Chapter 1

LIE GROUPS

1.1 Introduction

In this Chapter we discuss general features of Lie Groups, and expecially their relation to the corresponding Lie Algebras of infinitesimal generators.

Before plugging into a general analysis, it is convenient to discuss a preliminary simple example that while being elementary to follow, introduces many of the main concepts and helps establishing a concrete picture for the general theory. The example is the simplest instance of a Lie group, namely the group of proper rotations in two dimensions, isomorphic to SO(2).

The composition of the rotations by two angles θ_1 and θ_2 corresponds to a rotation of the angle $\theta_1 + \theta_2$:

$$R(\theta_2)R(\theta_1) \equiv R\left(\varphi(\theta_1, \theta_2)\right) = R(\theta_2 + \theta_1)\,, \tag{1.1.1}$$

so that the map φ describing the product law is continuous and differentiable:

$$\varphi(\theta_1, \theta_2) = \theta_1 + \theta_2\,. \tag{1.1.2}$$

The proper rotations form thus an Abelian Lie group of dimension 1. We have one more important property, namely that rotations of angles differeing by multiples of 2π are equivalent:

$$R(\theta + 2\pi) = R(\theta)\,. \tag{1.1.3}$$

As a manifold, this group is the circle S_1, a *compact* manifold.

The continuity and differentiability of the product law has a very profound consequence: the elements of the group are determined (up to a global property, the periodicity in the parameter) by the elements close to the identity, i.e. the infinitesimal transformations. Indeed, suppose we want to determine how a rotation $R(\theta)$ depends on θ. We look at how R changes upon an infinitesimal change of θ: we have

$$R(\theta + d\theta) \sim R(\theta) + d\theta \frac{dR(\theta)}{d\theta}\,. \tag{1.1.4}$$

In particular, for a transformation by an infinitesimal angle $d\theta$, we have

$$R(d\theta) \sim R(0) + d\theta\, \left.\frac{dR}{d\theta}\right|_{\theta=0} \equiv \mathbf{1} + d\theta\, J\,, \tag{1.1.5}$$

where we used the fact that in our parametrization $\theta = 0$ corresponds to the dientity element, and we introduced implicitely the definition of the *infinitesimal generator* J, which explicitely reads

$$J \equiv \left.\frac{dR}{d\theta}\right|_{\theta=0} . \tag{1.1.6}$$

By utilizing the explicit form Eq. (1.1.2) of the map φ, we can express the element $R(\theta + d\theta)$ as follows:

$$R(\theta + d\theta) = R\left(\varphi(d\theta, \theta)\right) = R(d\theta)R(\theta) = \left(\mathbf{1} + d\theta\, J\right) R(\theta) = R(\theta) + d\theta J R(\theta) . \tag{1.1.7}$$

This is pictorially expressed in Fig. 1.1.

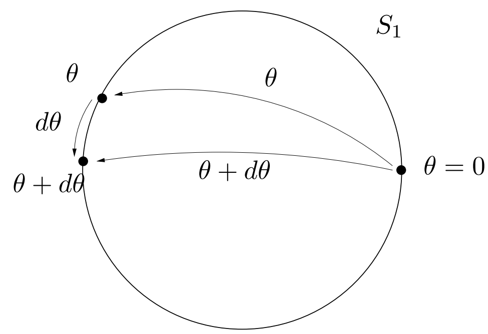

Figure 1.1. The rotation $R(\theta + d\theta)$ can be described also as the product $R(d\theta)R(\theta)$.

Comparing this expression with the one obtained in Eq. (1.1.4) we find a differential equation, accompanied by the initial condition $R(0) = \mathbf{1}$:

$$\begin{aligned} \frac{dR(\theta)}{d\theta} &= J\, R(\theta) \\ R(0) &= \mathbf{1} . \end{aligned} \tag{1.1.8}$$

The unique solution to this equation is

$$R(\theta) = \mathrm{e}^{\theta\, J} , \tag{1.1.9}$$

as can be verified utilizing the definition of the exponential of an operator as a power series.

When the rotations are realized as matrices of SO(2):

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.1.10}$$

we have an explicit parametrization of the group elements. The above discussion tells us that the knowledge of an infinitesimal rotation

$$R(d\theta) =\sim \begin{pmatrix} 1 & d\theta \\ -d\theta & 1 \end{pmatrix} = \mathbf{1} + d\theta\, J , \tag{1.1.11}$$

corresponding to the infinitesimal generator

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{1.1.12}$$

would be sufficient to determine the generic transformation by exponentiation:

$$R(\theta) = \exp(\theta\, J) = \exp\left[\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{1.1.13}$$

Let us notice that the rotation group can be regarded as a *group of transformations acting on its group manifold* S_1, parametrized by θ. Indeed, to every element of the group $R(\theta')$ is associated a map

$$\varphi_{\theta'}\ :\quad \theta \in S_1 \mapsto \varphi(\theta', \theta) = (\theta' + \theta) \in S_1\,, \tag{1.1.14}$$

defined through the group composition law $\varphi(\theta_1, \theta_2)$ defined in Eq. (1.1.2). Realizing the group in this way, infinitesimal elements are transformations

$$\varphi_{d\theta}\ :\quad \theta \mapsto \theta' + d\theta = (\mathbf{1} + d\theta\, \mathcal{J})\theta \tag{1.1.15}$$

so that the generator J is realized by

$$\mathcal{J} = \partial_\theta\,. \tag{1.1.16}$$

1.2 Definition and basic types of Lie groups

As already stated in the introductory part of this course, section **??**, Lie groups are first of all *continuous groups*. For a continuous group G, the parameters that label the elements of the group belong to a *topological space*, and the mapping $\phi\,:\, G \times G \to G$ such that $\forall x, y \in G$, $\phi(x, y) = xy^{-1}$ induced on the parameter space by the group product is *continuous*. If the group G is a *Lie group*, the parameter space is in addition a *differentiable manifold*, and the map ϕ above is infinitely *differentiable* (i.e., is *analytic*) in both arguments.

Keep in mind the introductory examples of the SO(2) Lie group discussed in sec. 1.1.

1.2.1 Lie groups of transformations

Let us now discuss a class of Lie groups that is of the greatest importance.

In many instances (and basically in all the cases we are interested in in Physics), we encounter Lie groups in the guise of groups of transformations (or, in particular, of linear transformations). Consider a group G of transformations T_α acting on some space V (which we suppose to be a n-dimensional manifold; in a local chart, let its coordinates be x^i) and depending on a set of continuous parameters α^μ:

$$x \in V \overset{T_\alpha}{\mapsto} f(\alpha, x) \in V\ , \qquad x \equiv x^i\ \ (i = 1, \ldots \dim V)\ , \qquad \alpha \equiv \alpha^\mu\ \ (\mu = 1, \ldots \dim G)\ . \tag{1.2.17}$$

We have already discussed groups of transformations: the product law is given by the composition of transformations. As an exercise, write out explicitely the request posed by the group axioms on the functions $f^i(x^j, \alpha^\mu)$. The composition of transformation defines a mapping $\varphi:\ G\times G \to G$ which represents the continuous analogue of the multiplication table for finite groups. Indeed, the composition

$$x \xmapsto{T_\alpha} f(\alpha, x) \xmapsto{T_\beta} f\left(\beta, f(\alpha, x)\right) \tag{1.2.18}$$

corresponds to the action on x of a new group element $T_{\varphi(\alpha,\beta)} \equiv T_\beta T_\alpha$:

$$x \xmapsto{T_\alpha} f(\varphi(\alpha, \beta), x)\ . \tag{1.2.19}$$

This defines (implicitely) the function $\varphi(\alpha, \beta)$, as by comparison with Eq. (1.2.18) we have $f(\varphi(\alpha, \beta), x) = f(\beta, f(\alpha, x))$.

Groups of transformations are *topological* (*Lie*) groups if the parameters α form a *topological space* (a *manifold*) G and the map $\varphi(\alpha, \beta)$ (as well as the map representing the inverse element) is *continuous* (*analytic*) in both arguments.

Example An example of a Lie groups of transformations os the group $\mathcal{C}$ of conformal trasformations of the real axis, namely the group of transformations

$$T_{a,b} \,:\, x \mapsto ax + b \ , \quad \forall\, x \in \mathbb{R} \ , \tag{1.2.20}$$

with $a, b \in \mathbb{R}$, and $a \neq 0$ (why this last requirement?). Thus the group-manifold is parametrized by $(a, b) \in \mathbb{R}^2 \setminus \mathbb{R}$ (where $\mathbb{R}$ is the line $a = 0$). This group is a Lie group, as the composition of the conformal transformations $[T_{c,d}]^{-1}$ and $T_{a,b}$ gives rise on the parameters to the mapping

$$\phi((a, b), (c, d)) = (a/c, -ad + b) \ , \tag{1.2.21}$$

which is continuous and has no poles since c cannot be zero. To check this, a convenient way is to describe the transformations Eq. (1.2.20) by means of triangular matrices acting on vectors $(x, 1)$:

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \xmapsto{T_{a,b}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ 1 \end{pmatrix} \ . \tag{1.2.22}$$

In this way, the composition of transformations becomes a matrix product.

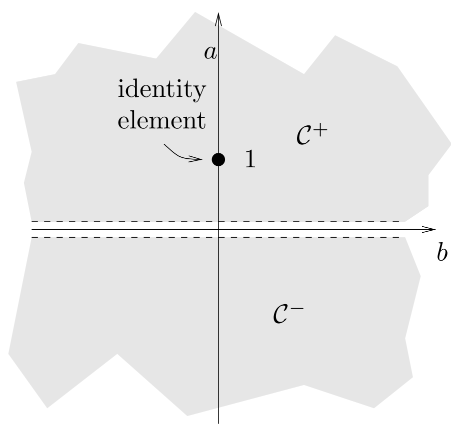

Figure 1.2. The (disconnected) group manifold of the real conformal group $\mathcal{C}$.

1.2.1.1 Matrix Lie groups

Particular cases of transformation groups are the groups of linear transformations acting on some vector space, namely the groups of matrices (if the space is finite-dimensional):

$$x^i \xmapsto{M} M^i{}_j \, x^j \ . \tag{1.2.23}$$

Composition of transformations corresponds to the matrix product.

Groups of matrices $M(\alpha)$ parametrized by a continuous set of parameters α and satisfying the requirements of *analiticity* in both arguments of the *composition law* $\varphi(\alpha, \beta)$ defined by

$$M(\alpha)M(\beta) = M(\varphi(\alpha, \beta)) \tag{1.2.24}$$

and of the mapping defined by taking the inverse are Lie groups, and are called[1] *Linear Lie groups.*

Notice that the linear transformation Eq. (1.2.23) can also be seen as the result of the application of the differential operator on the space of the x coordinates

$$\mathcal{M} = M^p{}_q\, x^q \frac{\partial}{\partial x^p} \ . \tag{1.2.25}$$

Indeed,

$$\mathcal{M}x^i = M^p{}_q\, x^q \frac{\partial x^i}{\partial x^p} = M^i{}_q\, x^q \ , \tag{1.2.26}$$

in accordance with Eq. (**??**). The operator $\mathcal{M}$ is a linear combination, with coefficients $M^p{}_q$, of a basis of operators

$$\mathcal{J}^q{}_p = x^q \frac{\partial}{\partial x^p} \ . \tag{1.2.27}$$

So, for groups of *linear* transformations on a space, we can use the language of differential operators Eq. (1.2.25) or that of their matrix components, according to our taste and convenience.

Example Consider the matrix group $\mathrm{GL}(n, \mathbb{R})$. The n^2 coordinates parametrizing the group elements g can be taken simply as the entries g_{ij} of the matrix (the double index (ij) plays the role of the index μ we generally use for coordinates α^μ on the group manifold. The left translation $h \mapsto gh$ of an element h by an element g is described explicitely by the matrix product:

$$h_{ij} \mapsto \varphi_{ij}(g, h) \equiv \sum_k g_{ik} h_{kj} \ , \tag{1.2.28}$$

This map clearly satisfies the requisites of continuity and differentiability.

1.2.2 Lie groups as transformation groups

Any Lie group G can be seen as a transformation group, the transformations acting on the group-manifold G itself. Indeed, the product induces two set of mappings, the *left* and *right translations* $L, R\,:\, G \times G \to G$ describing the transformations

$$\begin{aligned} L_g \ &: \ h \in G \mapsto L_g(h) = gh \ ; \\ R_g \ &: \ h \in G \mapsto R_g(h) = hg \ . \end{aligned} \tag{1.2.29}$$

These are two set of transformations acting on the points $h \in G$, and parametrized by the points $g \in G$. In a coordinate chart for G (i.e., when an explicit parametrization is given for the group elements) the left and right translations become explicit mappings. Denoting the coordinates of a point $h \in g$ in a given chart by α^μ, $\mu = 1, \ldots \dim G$ and those of g by β^μ, we write explicitely

$$\begin{aligned} \alpha^\mu &\xmapsto{L_g} \varphi^\mu(\beta, \alpha) \ , \\ \alpha^\mu &\xmapsto{R_g} \varphi^\mu(\alpha, \beta) \ . \end{aligned} \tag{1.2.30}$$

[1] In fact, the name of linear Lie group can be extended to all Lie groups G that possess at least one faithful matrix representation (which is tha case for all groups of interest for us, in fact).

1.3 Local properties: Lie groups and Lie algebras

The interplay between the topological and differential structure of Lie groups manifests itself in a crucial property: to every Lie group G is associated a finite-dimensional Lie algebra $\mathbb{G}$ (with $\dim \mathbb{G} = \dim G$) which encodes all the local properties of the group itself. The correspondence is not one-to-one: a single Lie algebra can correspond to several Lie groups, all locally isomorphic. In this section we will explain and investigate this relation in detail.

1.3.1 The role of infinitesimal transformations: Lie's theorems

1.3.1.1 Infinitesimal generators

Let us consider a Lie group G, and let us describe it as a group of transformations acting on he manifold G itself, by means of the left or right translation maps. Let us consider the effect of the (say) *left* translation of a group element $h = h(\alpha)$by an element $g = g(\delta\beta)$ close to the identity[2]. We have, using the differentiability of the group map φ,

$$\alpha^\mu \to \alpha^\mu + d\alpha^\mu = \varphi^\mu(\delta\beta, \alpha) = \alpha^\mu + \left.\frac{\partial\varphi^\mu(\eta,\alpha)}{\partial\eta^\nu}\right|_{\eta=0} \delta\beta^\nu \ . \tag{1.3.31}$$

We can also write Eq. (1.3.31) as follows:

$$\alpha^\mu + d\alpha^\mu = \left(\mathbf{1} + \delta\beta^\nu\, \widetilde{\mathcal{J}}^\rho{}_\nu(\alpha)\frac{\partial}{\partial\alpha^\rho}\right)\alpha^\mu \equiv \left(\mathbf{1} + \delta\beta^\nu\, \widetilde{\mathcal{J}}_\nu(\alpha)\right)\alpha^\mu \ . \tag{1.3.32}$$

The differential operators $\widetilde{\mathcal{J}}_\nu(\alpha)$, of components

$$\widetilde{\mathcal{J}}^\rho{}_\nu(\alpha) = \left.\frac{\partial\varphi^\rho(\eta,\alpha)}{\partial\eta^\nu}\right|_{\eta=0} \ , \tag{1.3.33}$$

are called *infinitesimal left generators.*

We can of course consider infinitesimal *right* translations, in which case we write

$$\alpha^\mu + d\alpha^\mu = \left(\mathbf{1} + \delta\beta^\nu\, \mathcal{J}^\rho{}_\nu(\alpha)\frac{\partial}{\partial\alpha^\rho}\right)\alpha^\mu \equiv \left(\mathbf{1} + \delta\beta^\nu\, \mathcal{J}_\nu(\alpha)\right)\alpha^\mu \ . \tag{1.3.34}$$

The differential operators $\mathcal{J}_\nu(\alpha)$, of components

$$\mathcal{J}^\rho{}_\nu(\alpha) = \left.\frac{\partial\varphi^\rho(\alpha,\eta)}{\partial\eta^\nu}\right|_{\eta=0} \ , \tag{1.3.35}$$

(notice the difference w.r.t. Eq. (1.3.33) in the order of the arguments of the map φ) are obviously called *infinitesimal right generators.*

Notice that, while it does not make any sense to sum two points of a manifold, by choosing local coordinates it makes sense to write $\alpha^\mu + d\alpha^\mu$ as we did in Eq. (1.3.31): we are actually working in the tangent plane $T_{\{\alpha\}}(G)$ at the point $\{\alpha\}$ to the manifold. The infinitesimal generators $\widetilde{\mathcal{J}}_\nu$ are *vector fields*, thus in every point $\{\alpha\}$ they belong to the tangent plane $T_{\{\alpha\}}(G)$. They act as infinitesimal transformations of the coordinates, while the group elements act as finite transformations. $\widetilde{\mathcal{J}}_\nu$ is not a group element, but $\left(\mathbf{1} + \delta\beta^\nu \widetilde{\mathcal{J}}_\nu\right)$ is, to first order in $\delta\beta$. We will discuss the properties of the vector fields $\widetilde{\mathcal{J}}_\nu$ in later Sec. 1.3.2

[2] In this discussion, we are assuming that coordinates $\beta = 0$ parametrize the identity of the group, so that elements with small cordinates correspond to group multiplications whose effect is close to the identity.

