \]

  Independence of the initial conditions on \( \eta_s \) entails \( v_{z, \text{fluid}} = z/t \) throughout the fluid evolution;

- For a \textit{purely longitudinal} Hubble-like expansion, entropy conservation implies:
  \[ s \tau = s_0 \tau_0 \quad \rightarrow \quad s_0 = (s \tau)/\tau_0 \]

- Entropy density is defined in the \textit{local fluid rest-frame}:
  \[ s \equiv \frac{dS}{dx_{\perp} dz} \bigg|_{z=0} = \frac{1}{\tau} \frac{dS}{dx_{\perp} d\eta_s} \quad \rightarrow \quad s \tau = \frac{dS}{dx_{\perp} d\eta_s} \]

- Entropy is related to the \textit{final multiplicity of charged particles} \((S \sim 3.6 N \text{ for pions})\), so that (at decoupling \( \eta \approx \eta_s \)):
  \[ s_0 \approx \frac{1}{\tau_0} \frac{3.6}{\pi R_A^2} \frac{dN_{\text{ch}}}{d\eta} \frac{3}{2} \]
"Bjorken" estimate: results

\[ s_0 \approx \frac{1}{\tau_0} \times \frac{3.6}{\pi R_A^2} \times \frac{dN_{\text{ch}}}{d\eta} \times \frac{3}{2} \]

- From \( dN_{\text{ch}}/d\eta \approx 1600 \) measured by ALICE at LHC and \( R_{\text{Pb}} \approx 6 \text{ fm} \) one gets:
  \[ s_0 \approx (80 \text{ fm}^{-2})/\tau_0 \]

- \( \tau_0 \) is found to be quite small (\( v_2 \) must develop early!):
  \[ 0.1 \lesssim \tau_0 \lesssim 1 \text{ fm} \rightarrow 80 \lesssim s_0 \lesssim 800 \text{ fm}^{-3} \]

- This should be compared with l-QCD
  \[ s(T = 200 \text{ MeV}) \approx 10 \text{ fm}^{-3} \]
Quantum harmonic oscillator at finite temperature

It is possible to study the quantum statistical mechanics of a single harmonic oscillator coupled to a thermal bath. The generalization to finite temperature QFT will be straightforward.

\[ H = \frac{p^2}{2} + \frac{1}{2} \omega_k^2 q^2 \]

Introducing the raising/lowering operators \(a^\dagger/a\) one gets:

\[ H = \omega_k \left( a^\dagger a + \frac{1}{2} \right), \quad \text{with} \quad q = \frac{1}{\sqrt{2\omega_k}} (a + a^\dagger) \quad \text{and} \quad [a, a^\dagger] = 1 \]

The partition function is easily evaluated:

\[ Z = \text{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta \omega_k (n+1/2)} = e^{-\beta \omega_k / 2} \sum_{n=0}^{\infty} e^{-\beta \omega_k n} = e^{-\beta \omega_k / 2} \frac{1}{1 - e^{-\beta \omega_k}} \]

One gets then for the thermodynamical potential \( \Omega \)

\[ Z = e^{-\beta \Omega} \quad \rightarrow \quad \Omega = \Omega_{\text{vac}} + \Omega_T = \frac{\omega_k}{2} + \ln \left(1 - e^{-\beta \omega_k}\right), \]

where we have separated the contribution of vacuum (responsible for UV divergences in QFT) and thermal fluctuations.
The vacuum and thermal fluctuations of the displacement are given by

\[ \langle q^2 \rangle = \frac{1}{2 \omega_k} \langle aa^\dagger + a^\dagger a \rangle = \frac{1 + 2N_k}{2 \omega_k}, \]

involving the Bose-Einstein distribution \( \langle a^\dagger a \rangle = N_k \equiv 1/(e^{\beta \omega_k} - 1) \).

