Chapter 1

BASICS OF GROUP REPRESENTATIONS

1.1 Group representations

A *linear representation* of a group G is identified by a *module*, that is by a couple

$$(D; V) \ , \tag{1.1.1}$$

where V is a vector space, over a field that for us will always be $\mathbb{R}$ or $\mathbb{C}$, called the *carrier space*, and D is an *homomorphism* from G to $D(G) \subset \mathrm{GL}\,(V)$, where $\mathrm{GL}(V)$ is the space of invertible linear operators on V, see Fig. **??**. Thus, we have a is a map

$$\forall g \in G \mapsto D(g) \in \mathrm{GL}(V) \ , \tag{1.1.2}$$

with the *linear operators*

$$D(g) \ : \ \mathbf{v} \in V \mapsto D(g)\mathbf{v} \in V \tag{1.1.3}$$

satisfying the properties

$$\begin{aligned} D(g_1 g_2) &= D(g_1) D(g_2) \ ; \\ D(g^{-1} &= [D(g)]^{-1} \ ; \\ D(e) &= \mathbf{1} \ . \end{aligned} \tag{1.1.4}$$

The product (and the inverse) on the righ hand sides above are those appropriate for operators: $D(g_1)D(g_2)$ corresponds to acting by $D(g_2)$ after having appplied $D(g_1)$. In the last line, $\mathbf{1}$ is the identity operator.

We can consider both finite and infinite-dimensional carrier spaces. In infinite-dimensional spaces, typically *spaces of functions*, linear operators will typically be linear differential operators.

Example Consider the infinite-dimensional space of infinitely-derivable functions ("functions of class $\mathcal{C}^\infty$") $\psi(x)$ defined on $\mathbb{R}$. Consider the group of translations by real numbers: $x \mapsto x + a$, with $a \in \mathbb{R}$. This group is evidently isomorphic to $\mathbb{R}, +$. As we discussed in sec **??**, these transformations induce an action $D(a)$ on the functions $\psi(x)$, defined by the requirement that the transformed function in the translated point $x + a$ equals the old original function in the point x, namely, that

$$D(a)\psi(x) = \psi(x - a) \ . \tag{1.1.5}$$

It follows that the translation group admits an infinite-dimensional representation over the space V of $\mathcal{C}^\infty$ functions given by the differential operators

$$D(a) = \exp\left(-a\frac{d}{dx}\right) = 1 - a\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2} + \dots \ . \tag{1.1.6}$$

In fact, we have

$$D(a)\psi(x) = \left[1 - a\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2} + \dots\right]\psi(x) = \psi(x) - a\frac{d\psi}{dx}(x) + \frac{a^2}{2}\frac{d^2\psi}{dx^2}(x) + \dots = \psi(x-a) \ . \tag{1.1.7}$$

In most of this chapter we will however be concerned with finite-dimensional representations.

1.1.1 Matrix representations

If the space V is finite-dimensional, with, dim $V = n$ (real or complex), linear operators $\mathcal{T}$ on V are equivalently described by the $n \times n$ matrices $T_i^{\ j}$ specifying their action[1]on a basis $\mathbf{e}_i$ of V:

$$\mathbf{e}_i \mapsto T^j_{\ i}\mathbf{e}_j \ . \tag{1.1.8}$$

Thus, a *finite-dimensional representation* (or matrix representation) of dimension n of the group G is an homomorphism of G into group of $n \times n$ matrices $[D(g)]^i_{\ j}$.

Examples

- Every group G admits the *trivial representation*, of dimension 1, which associates to every element the number 1:
$$\forall g \in G\,,\ D(g) = 1 \ . \tag{1.1.9}$$
- The group $\mathbb{Z}_n = \{e, a, \dots a^{n-1}\}$, $(a^n = e)$ has a non-trivial representation of complex dimension 1 by roots of unity:
$$D(a^k) = \mathrm{e}^{\frac{2\pi \mathrm{i} k}{n}} \ , \quad k = 0, \dots, n-1 \ . \tag{1.1.10}$$
- Consider again the group $\mathbb{Z}_2$, realizing this time its generator a geometrically as the rotation by π in the plane. This group maps the plane $\mathbb{R}^2$ into itself. Considering the action on the (Cartesian) basis vectors $\mathbf{e}_{1,2}$, we find that it is given by the matrices
$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad D(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{1.1.11}$$
namely, it furnishes a 2-dimensional representation of $\mathbb{Z}_2$.
- The group $\mathbb{Z}_2$ is isomorphic to the permutation group S_2, where the generator a is realized as the exchange (12). In sec. **??** we described the permutations in S_n as $n \times n$ matrices, see Eq. (**??**), namely we described an $n \times n$ representation of S_n. In the case of $S_2 \cong \mathbb{Z}_2$ we obtain the following 2-dimensional (real) representation:
$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{1.1.12}$$

[1] We take the convention that the matrix $T_i^{\ j}$ is the one acting on the basis vector. The action on the column vector v of the components v^i of a vector $\mathbf{v} = v^i\mathbf{e}_i = v^T\mathbf{e}$, is via the transpose matrix: $v^i \mapsto v^j T^i_j$, i.e. $\mathbf{v} \mapsto T^T v$.