Example The introductory example of the group SO(2), described in sec. 1.1, fits prefectly in the above discussion. From the composition map $\varphi(\theta, \theta')$ defined in Eq. (1.1.2) it follows that the infinitesimal generator is given by

$$\mathcal{J}(\theta) = \left. \frac{\partial \varphi(\theta, \eta)}{\partial \eta} \right|_{\eta=0} \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} \ , \tag{1.3.36}$$

in agreement with Eq. (1.1.16). The group is abelian, so there is no distinction between left and right infinitesimal generators.

Explicit realizations of infinitesimal generators We could repeat the above discussion using other explicit realizations of the group G, for instance as a group of transformations or as a matrix group. An element $g(\delta\beta)$ close to the identity can be written as

$$g(\delta\beta) = e + \delta\beta^{\nu}\, J_{\nu} \ , \tag{1.3.37}$$

where the infinitesimal generators J_{ν} are defined as

$$J_{\nu} \equiv \left. \frac{\partial g(\eta)}{\partial \eta^{\nu}} \right|_{\eta=0} \ . \tag{1.3.38}$$

Notice that an expression like $e + \delta\beta^{\nu}\, J_{\nu}$ has no direct meaning at the group level (the $+$ is not the group operation), but only on its suitably defined tangent space, where the addition makes sense. For instance, in case of matrices the addition is the natural matrix addition; for transformations acting on some target space, one can add locally add the result of transformations in the tangent to the target space. J_{ν} acts in the same way as a group element, e.g. by matrix multiplication or as a transformation on some target space; however, J_{ν} does not belong to the group, only $e + \delta\beta^{\nu} J_{\nu}$ does, to first order in $\delta\beta$.

Examples

- Consider again the introductory example of the group SO(2) in sec.1.1. When the group element are explicitely realized as 2×2 special orthogonal matrices $R(\theta)$, parametrized by an angle $\theta \in [0, 2\pi]$, as in Eq. (1.1.11), the infinitesimal generator is the matrix

$$J = \left. \frac{\partial R(\theta)}{\partial \theta} \right|_{\theta=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{1.3.39}$$

  The generator J is not a special orthogonal matrix (it is antisymmetric), but the matrix $\mathbf{1} + \delta\theta J$ is special orthogonal, to first order in $\delta\theta$: indeed,

$$(\mathbf{1} + \delta\theta J)^T (\mathbf{1} + \delta\theta J) = (\mathbf{1} - \delta\theta J)(\mathbf{1} + \delta\theta J) \sim \mathbf{1} + O(\delta\theta^2) \ . \tag{1.3.40}$$

- Consider the matrix group GL$(n, \mathbb{R})$, and take the entries g_{ij} of a matrix $g \in$ GL$(n, \mathbb{R})$ as its coordinates. The infinitesimal generators

$$J_{ij} = \left. \frac{\partial g}{\partial g_{ij}} \right|_{g_{ij}=0} \tag{1.3.41}$$

  are matrices whose single non-zero entry is a 1 in the i-th row and j-h column:

$$(J_{ij})_{kl} = \delta_{ik}\delta_{jl} \ . \tag{1.3.42}$$

  Notice that $J_{ij} \notin$ GL$(n, \mathbb{R})$, as $\det J_{ij} = 0$; however, $\mathbf{1} + \delta g^{ij} J_{ij} \in$ GL$(n, \mathbb{R})$, as its determinant is close to 1 (and thus non-zero).

1.3.1.2 First Lie theorem

Let us consider the expression in coordinates of the left translation by an element $g = g(\beta)$ of coordinates $\{\beta^\mu\}$ of an element $h = h(\alpha)$ of coordinates $\{\alpha^\mu\}$:

$$\alpha^\mu \xrightarrow{L_g} \varphi^\mu(\beta, \alpha) \ . \tag{1.3.43}$$

Consider now the effect of a small variation of β, i.e. of the element by which $h(\alpha)$ is multiplied on the left: $\beta \to \beta + d\beta$. The situation is depicted in Fig. 1.1, and is the generalization of that described in the introduction to this Chapter for the group SO(2), see Fig. 1.3.

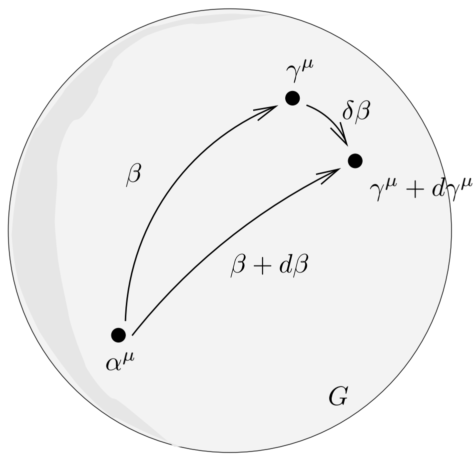

Figure 1.3. Effect of a small variation of the element β by which the element α is left-translated.

The resulting element undergoes a small change (due to the continuity properties of the group multiplication map φ)

$$\gamma^\mu + d\gamma^\mu \equiv \varphi^\mu(\beta + d\beta, \alpha) \ . \tag{1.3.44}$$

On the other hand, the result can be interpreted as the translation of γ by a smallelement $\delta\beta$, and is therefore determined by the generators of infinitesimal left translations, see Eq. (1.3.32):

$$\gamma^\mu + d\gamma^\mu = \varphi^\mu\left(\delta\beta, \gamma = \varphi(\beta,\alpha)\right) = \gamma^\mu + \left.\frac{\partial\varphi^\mu(\beta,\alpha)}{\partial\beta^\nu}\right|_{\beta=0} \delta\beta^\nu = \gamma^\mu + \widetilde{\mathcal{J}}^\mu{}_\nu \delta\beta^\nu \ . \tag{1.3.45}$$

Because of the associativity of the group law, the element $\beta + d\beta$ corresponds to the composition of the multiplication by β and $\delta\beta$:

$$\beta^\mu + d\beta^\mu = \varphi^\mu\left(\delta\beta, \beta\right) = \beta^\mu + \left.\frac{\partial\varphi^\mu(\eta,\alpha)}{\partial\eta^\nu}\right|_{\eta=0} \delta\beta^\nu = \beta^\mu + \widetilde{\mathcal{J}}^\mu{}_\nu(\beta) \delta\beta^\nu \ , \tag{1.3.46}$$

so that

$$d\beta^\mu = \widetilde{\mathcal{J}}^\mu{}_\nu(\beta)\delta\beta^\nu \ . \tag{1.3.47}$$

Due to the group properties, the matrix $\widetilde{\mathcal{J}}^\mu{}_\nu$ generating infinitesimal transformations is invertible in every point; by comparing Eq. (1.3.45), Eq. (**??**) and Eq. (1.3.47) we can write

$$d\gamma^\mu = \widetilde{\mathcal{J}}^\mu{}_\nu(\gamma)(\widetilde{\mathcal{J}}^{-1})^\nu{}_\rho(\beta) d\beta^\rho \ . \tag{1.3.48}$$

Thus, we have obtained a set of first-order differential equations for the functions

$$\gamma^\mu \equiv \varphi^\mu(\beta, \alpha) \ , \tag{1.3.49}$$

namely

$$\frac{d\gamma^\mu}{d\beta^\rho} = \widetilde{\mathcal{J}}^\mu_{\ \nu}(\gamma)(\widetilde{\mathcal{J}}^{-1})^\nu_{\ \rho}(\beta) \ . \tag{1.3.50}$$

If the integration of the above equation is possible, it yields the explicit expression of the map $\varphi(\beta, \alpha)$ for generic *finite* values of β, i.e., the effect of a generic left translation L_g on any fixed element $h(\alpha)$ from the knowledge of the *infinitesimal* group actions only, encoded in the infinitesimal generators $\widetilde{\mathcal{J}}^\mu_{\ \nu}$. As it is clear from the derivation, this may happen because of the interplay between the continuity and differentiability properties of the map φ and its group properties (associativity, invertibility). Thus, it is specific of Lie groups. This powerful statement goes under the name of *Lie's first theorem*

It is obvious that we could have repeated the same discussion considering the *right* translations of an element $h(\alpha)$ by an element $g(\beta)$ and the effect of the result of a small change in β (do it for exercise). We would have obtained a set of differential equations for the functions

$$\gamma^\mu \equiv \varphi^\mu(\alpha, \beta) \tag{1.3.51}$$

(notice the difference w.r.t. Eq. (1.3.49) in the order of the arguments of the map φ), namely

$$\frac{d\gamma^\mu}{d\beta^\rho} = \mathcal{J}^\mu_{\ \nu}(\gamma)(\mathcal{J}^{-1})^\nu_{\ \rho}(\beta) \ , \tag{1.3.52}$$

containing the generators of infinitesimal *right* translations.

1.3.1.3 Lie's second theorem

Let us now consider the integrability conditions for the differential equations Eq. (1.3.50):

$$\frac{\partial^2\gamma^\mu}{\partial\beta^\rho\partial\beta^\sigma} - (\rho \leftrightarrow \sigma) = 0 \ , \qquad \forall \rho, \sigma \ . \tag{1.3.53}$$

By explicit computation (exercise, or see appendix?) it turns out that, in order for Eq. (1.3.53) to old, all the quantities

$$c_{\alpha\beta}^{\ \ \gamma}(\beta) \equiv (\widetilde{\mathcal{J}}^{-1})^\gamma_{\ \sigma}(\beta) \left(\frac{\partial \widetilde{\mathcal{J}}^\sigma_{\ \alpha}(\beta)}{\partial\beta^\rho} \widetilde{\mathcal{J}}^\rho_{\ \beta}(\beta) - \frac{\partial \widetilde{\mathcal{J}}^\sigma_{\ \beta}(\beta)}{\partial\beta^\rho} \widetilde{\mathcal{J}}^\rho_{\ \alpha}(\beta) \right) \tag{1.3.54}$$

have to be *constant*, i.e., they should not depend on β.

These quantities will turn out to be the *structure constants* of the Lie algebra associated to the Lie group in question.

The two Lie's theorem described above can be recast in a even more geometrical by focusing on the properties of a class of vector fields on the group manifold, the *invariant vector fields* that, ss we will see, correspond to the generators of infinitesimal group translations.

1.3.2 Invariant vector fields on the group manifold

We can take advantage of the double nature of a Lie group G, which is a manifold *and* a group, by singling out, among the *vector fields* defined on it, a special class of vector fields which are *invariant* with rispect to the group action. The very definition of such invariant v.f. turns out to be equivalent to Lie's first theorem. On every manifold, the vector fields form are naturally endowed with the algebric structure of Lie algebra (which we will soon define). It is immediate to show that the invariant v.f. close a Lie algebra which is *finite-dimensional*. This algebra is the Lie algebra $\mathbb{G}$ associated to the Lie group G. So, the discussion of invariant vector fields provides a very general and direct path from the Lie group to its Lie algebra. As we will see, it also paves the way to the reverse process of reconstructing the (local behaviour of) the Lie group from its Lie algebra.

Let us now recall briefly the tranformation properties of vector fields on a manifold (see Appendix **??** for the main definitions about vector fields on a manifold).

Induced mappings of vector fields Consider a mapping[3]

$$\phi : \begin{array}{l} \mathcal{M} \to \mathcal{M} \ , \\ x \mapsto x' \equiv \phi(x) \ . \end{array} \tag{1.3.55}$$

Such a mapping induces a mapping on the space of vector fields $\mathrm{Diff}_0(\mathcal{M})$ which is sometimes indicated as $d\phi$, and called *differential* of the map ϕ:

$$d\phi : \begin{array}{l} \mathrm{Diff}_0(\mathcal{M}) \to \mathrm{Diff}_0(\mathcal{M}) \ , \\ X \mapsto X' \equiv d\phi(X) \ . \end{array} \tag{1.3.56}$$

The map $d\phi$ is defined by the requirement that $X'(x') = X(x)$, i.e., that the transformed vector field at the transformed point equals the original field at x. Thus the transformation of the components of the vector field involve the Jacobian matrix:

$$X'^{\mu}(x') \frac{\partial}{\partial x'^{\mu}} = X^{\mu}(x) \frac{\partial}{\partial xs^{\mu}} \quad \Rightarrow X'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} X^{\nu}(x) \ . \tag{1.3.57}$$

Left- and right-invariant vector fields The group structure of G provides a natural way to single out two subset of vector fields, the *left-invariant* and *right-invariant* vector fields. Indeed, $\forall g \in G$, the left- and right- translations L_g and R_g of Eqs. 1.2.29 and 1.3.79 induce an action on the vector fields. For instance, for left translations the induced action

$$X \longmapsto X_{L_g} \equiv dL_g(X) \tag{1.3.58}$$

reads explicitely in components, according to Eq. (1.3.57),

$$X^{\mu}_{L_g}\left(\varphi(\beta,\alpha)\right) = \frac{\partial \varphi^{\mu}(\beta,\alpha)}{\partial \alpha^{\nu}} X^{\nu}\left(\alpha\right) \ . \tag{1.3.59}$$

An analogous expression holds, of course, for the right translations.

[3] Considering the transformation $x \to x'$ as a mapping is an "active" point of view. One could take a passive point of view in which the transformation is considered to be a coordinate change. In this prespective, the induced mapping describes the transition properties of the vector fields between different coordinate charts.

We define *left-invariant* (*right-invariant*) vector fields $X_{L(R)}$ as vector fields which are *invariant in form* under the action induced by left (right) translations:

$$dL_g(X_L) = X_L \ , \quad dR_g(X_R) = X_R \ , \quad \forall g \in G \ . \tag{1.3.60}$$

A left- (right-)invariant vector field is thus characterized as follows (in an explicit chart):

$$\begin{aligned} X_L^\mu\left(\varphi^\mu(\beta,\alpha)\right) &= \frac{\partial\varphi^\mu(\beta,\alpha)}{\partial\alpha^\nu} X_L^\nu(\alpha) \ , \\ X_R^\mu\left(\varphi^\mu(\alpha,\beta)\right) &= \frac{\partial\varphi^\mu(\alpha,\beta)}{\partial\alpha^\nu} X_R^\nu(\alpha) \ , \end{aligned} \tag{1.3.61}$$

where α^μ and β^μ are the coordinates of any $h, g \in G$.