From the h.o. algebra

\[ [H, a] = -\omega_k a \quad [H, a^\dagger] = \omega_k a^\dagger \]

one gets the time-evolution of the raising/lowering operators

\[ a(t) = a \ e^{-i \omega_k t} \quad a^\dagger(t) = a^\dagger \ e^{i \omega_k t} \]

Dim:

\[ a(t) = e^{iHt} \ a \ e^{-iHt} \quad \frac{da}{dt} = i \ e^{iHt} \ [H, a] \ e^{-iHt} = -i \omega_k a(t) \]

In thermal field theory one often works with imaginary times \( t = -i \tau \)

\[ a(\tau) = a \ e^{-\omega_k \tau} \quad a^\dagger(\tau) = a^\dagger \ e^{\omega_k \tau} \quad \text{with} \quad \tau \in [0, \beta] \]
Temporal propagators

The propagator along the real-time axis is defined as:

\[ D(t_1 - t_2) = \theta(t_1 - t_2)D^>(t_1 - t_2) + \theta(t_2 - t_1)D^<(t_1 - t_2), \]

where (from the algebra of the raising/lowering operators):

\[ D^>(t_1 - t_2) \equiv \langle q(t_1)q(t_2) \rangle = \frac{1}{2\omega_k} \left[ (1 + N_k)e^{-i\omega_k(t_1-t_2)} + N_k e^{i\omega_k(t_1-t_2)} \right] \]
\[ D^<(t_1 - t_2) \equiv \langle q(t_2)q(t_1) \rangle = \frac{1}{2\omega_k} \left[ (1 + N_k)e^{i\omega_k(t_1-t_2)} + N_k e^{-i\omega_k(t_1-t_2)} \right] \]

For the imaginary-time propagator \( \mathcal{D}(\tau) \equiv D(t=-i\tau) \) one has:

\[ \mathcal{D}(\tau) = \frac{1}{2\omega_k} \left[ (1 + N_k)e^{-\omega_k|\tau|} + N_k e^{\omega_k|\tau|} \right], \quad \text{with} \quad \tau \in [-\beta, \beta] \]

for which the symmetry property \( \mathcal{D}(\tau - \beta) = \mathcal{D}(\tau) \) (with \( 0 \leq \tau \leq \beta \)) holds.

The propagator can then be expanded in Matsubara frequencies

\[ \mathcal{D}(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n\tau} \Delta(i\omega_n) \quad \text{with} \quad \omega_n \equiv (2\pi/\beta)n \]
Path integrals in statistical QM

We wish to provide a path-integral representation for the partition function of a quantum system in equilibrium with a thermal bath

\[ Z \equiv \text{Tr} e^{-\beta H} = \int dx \langle x | e^{-\beta H} | x \rangle \]

here written in the basis of position-eigenstates \( \{|x\rangle\} \).

The starting point is the transition amplitude \( H \langle x_f, t_f | x_i, t_i \rangle \)

\[ \langle x_f | e^{-iH(t_f-t_i)} | x_i \rangle = \int_{x(t_i)=x_i, x(t_f)=x_f} [\mathcal{D}x(t)] e^{i \int_{t_i}^{t_f} dt [1/2(dx/dt)^2-V(x)]} \]

Moving to imaginary times \( t = -i\tau \) one has

\[ \langle x_f | e^{-H(\tau_f-\tau_i)} | x_i \rangle = \int_{x(\tau_i)=x_i, x(\tau_f)=x_f} [\mathcal{D}x(\tau)] e^{- \int_{\tau_i}^{\tau_f} d\tau [1/2(dx/d\tau)^2+V(x)]} \]

Hence, setting \( \tau_i = 0 \) and \( \tau_f = \beta \) and periodic boundary conditions,

\[ Z = \int_{x(0)=x(\beta)} [\mathcal{D}x(\tau)] e^{- \int_{0}^{\beta} d\tau [1/2(dx/d\tau)^2+V(x)]} \]
Harmonic Oscillator partition function: path-integral

The h.o. partition function reads then (after integration by parts)

\[
Z = \int_{q(0)=q(\beta)} [Dq(\tau)] e^{-\int_0^\beta d\tau (1/2)q(\tau)(-\partial_\tau^2 + \omega_k^2)q(\tau)}
\]

Such a representation turns out to be convenient

- for the evaluation of correlation functions
- as a starting point for a perturbative expansion in the presence of interactions (i.e. non-gaussian terms in the action)