- Consider the group $\mathbb{D}_2$ (the subgroup of O(2) that leaves invariant a segment in the plane). Again, consider its action on the basis vectors $\mathbf{e}_{1,2}$ to find a real two-dimensional representation. Using the notation in sec. **??**, see Eq. (**??**), we find

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;,\; D(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; D(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \;,\; D(ab) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{1.1.13}$$

- Notice that of course it would be sufficient to specify the matrix representing the generators only (a and b in this case), as then the remaining matrices are determined by the homomorphic nature of the map: $D(ab) = D(a)D(b)$.
- Consider the group $\mathbb{D}_3$, see sec. **??** for notations. Its action on the basis vectors determines the two-dimensional faithful representation in which the generators a (generating the $\mathbb{Z}_3$ subgroup) and b (a reflection) are represented as follows:

$$D(a) = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} \;,\quad D(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{1.1.14}$$

  Since $\mathbb{D}_3 \cong S_3$, Eq. (1.2.106) furnishes also a 2-dimensional representation of the permutation group S_3.
- In fact any dihedral group $\mathbb{D}_n$ admits a 2-dimensional representation, the defining representation by which it is geometrically described as a symmetry group in the plane, in which the two generators a,b are given by

$$D(a) = \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \;,\quad D(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{1.1.15}$$

- As discussed in sec. **??**, the continuous group SO(2), the group of proper rotations in two dimensions, is isomorphically described as the group of orthogonal 2×2 matrices with unit determinants. Elements are parametrized by $\theta \in [0, 2\pi]$, and they have the 2-dimensional representation of Eq. (**??**):

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,. \tag{1.1.16}$$

  The 2-dimensional representation of O(2) contains the above matrices and their products with $-\mathbf{1}$. In the previous examples we described 2-dimensional representations of discrete subgroups of O(2), which were obviously given by discrete subset of the 2-dimensional O(2) matrices.
- The S_3 group, which is isomorphic to $\mathbb{D}_3$, can be presented by means of the two generators $a = (123)$ and $b = (13)$. Using again the representation of S_n by means of $n \times n$ matrices given in Eq. (**??**), we obtain the following 3-dimensional representation:

$$D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;,\quad D(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \,. \tag{1.1.17}$$

1.1.2 Faithful representations

A representation (D, V) is *faithful* if D is an *isomorphism*. Genericallyally, a representation is not faithful: ker $D \neq \emptyset$, and many elements are mapped into the identity matrix $\mathbf{1}$. By the first isomorphism theorem, see sec. **??**, when D is reduced to the factor group $G/\ker D$, it becomes as isomorphism:

$$D \,:\, G/\ker D \longleftrightarrow D(G) \ . \tag{1.1.18}$$

Thus $D(G)$ is a faithful representation of $G/\ker D$.

Example Al the examples of representations given in the previous section 1.1.1 are faithful, except of course the trivial representation of Eq. (1.1.9). In all cases, there is a distinct matrix for each group element.

1.1.3 Equivalent representations

The explicit form of the matrices $D(g)$ in a representation D of the group G depends of course of the choice of a basis in the carrier space V.

Matrix representatives of the same linear operator in different bases are related by a similarity transformation $T' = STS^{-1}$, where S is the non-singular matrix implementing the basis change: $\mathbf{e}'_i = S_i{}^j \mathbf{e}_j$.

Two matrix reresentations D' and D (over the same carrier space) are *equivalent* if all its matrices are related by the same change of basis :

$$\forall g \in G\,,\ D'(g) = SD(g)S^{-1} \ . \tag{1.1.19}$$

Notice that an inner automorphism of G, namely the conjugation by a fixed element $h \in G$, corresponds in a representation D to the particular change of basis with the matrix $S = D(h)$, and gives obviously rise to an equivalent representation. Indeed

$$D(hgh^{-1}) = D(h)D(g)[D(h)]^{-1} \ , \tag{1.1.20}$$

since D is an homomorphism. It is clear that the relevant properties of a representation must be independent of the explicit choice of basis. Therefore it will be useful to characterize a representation by means of basis-independent quantities. For instance, we may consider the traces of the matrices $D(g)$.