Examples

- Consider the group $(\mathbb{R}, +)$. The single coordinate parametrizing the group is the element $x \in \mathbb{R}$ itself. The left translation by an element $a \in \mathbb{R}$ acts by $x \mapsto x + a$ (and coincides obviously with the right translation). The single component $X(x)$ of an invariant vector field must satisfy Eq. (1.3.146), which in this case reads
$$X_L(a+x) = \frac{\partial(a+x)}{\partial x} X(x) = X(x) \ , \tag{1.3.62}$$
implying that $X_L(x)$ has to be a constant. The set of left-invariant vector fields
$$X_L = k\,\frac{\partial}{\partial x} \quad (k \text{ a real constant}) \tag{1.3.63}$$
is thus a vector space of dimension 1 (and no longer infinite-dimensional as the space $\mathrm{diff}(\mathbb{R})$ of all vector fields: the *functional* dependence of $X(x)$ has been fixed by Eq. (1.3.148). The basis element (or *generator*) of this vector space is simply $\mathcal{J} = \partial/\partial x$. The story is exactly identical for right translations, as the group is abelian.
- Consider $\mathbb{R}^+$ with the multiplication as group product. The group is abelian, so again left- or right-invariant vector fields coincide. The "translation" by an element $a \in \mathbb{R}^+$ acts on a point $x \in \mathbb{R}^+$ by $x \mapsto ax$, and the signle component of an invariant vector field must satisfy
$$X_L(ax) = \frac{\partial(ax)}{\partial x} X(x) = aX(x) \ , \tag{1.3.64}$$
namely $X_L(x)$ is homogeneous of degree 1. The set of left-invariant vector fields
$$X_L = k\,x\frac{\partial}{\partial x} \quad (k \text{ a real constant}) \tag{1.3.65}$$
is a vector space of dimension 1, spanned by the generator $\mathcal{J} = x\,\partial/\partial x$.
- Consider the matrix group $\mathrm{GL}(n,\mathbb{R})$, parametrized by the entries g_{ij} of its elements. The group map $\varphi_{ij}(g,h)$ has been given in Eq. (1.3.65). Correspondingly, the components of left-invariant vector fields satisfy therefore the relation
$$X_L^{ij}(gh) = \frac{\partial \sum_k (g_{ik}h_{kj})}{\partial h_{lm}} X_L^{lm}(h) = \sum_k g_{ik}\, X_L^{kj} = (gX_L)^{ij} \ , \tag{1.3.66}$$
namely, the value of the vector field at a translated point is obtained simply by the matrix action gX of the group element on the vector field. Eq. (1.3.66) generalizes the uni-dimensional case of Eq. (1.3.147)

The invariant vector fields and the tangent space to the identity The vector spaces of both left- and right-invariant vector fields are isomorphic to the tangent space $T_e(G)$ to the origin of the group-manifold. Indeed, the condition of left- (or right-)invariance, Eqs. (1.3.147,1.3.148), allows to associate a left- (or right-)invariant vector field to a tangent vector in a specific point (for instance, the identity): in components,

$$\begin{aligned} X_L^\mu(\beta) &= \left.\frac{\partial\varphi^\mu(\beta,\alpha)}{\partial\alpha^\nu}\right|_{\alpha=\alpha_0} X_L^\nu(\alpha_0) = \mathcal{J}^\mu{}_\nu(\beta) X_L^\nu(\alpha_0) \ ; \\ X_R^\mu(\beta) &= \left.\frac{\partial\varphi^\mu(\alpha,\beta)}{\partial\alpha^\nu}\right|_{\alpha=\alpha_0} X_R^\nu(\alpha_0) = \widetilde{\mathcal{J}}^\mu{}_\nu(\beta) X_R^\nu(\alpha_0) \ , \end{aligned} \tag{1.3.67}$$

where α_0^μ are the coordinates of the identity element and we use the notation introduced in Eq. (**??**) and Eq. (**??**). The constants $X_{L,R}^\nu(\alpha_0)$ are the components of a tangent vector to the identity. Eq. (1.3.67) establishes a bijective correspondence between the the tangent space to the identity and the space of left- or right-invariant vector fields, thanks to the properties of differentiablilty and invertibility of the function $\varphi(\beta,\alpha)$ describing the group product.

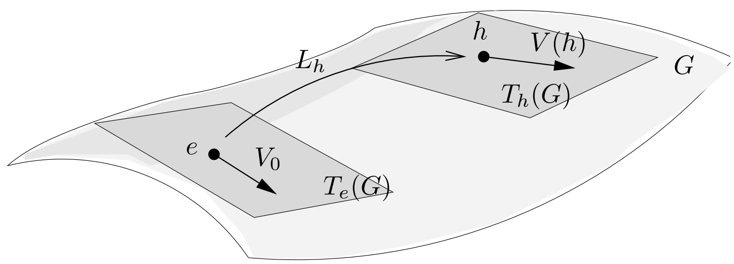

Figure 1.4. Given any tangent vector at the identity V_0, the globally defined, differentiable, invertible map provided by the group composition law allows to associate uniquely a tangent vector in any point $h \in G$, i.e., allows to define a vector field V.

We can obviously look at the correspondence between $T_e(G)$ and the invariant vector fields the other way round. On a manifold G which is also a Lie group, take any tangent vector $V_0 = v^\mu \frac{\partial}{\partial\alpha^\mu}$ in a fixed point, e.g., the identity (this point corresponds to the origin of the coordinate system $\{\alpha^\mu\}$). We can associate to V_0 a *globally defined* and everywhere *non-vanishing* vector field V by defining its components in every point via the derivatives of the group map $\varphi(\beta,\alpha)$:

$$V^\mu(\beta) = v^\nu \left.\frac{\partial\varphi^\mu(\beta,\alpha)}{\partial\alpha^\nu}\right|_{\alpha=0} = v^\nu \mathcal{J}^\mu{}_\nu(\beta) \ . \tag{1.3.68}$$

That is, we define a vector field V at the point $h(\beta)$ via the differential of the left translation map $L_h : e \to h$, i.e. we set

$$V(h) \equiv dL_h(V(0)) \ . \tag{1.3.69}$$

The vector field $V(h)$ defined in this way is left-invariant by construction, as it is clear by considering Eq. (1.3.145) and Eq. (1.3.147). It the very existence of the globally defined, differentiable and invertible left translation map $L_h : g \mapsto hg$, i.e. $\alpha^\mu \mapsto \varphi^\mu(\beta,\alpha)$ that allows to associate to every tangent vector to the identity a vector field. Of course, we could repeat the discussion associating to every tangent vector a right-invariant vector field.

We can reformulate this correspondence in language that will be useful later on. Let us first recall the meaning of a tangent vector v in point p of a manifold (see the section on manifolds, sec. **??** of Chapter **??** for a more detailed review).

The tangent vector identifies the *initial direction* of a class of curves through the point p. Let such a curve $\sigma(t)$ be parametrized so that $\sigma(0) = p$. Then, for any function f locally defined at p on the manifold, we can consider its derivative along the curve and view it as the result of acting with a first-order differential operator v on the function f:

$$\frac{d}{dt}\left[f\left(\sigma(t)\right)\right]\Big|_{t=0} \equiv v[f](p) \ . \tag{1.3.70}$$

In an explicit coordinate chart$\{\alpha^\mu\}$ (with α_0^μ being, in particular, the coordinates of the point p and $\sigma^\mu(t)$ the coordinate expression of the curve $\sigma(t)$) we have from Eq. (1.3.70)

$$v = v^\mu \frac{\partial}{\partial\alpha^\mu} \ , \qquad v^\mu = \frac{d\sigma^\mu(t)}{dt}\Big|_{t=0} \ . \tag{1.3.71}$$

For any tangent vector v at the identity point of G, we have, according to Eq. (1.3.70),

$$v[f](e) = \frac{d}{dt}\left[f\left(\sigma(t)\right)\right]\Big|_{t=0} \ , \tag{1.3.72}$$

where the curve $\sigma(t)$, such that $\sigma(0) = e$, has its initial direction specified by v according to Eq. (1.3.71). Statting from v we can define a *left-invariant* vector field V_L giving its value at any point $g \in G$ as follows:

$$V_L[f](g) = \frac{d}{dt}\left[f\left(g\sigma(t)\right)\right]\Big|_{t=0} \ , \tag{1.3.73}$$

where $g\sigma(t)$ is the group product of g and $\sigma(t)$. Indeed, let $\{\beta^\mu\}$ be the coordinates of g, and $\sigma^\mu(t)$ those of $\sigma(t)$. The l.h.s. of the definition Eq. (1.3.73) reads then

$$V_L^\mu \frac{\partial f}{\partial\beta^\mu}(\beta) \ , \tag{1.3.74}$$

while the r.h.s. is explicited as follows (φ being, as usual, the group product map):

$$\frac{\partial f}{\partial\varphi^\mu(\beta,\sigma(0))} \frac{\partial\varphi^\mu(\beta,\sigma(t))}{\partial\sigma^\nu(t)}\Big|_{t=0} \frac{d\sigma^\mu(t)}{dt}\Big|_{t=0} = \frac{\partial f}{\partial\beta^\mu} \frac{\partial\varphi^\mu(\beta,\sigma)}{\partial\sigma^\nu}\Big|_{\sigma=0} v^\nu \ , \tag{1.3.75}$$

where in the second step we used Eq. (1.3.71) and the fact that the coordinates of the identity element $\sigma(0) = e$ are chosen to be zero. Equating the two sides, Eq. (1.3.74) and Eq. (1.3.75), we desume the expression of the components of the l.i.v.f. V_L at the point $g(\beta)$:

$$V_L^\mu(\beta) = \frac{\partial\varphi^\mu(\beta,\sigma)}{\partial\sigma^\nu}\Big|_{\sigma=0} v^\nu = \mathcal{J}^\mu_{\ \nu}(\beta) v^\nu \ , \tag{1.3.76}$$

in perfect agreement with Eq. (1.3.68).

Similarly we can define the *right-invariant* vector field V_R:

$$V_R[f](g) = \frac{d}{dt}\left[f\left(\sigma(t)g\right)\right]\Big|_{t=0} \ . \tag{1.3.77}$$

Example

- Consider again $\mathrm{GL}(n, \mathbb{R})$, parametrized by the entries g_{ij} of its elements. We described the condition of left-invariance for vector fields on $\mathrm{GL}(n, \mathbb{R})$ in Eq. (1.3.66). It follows from that property that, if V_0 (of components v^{ij}) is a tangent vector to the identity, the corresponding left-invariant vector fields reads

$$X_{L,V_0}(g) = (gv)^{ij} \frac{\partial}{\partial g^{ij}} \ . \tag{1.3.78}$$

Invariant vector fields and infinitesimal generators The spaces of left- or right-invariant vector fields is therefore a vector space of finite dimension $\dim G$, isomorphic to $T_e(G)$. A natural basis of generators for this vector space is given by the invariant vector fields $\mathcal{J}_\lambda$ ($\widetilde{\mathcal{J}}_\lambda$) associated via Eq. (1.3.67) to the natural basis elements of $T_e(G)$, namely the derivatives $\partial/\partial\beta^\lambda\big|_{\beta=\beta_0}$ of components δ^μ_λ:

$$\begin{aligned} \mathcal{J}_\lambda(\beta) &= \mathcal{J}^\mu_\lambda(\beta) \frac{\partial}{\partial \beta^\mu} \ , \\ \widetilde{\mathcal{J}}_\lambda(\beta) &= \widetilde{\mathcal{J}}^\mu_\lambda(\beta) \frac{\partial}{\partial \beta^\mu} \ . \end{aligned} \tag{1.3.79}$$

The Lie group G, admits thus at least $\dim G$ globally defined vector fields. $\widetilde{\mathcal{J}}_\nu$. While in a neighbourhood of a point of a manifold the space of vector fields is infinite-dimensional, the requirement of being globally defined is often so restrictive that no vector field obeys it. For instance, as well known, no vector field exists on the two-sphere S_2, as you cannot comb a tennis ball :-) ; A vector fields exists instead on the circle S_1, as it is easy to visualize. In this respect, group manifolds are particular; for instance, S_2 cannot be a group manifold (while S_1 is the group manifold of SO(2)).

Eq. (1.3.79) states that the natural basis for the space of *left-invariant vector fields* is provided by the generators of ininitesimal *right* translations $\mathcal{J}_\lambda$. Analogously, the natural basis for the space of *right-invariant vector fields* is provided by the generators of ininitesimal *left* translations $\widetilde{\mathcal{J}}_\lambda$. This relation can be understood as indicated schematically in Fig. 1.5.

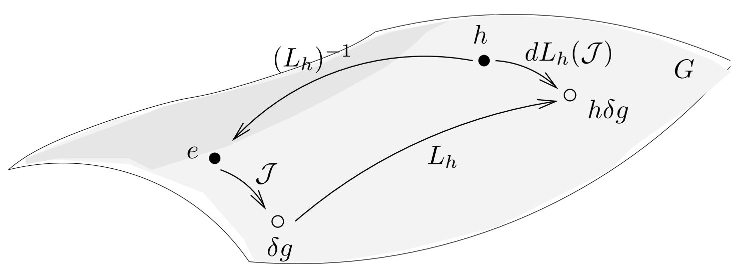

Figure 1.5. The effect of a small right translation of an element h is described via the effect on tangent vectors induced by the left mapping of h to the identity.

As we already discussed, the coordinate change $d\alpha$ of an element $h(\alpha)$ of the group under the (say) *right* action of an element $\delta g(\delta\beta)$ close to the identity correspond to the action of a vector in the tangent space $T_h(G)$. In turn, this change is determined, because of the group composition, by the action of $\delta g(\delta\beta)$ at the identity, which corresponds simply to a tangent vector in $T_e(G)$ of components $\delta\beta^\nu$. This is effected by mapping h to e with a *left* translation. The tangent vectors are then related by the differential of this group translation.

Invariant vector fields and Lie's first theorem Consider the invariance property of vector fields, Eq. (1.3.146), in the case of the basis Eq. (1.3.79): for instance, for the components of left-invariant v.f. we have

$$\widetilde{\mathcal{J}}^\mu{}_\lambda(\gamma = \varphi(\beta,\alpha)) = \frac{\partial\varphi^\mu(\beta,\alpha)}{\partial\beta^\nu}\widetilde{\mathcal{J}}^\nu{}_\lambda(\beta) \ . \tag{1.3.80}$$

We can rewrite this relation as follows:

$$\frac{\partial\varphi^\mu(\beta,\alpha)}{\partial\beta^\nu} = \frac{\partial\gamma^\mu}{\partial\beta^\nu} = \widetilde{\mathcal{J}}^\mu{}_\lambda(\gamma)(\widetilde{\mathcal{J}}^{-1})^\mu{}_\lambda(\beta) \ . \tag{1.3.81}$$

This is precisely the statement of Lie's first theorem, Eq. (1.3.50).

1.3.3 Integral curves of invariant vector fields and one-parameter subgroups

We have emphasized in the introduction to this Chapter, and in the discussion of the first Lie theorem, sec. 1.3.1.2 the following extremely important point. The fact that Lie groups are at the same time e groups and manifolds, i.e. that the group composition map $\varphi(\beta,\alpha)$ is continuous, differentiable (and invertible) in both arguments, allows to derive the expression of *finite* group transformations from the generators of *infinitesimal* transformations.

1.3.3.1 The exponential map

In the introduction to this Chapter we saw how the differentiability of the product law of the group SO(2) leads to a differential equation, Eq. (**??**), whose solution expresses the finite transformation $R(\theta)$ as the exponential of the *infinitesimal generator* J, as in Eq. (1.1.8):

$$R(\theta) = \mathrm{e}^{\theta\, J} \ . \tag{1.3.82}$$

The exponential expression of the finite transformation basically comes from the fact that, be-cause, of the group property, a finite group element can be obtained by repeated products of an element close to the identity (see Fig. 1.6):

$$\lim_{n\to\infty}\left(1+\frac{\theta}{n}J\right)^n \tag{1.3.83}$$

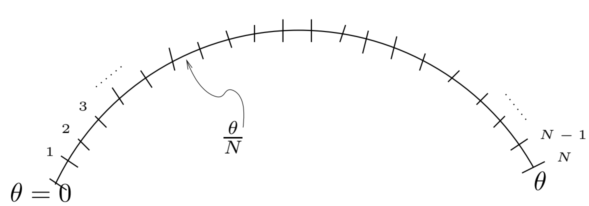

Figure 1.6. The effect of a small right translation of an element h is described via the effect on tangent vectors induced by the left mapping of h to the identity.

The exponential expression Eq. (1.3.82) of finite group elements generalizes to any matrix group. In fact, consider the endomorphisms $\mathrm{End}(V)$ of a vector space$V$, which in a basis are matrices of dimension $\dim V$. Within $\mathrm{End}(V)$ is well defined, via series expansion, the exponential map

$$m \mapsto M = \exp(m) = \sum_{k=0}^{\infty} \frac{1}{k!} m^k \ . \tag{1.3.84}$$

This map maps in a differentiable way a neighbourhood of the zero matrix $m = \mathbf{0}$ to a neighbourhood of the identity matrix $M = \mathbf{1}$. For matrices M in a suitable neighbourhood of the identity the map is also invertible.

Properties of the matrix exponential The matrix exponential map enjoys many properties of the usual exponential function, with some important differences due to the non-commutative nature of the matrices. For instance, with m, n square matrices, one has

$$\exp(m) \exp(n) = \exp(m+n) \quad \Leftrightarrow \quad [m,n] = 0 \ , \tag{1.3.85}$$

i.e. only if the two matrices commute. Otherwise, Eq. (1.3.85) generalizes to the so-called *Campbell-Hausdorff* formula

$$\exp(m) \exp(n) = \exp\left(m + n + \frac{1}{2} [m,n] + \frac{1}{12} ([m,[m,n]] + [[m,n]\,,n]) + \ldots \right) \ . \tag{1.3.86}$$

Deduce directly the first terms in the expansion above. A property which is immediate to show (check it) is that

$$\exp(U^{-1} m U) = U^{-1} \exp(m)\, U \ . \tag{1.3.87}$$

A very useful property of the determinants is the following:

$$\det\left(\exp(m)\right) = \exp\left(\mathrm{tr}\, m\right) \ , \tag{1.3.88}$$

or equivalently, by taking the logarithm (which for matrices is defined via a formal power series)

$$\det M = \exp\left(\mathrm{tr}(\ln M)\right) \ . \tag{1.3.89}$$

These relations can be proven easily for a diagonalizable matrix; indeed, determinant and trace are invariant under change of basis, so we can think of having diagonalized M. If λ_i are the eigenvalues of M, we have then

$$\det M = \prod_i \lambda_i = \exp\big(\sum_i \ln \lambda_i\big) = \exp\left(\mathrm{tr}(\ln M)\right) \ . \tag{1.3.90}$$

The result can then be extended to generic matrices as it can be argued that every matrix can be approximated to any chosen accuracy by diagonalizable matrices.