It is convenient to introduce a source term \( j(\tau) \)

\[
Z[j] = \int_{q(0)=q(\beta)} [Dq(\tau)] e^{-\int_0^\beta d\tau [(1/2)q(\tau)(-\partial_\tau^2 + \omega_k^2)q(\tau) - j(\tau)q(\tau)]}
\]  

Let us now introduce the Green function of the differential operator

\[
(-\partial_\tau^2 + \omega_k^2)D(\tau - \tau') = \delta(\tau - \tau')
\]

We will show that it actually coincides with the previously introduced thermal propagator.
Performing the change of variables

\[ q(\tau) = q'(\tau) + \int d\tau' D(\tau - \tau') j(\tau') \]

the generating functional \( Z[j] \) can be written as

\[ Z[j] = Z[0] \times \exp \left[ \frac{1}{2} \int d\tau \int d\tau' j(\tau) D(\tau - \tau') j(\tau') \right] \tag{2} \]

Combining Eqs. (1) and (2) one gets

\[ \frac{\delta^2 \ln Z[j]}{\delta j(\tau_1) \delta j(\tau_2)} \bigg|_{j=0} = D(\tau_1 - \tau_2) = \langle T_\tau q(\tau_1)q(\tau_2) \rangle \]

The equivalence can be verified also decomposing the Green function \( D(\tau - \tau') \) in its Matsubara modes \( \Delta(i\omega_n) \)

\[ (-\partial^2 + \omega_k^2) \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n(\tau-\tau')} \Delta(i\omega_n) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n(\tau-\tau')} \]

One gets then

\[ \Delta(i\omega_n) = \frac{1}{\omega_n^2 + \omega_k^2} \]
The sum over Matsubara frequencies

The function to integrate has simple poles

- at $z = i\omega_n \ (n = 0, \pm 1, \pm 2...)$
- at $z = \pm \omega_k$

The Matsubara sum can be expressed as (for $0 \leq \tau \leq \beta$)

$$\frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \Delta(i\omega_n) = \int e^{-z\tau} \frac{dz}{2\pi i} \frac{e^{-z\tau}}{1 - e^{-\beta z}} \frac{-1}{(z - \omega_k)(z + \omega_k)}$$

as can be verified expanding $(1 - e^{-\beta z})$ around $z = i\omega_n$.

The integral can be also evaluated taking the residues of the poles outside the $C$ contour at $z = \pm \omega_k$. One gets:

$$\mathcal{D}(\tau > 0) = \frac{1}{2\omega_k} \left[(1 + N_k)e^{-\omega_k \tau} + N_k e^{\omega_k \tau}\right]$$

which coincides with what previously found!
Thermodynamics of the free scalar field

Let us now promote the displacement operator to a *scalar field* function of space and time:

\[ q(t) \longrightarrow \phi(t, \vec{x}) \]

The free Lagrangian/Hamiltonian involves also spatial gradients:

\[
L_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2, \quad H_0 = \frac{\pi^2}{2} + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2
\]

The raising/lowering operators now create/destroy particles with momentum \( \vec{k} \) and energy \( \omega_k \equiv \sqrt{k^2 + m^2} \) and one has

\[
H_0 = \sum_{\vec{k}} \omega_k \left( a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2} \right) \longrightarrow \Omega_0 = \sum_{\vec{k}} \frac{\omega_k}{2} + \sum_{\vec{k}} \ln \left( 1 - e^{-\beta \omega_k} \right)
\]

The (UV divergent!) contribution of *vacuum-fluctuations* does not depend on \( T \) and can be dropped (if we are not interested in gravity, cosmological constant....). In the infinite-volume limit one gets:

\[
\sum_{\vec{k}} \longrightarrow \sqrt{\int \frac{d\vec{k}}{(2\pi)^3}} \quad \Rightarrow \quad P_0 = -\frac{\Omega_0}{V} = -\frac{1}{\beta} \int \frac{d\vec{k}}{(2\pi)^3} \ln \left( 1 - e^{-\beta \omega_k} \right) = \frac{\pi^2}{90} T^4
\]
From the “box-quantization” of the field
\[
\phi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[ a_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} + a_{\vec{k}}^\dagger e^{-i\vec{k} \cdot \vec{x}} \right]
\]
with \([a_{\vec{k}}, a_{\vec{k}}^\dagger] = \delta_{\vec{k},\vec{k}'}\),