1.1.4 Definition of characters

The *character* $\chi_D(g)$ of an element $g \in G$ in the representation D is the trace of its matrix representative $D(g)$:

$$\chi_D(g) = \mathrm{tr}\, D(g) \ . \tag{1.1.21}$$

Characters are clearly independent from the explicit choice of basis. Thus all equivalent representations have the same characters. Notice also that, the character $\chi_D(g)$ depends only on the conjugacy class of g: we say that the characters are *class functions*. Indeed, for any element hgh^{-1} conjugate to g, by taking the trace of Eq. (1.1.20) we have

$$\chi_D(hgh^{-1}) = \mathrm{tr}\left(D(hgh^{-1})\right) = \mathrm{tr}\left(D(h)D(g)[D(h)]^{-1}\right) = \mathrm{tr}\, D(g) = \chi_D(g) \ . \tag{1.1.22}$$

Let us notice that the character of the identity element in a representation D is always equal to the dimension of the representation:

$$\chi(e) = \mathrm{tr}\, D(e) = \mathrm{tr}\, \mathbf{1} = \dim D \ . \tag{1.1.23}$$

It is clear that we want to describe the representations of a given group up to equivalence. Moreover, representations of higher dimension can be built in certain elementary ways (in particular, by direct sums) out of smaller ones. Therefore, the essential task will be the characterization of those representations that cannot be further decomposed into simpler blocks. This is what we investigate in the next section.

1.1.5 Invariant subspaces, reducible and irreducible representations

Invariant subspaces A linear subspace $W \subset V$ of the carrier space V of a representation D of a group G is an *invariant subspace* if it is preserved by the action of all operators of the representation:

$$\forall g \in G\,,\ \forall w \in W\,,\quad D(g)\,: w \in W \mapsto D(g)w \in W\ . \tag{1.1.24}$$

If we choose a basis of V adapted to the (orthogonal) decomposition $V = W \oplus W^{\perp}$, where $W^{\perp}$ is the (orthogonal) complement of W, all the matrix representatives $D(g)$ must be of block-triangular form:

$$\forall g \in G\,,\quad D(g) = \begin{pmatrix} D_1(g) & A(g) \\ \mathbf{0} & D_2(g) \end{pmatrix}\,,\quad \text{acting on} \quad \begin{pmatrix} W^{\perp} \\ W \end{pmatrix} \tag{1.1.25}$$

for W to be mapped in W. Both the $D_1(g)$ and $D_2(g)$ matrices form representations of G. Indeed, using Eq. (1.1.25),

$$D(g_1)D(g_2) = \begin{pmatrix} D_1(g_1)D_2(g_2) & D_1(g_1)A(g_2) + A(g_1)D_2(g_2) \\ \mathbf{0} & D_2(g_1)D_2(g_2) \end{pmatrix}\ . \tag{1.1.26}$$

Therefore $D(g_1)D(g_2) = D(g_1g_2)$ only if both D_1 and D_2 are homomorphisms: $D_i(g_1)D_i(g_2) = D_i(g_1g_2)$. No particular request arises for the blocks $A(g)$, since no request is made on the complement space $W^{\perp}$.

Irreducible representations A representation D is called *irreducible* if it does not admit any invariant subspace.

Reducible representations A representation D is *reducible* if it admits an invariant subspace. Equivalently, D is reducible if by means of a change of basis every matrix $D(g)$, $\forall g \in G$, can be put in the block-triangular form of Eq. (1.1.25).

In turn, the representations D_1 and D_2 of Eq. (1.1.25) could be reducible, and so on, so that in general reducible representations are equivalente to a representation by block triangular matrices with many blocks.

Example Referring to sec. **??**, consider the subgroup of the modular group $\mathfrak{M} \cong \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ generated by the translation $T : \tau \mapsto \tau + 1$. This Abelian subgroup contains all the translations by integers n, and is isomorphic to $\mathbb{Z}$. As a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ its elements are represented by

$$D(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\ . \tag{1.1.27}$$

This 2-dimensional representation of $\mathbb{Z}$ is reducible. On the diagonal appears twice the trivial representation.