From infinitesimal generators to group elements We have already remarked that, for matrix groups, the tangent space to the identity (i.e., to the matrix $\mathbf{1}$), is again given by the set of matrices, forming a vector space w.r.t. to the addition of matrices. So it makes sense to write an element close to the identity as $\mathbf{1} + \delta\beta J$, with $\delta\beta$ small. The infinitesimal generator J, belonging to the tangent plane in $\mathbf{1}$, is a matrix constrained only by the fact that $\mathbf{1} + \delta\beta J$ must belong to the matrix group in question, to first order in $\delta\beta$ (for instance, be special orthogonal).

Thus, for any matrix group, the matrix exponential realizes a map from the tangent plane to the identiy, i.e. the space of *infinitesimal generators*, to a suitable part of the entire group, the part "sufficiently close" to the identity (we will discuss later the meaning of this remark). To any infinitesimal generator J we can in fact associate, via the exponential map, a *one-parameter subgroup* of matrices

$$M(\beta) = \exp(\beta\, J)\ . \tag{1.3.91}$$

with the simple composition law

$$M(\alpha)M(\beta) = M(\alpha + \beta)\ . \tag{1.3.92}$$

A generic element of the group (in the region connected to the identity) can be written as

$$g(\beta) = \exp(\beta^\nu J_\nu)\ , \tag{1.3.93}$$

where $\{J_\nu\}$ is a basis of $\dim G$ infinitesimal generators. The parameters β^ν can be taken as coordinates on the group manifold; this is often a convenient choice. Notice that the group product of elements writen in the exponential parametrization Eq. (1.3.93) is determined, via the Campbell-Hausdorff formula Eq. (1.3.86), by the commutators

$$[J_\mu, J_\nu]\ . \tag{1.3.94}$$

The infinitesimal generators form, rispect to the commutation, a *Lie algebra*, as we will discuss at length. Thus, the exponential parametrization of the group elements relates the group structure to the structure of the Lie algebra of its generators.

From infinitesimal transformations to finite ones The exponential map that we just discussed for matrices generalizes also to operators on infinite-dimensional spaces. Thus we can always use an exponential parametrization when the Lie group G is a group of transformations. For instance, consider the group $(\mathbb{R}, +)$ seen as a group of transformations acting on $\mathbb{R}$ itself: the element $a \in \mathbb{R}$ acts by $x \mapsto x + a$, for every $x \in \mathbb{R}$. The infinitesimal translations are generated, on any function $f(x)$, by the invariant vector field $\mathcal{J} = \partial_x$, see Eq. (1.3.147). The finite translation by a, i.e. the finite group element, are effected on the functions by $\exp(-a\partial_x)$[4]. Indeed, we have

$$\exp\left(-a\frac{d}{dx}\right) f(x) = \left(1 - a\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2} - \ldots\right) f(x) = f(x) - a\frac{df}{dx} + \frac{a^2}{2}\frac{d^2 f}{dx^2} - \ldots = f(x-a)\ . \tag{1.3.95}$$

We have seen in sec. 1.2.2 that every Lie group G can be seen as a transformation group over the group manifold G itself; we have discussed how the infinitesimal generators of the group are given, in this framework, by the invariant vector fields. We will now discuss in this geometric framework, which is applicable to *any* Lie group, how the finite transformations can be recovered from the expression of the infinitesimal generators.

1.3.3.2 Vector fields and fluxes

We recalled in previous section 1.3.2 the definition of tangent vectors as specifying the directional derivative of functions in the initial direction of a curve $\sigma(t)$, see Eq.s (1.3.70,1.3.70).

[4] As we discussed in sec **??**, given a group of transformations on a target spaceof coordinates x, the group induces an action on the space of functions $\psi(x)$, defined by the requirement that the transformed function in the transformed point ($x + a$ in our case) equals the original function in the point x, namely, that $\psi'(x) = \psi(x - a)$.

Generalizing this definition, a vector field V represents in each point the tangent of a *flux line*, a.k.a. as an *integral curve* of the vector field (think of fluido-dynamics or electro-magnetism). The ensemble of the flux lines of a vector field is simply called the *flux* of the vector field. The flux $\sigma(t, p_0)$ is a family of curves parametrized by t, passing through the point p_0 at $t = 0$, whose tangent is, in any point, representend by the value of V at that point. See Fig. 1.7 for a

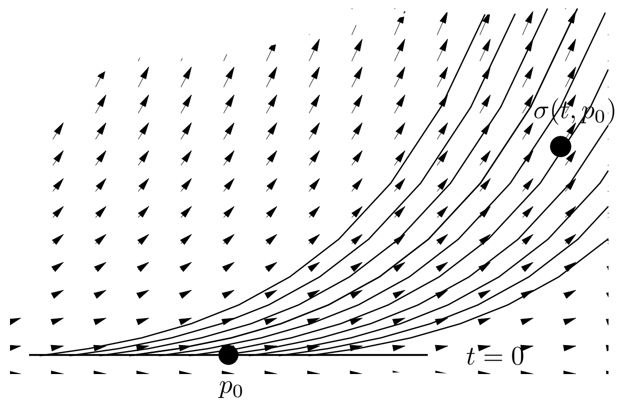

Figure 1.7. The integral curves $\sigma(t, p_0)$ form the flux of a vector field.

graphicalexample. Namely, for any function f, we have

$$\begin{aligned} \frac{d}{dt}\left[f\left(\sigma(t, p_0)\right)\right] &= V[f]\left(\sigma(t, p_0)\right) \ , \\ \sigma(0, p_0) &= p_0 \ . \end{aligned} \tag{1.3.96}$$

In coordinates, this means

$$\begin{aligned} \frac{d\sigma^\mu(t, p_0)}{dt} &= V^\mu\left(\sigma(t, p_0)\right) \ , \\ \sigma^\mu(0, p_0) &= \alpha^\mu(p_0) \ . \end{aligned} \tag{1.3.97}$$

Thus, $\sigma^\mu(t, p_0)$ s given by the unique solution of the system Eq. (1.3.97) of ODE with specified initial conditions.

For a given p_0, $\sigma(t, p_0)$ is an integral curve through p_0. We can also view $\sigma(t, p_0)$ as defining a set of transformations σ_t, parametrized by t, acting on the manifold:

$$\sigma_t \ : p_0 \mapsto \sigma_t(p_0) = \sigma(t, p_0) \ . \tag{1.3.98}$$

These transformation form an abelian one-dimensional group w.r.t. to the composition of transformations. Indeed we have

$$\sigma\left(t', \sigma(t, p_0)\right) = \sigma(t + t', p_0) \ , \tag{1.3.99}$$

which means that the group law is simply

$$\sigma_{t'} \cdot \sigma_t = \sigma_{t+t'} \ . \tag{1.3.100}$$

The inverse is clearly given by $(\sigma_t)^{-1} = \sigma_{-t}$, the identity by σ_0. So, we have a group of transformations locally isomorphic to $(\mathbb{R}, +)$.

Exponential map and one-parameter subgroups of a Lie group On a group manifold G, to every tangent vector to the identity v, i.e., to every infinitesimal generator, we can associate a left-invariant (or a right-invariant) vector field V_L (or V_R), via Eq. (1.3.73) (or Eq. (1.3.77)), and therefore a one-parameter group of transformations $\{\sigma_t\}$. These transformations map a group element h to a new element $\sigma_t(h)$. We can view these transformations as a one-parameter subgroup of G itself by interpreting $\sigma_t(h)$ as specifying the left (or right) translation of h by a group element $\hat{\sigma}(t)$:

$$\sigma_t(h) \leftrightarrow L_{\hat{\sigma}(t)} h = \hat{\sigma}(t) h \ , \tag{1.3.101}$$

or $\sigma_t(h) = R_{\hat{\sigma}(t)} h = h\hat{\sigma}(t)$. In other words, we have

$$\sigma^\mu(t,h) = \varphi^\mu(\hat{\sigma}(t), h) = \varphi^\mu(\sigma(t,e),\alpha) \ , \tag{1.3.102}$$

in terms of the group composition map φ, where we also noted that for the group element $\hat{\sigma}(t)$ we have simply

$$\hat{\sigma}(t) \leftrightarrow \sigma_t(e) \ , \tag{1.3.103}$$

so that the its coordinates are simply given by $\sigma^\mu(t,e)$.

The action of the *finite* transformations σ_t on each point $h = h(\alpha)$ of the group manifold are determined by the *infinitesimal* transformations generated by the left- (or right-)invariant vector field V_L (or V_R) of which $\sigma(t,h)$ represents the flux. They can be given an explicit expression via the operatorial *exponential map*:

$$\sigma_t = \mathrm{e}^{t\, V_L} \ , \tag{1.3.104}$$

or $\sigma_t = \exp(t\, V_R)$. When acting on a point $h(\alpha)$, the vector field V_L is of course evaluated at that point:

$$\sigma^\mu(t,h(\alpha)) = (\exp(tV_L(\alpha))\alpha)^\mu \ . \tag{1.3.105}$$

Indeed, let us show that the flux $\sigma(t,h)$ defined by Eq. (1.3.104) is really the flux of the invariant vector field V_L. We have from Eq. (1.3.105)

$$\begin{aligned} \frac{d\sigma^\mu(t,\alpha)}{dt} &= (V_L(\alpha)\exp(tV_L(\alpha))\alpha)^\mu = V_L^\nu(\alpha)\frac{\partial}{\partial\alpha^\nu}\sigma^\mu(t,\alpha) \\ &= V_L^\nu(\alpha)\frac{\partial\varphi^\mu(\sigma(t,e),\alpha)}{\partial\alpha^\nu} = V_L^\mu(\varphi(\sigma(t,e),\alpha)) = V_L^\mu(\sigma^\mu(t,\alpha)) \ . \end{aligned} \tag{1.3.106}$$

In the second line we used Eq. (1.3.102) and the definition of left-invariant vector field, Eq. (1.3.68).

Exponential parametrization It follows from the above discussion that a possible (and often convenient) parametrization of a Lie group is obtained expressing every element (connected to the identity, as we will see) as the exponential of a suitable infinitesiaml generator, i.e., of a suitable linear combination of the infinitesimal generators $\mathcal{J}_\lambda$, see Eq. (1.3.79):

$$h(\alpha) = \mathrm{e}^{\alpha^\lambda \mathcal{J}_\lambda} \ , \tag{1.3.107}$$

and using the parameters α^λ themselves as coordinates on the group manifold. In Eq. (1.3.107) we are viewing the finite group elements as finite *right* translations; of course, using the exponentials of the $\widetilde{\mathcal{J}}_\lambda$ right-invariant vector fields we could write them as finite *left* translations. It is always possible to express group elements as left- or right- translations, see sec. 1.2.2. Thus, Eq. (1.3.107) has a general validity as it applies to any Lie group. More that this, the exponential parametrization of finite group elements in terms of the infinitesimal generators applies to *any* explicit realization of the Lie group elements. We have already seen this in the case of matrix Lie groups in Eq. (1.3.93).

Group product law from infinitesimal generators The product of two group elements seen as transformations on the group manifold consists in the composition of the two transformations. The parametrization Eq. (1.3.107) realizes them as exponentials of differential operators. The exponential map of differential operators, see e.g. Eq. (**??**), enjoys the same properties as the exponential map for matrices. In particular, the product of two exponentials is given by the Campbell-Hausdorff formula Eq. (1.3.86). We see therefore that the group product law is determined, upon use of the Campbell-Hausdorff formula, by the *commutators* of the infinitesimal generators, i.e. by the commutators of the invariant vector fields

$$[\mathcal{J}_\lambda, \mathcal{J}_\rho] = \left[\mathcal{J}(\alpha)^\sigma{}_\lambda \frac{\partial}{\partial\alpha^\sigma}, \mathcal{J}^\tau{}_\rho(\alpha)\frac{\partial}{\partial\alpha^\tau}\right] \tag{1.3.108}$$

As we will discuss in detail later, the space of invariant vector fields endowed with the commutator forms an algebric structure called a Lie algebra. Thus, the group structure of a Lie group is etirely determined, via the exponential map, from the Lie algebra of its infinitesimal generators.

Let us now side-track for a while and describe what a Lie algebra is, before discussing in detail the Lie algebra of invariant vector fields.

1.3.4 Lie algebras

A Lie algebra $\mathbb{G}$ is first of all a *vector space* over $\mathbb{R}$ (if $\mathbb{G}$ as a vector space is taken over $\mathbb{C}$ we talk of a *complexified* or complex Lie algebra); let the group operation in $\mathbb{G}$ be denoted as $+$. There is an additional operation[5] $[\ ,\ ] : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, called the *Lie product* or *Lie bracket*:

$$\forall\, x, y \in \mathbb{G}\ , \quad x, y \mapsto [x, y]\ . \tag{1.3.109}$$

The Lie product satisfies the following properties.

i) Linearity:

$$[x, \alpha y + \beta z] = \alpha\,[x, y] + \beta\,[x, z]\ , \tag{1.3.110}$$

where $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$ for complexified Lie algebras).

ii) Antisymmetry:

$$[x, y] = -\,[y, x]\ . \tag{1.3.111}$$

iii) Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\ . \tag{1.3.112}$$

Notice that i) and ii) imply the linearity also in the first argument and that the Lie product is *not associative*. Indeed we have $[x, [y, z]] \neq [[x, y], z]$, as from the Jacobi identity we get $[x, [y, z]] - [[x, y], z] = -\,[y, [z, x]]$, which is generically non-zero.

The dimension of $\mathbb{G}$ as a vector space is also called the *dimension* or order of the Lie algebra, and denoted as $\dim \mathbb{G}$.

Fixing a basis $\{t_i\}$ of vectors, on which every element of $\mathbb{G}$ can be expanded: $x = x^i t_i$, the Lie product structure of $\mathbb{G}$ is encoded in a set of $(\dim\mathbb{G})^3$ constants $c_{ij}{}^k$, called the *structure constants* of the algebra:

$$[t_i, t_j] = c_{ij}{}^k\, t_k\ . \tag{1.3.113}$$

The basis vectors t_i are called the *generators* of the Lie algebra.

[5] The Lie product should not be automatically thought of as a *commutator* constructed with some other product; we could have used any other symbol to denote it. In many important cases, though, as we will discuss, the Lie bracket will indeed be a commutator.

As a consequence of the properties of the Lie product, the structure constants obey the following relations.

i) Antisimmetry:

$$c_{ij}{}^{k} = -c_{ji}{}^{k} \ . \tag{1.3.114}$$

ii) Jacobi identity:

$$c_{im}{}^{s} c_{jk}{}^{m} + c_{jm}{}^{s} c_{ki}{}^{m} + c_{km}{}^{s} c_{ij}{}^{m} = 0 \ . \tag{1.3.115}$$

Of course, the explicit set of structure constants for a given Lie algebra is unique only up to the effect of changes of basis. Under a change of basis

$$t_i \mapsto t'_i = S_i{}^{j} t_j \ , \tag{1.3.116}$$

the structure constants change tensorially:

$$c'{}_{ij}{}^{k} = S_i{}^{m} S_j{}^{n} c_{mn}^{\ p} \left(S^{-1}\right)_p{}^{k} \ . \tag{1.3.117}$$

For instance, the structure constants change sign if we change of sign all the generators (i.e. we set $S_i{}^{j} = -\delta_i{}^{j}$). So Lie algebras with opposite structure constants are isomorphic.

For the Lie algebras we can introduce important concepts related to the various ways in which a Lie algebra, abstractly defined by the Lie product "multiplication table", i.e. by the structure constants, can be realized concretely, in parallel with what we did for groups in the previous Chapter.