one can estimate its fluctuations (\(\langle a_{\vec{k}}^\dagger a_{\vec{k}'} \rangle = \delta_{\vec{k},\vec{k}'} N_k\))

\[
\langle \phi^2(\vec{x}) \rangle = \frac{1}{V} \sum_{\vec{k}} \frac{1 + 2N_k}{2\omega_k} \xrightarrow{\text{inf. volume}} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k}
\]

where one can isolate the (UV divergent!) vacuum fluctuations

\[
\langle \phi^2 \rangle_{\text{vac.}} \sim \int^{\Lambda} k^2 dk \sim \Lambda^2
\]

From the commutation relations

\[
[H, a_{\vec{k}}] = -\omega_k a_{\vec{k}} \quad [H, a_{\vec{k}}^\dagger] = \omega_k a_{\vec{k}}^\dagger
\]

one gets that the evolution of the \(a_{\vec{k}} / a_{\vec{k}}^\dagger\) operators is

\[
a_{\vec{k}}(\tau) = a_{\vec{k}} e^{-\omega_k \tau} \quad a_{\vec{k}}^\dagger(\tau) = a_{\vec{k}}^\dagger e^{\omega_k \tau} \quad \text{with} \quad \tau \in [0, \beta]
\]

The propagator is then a generalization of the harmonic oscillator result

\[
D_0(\tau, \vec{k}) = \frac{1}{2\omega_k} \left[(1 + N_k)e^{-\omega_k |\tau|} + N_k e^{\omega_k |\tau|}\right]\text{, with} \quad \tau \in [-\beta, \beta]
\]
The partition function of the free scalar field is given by

$$Z_0 = \int_{\phi(0, x) = \phi(\beta, x)} [\mathcal{D}\phi(\tau, x)] e^{-\int_0^\beta d\tau dx \left(\frac{1}{2}[\frac{\partial^2 \phi}{\partial \tau^2} + (\nabla \phi)^2 + m^2 \phi]^2\right)}$$

After partial integration...

$$Z_0 = \int_{\phi(0, x) = \phi(\beta, x)} [\mathcal{D}\phi(\tau, x)] e^{-\int_0^\beta d\tau dx \left(\frac{1}{2}\phi[-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2]\right)} \equiv e^{-S_E^0[\phi]}$$

The euclidean propagator can then be viewed as the Green function

$$\left[-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2\right]D_0(\tau - \tau') = \delta(\tau - \tau')\delta(x - x')$$

whose Matsubara-Fourier components are given by

$$\Delta_0(i\omega_n, \vec{k}) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} \equiv \frac{1}{\omega_n^2 + \omega_k^2},$$

as in the case of the harmonic oscillator.
The interacting scalar field: thermodynamics

In order to illustrate the effects of the interaction on the thermodynamics we will consider the usual case of $\lambda \phi^4$ theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

whose partition function reads

$$Z = \int_{\phi(0,x)=\phi(\beta,x)} [\mathcal{D}\phi(\tau,x)] e^{-\int_0^\beta d\tau d\mathbf{x} \left[ \frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]}$$

One can attempt a *perturbative expansion* around the free result

$$Z = \int_{\phi(0,x)=\phi(\beta,x)} [\mathcal{D}\phi(\tau,x)] e^{-\mathcal{S}_0^0[\phi]} \times \left[ 1 - \left( \frac{\lambda}{4!} \right) \beta V \langle \phi^4 \rangle_0 + \cdots \right]$$

Using the free action to evaluate the thermal averages one gets

$$Z = Z_0 - Z_0 \left( \frac{\lambda}{4!} \right) \beta V \langle \phi^4 \rangle_0 + \cdots = Z_0 \left[ 1 - \left( \frac{\lambda}{4!} \right) \beta V \langle \phi^4 \rangle_0 + \cdots \right]$$