Completely reducible representations If a representation $(D;V)$ admits an invariant subspace $W \subset V$ and moreover also the *complement* $W^\perp$ of W in V is *invariant*, then in a basis adapted to the decomposition $V = W \oplus W^\perp$ all the matrices $D(g)$ must be block-diagonal:

$$D(g) = \begin{pmatrix} D_1(g) & \mathbf{0} \\ \mathbf{0} & D_2(g) \end{pmatrix} \,, \quad \text{acting on} \quad \begin{pmatrix} W^\perp \\ W \end{pmatrix} \,, \tag{1.1.28}$$

with D_1 and D_2 furnishing two representations of G. We say that the representation D is the *direct sum* of the two representations D_1, D_2 and we write

$$D = D_1 \oplus D_2 \,. \tag{1.1.29}$$

Indeed, all the reasonings given after Eq. (1.1.25) apply, the only difference being that the upper block $A(g)$ in Eq. (1.1.25) has to vanisg for $W^\perp$ to be always mapped into itself by these matrices.

In turn, W and $W^\perp$ could be splitted into smaller invariant subspaces. Thus, in general, for a *completely reducible representation* the carrier space V decomposes into a *direct sum of invariant subspaces*:

$$V = \oplus_\mu V_\mu \,, \qquad \forall g \in G \,, \forall \mu \,, \; D(g) : V_\mu \to V_\mu \,, \tag{1.1.30}$$

where the V_μ do not contain any smaller invariant subspace. The representation D is then equivalent to a representation in which all the matrices $D(g)$ have a block-diagonal form:

$$\forall g \in G \,, \; D(g) = \begin{pmatrix} D_1(g) & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & D_2(g) & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & D_3(g) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \,, \tag{1.1.31}$$

namely D decomposes into a direct sum of (irreducible) representations:

$$D = \oplus_\mu a_\mu D_\mu \,, \tag{1.1.32}$$

where the integer a_μ denotes how many times the irreducible representation D_μ appears in the decomposition.

Examples

- Consider the two-dimensional representation of $\mathbb{Z}_2$ given in Eq. (1.1.11). All the matrices are already in block-diagonal form, and we have thus evidently $D = D_1 \oplus D_1$, where D_1 is the 1-dimensional representation of Eq. (1.1.10): $D_1(e) = 1$, $D_1(a) = -1$.
- Consider the 2-dimensional representation of $\mathbb{Z}_2$ given in Eq. (1.1.12). It acts on a 2-dimensional vector space $V = \mathbb{R}^2$, with a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. It is easy to see that the 1-dimensional subspace W generated by $\mathbf{e}_+ = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$ is invariant. Its orthogonal complement $W^\perp$, generated by $\mathbf{e}_- = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$, is also invariant. Indeed, effecting the change of basis with the matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.1.33}$$

  we obtain an equivalent representation D' in which all the matrices $D'(g) = SD(g)S^{-1}$ have been diagonalized:

$$D'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \quad D'(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{1.1.34}$$

We see that therefore the representation D is completely reducible:

$$D = D_0 \oplus D_1 \ , \tag{1.1.35}$$

where D_0 is the trivial representation and D_1 is the representation of Eq. (1.1.10).

- Consider the 3-dimensional representation of S_3 described in Eq. (1.1.17). Denoting as $\{\mathbf{e}_i\}$, $i = 1, 2, 3$ be the basis of the $\mathbb{R}^3$ carrier space of the representation, similarly to the previous example one finds that the subspace generated by $\mathbf{e}_+ = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$ is invariant, and so is its orthogonal complement. As an exercise, complete the calculation and show that the decomposition of this representation is

$$D = D_0 + D_2 \ , \tag{1.1.36}$$

where D_0 is the trivial representation and D_2 is the 2-dimensional representation of S_3 defined in Eq. (1.2.107).

- The 2-dimensional representation Eq. (1.1.16) of SO(2) can be diagonalized by the complex change of basis

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \ . \tag{1.1.37}$$

One finds indeed

$$D'(\theta) = S D(\theta) S^{-1} = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}\right) \ . \tag{1.1.38}$$

The 2-dimensional representation D is thus a direct product:

$$D = D_1 \oplus D_{-1} \ , \quad \text{where } \forall\theta\,, \ D_1(\theta) = \mathrm{e}^{\mathrm{i}\theta}\,, \ D_{-1}(\theta) = \mathrm{e}^{-\mathrm{i}\theta} \ . \tag{1.1.39}$$

The representation D_1 establishes explicitely the well-known isomorphism between SO(2) and U(1).

The importance of being irreducible In classifying the representations of a group, the essential and hard part is to classify the *irreducible* representations. Using irreducible representations as building blocks, one can them construct (via direct sums, typically) other larger representations which are reducible[2].

It is therefore important to establish efficient *irreducibility criteria* which allows to determine whether a representation is irreducible or not. Very important in this regard are the Schur's Lemma and in general the orthogonality properties of the matrix elements of the representation matrices, that we will discuss later in this chapter.