Homomorphism Two Lie algebras $\mathbb{G}$ and $\mathbb{K}$ are *homomorphic* if there exists an *homomorphism* between the two, namely a map $\phi : \mathbb{G} \to \mathbb{K}$ that preserves the Lie product:

$$[\phi(x), \phi(y)] = \phi\left([x, y]\right) \ . \tag{1.3.118}$$

Representation A Lie algebra can be realized by a set of square matrices (forming a vector space), the Lie product being defined as the commutator in the matrix sense. If a Lie algebra $\mathbb{G}$ is *homomorphic* to some *matrix Lie algebra* $\mathcal{D}(\mathbb{G})$, then $\mathcal{D}(\mathbb{G})$ is said to give a matrix *representation* of $\mathbb{G}$.

Isomorphism Two Lie algebras between which there exists an *invertible homomorphism*, i.e., an *isomorphism*, are said to be *isomorphic*. They correspond to two different realizations of the same abstract Lie algebra. An isomorphism of $\mathbb{G}$ into some matrix Lie algebra $\mathcal{D}(\mathbb{G})$ is called a *faithful* representation.

Adjoint representation We have seen that, given a Lie algebra $\mathbb{G}$, we can associate to it a set of structure constants (unique only up to changes of basis). The converse is also true. Indeed, given a set of n^3 constants $c_{ij}{}^{k}$, the conditions Eq. (1.3.114) and Eq. (1.3.115) are necessary and sufficient for the set of $c_{ij}{}^{k}$ to be the structure constant of a Lie algebra. The necessity was argued before. The sufficience is exhibited by constructing explicitely a *matrix* Lie algebra with structure constants $c_{ij}{}^{k}$. Define in fact the n square matrices T_i, with matrix elements

$$(T_i)_j{}^{k} \equiv c_{ij}{}^{k} \ . \tag{1.3.119}$$

The matrix commutator of two such matrices can be computed by making use of the antisimmetry and of the Jacobi identity (do it!):

$$[T_i, T_j]_p{}^q = (T_i)_p{}^s (T_j)_s{}^q - (i \leftrightarrow j) = \ldots = -c_{ij}{}^k \, (T_i)_p{}^q \, . \tag{1.3.120}$$

Thus, the matrices obtained as real linear combinations of the T_i's form a Lie algebra of dimension n, with structure constants $-c_{ij}{}^k$ (or, equivalently, the matrices $-T_i$ generate and algebra with structure constants $c_{ij}{}^k$).

The above construction tells us also that, given a Lie algebra $\mathbb{G}$ with structure constants $c_{ij}{}^k$, it always possess a faithful representation in terms of $\dim \mathbb{G} \times \dim \mathbb{G}$ matrices, called the *adjoint representation*, generated by the matrices T_i of Eq. (1.3.119).

Examples

- *The vector product in* $\mathbb{R}^3$*.* The vector space $\mathbb{R}^3$ becomes a Lie algebra when equipped with the ordinary *vector product*. Namely, the Lie product in this example is defined as $[\vec{x}, \vec{y}] \equiv \vec{x} \wedge \vec{y}$. Linearity and antisymmetry are immediate, and the Jacobi identity is satisfied (show it) because of the well-known identity

$$\vec{x} \wedge (\vec{y} \wedge z) = \vec{y} \, (\vec{x} \cdot z) - \vec{z} \, (\vec{x} \cdot y) \tag{1.3.121}$$

  In an orthonormanl basis $\{\vec{e}_i\}$, we find the structure constants of this algebra to be given by the Levi-Civita antisymmetric tensor:

$$\vec{e}_i \wedge \vec{e}_j = \epsilon_{ijk} \, \vec{e}_k \quad \Rightarrow \quad c_{ij}{}^k = \epsilon_{ijk} \ , \tag{1.3.122}$$

  where $\epsilon_{123} = 1$ and $\epsilon_{P(1)P(2)P(3)} = (-)^{\sigma(P)}$ for any permutation $P \in S_3$ ($\sigma(P)$ being 1 if P is even or -1 if P is odd); all other components of the tensor vanish.
- *The Lie Algebra* so(3) Consider the set of antisymmetric 3×3 real matrices, which is usually denoted as so(3). It forms a Lie Algebra, the Lie bracket being defined as the commutator of the matrix product:

$$\forall m, m \in \mathrm{so}(3) \, , \quad [m, n] \equiv mn - nm \in \mathrm{so}(3) \, . \tag{1.3.123}$$

  Indeed it is easy to verify that when the Lie bracket is defined as a commutator it satisfies automatically antisymmetry and Jacobi properties. Moreover the commutator of two antisymmetric matrices is still antisymmetric:

$$[m, n]^T = (mn - nm)^T = n^T m^T - m^T n^T = nm - mn = -\,[m, n] \ , \tag{1.3.124}$$

  since $m^T = -m$, $n^T = -m$. Matrices $m \in \mathrm{so}(3)$ depend on 3 real parameters, which for instance can be taken as its elements in the upper triangular block, m_{ij} with $j < j$. It is more costumary to relabel the 3 parameters as $\alpha^i = \epsilon^{ijk} \, m_{jk}$, namely to set

$$m(\alpha) = \begin{pmatrix} 0 & \alpha^3 & -\alpha^2 \\ -\alpha^3 & 0 & \alpha^1 \\ \alpha^2 & -\alpha^1 & 0 \end{pmatrix} = \alpha^i \, L_i \ , \tag{1.3.125}$$

  the generators L_i being given by

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \ , \ L_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ L_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{1.3.126}$$

namely by

$$(L_i)_j{}^k = \epsilon_{ijk} \,. \tag{1.3.127}$$

The matrix L_i generates infinitesimal rotations around the i-th axis (compare with Eq.s (1.1.11,1.1.12) for the SO(2) case). Notice that the expression Eq. (1.3.127) means that, according to the definition Eq. (1.3.119), the L_i are the generators of the adjoint representation of the abstract Lie algebra with structure constants $c_{ij}{}^k = \epsilon_{ijk}$ (which is for instance realized also by $\mathbb{R}^3$ with the external product, as we saw in the previous example, or by the set of antihermitean matrices, su(2), as we will see in next example). We have therefore

$$[L_i, L_j] = -c_{ij}{}^k \,, \tag{1.3.128}$$

as of course can be checked directly by computing the matrix commutatators.

- *The Lie algebra* su(2) The set of 2×2 antihermitean traceless matrices is usually denoted as su(2). It is closed under the matrix commutator (the computation is perfectly analogous to Eq. (1.3.124)), so it forms a Lie algebra. Such matrices depend on 3 real parameters. A set of three independent such matrices is provided, for instance, by

$$t_i = -\frac{\mathrm{i}}{2}\sigma_i \,, \quad (i = 1, 2, 3) \,, \tag{1.3.129}$$

where the σ_i are the Pauli matrices. Therefore every matrix $u \in \mathrm{su}(2)$ can be written as $u = \beta^i\, t_i$, with $\beta^i \in \mathbb{R}$. From the well-known products of the Pauli matrices

$$\sigma_i \sigma_j = \delta_{ij} + \mathrm{i}\, \epsilon_{ijk}\, \sigma_k \tag{1.3.130}$$

follow the commutators

$$[t_i, t_j] = \epsilon_{ijk}\, t_k \,. \tag{1.3.131}$$

The su(2) Lie algebra is thus isomorphic to so(3) and to $\mathbb{R}^3$ with the external product.

1.3.5 The Lie algebra of a Lie group

1.3.5.1 The Lie algebra of vector fields

In the mathematical appendix **??** the concept and main properties of vector fields on a manifold $\mathcal{M}$ are reviewed.

The composition of two vector fields is no longer a vector field. In fact, applying first the vector field Y then X to a function f gives

$$X\left(Y(f)\right) = X^\nu \partial_\nu \left(Y^\mu \partial_\mu f\right) = (X^\nu \partial_\nu Y^\mu)\partial_\mu f + X^\nu Y^\mu \partial_{\nu\mu} f \;. \tag{1.3.132}$$

The presence of the term with second derivatives of f forbids us to write the result as due to the application of a new vector field. So the space of vector fields $\mathrm{Diff}_0(\mathcal{M})$ cannot be given the structure of an algebra using the composition as the product.

However, $\mathrm{Diff}_0(\mathcal{M})$ can be given the structure of a *Lie algebra*, by defining the Lie product as the "commutator" of compositions:

$$[X, Y] \equiv X\left(Y(f)\right) - Y\left(X(f)\right) \;. \tag{1.3.133}$$

Indeed, the second derivative term in Eq. (1.3.132) cancel when subtrating the same expression with $X \leftrightarrow Y$ and thus $[X, Y]$ is again a vector field:

$$[X, Y] = (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu)\partial_\mu \;. \tag{1.3.134}$$

As an exercise, check that the Lie bracket $[X, Y]$ as defined in Eq. (1.3.133) satisfies the Jacobi identities. Thus, to any manifold G is naturally associated a Lie algebra, the Lie algebra $\mathrm{Diff}_0(G)$ of the vector fields. This algebra is however infinite-dimensional. However, we have seen that the group structure of G allows to single out a subclass of vector fields, the (left- or right-)invariant vector fields

We have recalled in sec. 1.3.2 the mappings which are induced on the vector fields by a mapping on the manifold. The induced mappings of vector fields Eq. (1.3.57) have the important properties of *preserving the Lie brackets* of vector fields:

$$[X, Y]' = [X', Y'] \ . \tag{1.3.135}$$

In other words, the differential mappings $d\phi$ are isomorphisms (if ϕ is invertible, of course) of the Lie algebra of vector fields, $\mathrm{Diff}_0(\mathcal{M})$. Indeed, from Eq. (1.3.134)

$$[X, Y]'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}}(X^{\nu}\partial_{\nu}Y^{\rho} - Y^{\nu}\partial_{\nu}X^{\rho})(x) \ . \tag{1.3.136}$$

On the other hand,

$$\begin{aligned}
[X', Y']^{\mu}(x') &= X'^{\nu}(x')\frac{\partial}{\partial x'^{\nu}}Y'^{\mu}(x') - (X \leftrightarrow Y) \\
&= X^{\rho}(x)\frac{\partial x'^{\nu}}{\partial x^{\rho}}\frac{\partial}{\partial x'^{\nu}}\left(Y^{\sigma}(x)\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\right) - (X \leftrightarrow Y) \\
&= X^{\rho}(x)\frac{\partial}{\partial x^{\rho}}\left(Y^{\sigma}(x)\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\right) - (X \leftrightarrow Y) \\
&= \left(X^{\rho}(x)\frac{\partial Y^{\sigma}(x)}{\partial x^{\rho}} - (X \leftrightarrow Y)\right)\frac{\partial x'^{\mu}}{\partial x^{\sigma}} \ ,
\end{aligned} \tag{1.3.137}$$

which coincides with the r.h.s. of Eq. (1.3.136).

Since the induced mappings on vector fields have the property of preserving the Lie bracket, see Eq. (1.3.135), the subset of left- or right-invariant vector fields, beside a vector space isomorphic to the tangent space to the identity $T_e(G)$, is also a *Lie subalgebra* of the Lie algebra of vector fields. In fact, the Lie bracket of two, say, left-invariant vector fields is again a left-invariant vector field: upon the left translation by g,

$$dL_g([X_L, Y_L]) = [dL_g(X_L), dL_g(Y_L)] = [X_L, Y_L] \ . \tag{1.3.138}$$

Analogously for right-invariant vector fields.

Thus, we see that the *left- or right-invariant vector fields* form a *Lie algebras of dimension* $\dim G$, that we may call respectively $\mathbb{G}_{L,R}$.

The Lie algebras $\mathbb{G}_{R(L)}$ in this basis take the form

$$\begin{aligned}
\left[\widetilde{\mathcal{J}}_{\lambda}, \widetilde{\mathcal{J}}_{\mu}\right] &= c_{\lambda\mu}^{(R)\,\nu}\,\widetilde{\mathcal{J}}_{\nu} \ . \\
[\mathcal{J}_{\lambda}, \mathcal{J}_{\mu}] &= c_{\lambda\mu}^{(L)\,\nu}\,\mathcal{J}_{\nu} \ ,
\end{aligned} \tag{1.3.139}$$

The structure constants can be expressed in terms of the functions $\varphi^{\mu}(\alpha, \beta)$ using Eq. (1.3.79), Eq. (1.3.46) and Eq. (1.3.134). For instance, for the right-invariant vector fields we have

$$\begin{aligned}
\left[\widetilde{\mathcal{J}}_{\lambda}, \widetilde{\mathcal{J}}_{\rho}\right]^{\mu}(\beta)\,\partial_{\mu} &= \left(\widetilde{\mathcal{J}}^{\nu}{}_{\lambda}\partial_{\nu}\widetilde{\mathcal{J}}^{\mu}{}_{\rho} - (\lambda \leftrightarrow \rho)\right)(\beta)\,\partial_{\mu} \\
&= \left(\widetilde{\mathcal{J}}^{\nu}{}_{\lambda}\partial_{\nu}\widetilde{\mathcal{J}}^{\mu}{}_{\rho} - (\lambda \leftrightarrow \rho)\right)(\widetilde{\mathcal{J}}^{-1})^{\tau}{}_{\mu}\,\widetilde{\mathcal{J}}^{\omega}{}_{\tau}\partial_{\omega} \ .
\end{aligned} \tag{1.3.140}$$

Comparing with Eq. (1.3.139), we find

$$c^{(R)\,\tau}_{\lambda\mu} = \left(\widetilde{\mathcal{J}}^{\nu}{}_{\lambda}\partial_{\nu}\widetilde{\mathcal{J}}^{\mu}{}_{\rho} - (\lambda \leftrightarrow \rho)\right)(\widetilde{\mathcal{J}}^{-1})^{\tau}{}_{\mu}\,, \tag{1.3.141}$$

in agreement with the expression Eq. (**??**) of the second Lie theorem. Similarly, we can compute the structure constants for the $\mathbb{G}_L$ algebra. It can be shown, see later ..., that the two algebras are isomorphic, and indeed one has simply $c^{(L)\,\tau}_{\lambda\mu} = -c^{(R)\,\tau}_{\lambda\mu}$. These expressions must be, in spite of their apparent dependence on the coordinate β, constants, in force of the above discussion.

1.3.5.2 Invariant vector fields and Lie algebras for Matrix Lie groups

Consider the matrix group $\mathrm{GL}(n,\mathbb{R})$, parametrized parametrizing by the n^2 entries g_{ij} of the matrix. The left translation $h \mapsto gh$ of an element h by an element g is described explicitely by the matrix product, see Eq. (1.2.28). We have discussed in previous sections how invariant vector fields on $\mathrm{GL}(n,\mathbb{R})$ are determined by tangent vectors to the identity, according to Eq. (1.3.67). The result, given in Eq. (1.3.78), is that if v (of components v^{ij}) is a tangent vector to the identity, the corresponding left-invariant vector fields reads $X_{L,v}(g) = (gv)^{ij}\frac{\partial}{\partial g^{ij}}$. Computing the Lie bracket of two invariant vector fields gives then (check it for exercise):

$$\left[X_{L,v}, X_{L,w}\right](g) = (g\,[v,w])^{ij}\,\frac{\partial}{\partial g^{ij}}\ . \tag{1.3.142}$$

The Lie algebra $\mathbb{G}_L$ is thus mapped to the matrix Lie algebra of the components of tangent vectors v to the identity. The matrices v are *unconstrained*, and they clearly form a Lie algebra as the commutator of two matrices is again a matrix. This algebra is named $\mathrm{gl}(n,\mathbb{R})$ A basis of generators J_{ij} for this algebra is naturally given by the matrices

$$(J_{ij})_{lm} = \delta_{ij}\delta_{jm}\,, \tag{1.3.143}$$

namely by matrices J_{ij} having the element in the i-th row and j-th column equal to 1, al the others being zero. These are exactly the infinitesimal generators obtained directly from their definition Eq. (1.3.41) as derivatives of the group element taken at the origin, see Eq. (1.3.42). Analogously things work for right-invariant vector fields.

For other matrix groups, the entries of the matrix g give an over-complete parametrization. The conditions that define the matrix group in question (for instance, the condition of being orthogonal) correpond to equations on the matrix entries, and the group manifold is embedded as a hypersurface in the flat $\mathbb{R}^{n^2}$ (or $\mathbb{C}^{n^2}$) space parametrized by the matrix entries. Nevertheless, we can use the same scheme as above, and express the tangent vectors and invariant vector fields in these coordinates: it is like describing the tangent vector fields to a sphere S^2 immersed in $\mathbb{R}^3$ in terms of the cartesian coordinates of the latter. Of course, the components of the tangent vectors in such a coordinate system cannot be arbitrary (the tangent vectors have after all to be tangent to the hypersurface!), and one will have to impose the appropriate conditions on them. Still, the mapping of the Lie algebras $\mathbb{G}_{L,R}$ to the *matrix Lie algebra* of the components of the tangent vectors to the identity holds true.