One gets then for the pressure $P \equiv \ln(Z)/\beta V$

$$P = P_0 + P_1 + \ldots = P_0 + \ln \left[ 1 - \left( \frac{\lambda}{4!} \right) \beta V \langle \phi^4 \rangle_0 + \ldots \right] / \beta V$$

$$= P_0 - \left( \frac{\lambda}{4!} \right) \langle \phi^4 \rangle_0 + \ldots.$$
The pressure: the first perturbative correction

Before addressing its exact evaluation we see that the first perturbative correction to the pressure $P_1 = -\left(\frac{\lambda}{4!}\right)\langle \phi^4 \rangle_0$ is negative: we expect the interacting result to stay below the Stefan-Boltzmann limit. The correction can be computed through Wick’s theorem:

$$P_1 = -\left(\frac{\lambda}{4!}\right) 3 \left(\langle \phi^2 \rangle_0\right)^2 = -\left(\frac{\lambda}{4!}\right) 3 \left[ D_0(\tau = 0, x = 0) \right]^2$$

$$= -\frac{\lambda}{8} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k} \right]^2$$

The calculation is plagued by UV divergences due to vacuum fluctuations. However, the UV divergent term in the integral does not depend on $T$ and, for the moment, one can simply subtract it. The final result, including the first perturbative correction is

$$P_0 + P_1 = -\frac{1}{\beta} \int \frac{d\vec{k}}{(2\pi)^3} \ln \left(1 - e^{-\beta\omega_k}\right) - \frac{\lambda}{8} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{N_k}{\omega_k} \right]^2$$

$$= \left(\frac{\pi^2}{90} - \frac{\lambda}{1152}\right) T^4$$

Naively one expects higher order corrections to be $O(\lambda^2)$.
A closer look at the UV divergences

The perturbative correction $P_1$ to the pressure arises from the above diagram and can be recast as

$$P_1 = -\frac{\lambda}{8} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \right]^2 - \frac{\lambda}{8} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{N_k}{\omega_k} \right]^2 - \frac{\lambda}{4} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} \right] \left[ \int \frac{d\vec{k}'}{(2\pi)^3} \frac{N_{k'}}{\omega_{k'}} \right]$$

If the first term can be simply dropped, being $T$-independent, the third term is in principle more subtle, since vacuum (UV divergent!) and thermal fluctuations look apparently coupled.
A closer look at the UV divergences

The source of the **UV divergence** is in the **self-energy** correction to the single-particle propagator

\[
\Delta^{-1} = \Delta_0^{-1} + \Sigma = \omega_n^2 + \omega_k^2 + \frac{\lambda}{2} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k},
\]

where the statistical factor in the self-energy arises from the 12 different possible contractions from the Wick’s theorem \((\int d^4z \equiv \int d\tau z d\vec{z})\)

\[
\int [\mathcal{D}\phi] \phi(x)\phi(y) \left( -\frac{\lambda}{4!} \right) \int_0^\beta d^4z \phi(z)\phi(z)\phi(z)\phi(z) e^{-S_E[\phi]}
\]

\[
\sim -\frac{\lambda}{2} \int_0^\beta d^4z \mathcal{D}_0(x - z)\mathcal{D}_0(z - y)\mathcal{D}_0(\tau = 0, \vec{x} = 0)
\]
The UV divergence can be cured adding a *mass counterterm* to $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{ct} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 - \frac{1}{2} \delta m^2 \phi^2 + \ldots$$

so that the self-energy reads now

$$\Sigma = \Sigma_{\text{vac}} + \Sigma_T = \frac{\lambda}{2} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k} + \delta m^2$$

One can now impose the *renormalization condition*

$$\Sigma_{\text{vac}} = \frac{\lambda}{2} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k} + \delta m^2 = 0 \quad \rightarrow \quad \delta m^2 = -\frac{\lambda}{2} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{2\omega_k}$$

The self-energy is now finite and can be identified with a *thermal mass*

$$\Sigma_T = \frac{\lambda}{2} \int \frac{d\vec{k}}{(2\pi)^3} \frac{N_k}{\omega_k} \quad m \to 0 \quad \equiv m_T^2 \quad \equiv m_T^2$$
A closer look at the UV divergences