1.1.6 Construction of representations of transformation groups

Consider a tranformation group G acting on some space $\mathcal{V}$, (typically, finite dimensional):

$$g \in G\,; \ \mathbf{x} \in \mathcal{V} \mapsto g\mathbf{x} \in \mathcal{V} \ . \tag{1.1.40}$$

As an example, consider for instance of a group of trasformation of our three-dimensional space $\mathbb{R}^3$. Different representations of G arise by considering spaces of *functions* $\psi(\mathbf{x})$ defined over $\mathcal{V}$ (in the example above, think of spaces of "wave functions" $\psi(\mathbf{x})$, for $\mathbf{x} \in \mathbb{R}^3$), which we regard as states $|\psi\rangle$ in an Hilbert space).

[2] This is analogous to the fact, discussed in **??**, that in classifying the possible groups the "hard", essential part is the classification of simple groups.

We have already discussed in sec. **??** how the action Eq. (1.1.40) of G on $\mathcal{V}$ induces homomorphically (see Eq. (**??**)) an action on the functions by means of linear operators $O_g : \psi \mapsto O_g\psi$ such that

$$\left(O_g\psi\right)(g\mathbf{x}) = \psi(\mathbf{x}) \ . \tag{1.1.41}$$

See also the first example in sec. 1.1. If we are able to single out a finite-dimensional invariant subspace V of functions, on the functions belonging to this subspace the linear operators O_g act as matrices; then we have constructed a matrix representation of G.

Consider a function $\psi(\mathbf{x})$, generic or with some particular properties, and apply to it all the elements of G. Not all of the transformed functions $O_g\psi$ will be linearly independent; denote as ψ_ν of them ($\nu = 1, \ldots n$) a maximal set of transformed functions which are independent. Take $\{\psi_\nu\}$ as basis vectors of a n-dimensional subspace V of our function space. This space is *invariant* under G, as the transformed functions can be reexpressed in the basis:

$$\forall g \in G\,,\ O_g\psi_\mu = \left[D(g)\right]_\mu{}^\nu \psi_\nu \ . \tag{1.1.42}$$

Indeed, suppose that $\psi_\mu = O_{g_\mu}\psi$, for certain elements g_ν. Then $O_g\psi_\mu = O_{gg_\nu}\psi$, so it is one of the trasnformed images of ψ, which by hypothesis can be expressed on the $\{\psi_\mu\}$ independent functions. In this way we obtain a representation (D, V) of the group G.

Examples

- Consider the group $\mathbb{Z}_2$ realized on $\mathbb{R}^3$ as the group of space inversions:

$$e\,:\ \mathbf{x} \mapsto \mathbf{x}\ , \quad a\,:\ \mathbf{x} \mapsto -\mathbf{x}\ . \tag{1.1.43}$$

Let us construct some representations of this group by considering different types of functions.

  - Start from a generic function $\psi_1(\mathbf{x})$. The action of $\mathbb{Z}_2$ on the function is defined according to Eq. (1.1.41), so we have for the generator a,

$$O_a\psi_1(\mathbf{x}) = \psi(-\mathbf{x}) \equiv \psi_2(\mathbf{x}) \ \Rightarrow\ O_a\psi_1 = \psi_2 \ . \tag{1.1.44}$$

The identity e of course leaves the function invariant. So the transformed function $O_a\psi_1 \equiv \psi_2$ is linearly independent from ψ_1, and $\psi_{1,2}$ span a two-dimensional invariant subspace. Indeed, acting on ψ_2 with O_a we have (as obvious by the homomorphic nature of the map)

$$O_a\psi_2(\mathbf{x}) = \psi_2(-\mathbf{x}) = \psi_1(\mathbf{x}) \ \Rightarrow\ O_a\psi_2 = \psi_1 \ . \tag{1.1.45}$$

From Eq. (1.1.42) we see that on the space V generated by $\psi_1, 2$ the group of space inversions $\mathbb{Z}_2$ acts via the 2-dimensional representation of Eq. (1.1.12): the generator a becomes

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \overset{a}{\mapsto} \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = D(a)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \ . \tag{1.1.46}$$

The identity e is of course represented, as always, by the identity matrix.

  - Start from an *even* function $\psi(\mathbf{x}) = \psi(-\mathbf{x})$. We have now

$$O_a\psi(\mathbf{x}) = \psi(-\mathbf{x}) = \psi(-\mathbf{x}) \ \Rightarrow\ O_a\psi = \psi \ . \tag{1.1.47}$$

The transformed function coincides with the original one, so that ψ spans an unidimensional invariant space which is the carrier space of a trivial representation

$$D_0(e) = 1\ , \quad D_0(a) = 1\ , \tag{1.1.48}$$

as it follows from Eq. (1.1.42). In other words, any *even* function $\psi(\mathbf{x})$ is invariant under the $\mathbb{Z}_2$ group of space inversions and individuates an invariant subspace within the representation associated to a generic function.