1.3.6 Lie groups from Lie algebras

The outshot of all our discussions so far is that a connected Lie group G can be described in terms of its Lie algebra $\mathbb{G}$, which is the algebra of the infinitesimal generators of G, i.e., the algebra of invariant vector fields, via the exponential map:

$$G = \exp \mathbb{G}\,. \qquad (1.3.144)$$

Describing Lie group G via the exponential parametrization Eq. (1.3.144) is the continuous analogous of describing a finite group via a presentation, namely via a set of generators and relations (see Sec.s **??** and **??**).

Any element $x \in \mathbb{G}$, a generator, gives rise to a one-parameter subgroup of G, containing the elements

$$\mathrm{e}^{\alpha\, x}\,, \quad \alpha \in \mathbb{R}\,,\; x \in \mathbb{G}\,. \qquad (1.3.145)$$

This is analogous to the subgroup $\{a^n\}$, where a is the generator of a finite group. Notice that this subgroup can be compact or non-compact:

i) if the generator x (which we suppose to be represented as an operator, or a matrix) is *hermitean*, then it generates a *non-compact* subgroup. Heuristically, we can see it because in the basis of its eigenvectors, $x \to \mathrm{diag}(\lambda_i)$, with $\lambda_i \in \mathbb{R}$, so that $\mathrm{e}^{\alpha x} \to \mathrm{diag}(\mathrm{e}^{\alpha\lambda_i})$, and shows no periodicity in α;
ii) if the generator x (which we suppose to be represented as an operator, or a matrix) is *anti-hermitean*, then it generates a *compact* subgroup. Heuristically, we can see it because in the basis of its eigenvectors, $x \to \mathrm{diag}(\mathrm{i}\lambda_i)$, with $\lambda_i \in \mathbb{R}$, so that $\mathrm{e}^{\alpha x} \to \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha\lambda_i})$, and will be generically periodic in α.

This kind of properties of the generators are the analogue to the presence or absence of relations of the form $a^m = e$ that specify the order of a generator.

If two generators commute in the algebraic sense, the associated one-parameter subgroups commute in the group-theoretical sense:

$$\mathrm{e}^{\alpha\, x}\mathrm{e}^{\beta\, y} = \mathrm{e}^{\beta\, y}\mathrm{e}^{\alpha, x} \;\;\Leftrightarrow [x,y] = 0\,, \qquad (1.3.146)$$

as it follows from the Baker-Campbell-Hausdorff formula Eq. (**??**). If all the generators $\{x_i\}$ ($i = 1, \ldots d$ where d is the dimension of G) commute, then the group is the direct product of the d abelian one-parameter subgroups In general, the generators $\{x_i\}$ do not commute, and form a non-trivial Lie algebra:

$$[x_i, x_j] = c_{ij}{}^k x_k\,. \qquad (1.3.147)$$

These relations are the equivalent of the extra relations that can exist between the generators, see Eq.s (**??**,**??**).

In the same way as a finite group is determined by its presentation in terms of genrators and relations, the group product in the Lie group G law is completely determined by its Lie algebra Eq. (1.3.147): for two generic elements of G $g = \mathrm{e}^{\hat{g}}$ and $h = \mathrm{e}^{\hat{h}}$, where $\hat{g} = \alpha^i x_i$ and $\hat{h} = \beta^i x_i$ are two generic elements of $\mathbb{G}$, we have by the BCH formula Eq. (**??**)

$$gh = \mathrm{e}^{\hat{g}}\mathrm{e}^{\hat{h}} = \mathrm{e}^{\hat{g}+\hat{h}+\frac{1}{2}[\hat{g},\hat{h}]+\frac{1}{12}([\hat{g},[\hat{g},\hat{h}]]+[[\hat{g},\hat{h}],\hat{h}])+\cdots}\,, \qquad (1.3.148)$$

where the exponent in the r.h.s. is again a Lie algebra element.

The exponential map allows to associate many properties and structures of a Lie algebra $\mathbb{G}$ to properties and structures of the corresponding Lie groups. We will often utilize this approach when discussing in Lie algebras in more detail in the next Chapter.

1.3.7 The "classical" matrix Lie groups and their Lie algebras

A large class of Lie Groups (and basically all of the Lie groups we will consider) can be thought of as matrix groups, either because they are directly defined in this way or because they possess at least a *faithful finite-dimensional representation*; that is, there exists at least an *isomorphic* mapping (the representation being faithful) of the group onto a matrix group. Thus it is particularly important to discuss these groups and their Lie algebras.

1.3.7.1 The general linear groups, $\mathrm{GL}(n, \mathbb{R})$ *or* $\mathrm{GL}(n, \mathbb{C})$

Consider the multiplicative group of real numbers $(\mathbb{R} \setminus \{0\}, \cdot)$ which can also be described as the group $\mathrm{Gl}(1)$ of 1×1 invertible matrices. A parametrization of the single matrix element $x \in \mathbb{R} \setminus \{0\}$ which complies with a),b) above is obviously

$$x(\alpha) = \exp(\alpha) \ . \tag{1.3.149}$$

Any neighbourhood of $x = 1$ is mapped one-to-one by the exponential map in a neighbourhood of $\alpha = 0$. The infinitesimal generator is just the number

$$J = \left.\frac{\partial x}{\partial \alpha}\right|_{\alpha=0} = 1 \ . \tag{1.3.150}$$

Notice the whole of $\mathrm{GL}(1)$ decomposes into the component $GL(1)_0$ composed of all $x > 0$, containing the identity, which is covered by the above parametrization, and the "disconnected" component $x < 0$. The latter is not covered by the exponential parametrization, but all $x < 0$ can be written as $(-1)|x|$, with $|x| \in GL(1)_0$.

The above can be generalized to $\mathrm{GL}(n, \mathbb{R})$ (but everything can be straightforwardly repeated for $\mathrm{GL}(n, \mathbb{C})$). The component of $\mathrm{GL}(n, \mathbb{R})$ connected to the identity, $\mathrm{GL}(n, \mathbb{R})_0 = \{M : \ \det M > 0\}$, can be expressed via the exponential map: from the Lie algebra $\mathrm{gl}(n, \mathbb{R})$:

$$M = \mathrm{e}^{m} = \mathrm{e}^{m^{ij}\, J_{ij}} \ . \tag{1.3.151}$$

Notice that we have $\det M = \exp \mathrm{tr}\, m > 0$, and the matrix m is unrestricted. The space of unconstrained matrices form a Lie algebra, called $\mathrm{gl}(n, \mathbb{R})$: the commutator of two matrices is again a matrix.

Choosing as coordinates the entries m^{ij} of m, the n^2 infinitesimal generators J_{ij} appearing in Eq. (1.3.151) were already given in Eq. (1.3.143), which we repeat here for convenience:

$$(J_{ij})_{kl} = \delta_{ik} \delta jl \ . \tag{1.3.152}$$

The full group $\mathrm{GL}(n, \mathbb{R})$ contains two disconnected components (with positive or negative determinant) and is then obtained as the direct product $\mathrm{GL}(n, \mathbb{R})_0 \otimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by $M \to -M$.

1.3.7.2 The special linear groups, $\mathrm{SL}(n.\mathbb{R})$ *or* $\mathrm{SL}(n.\mathbb{C})$

A matrix $M \in \mathrm{SL}(n.\mathbb{R})$ must have $\det M = 1$. Writing it as $M = \exp m$, we see using the property Eq. (1.3.88) of the matrix exponential function, we must require

$$\mathrm{tr}\, m = 0 \, . \tag{1.3.153}$$

Traceless $n \times n$ matrices form a Lie algebra (show it as an exercise), which is called $\mathrm{sl}(n, \mathbb{R})$. The algebra is clearly n^2-1-dimensional. Different choices of explicit bases of generators are possible;

it is of course natural to still consider the same J_{ij} as in $\mathrm{gl}(n, \mathbb{R})$, Eq. (**??**), when $i \neq j$, but the generators with diagonal entries must be traceless (so there are only $n-1$ independent ones).

For instance, a basis of generators for $\mathrm{sl}(2, \mathbb{R})$ is given by (the names are traditional)

$$L_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad L_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{1.3.154}$$

The explicit form of the Lie algebra, namely the structure constants, are encoded in the commutators

$$[L_0, L_{\pm 1}] = \pm 2\, L_{\pm 1} , \quad [L_1, L_{-1}] = L_0 . \tag{1.3.155}$$

1.3.7.3 The unitary groups U(n)

Writing an unitary matrix U as

$$U = \mathrm{e}^u \tag{1.3.156}$$

requires, considering U close to the identity, i.e., u small, that u be *anti-hermitean*: indeed,

$$\mathbf{1} = U^\dagger U = (\mathrm{e}^u)^\dagger \mathrm{e}^u = (\mathbf{1} + u^\dagger + \ldots)(\mathbf{1} + u + \ldots) = \mathbf{1} + u^\dagger + u + \ldots \ , \tag{1.3.157}$$

so that to first order in u we have

$$u^\dagger + u = \mathbf{0} \ . \tag{1.3.158}$$

The anti-hermitean matrices form a Lie algebra: if u, v are antihermitean, their commutator is anti-hermitean:

$$[u, v]^\dagger = (uv - vu)^\dagger = (v^\dagger u^\dagger - u^\dagger v^\dagger) = vu - uv = -\,[u, v] \ . \tag{1.3.159}$$

Notice also that anti-hermitean $n \times n$ matrices depend on n^2 real parameters, just as the unitary ones. In fact, the n diagonal elements of u must be immaginary, and the $n(n-1)/2$ complex upper trangular components, u_{ij} with $j > i$, determines the lower trangular components u_{ij} with $j < i$, because $(u^\dagger)_{ij} = u^*_{ji}$.

It is then possible to give an explicit real parametrization of u (in terms of suitable combinations of a subset of the real and imaginary parts of its matrix elements), i.e., to choose an explicit basis of generators. We will consider in the sequel some natural choices.

1.3.7.4 The special unitary groups SU(n)

The group of special unitary matrices contains the unitary matrices U with unit determinant. In the exponential parametrization Eq. (1.3.156) this implies, using the property Eq. (1.3.88) of the matrix exponential,

$$\mathrm{tr}\, u = 0 \ . \tag{1.3.160}$$

The *traceless, anti-hermitean* matrices form a Lie algebra of real dimension $n^2 - 1$ (we impose the single, real, condition Eq. (**??**) on the n^2-dimensional space of anti-hermitean matrices), called $\mathrm{su}(n)$.

1.3.7.5 The groups U(2) *and* SU(2)

Consider $U = \mathrm{e}^u \in \mathrm{U}(2)$. Imposing $u^\dagger + u = \mathbf{0}$ restricts u so that it can be parametrized by four real parameters α_A $(A = 0, 1, 2, 3)$ as follows:

$$u(\alpha) = \frac{1}{2}\begin{pmatrix} \mathrm{i}(\alpha_0+\alpha_3) & \alpha_2+\mathrm{i}\alpha_1 \\ -\alpha_2+\mathrm{i}\alpha_1 & \mathrm{i}(\alpha_0-\alpha_3) \end{pmatrix} = \frac{\mathrm{i}}{2}\left(\alpha_0\,\mathbf{1} + \alpha_i\,\sigma^i\right) \tag{1.3.161}$$

(the factor 1/2 is just a traditional, inessential, choice). The four generators generators are $\frac{\mathrm{i}}{2}\mathbf{1}$ and $\frac{\mathrm{i}}{2}\,\sigma^i$, where σ^i are the Pauli matrices. They form the u(2) Lie algebra, whose only non-trivial part is the su(2) Lie algebra generated by the Pauli matrices, which we already discussed in sec. 1.3.4, see Eq.s (1.3.129-1.3.131).

One finds then

$$U(\alpha) = \mathrm{e}^{\mathrm{i}\alpha_0/2}\left(\cos\frac{|\vec{\alpha}|}{2}\mathbf{1} + \mathrm{i}\sin\frac{|\vec{\alpha}|}{2}\,\hat{\alpha}_i\sigma^i\right)\;, \tag{1.3.162}$$

where $\hat{\alpha}$ is the versor of components has components $\hat{\alpha}_i = \alpha_i/|\vec{\alpha}|$. To see this, note that $\mathbf{1}$ obviously commutes with $\alpha_i\sigma^i$, so we can first of all split the exponential according to Eq. (**??**). The matrix exponential $\exp(\mathrm{i}\alpha_i\sigma^i/2$ can then be directly computed by means of the definition Eq. (**??**), since $(\mathrm{i}\alpha_i\sigma^i/2)^2 = -|\vec{\alpha}|^2/4\,\mathbf{1}$ and therefore only the two matrix structures $\mathrm{i}\alpha_i\sigma^i$ and $\mathbf{1}$ appear in the expansion. The matrix within brackets in Eq. (1.3.162) is un fact an SU(2) matrix (its determinant is 1) while the factor $\mathrm{e}^{\mathrm{i}\alpha_0}$ is $\det U$. An SU(2) matrix is thus parametrized by the points of a sphere S^3, defined by a versor $\hat{\alpha}$ and an angle $\psi = |\vec{\alpha}|$, with $\psi \in [0, 2\pi]$. Indeed for $\psi = 2\pi$ we get again, for all values of $\hat{\alpha}$, a single element, namely $-\mathbf{1}$, of the group, see Fig. 1.8.

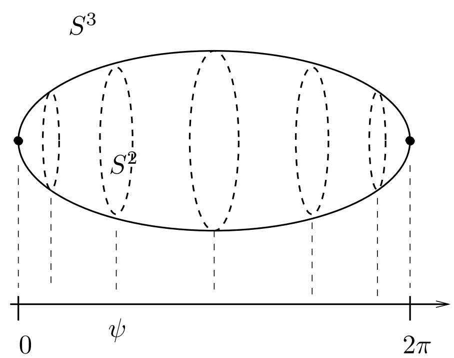

Figure 1.8. The group manifold of SU(2). For each value of $\psi \in [0, 2\pi]$ there is an S^2 of radius $\sin\frac{\psi}{2}$; this defines clearly an S^3 (the drawing is, for obvious reasons, represented in one dimension less).

1.3.7.6 The su(3) *Lie algebra and the Gell-Mann matrices*

Let's consider the next explicit example of special unitary group, the group SU(3) which, by the way, is very important in particle Physics: for instance, it is the local symmetry (or "gauge" symmetry) that rotates the three "colors" of the quarks (the components of protons, neutrons etc.) and accounts for the strong interactions of these particles.

For 3×3 matrices satisfying the anti-hermiticity condition Eq. (1.3.158) and traceless, as in Eq. (1.3.160), it is traditional (at least for particle Physicists) to choose the following basis of

$8 = 3^2 - 1$ generators:

$$t_a = -\frac{\mathrm{i}}{2}\lambda_a \ , \ \ (a = 1, 2, \ldots, 8) \ , \tag{1.3.163}$$

where

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \ ,$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \ ,$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.3.164}$$

are known as Gell-Mann matrices, and are the analogue in the su(3) context of the Pauli matrices for su(2). As an exercise, compute the structure constants of su(3) in this basis. The basis is evedently such that the generators t_i, $i = 1, 2, 3$, close an su(2) subalgebra, so $[t_i, t_j] = \epsilon_{ijk} t_k$. Find the remaining commutations.

1.3.7.7 The orthogonal and special orthogonal groups O(n) *and* SO(n)

Parametrize exponentially a real orthogonal matrix:

$$M = \exp m \ . \tag{1.3.165}$$

The orthogonality condition $M^T M = \mathbf{1}$ turns into the requirement that m be *antisymmetric*:

$$m + m^T = \mathbf{0} \ . \tag{1.3.166}$$

As it immediate to check (it goes exactly as in Eq. (1.3.159) replacing the hermitean conjugate with the transpose) the antisymmetric matrices form a Lie algebra called so(n).