This suffices to cure also the UV divergence in the pressure. In fact now

\[ Z = \int [\mathcal{D}\phi(\tau, x)] \ e^{-S^0_\Box[\phi]} \times \left[ 1 - \int_0^\beta d\tau d\mathbf{x} \left( \frac{\lambda}{4!} \phi^4(\tau, \mathbf{x}) + \frac{1}{2} \delta m^2 \phi^2(\tau, \mathbf{x}) \right) + \ldots \right] \]

so that

\[ P_1 = -\frac{\lambda}{8} [\mathcal{D}_0(\tau = x = 0)]^2 - \frac{1}{2} \delta m^2 \mathcal{D}_0(\tau = x = 0) \]

\[ = -\frac{\lambda}{8} \left[ \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k} \right]^2 - \frac{1}{2} \left[ -\frac{\lambda}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_k} \right] \left[ \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1 + 2N_k}{2\omega_k} \right] \]

where the dangerous “vacuum × thermal” term cancels.
Higher order corrections to the pressure arise from diagrams like

\[
\int [\mathcal{D} \phi(x)] e^{-S^0_E[\phi]} \int_0^\beta d^4 z \, d^4 z' \frac{1}{2} \left( \frac{\lambda}{4!} \right)^2 \phi^4(z) \phi^4(z')
\]

From \( PV \equiv (1/\beta) \ln Z \) and the Wick’s contractions one gets

\[
P^\text{ring}_2 = \frac{\lambda^2}{16} [\mathcal{D}_0(0, 0)]^2 \frac{1}{\beta V} \int_0^\beta d^4 z \, d^4 z' [\mathcal{D}_0(\tau_z - \tau'_{z'}, z - z')]^2
\]

Since

\[
(1/\beta) \int_0^\beta d\tau e^{-i(\omega_n - \omega_{n'})\tau} = \delta_{n,n'} \quad \text{and} \quad \int d\vec{x} e^{i(\vec{k} - \vec{k}') \cdot \vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}')
\]

one gets

\[
P^\text{ring}_2 = \frac{\lambda^2}{16} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{N_k}{\omega_k} \right]^2 \frac{1}{\beta} \sum_{n=-\infty}^\infty \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{(\omega_n^2 + \omega_k^2)^2}
\]
The expression we got for the “ring” diagram

\[ P^\text{ring} = \frac{\lambda^2}{16} \left[ \int \frac{d\vec{k}}{(2\pi)^3} \frac{N_k}{\omega_k} \right]^2 \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{(\omega_n^2 + \omega_k^2)^2} \]

in the massless limit \( \omega_k = k \) is plagued by an IR divergence \( \sim \int \frac{dk}{k^2} \) arising from the \( \omega_n = 0 \) Matsubara mode.
Resummation of ring diagrams

The situation is even worse for higher-order similar diagrams, with a behaviour of the zero Matsubara mode $\sim \int \frac{dk}{k^{2(n-1)}}$. However the whole set of ring diagrams can be resummed (check!):

$$P_{\text{ring}} = -\frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d\vec{k}}{(2\pi)^3} \left[ \ln(1 + \Sigma \Delta_0(i\omega_n, \vec{k})) - \Sigma \Delta_0(i\omega_n, \vec{k}) \right]$$

In the above $\Sigma = \lambda T^2/24 \equiv m_T^2$ (analogous of Debye screening mass in QED/QCD) and the dominant contribution come from the $n=0$ mode

$$P_{\text{ring}}^{n=0} = -\frac{1}{2\beta} \int \frac{d\vec{k}}{(2\pi)^3} \left[ \ln \left( 1 + \frac{m_T^2}{k^2} \right) - \frac{m_T^2}{k^2} \right] = \frac{m_T^3 T}{12\pi} \sim \lambda^{3/2} T^4$$

Notice the non-analytic behaviour in the coupling with an expansion $P \sim P_0 + P_1 + P_{3/2} + ...$ arising from the resummation of an infinite set of diagrams.
With proper resummations the expansion can be pushed to higher orders$^3$. Notice that, if the coupling is not sufficiently weak, the convergence is poor.

Numerical results: scalar field theory

The situation looks even worse in QCD\(^4\) and one has to develop more clever and powerful resummation schemes.