– Start now from an *odd* function $\psi(\mathbf{x}) = -\psi(-\mathbf{x})$. The action of the $\mathbb{Z}_2$ generator on such a function is given by

$$O_a\psi(\mathbf{x}) = \psi(-\mathbf{x}) = -\psi(-\mathbf{x}) \;\Rightarrow\; O_a\psi = -\psi \;. \tag{1.1.49}$$

Again, the transformed functions are linearly dependent from ψ, so that ψ spans an unidimensional invariant space. However now $\mathbb{Z}_2$ acts on this space by the non-trivial representation of Eq. (1.1.10):

$$D_1(e) = 1 \;, \quad D_1(a) = -1 \;. \tag{1.1.50}$$

In other words, any *odd* function $\psi(\mathbf{x})$ transforms non-trivially under space-inversions, but still forms a "singlet".

We have thus retrieved the decomposition of the two-dimensional representation D of Eq. (1.1.46) associated to a generic function into a direct sum:

$$D = D_0 \oplus D_1 \;, \tag{1.1.51}$$

namely the same decomposition we discussed in Eq.s (1.1.33,1.1.35). The decomposition corresponds to the existence of invariant subspaces corresponding to functions with *definite symmetry properties* under the $\mathbb{Z}_2$ action, namely to even or odd functions. Although the example is quite simple, this is the general pattern.

- Consider the group of discrete translations $T(n)$, $n \in \mathbb{Z}$, of the real axis:

$$T(n) \;:\; x \in \mathbb{R} \mapsto (x+n) \in \mathbb{R} \;. \tag{1.1.52}$$

This group is clearly isomorphic to $\mathbb{Z}$, with $T(n) \leftrightarrow n$.

– If we start from a generic complex function $\psi_0(x)$ and act with the translation group $\mathbb{Z}$ we obtain infinite independent functions:

$$\left(O_n\psi_0\right)(x+n) = \psi_0(x) \;\Rightarrow\; O_n\psi_0 = \psi_n \;, \text{ such that } \psi_n(x) = \psi_0(x-n) \;. \tag{1.1.53}$$

We obtain then an infinite-dimensional representation of $\mathbb{Z}$, in which the generator 1 (i.e. the translation by 1) is represented as a "cyclic permutation of infinite order":

$$D(1) = \begin{pmatrix} \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \;; \tag{1.1.54}$$

comparing it with the $N \times N$ representation of a cyclic permutation $(12\dots N) \in S_N$ according to Eq. (**??**) we see indeed that we would retrieve Eq. (1.1.54) for $N \to \infty$. Indeed, $\mathbb{Z}$ is also often referred to as the infinite cyclic group.

– If we start from a *periodic* function $\psi(x)$:

$$\psi(x+n) = \psi(x) \;, \tag{1.1.55}$$

all transformed functions coincide with the original one: $O_n\psi = \psi$, and we get the trivial uni-dimensional representation of $\mathbb{Z}$.

- We can more generally consider "quasi-periodic" functions, i.e. functions which are periodic up to a phase:
$$\psi(x+n) = \mathrm{e}^{-\mathrm{i}\theta n}\psi(x) \ . \tag{1.1.56}$$
Such functions are of the form $\psi(x) = \psi_P(x)\mathrm{e}^{-\mathrm{i}\theta x}$, where ψ_P is a periodic function. The transformed functions $O_n\psi$ are still all linearly dependent, so we obtain uni-dimensional representations. In fact we obtain the unitary representations of $\mathbb{Z}$, labeled by the angle θ, described before in Eq. (1.1.67):
$$O_n\psi = D_\theta(n)\psi = \mathrm{e}^{\mathrm{i}n\theta}\psi \ . \tag{1.1.57}$$

1.1.7 Irreducible representations of the symmetry group and energy eigenstates

We have introduced in sec. **??** the symmetry group of a quantum mechanicsl system. It is the group G of operators O_g on the Hilbert space, i.e. on the wave-functions $psi(\mathbf{x})$: $\psi \mapsto O_g\psi$, that leave the Hamiltonian operator $H(\mathbf{x})$ invariant:
$$O_g H O_g^{-1} = H \ . \tag{1.1.58}$$
Consider a maximal set of linearly independent eigenfunctions ψ_n $(n = 1,\ldots N)$ associated to a single energy level ϵ, possibly degenerated if $N > 1$:
$$H\psi_n = \epsilon\psi_n \ . \tag{1.1.59}$$
An extremely important result for quantum mechanics is that the *energy eigefunctions ψ_n* form a basis for an *irreducible representation of the symmetry group G.*