The matrix m can be parametrized in terms of $n(n-1)/2$ parameters, for instance the entries m_{ij} with $j > i$, i.e. the upper triangular ones. The remaining entries m_{ij} with $j < i$are infact then simply given by $m_{ij} = -m_{ji}$. Thus, we can write

$$m = \sum_{i<j} m^{ij} J_{ij} \tag{1.3.167}$$

where the generators J_{ij} (defined for $i < j$) are given by

$$(J_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} \ , \tag{1.3.168}$$

i.e., in matrix notation,

$$J_{ij} = \begin{array}{cc} \begin{array}{ccccc} & i & & j & \end{array} & \\ \begin{pmatrix} 0 & \cdots & \cdots & 0 & \cdots \\ \vdots & \ddots & 0 & 1 & 0 \\ \vdots & 0 & \ddots & 0 & \cdots \\ 0 & -1 & 0 & \ddots & \cdots \\ \vdots & 0 & \vdots & \vdots & \ddots \end{pmatrix} & \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \end{array} \tag{1.3.169}$$

As we remarked in sec. 1.2.1.1, the linear transformations effected by the matrices J_{ij} on an n-dimansional space spanned by coordinates x^i can equivalently be effected by differential operators

$$\mathcal{J}_{ij} = x_j \partial_i - x_i \partial_j \ . \tag{1.3.170}$$

The structure constants of the so(n) Lie algebra can be extracted by computing the commutators of the J_{ij} generators or those of their differential counterparts $\mathcal{J}_{ij}$ (check this explicitely as an exercise). One gets

$$[J_{ij}, J_{kl}] = \delta_{il} J_{jk} - \delta_{ik} J_{jl} + \delta_{jk} J_{il} - \delta_{jl} J_{ik} \ , \tag{1.3.171}$$

Let us notice that the exponential parametrization Eq. (1.3.165) produces orthogonal matrices with unit determinant, i.e. elements of SO(n). Indeed,

$$\det M = \mathrm{e}^{\mathrm{tr}\, m} = \mathrm{e}^0 = 1 \ . \tag{1.3.172}$$

The group of ortogonal matrices O(n) consists of twp *disconnected* components, corresponding to matrices with determinant ± 1. Indeed, any element with determinant -1 cannot be continuosly reached from the identity, as the determinant should jump from 1 to -1. The exponential parametrization only covers the component connected to the identity, that is SO(n).

1.3.7.8 (Special) pseudo-orthogonal groups

The pseudo-orthogonal group O(p, q), see sec. **??**, contains the real $(p+q) \times (p+q)$ matrices $\Lambda^\mu{}_\nu$ that preserve a metric of signature (p, q), which we may take to be

$$\eta_{\mu\nu} = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{p}, \underbrace{1, \ldots, 1}_{q}) \ . \tag{1.3.173}$$

So we must have

$$\Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu = \eta_{\mu\nu} \ , \tag{1.3.174}$$

i.e., in matrix notation, $\Lambda^T \eta \Lambda = \eta$ or

$$\Lambda^{-1} = \eta^{-1} \Lambda^T \eta \ , \tag{1.3.175}$$

which generalizes the usual orthogonality condition $\Lambda^{-1} = \Lambda^T$ of the case $p = 0$ with euclidean signature. We adopt the usual convention about raising and lowering indices, and in particular we write $\eta^{\mu\nu}$ for the inverse metric.

The determinant of Λ can be ± 1. Restricting to the subgroup with unit determinant defines the SO(p, q) group.

For an element close to the identity, $\Lambda = \mathbf{1} + \lambda$, Eq. (1.3.175) imposes

$$(\mathbf{1} - \lambda) + \eta^{-1} (\mathbf{1} + \lambda^T) \eta \ \ \Rightarrow \ \ (\lambda \eta^{-1})^T + \lambda \eta^{-1} = \mathbf{0} \ , \tag{1.3.176}$$

namely that $\lambda \eta^{-1}$ be *anti-symmetric*. This condition defines the Lie algebra so(p, q). Thus a basis of infinitesimal generators $J_{\rho\sigma}$ is obtained simply multiplying by η a basis for anti-symmetric $(p + q)$ matrices, such as the one given in Eq. (1.3.168):

$$(J_{\rho\sigma})^\mu{}_\nu = (\delta^\mu{}_\rho \delta^\tau{}_\sigma - \delta^\tau{}_\rho \delta^\mu{}_\sigma) \eta_{\tau\nu} = \delta^\mu{}_\rho \eta_{\sigma\nu} - \delta^\mu{}_\sigma \eta_{\rho\nu} \ . \tag{1.3.177}$$

The commutation relations take the form

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} \ , \tag{1.3.178}$$

generalizing Eq. (1.3.171) for the generators of so(n).

A generic matrix in so(p, q) can be written as

$$\lambda(\omega) = \omega^{\mu\nu} J_{\mu\nu} \ , \tag{1.3.179}$$

and is thus parametrized by the $(p+q)(p+q-1)/2$ parameters $\omega^{\mu\nu}$, which form an antisymmetric tensor: $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

The part connected to the identity of the group SO(p, q) is then obtained via the exponential map:

$$\Lambda(\omega) = \mathrm{e}^{\omega^{\mu\nu} J_{\mu\nu}} \ . \tag{1.3.180}$$

1.3.7.9 The group SO(3).

We have already discussed the so(3) Lie algebra of real 3×3 anti-symmetric matrices in sec. 1.3.4, Eqs. (1.3.123-1.3.128). We saw this Lie algebra is isomorphic to the su(2) Lie algebra. Let us just remark here that the traditional choice of generators L_i in Eq. (1.3.126) is related to the general choice Eq. (**??**) of generators J_{ij} by

$$L_i = \frac{1}{2} \epsilon_{ijk} J_{jk} \ . \tag{1.3.181}$$

While the notation J_{jk} refers to the fact that J_{jk} generates a rotation along the $i - j$ plane, the notation L_i refers to the fact that L_i generates a rotation around the i-th axis. In three dimensions, the two ways of labeling the rotations are possible. According to the choice Eq. (1.3.181) we parametrize the anti-symmetric matrix m as follows:

$$m(\alpha) = \begin{pmatrix} 0 & \alpha^3 & -\alpha^2 \\ -\alpha^3 & 0 & \alpha^1 \\ \alpha^2 & -\alpha^1 & 0 \end{pmatrix} = \alpha^i \, L_i \ . \tag{1.3.182}$$

Using the fact that the matrix m satisfies the relation[6] $m^3 = -|\vec{\alpha}|^2 \, m$, the matrix exponential of o gives

$$M(\alpha) = \mathrm{e}^{m(\alpha)} = \mathbf{1} + \sin|\vec{\alpha}| \, (\hat{\alpha}^i L_i) + (1 - \cos|\vec{\alpha}|) \, (\hat{\alpha}^i L_i)^2 \ , \tag{1.3.183}$$

where one has explicitely[7]

$$(\hat{\alpha}^i L_i)_{jk} = \hat{\alpha}^i \, \epsilon_{ijk} \ , \quad [(\hat{\alpha}^i L_i)^2]_{jk} = \hat{\alpha}^i \hat{\alpha}^l \epsilon_{ijm} \epsilon_{lmk} = \hat{\alpha}^j \hat{\alpha}^k - \delta^{pq} \ . \tag{1.3.184}$$

Eq. (1.3.183) describes a rotation of an angle $\psi = |\vec{\alpha}|$ around an axis individuated by the versor $\hat{\alpha}$. Indeed, applying the matrix $M(\alpha)$ to a vector $\vec{x}$ gives a rotated vector of components

$$[M(\alpha)]^i{}_j x^j = [\cos\psi \, \delta^i{}_j + \sin\psi \, \hat{\alpha}^k \epsilon_k{}^i{}_j + (1 - \cos\psi \, \hat{\alpha}^i \hat{\alpha}_j] \, x^j \ , \tag{1.3.185}$$

where we used Eq. (1.3.184). In vector notation, the above can be written as

$$\vec{x}\,' = M(\alpha)\vec{x} = \cos\psi \, \vec{x} - \sin\psi \, \hat{\alpha} \wedge \vec{x} + (1 - \cos\psi)(\hat{\alpha} \cdot \vec{x})\hat{\alpha} \ . \tag{1.3.186}$$

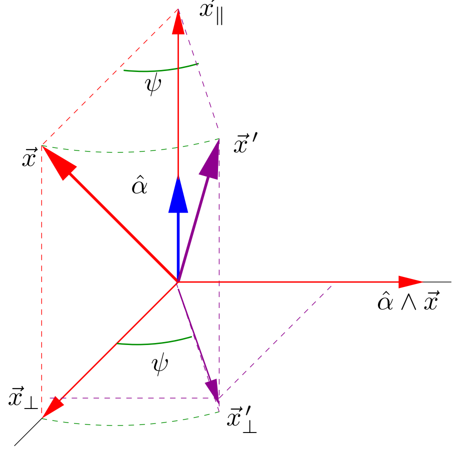

Figure 1.9. The rotation of a vector $\vec{x}$ in terms of an axis, $\hat{\alpha}$, and an angle, ψ.

This is the correct expression found by geometric considerations. Indeed (see Fig. 1.9), decompose $\vec{x}$ into a parallel and perpendicular part with respect to $\hat{\alpha}$: $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$, with $\vec{x}_\parallel = (\vec{x}\cdot\hat{\alpha})\hat{\alpha}$ and $\vec{x}_\perp = \vec{x} - (\vec{x}\cdot\hat{\alpha})\hat{\alpha}$. A third orthogonal direction laying in the plane normal to $\hat{\alpha}$ is given by $\hat{\alpha}\wedge\vec{x} =$. A rotation of an angle ψ around the axis $\hat{\alpha}$ leaves of course $\vec{x}_\parallel$ unchanged, while $\vec{x}\,'_\perp = \cos\psi\,\vec{x}_\perp + \sin\psi\,\hat{\alpha}\wedge\vec{x}$. Altogether we get thus $\vec{x}\,' = \vec{x}_\parallel + \vec{x}\,'_\perp = (\hat{\alpha}\cdot\vec{x})\hat{\alpha} + \sin\psi\,\hat{\alpha}\wedge\vec{x}$ $+\cos\psi\,[\vec{x} - (\hat{\alpha}\cdot\vec{x})\hat{\alpha}]$, which coincides with Eq. (1.3.186).

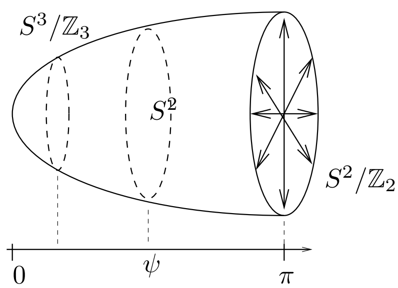

Figure 1.10. The group manifold of S)(3). For each value of $\psi \in [0,\pi[$ there is an S^2 of radius $\sin\frac{\psi}{2}$; however at $\psi = \pi$ we have an $S^2/\mathbb{Z}_2$ (antipodal points are identified). Altogether this defines an $S^3/\mathbb{Z}_2$: comparing with Fig. **??** we see that we have now exactly one representative of that S^3 for each pair of opposite points, which are identified under $\mathbb{Z}_2$. (The drawing is, for obvious reasons, represented in one dimension less).

We see that the exponential parametrization covers the entire SO(3) group, and that the range of parameters is as follows. The versor $\hat{\alpha}$ individuates a point on S^2, namely individuates

[6] This relation arises from the fact that the secular equation $\det(m - \lambda\mathbf{1}) = 0$ reads explicitely $\lambda^3 + |\vec{\alpha}|^2\,\lambda + 0$. We have $m = S^{-1}\Lambda S$, where S is the eigenvector matrix and $\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, where λ_i are the eigenvalues of m. Then the secular equation implies that $\Lambda^3 + \vec{\alpha}|^2\,\Lambda = \mathbf{0}$, which changing back to the original basis, gives $m^3 = -|\vec{\alpha}|^2\,m$.

[7] Take into account the "contraction" of the Levi-Civita tensor $\epsilon_{ijk}\epsilon^{ipq} = (2!)\delta^{pq}_{jk}$, where we use the antisymmetrized δ-symbol $\delta^{pq}_{jk} = \frac{1}{2}(\delta^p_j\delta^q_k - \delta^p_k\delta^q_j)$.

the rotation axis including the direction. The rotation angle then satisfies $\phi \in [0, \pi]$, as a rotation by an angle $\phi + \pi$ around an oriented axis $\hat{\alpha}$ is the same as a rotation by ϕ around the axis $-\hat{\alpha}$. Notice that for $\phi = |\vec{\alpha}| = 0$ we get the identity element, for $\phi = \pi$ we get group elements $\mathbf{1} - (\hat{\alpha}^i L_i)^2$ which are parametrized not by $\hat{\alpha}$, i.e. by an S^2, but by $\hat{\alpha}$ up to its sign, namely by an S^2 with opposite (anti-podal) points identified, i.e. by an $S^2/\mathbb{Z}_2$. See Fig. 1.10.

So, comparing with section 1.3.7.5, we see that, despite the fact that the Lie algebras su(2) and so(3) are isomorphic, using the exponential map we obtain two Lie groups, SU(2) and SO(3), which differ *globally* as manifolds: while SU(2) is a two-sphere S_2, SO(3) is a quotient $S_2/\mathbb{Z}_2$.

This interesting pattern is in fact general: the Lie groups are determined by their Lie algebras only *locally*. We will investigate this fact in the next section.

1.3.7.10 The symplectic groups $\mathrm{Sp}(2n, \mathbb{R})$

In sec.s **??** and **??** we introduced the symplectic groups $\mathrm{Sp}(2n, \mathbb{R})$ containing real $2n \times 2n$ matrices A such that they preserve the "symplectic form" Ω, as described in Eq. (**??**). For elements close to the identity, $A \sim \mathbf{1} + a$ with a small, the condition Eq. (**??**) implies

$$(\mathbf{1} + a^T)\Omega(1 + a) = \Omega \quad \Rightarrow \quad a^T\Omega + \Omega a = \mathbf{0} \tag{1.3.187}$$

that is,

$$(\Omega a)^T = \Omega a \ , \tag{1.3.188}$$

where the last step follows because Ω is anti-symmetric. The $2n \times 2n$ matrices a such that Ωa is *symmetric* form the Lie algebra $\mathrm{sp}(n, \mathbb{R})$. Check indeed that, if a and b both satisfy Eq. (1.3.188) then also their commutator does (hint: use the property $\Omega^2 = -\mathbf{1}$ of the symplectic form).

1.4 Global properties of Lie groups

We have seen in the previous section how the double nature of Lie groups, which are at the same time differentiable manifolds and groups, allows, at least *locally* in a neighbourhood of the identity, to determine the group law for finite elements from the Lie algebra of the *infinitesimal* generators via the exponential map.

We have also seen however, comparing the exponentiation of su(2) and so(3), how isomorphic Lie algebras may give rise to Lie groups which are only locally isomorphic, but differ *globally*. This possibility is described precisely in the so-called *3-rd Lie theorem*, which we will discuss in this section.

More in general, in this section we will discuss some of the global *properties* that Lie groups may (or may not) possess when they are thought of as *topological spaces* (in particular, differential manifolds).

1.4.1 Connectedness

A topological space $\mathcal{M}$ is *arc-wise connected*[8] if any two points of $\mathcal{M}$ can be connected by a *continuous curve*. We will call *Lie group* also a group G which is a *disconnected* manifold composed of a *finite set* of *disconnected components*

$$G = G_0 \sqcup G_1 \ldots \sqcup G_n \ , \tag{1.4.189}$$

[8] The notion of arc-wise connectedness is stronger (and much more intuitive) than the standard notion of connectedness for a topological space, which sounds as follows: $\mathcal{M}$ is connected iff $\mathcal{M}$ and the null set $\emptyset$ are the only subset of $\mathcal{M}$ which are at the same time open and closed.

with G_0 being the connected component that contains the identity element of G. The mapping $\phi(g_{(i)}, h_{(j)}) = g_{(i)} h_{(j)}^{-1}$, with $g_{(i)}, h_{(j)} \in G_i, G_j$ is of course still required to be analytic in both arguments, as long as they vary continuously within the connected component they belong to. Notice that some text-books reserve the name "Lie group" to the component connected to the identity only.

It is a very important fact that the component connected to the identity G_0 is an *invariant subgroup* of G. First of all, it is a subgroup. Indeed, if $g_1, g_2 \in G_0$, then this means that there exist two continuous curves $g_1(t), g_2(t)$, with $t \in [0,1]$, connecting them to the identity: $g_1(0) = g_2(0) = e$, $g_1(1) = g_1$ and $g_2(1) = x_2$. Also the curve $g_3(t) \equiv \phi(g_1(t), g_2(t)$ is then continuous (thanks to the properties of the map ϕ) and connects the element $g_1 g_2^{-1}$ to the identity. So $g_1 g_2^{-1} \in G_0$, $\forall\, g_1, g_2 \in G_0$ and G_0 is therefore a subgroup. It is furthermore invariant, for if $h \in G_0$, then there is a continuous curve $h(t)$ connecting it to the identity. But then, $\forall\, g \in G$, the path described by $g^{-1} h(t) g$ is continuous (thanks to the continuity of the group product) and connects $g^{-1} h g$ to the identity, so that $g^{-1} h g$ still belongs to G_0.