Indeed, on the one hand
$$O_g H\psi_n = \epsilon O_g\psi_n \ ; \tag{1.1.60}$$
on the other hand
$$O_g H\psi_n = O_g H O_g^{-1} O_g\psi_n = H O_g\psi_n, \tag{1.1.61}$$
for every $g \in G$. Comparing Eq.s (1.1.60,1.1.61) we see that $O_g\psi_n$ is still an energy eigenfunction with energy ϵ, so by hypothesis it can be expanded in the basis $\{\psi_n\}$. Thus we have
$$O_g\psi_n = \psi_m[D(g)]^m_{\ n} \ . \tag{1.1.62}$$

Examples

- Consider a one-dimensional quantum system with a symmetric potential. It enjoys a $\mathbb{Z}_2$ reflection symmetry $x \mapsto -x$ under which the Hamiltonian is invariant. It follows that energy eigenfunctions $\psi(x)$ must transform in an irreducible representation of this group. We have already encountered in the first example in sec. 1.1.6 two irreducible representations, D_0 and D_1 of $\mathbb{Z}_2$ corresponding to *even* and *odd* functions. Let us anticipate a result that we will find in a while, namely that these are the *only* irrepses. It follows that every energy eigenfunction $\psi(x)$ must have a *definite parity*, i.e. in must be either even or odd. Think for instance of the infinite well with its cos and sin eigenfunctions.
- Consider an particle in a uni-dimensional periodic potential (modelling e.g. an electron in a crystal)....
- In a quantum mechanical symmetry with rotational invariance ....

1.1.8 Unitary representations

Suppose that on the carrier space V of a representation (D, V) is a scalar product is defined,

$$\forall \mathbf{v}, \mathbf{y} \in V\,,\ (\mathbf{v}, \mathbf{y}) \in \mathbb{C}\ , \tag{1.1.63}$$

with the property of being (anti)-linear in the second (first) argument, and of providing a positive norm $(\mathbf{v}, \mathbf{v}) \geq 0$. This is the case usually of interest in physics, as V is a finite-dimensional space or an infinite-dimensional Hilbert space of some quantum system.

The representation is *unitary* if every operator $D(g)$ is unitary with respect to te scalar product, namely if

$$\forall g \in G\,, \forall \mathbf{v}, \mathbf{y} \in V\,,\ (D(g)\mathbf{v}, D(g)\mathbf{y}) = (D(g)^\dagger D(g)\mathbf{v}, \mathbf{y}) = (\mathbf{v}, \mathbf{y}) =\ \ , \tag{1.1.64}$$

where in the second step we used the definition of the hermitian conjugate operator $D(g)^\dagger$, so that the requirement of unitarity becomes that

$$\forall g \in G\,,\ D(g)^\dagger D(g) = \mathbf{1}\ . \tag{1.1.65}$$

For V finite-dimensional, in a basis adapted to the scalar product, namely in a basis $\{\mathbf{e}_i\}$ such that $(\mathbf{e}_,\mathbf{e}_j) = \delta_{ij}$, the request Eq. (1.1.64) becomes simply that all the *matrices* $D(g)$ be unitary. Thus, a matrix representation is unitary iff it is *equivalent* to a representation made entirely of *unitary matrices.*

Examples

- The group $\mathrm{SO}(2) \cong \mathrm{U}(1) = \{\mathrm{e}^{\mathrm{i}\theta}\}$, parametrized by $\theta \in [0, 2\pi]$, admits an infinite series of 1-dimensional *unitary* representations D_n, for $n \in \mathbb{Z}$, given by

$$D_n(\theta) = \mathrm{e}^{\mathrm{i}n\theta}\ . \tag{1.1.66}$$

  It is immediate to see that these are representations: $D_n(\theta_1)D_n(\theta_2) = D_n(\theta_1 + \theta_2)$ and that they are faithful. Moreover, a complex number, i.e. a 1×1 matrix, is unitary if it is of unit modulus. In Eq. (1.1.66), n is a representation label, while θ parametrizes the group element.
- The group $\mathbb{Z}$ admits a continuous series of 1-dimensional *unitary* representations D_θ, with $\theta \in [0, 2\pi]$, given by

$$D_\theta(n) = \mathrm{e}^{\mathrm{i}n\theta}\ , \tag{1.1.67}$$

  where θ labels the representation, while n is the group element of $\mathbb{Z}$.