The fact that a finite-dimensional Lie group consists in general of a finite set of disconnected components can be (more precisely) stated as follows: the *factor group*

$$D_0 \equiv G/G_0 \tag{1.4.190}$$

is a *finite group*. The order $|D_0|$ gives the number of disconnected components of G. Every element $g \in G$ can be uniquely written as a product

$$g = \hat{g}\, \tilde{g}\ ,\quad \hat{g} \in D_0\ ,\ \tilde{g} \in G_0\ . \tag{1.4.191}$$

Namely, G is a splitting extension of G_0 by D_0, and the structure of group products is G is determined from the knowledge of the group product structure of G_0, of that of D_0, and of the "adjoint" action of D_0 on G_0. Indeed

$$g_1 g_2 = \hat{g}_1 \tilde{g}_1 \hat{g}_2 \tilde{g}_2 = \hat{g}_1 \hat{g}_2\, (\hat{g}_2^{-1} \tilde{g}_1 \hat{g}_2) \tilde{g}_2\ , \tag{1.4.192}$$

where $\hat{g}_1 \hat{g}_2$ is a product in D_0 and $(\hat{g}_2^{-1} \tilde{g}_1 \hat{g}_2) \tilde{g}_2$ a product in G_0, given the invariance property.

Thus, the crucial point in the study of Lie groups is the study of the connected Lie groups, as general (disconnected) groups can be constructed as splitting extensions (namely, as direct or semi-direct products) of the connected ones by finite groups.

Examples

- We have already discussed in sec. 1.3.7.1 how the general linear groups $\mathrm{GL}(n, \mathbb{R})$ are not connected, as elements with negative determinant cannot be continuously reached from those with positive determinant without passing through the forbidden value zero (i.e. without leaving the group).
  Consider for instance the group $\mathrm{GL}(1, \mathbb{R}$, namely $\mathbb{R} \setminus \{0\}$ with the ordinary multiplication. The two disconnected components are the positive and negative real axis. The positive real axis $\mathbb{R}^+$ is the component connected to the identity element 1, and it is an invariant subgroup. Since every element x of the negative real axis can be written as $(-1)|x|$, with $|x| \in \mathbb{R}^+$, the factor group is
  $$\mathrm{GL}(1)/\mathbb{R}^+ \sim \mathbb{Z}_2\ . \tag{1.4.193}$$
  Moreover, $\mathrm{GL}(1) \sim \mathbb{Z}_2 \otimes \mathbb{R}^+$ is simply a direct product, as the elements of $\mathbb{Z}_{\not\approx}$ have trivial adjoint action on $\mathbb{R}^+$.

For higher dimensions, the component connected with the identity $\mathrm{GL}(n,\mathbb{R})_0$ is the special linear group $\mathrm{SL}(n,\mathbb{R})$. The elements of the other component can all be written in the form Mg, where $g \in \mathrm{SL}(n,\mathbb{R})$ and M is a fixed matrix of determinant -1 and such that $M^2 = 1$. So, the factor group is

$$\mathrm{GL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{R}) \sim \mathbb{Z}_2 \ . \tag{1.4.194}$$

- Similarly to the above example, we have already seen in sec. **??** how the orthogonal groups $\mathrm{O}(n)$ are made of two disconnected components, corresponding to having ± 1 determinant. The component connected to the identity is $\mathrm{SO}(n)$, corresponding to determinant $+1$, which is an invariant subgroup. The factor group is again

$$\mathrm{O}(n)/\mathrm{SO}(n) \sim \mathbb{Z}_2 \ . \tag{1.4.195}$$

- Consider the group $\mathcal{C}$ defined in Eq. (1.2.20). Its group manifold $\mathbb{R}^2 \setminus \mathbb{R}$ has two disconnected components, $\mathcal{C}^{\pm}$, containing the points with $a > 0$ and $a < 0$ respectively; see Fig. 1.2. Of course $\mathcal{C}^+$ is the part connected to the identity: $(1,0) \in \mathcal{C}^+$. The factor group $\mathcal{C}/\mathcal{C}^+$ is the $\mathbb{Z}_2$ group generated by the transformation $x \mapsto -x$, namely, in the language of Eq. (**??**), by the matrix $-\sigma^3$; indeed, any matrix in $\mathcal{C}^-$ can be written as the product of σ^3 times a matrix in $\mathcal{C}^+$:

$$\forall\, a < 0 \ , \qquad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a & -b \\ 0 & 1 \end{pmatrix} \ . \tag{1.4.196}$$

So we have

$$\mathcal{C}/\mathcal{C}^+ = \mathbb{Z}_2 \ . \tag{1.4.197}$$

However, notice that $\mathcal{C}$ is *not* the direct product of $\mathbb{Z}_2$ and $\mathcal{C}^+$. Indeed, the adjoint action of $\mathbb{Z}_2$ on the invariant subgroup $\mathcal{C}^+$ is non-trivial: for every transformation $T_{a,b} \in \mathcal{C}^+$, we have (in the matrix language)

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix} \ , \tag{1.4.198}$$

namely, by conjugating with the $\mathbb{Z}_2$ generator we get in general a different element of $\mathcal{C}^+$.

1.4.2 Compactness of a Lie group

The group manifold of a Lie group can be *compact* or not. When the manifold can be seen as an hypersurface embedded in a higher-dimensional flat space (for instance, S^2 can be seen as the surface $x^2 + y^2 + z^2 = 1$ within $\mathbb{R}^3$, the notion of compactness is simply[9] that the immersion of the manifold stays within a ball $\mathcal{B}$ of finite radius.

Examples

- We utilized in sec. **??** as preliminary examples a compact group, $\mathrm{SO}(2)$, whose group manifold is the circle S_1
- The additive group $(\mathbb{R},+)$ as a manifold is simply the real line $\mathbb{R}$, so it is non-compact.
- The group $\mathrm{SO}(1,1)$ is parametrized as in Eq. (**??**) by a coordinate $\nu \in \mathbb{R}$, on which no periodicity is imposed. So the group-manifold is $\mathbb{R}$, which is non-compact.

[9] In general the technical definition of compactness of a topological space is that the accumulation point of any sub-succession extracted from a Cauchy succession lies within the space itself. When the space is given a metric, the natural definition of compactness is that the distance within any two points is finite.

- The group $\mathcal{C}^+$ (the part connected to the identity of the group of conformal transformations of the real line) as a manifold is $\mathbb{R} \times \mathbb{R}^+$ (see FIg. 1.2, which is non-compact.

We will see later that, at least for an important class of Lie groups, the *semisimple* ones, the compactness of the Lie group is determined by an algebric property of the corresponding Lie algebra.

1.4.3 Simple connectedness and local isomorphisms: the 3rd Lie theorem

A Lie group G is *simply-connected* iff, as a manifold, its *fundamental group* is trivial:

$$\pi_1(G) = \{e\} \ , \tag{1.4.199}$$

namely if all loops drawn on the manifold G are contractible.

Intuitively, this property depends on "large" group transformations, as the loops that are not contractible cannot be drawn on all neighbourhoods of a point: indeed, any point $g \in G$ must belong to some neighbourhood $\mathcal{U}_g$ homeomorphic to an open neighbourhood of $\mathbb{R}^{\dim G}$, in which no contractible loop can be drawn.

This remarks lead naturally to the concept of *local isomorphism* between Lie groups. Two Lie groups G_1 and G_2 are locally isomorphic if there exists an *isomorphic* mapping

$$\omega \ : \ \mathcal{U}_{e_1} \ \to \ \mathcal{U}_{e_2} \tag{1.4.200}$$

between two neighbourhoods of the identity elements e_1, e_2 of the two groups. This relation is an equivalence relation.

The *3rd Lie theorem* states that the general situation is the following. Within a given class of locally isomorphic groups, there is a single representative G which is simply-connected, i.e. such that $\pi_1(G) = \{e\}$. Any other group G_i in the same class, that is, any other group G_i which is locally isomorphic but not isomorphic to G, is as *homomorphic* image of G by an homomorphism ω_i such that

$$\mathcal{D}_i \equiv \ker \omega_i \tag{1.4.201}$$

is a *discrete* group (and, being the kernel of an homomorphism, is an invariant subgroup of G). In other words, we have

$$G_i = G/\mathcal{D}_i \ . \tag{1.4.202}$$

Since G is simply-connected, we have

$$\pi_1(G_i) = \pi_1\left(G/\mathcal{D}_i\right) \sim \mathcal{D}_i \ . \tag{1.4.203}$$

This relation states that the non-trivial closed loops of G_i are exactly those paths that would be open in the manifold G, and become closed only because of the identification between different points that correspond to taking the quotient by $\mathcal{D}_i$.

The simply connected group G is called the *universal cover* of the multiply-connected groups G_i that are locally isomorphic to it. The name refers to the fact that the fundamental group $\pi_1(G_i)$ can be realized as a group of discrete transformations on G.

In practice, to construct groups locally isomorphic (but not isomorphic) to a simply-connected Lie group G, one looks for possible discrete invariant subgroups of G, and contructs the factor groups by these.

Example Consider again the group $(\mathbb{R},+)$, that we may view as a group of translations T_a ($a \in \mathbb{R}$) acting on the real axis:

$$x \overset{T_a}{\longmapsto} x + a \ . \tag{1.4.204}$$

Thsi group is of course simply-connected (the only closed loop drawable on $\mathbb{R}$ is a point). The group $(\mathbb{R},+)$ possesses a discrete subgroup (which is of course invariant, since the group $(\mathbb{R},+)$ is abelian), namely the subgroup $(\mathbb{Z},+)$ of translations by integers only:

$$x \overset{T_n}{\longmapsto} x + n \ , \quad n \in \mathbb{Z} \ . \tag{1.4.205}$$

For the factor group we have

$$(\mathbb{R},+)/(\mathbb{Z},+) \sim \mathrm{SO}(2) \sim \mathrm{U}(1) \,, \tag{1.4.206}$$

and as a manifold it is the *compact* manifold S^1. In fact, translations that differ by integers are to be identified, so that $T_a \sim T_{a+1}$ and all inequivalent translations (the elements of the factor group) are labeled by $a \in [0,1]$ with periodic conditions, i.e. by points of S^1. The composition law is the periodic sum of parameters. The groups $(\mathbb{R},+)$ and SO(2) are locally isomorphic.

1.4.3.1 Homomorphisms between U(2)*,* SU(2) *and* SO(3)

Let us consider some homomorphic relations between matrix groups which illustrate some of the above concepts and, on the other hand, are important in physical applications. Consider an anti-hermitean traceless 2×2 matrix h (as we will see, this is an element of the Lie algebra su(2)). Any such matrix can be written as

$$h = \mathrm{i}\, x_i \sigma^i = \mathrm{i} \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{pmatrix} \ , \tag{1.4.207}$$

where σ_i are the Pauli matrices (we will see that σ_i represent the generators of the su(2) algebra); indeed one has then automatically $h = -h^\dagger$ and $\mathrm{tr}\, h = 0$. The determinant takes the value $-\det\, h = \sum_i x_i^2$.

Given a matrix $U \in \mathrm{SU}(2)$ we can make it act on h as follows (this is called, as we will see, the *adjoint* action of the group element on the Lie algebra):

$$h :\mapsto h' = U^\dagger h U \ . \tag{1.4.208}$$

It is immediate to verify that h' is still anti-hermitean and traceless: indeed, $(h')^\dagger = U^\dagger h^\dagger U = -U^\dagger h U = -h'$ and $\mathrm{tr}\, h' = \mathrm{tr}(U^\dagger h U) = \mathrm{tr}\, h = 0$. Therefore h' can be expanded on the Pauli matrices: $h' = \mathrm{i}\, x'_i \sigma^i$. Also the determinant is obviously preserved under the adjoint action: $\det\, h' = \det\, h$, so that $\sum_i (x'_i)^2 = \sum_i x_i^2$. This means that the parameters x'_i (parametrizing h' are related to the x_i parameters defining h by a proper rotation, i.e. by en element $\Lambda(U) \in \mathrm{SO}(3)$:

$$x'_i = x_j [\Lambda(U)]^j{}_i \ . \tag{1.4.209}$$

Notice that this definition of $\Lambda(U)$ is equivalent to the definition

$$U^\dagger \sigma^i U = \Lambda(U)^i{}_j \sigma^j \ . \tag{1.4.210}$$

The map ω defined, via Eq. (1.4.210), by

$$U \in \mathrm{SU}(2) \overset{\omega}{\longrightarrow} \Lambda(U) \in \mathrm{SO}(3) \tag{1.4.211}$$

is an *homomorphism*. Indeed, the adjoint action of U_1U_2 is represented by $x_i \mapsto \Lambda(U_1U_2)_i{}^j x_j$, but on the other hand we have

$$U_2^\dagger U_1^\dagger \sigma^i U_1 U_2 = [\Lambda(U_1)]^i_j U_2^\dagger \sigma^j U_2 = [\Lambda(U_1)]^i_j [\Lambda(U_2)]^j_k \sigma^k = [\Lambda(U_1)\Lambda(U_2)]^i_k \sigma^k \ . \tag{1.4.212}$$

Thus, $\Lambda(U_1U_2) = \Lambda(U_1)\Lambda(U_2)$. The kernel of this homomorphism is clearly given by

$$\ker \omega = \{\mathbf{1}, -\mathbf{1}\} \ , \tag{1.4.213}$$

i.e. by the centre, $\mathbb{Z}_2$, of SU(2). The group SO(3) is thus *isomorphic* to the factor group of SU(2) by $ker\, \omega$:

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2 \ . \tag{1.4.214}$$

Indeed a sign change in U cancels in the definition Eq. (1.4.210) of $\Lambda(U)$. The isomorphism Eq. (1.4.213) is of great relevance in Physics, as it is intimately related to concept of "spinor" representations of the rotation group.

Notice that in the above discussion, the fact that $\det U = 1$ does not play any role, only the unitarity of U mattered. Thus we may repeat the whole construction to obtain an homomorphism

$$\psi : \ U \in \mathrm{U}(2) \mapsto \Lambda(U) \in \mathrm{SO}(3) \ . \tag{1.4.215}$$

In the definition Eq. (1.4.210) of the homomorphic image $\Lambda(U)$ any phase factor in U does not matter. Thus,

$$\ker \psi = \{\mathrm{e}^{\mathrm{i}\alpha}\mathbf{1} \ , \ \alpha \in [0, 2\pi]\} \ , \tag{1.4.216}$$

namely the kernel is the U(1) centre of U(2), and we have

$$\mathrm{SO}(3) \cong \mathrm{U}(2)/\mathrm{U}(1) \ . \tag{1.4.217}$$

We have also an homomorphism between U(2) and SU(2) which can be described as follows. Any matrix $U \in \mathrm{U}(2)$ can be parametrized as

$$U = \mathrm{e}^{\mathrm{i}\alpha}\, u \ , \tag{1.4.218}$$

with $u \in \mathrm{SU}(2)$ and $\mathrm{e}^{\mathrm{i}\alpha} \in \mathrm{U}(1)/\mathbb{Z}_2$, namely α is identified with $\alpha + \pi$. If we allowed $\mathrm{e}^{\mathrm{i}\alpha} \in \mathrm{U}(1)$, i.e. we simply took α to be periodic of period 2π, we would be over-parametrizing U(2), as $\mathrm{e}^{\mathrm{i}\alpha}(-u)$ and $\mathrm{e}^{\mathrm{i}\alpha+\pi}u$ give the same unitary matrix U. The homomorphism $\chi : \ \mathrm{U}(2) \mapsto \mathrm{SU}(2)$ is thus simply described by

$$U = \mathrm{e}^{\mathrm{i}\alpha}\, u \in \mathrm{U}(2) \mapsto u \in \mathrm{SU}(2) \ . \tag{1.4.219}$$

The kernel of χ is thus clearly given by

$$\ker \chi = \{\mathrm{e}^{\mathrm{i}\alpha}\mathbf{1} \ , \ \alpha \sim \alpha + \pi\} \ \cong \mathrm{U}(1)/\mathbb{Z}_2 \ . \tag{1.4.220}$$

We can altogether summarize the homomorphism we have being discussed as follows:

$$\mathrm{U}(2) \xrightarrow{\chi} \mathrm{SU}(2) \cong \mathrm{U}(2)/(\mathrm{U}(1)/\mathbb{Z}_2) \xrightarrow{\omega} \mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2 \cong \ (\mathrm{U}(2)/(\mathrm{U}(1)/\mathbb{Z}_2))/\,\mathbb{Z}_2 \cong \mathrm{U}(2)/\mathrm{U}(1)\,. \tag{1.4.221}$$

The homomorphism ψ of Eq. (1.4.215) is the composition of χ and ω.