1.1.8.1 Unitary representations and reducibility

If unitary representation is reducible, then it is *completely reducible.* This is immediately seen in the case of finite-dimensional representations: if D is reducible then every $D(g)$ it is equivalent to a block-triangular matrix as in Eq. (1.1.26). Such matrices cannot be equivalent to unitary matrices unless the block $A(g)$ vanishes, in which case D is completely reducible. As an exercise, formalize better the proof and extend it to the infinite-dimensional case.

1.1.9 Induced representations

Conversely, a representation $D(G/H)$ of the factor group of G by a normal subgroup H, gives automatically also a representation of G, called an *induced representation*, such that $D(hg) = D(g)$, $\forall g \in G, h \in H$ (namely, all cosets are mapped in the same matrix).

Examples

- The symmetric group S_3 admits the normal subgroup $A_3 = \{e, (123), (132)\} \cong \mathbb{Z}_3$. The factor group S_3/A_3 contains the classes $[e] = A_3$ end $[(12)] = \{(12), (13), (23)\}$. We have $S_3/A_3 \cong \mathbb{Z}_2$, so the factor groups admits the 1-dimensional representation of Eq. (1.1.10), namely $D([e]) = 1\,, D([(12)]) = -1$. This representation induces a 1-dimensional representation of S_3. given by

$$\begin{aligned} D(e) = D\left((123)\right) = D\left((132)\right) = 1\ , \\ D\left((12)\right) = D\left((13)\right) = D\left((23)\right) = -1\ . \end{aligned} \tag{1.1.68}$$

- The above discussion generalizes to the case of S_n. Since $S_n/A_n \cong \mathbb{Z}_2$, the representation of $\mathbb{Z}_2$ as roots of unity induces the non-trivial representation of S_n in which we assign 1 to even and -1 to odd permutations:

$$D(P) = (-1)^{\sigma(P)}\ , \tag{1.1.69}$$

  where $\sigma(P)$ is the sign of the permutation.

1.2 Important properties of group representations

Bla bla ...

1.2.1 Unitarity of the representations of finite groups

Every representation (D, V) of a *finite* group G is equivalent to a *unitary* representation.

The proof of this very important result goes schematically as follows. Using the inner product $(\mathbf{v}, \mathbf{y})$ defined on the carrier space V, one can define a new scalar product by "averaging" over the group action:

$$(\mathbf{v}, \mathbf{y})_G \equiv \frac{1}{|G|}\sum_{g\in G}(D(g)\mathbf{v}, D(g)\mathbf{y})\ . \tag{1.2.70}$$

It is possible that this new inner product satisfies all the properties required of a inner product, namely (anti)-linearity and positivity, given that the original inner product does (check it explicitely as an exercise). The representation D preserves the inner product $(,)_G$ as, $\forall h \in G\,, \forall \mathbf{v}, \mathbf{y} \in V$,

$$\begin{aligned} (D(h)\mathbf{v}, D(h)\mathbf{y})_G = \frac{1}{|G|}\sum_{g\in G}(D(g)D(h)\mathbf{v}, D(g)D(h)\mathbf{y}) = \\ \frac{1}{|G|}\sum_{g\in G}(D(gh)\mathbf{v}, D(gh)\mathbf{y}) = \frac{1}{|G|}\sum_{g\in G}(D(g)\mathbf{v}, D(gh)\mathbf{y}) = (\mathbf{v}, \mathbf{y})_G\ . \end{aligned} \tag{1.2.71}$$

We have used the fact that D is an homomorphism and the invariance property of the finite sum over the elements of the group:

$$\sum_{g\in G} f(gh) = \sum_{gh\in G} f(gh) = \sum_{g\in G} f(g)\ . \tag{1.2.72}$$

Indeed, we have the *right-invariance* property $\sum_{gh\in G} = \sum_{g\in G}$ because as multiplying on the right every element of $g \in G$ by a fixed h one obtains again all the elements of G (this is the property of the rows of the multiplication table of G). Of course, an analogous *left-invariace* property holds as well.

It is now sufficient to change in V from the basis $\{\mathbf{e}_i\}$ adapted to the original scalar product: $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$, to a basis $\{\mathbf{f}_i\}$ such that $(\mathbf{f}_i, \mathbf{f}_j)_G = \delta_{ij}$. If S is the change of basis matrix, $\mathbf{f}_i = S_i{}^j \mathbf{f}_j$, the equivalent representation D' given by the matrices $D'(g) = SD(g)S^{-1}$ is *unitary* with respect to the original scalar product, as it is easy to show starting from the fact that $(S\mathbf{v}, S\mathbf{y})_G = (\mathbf{v}, \mathbf{y})$.

We will see later, in sec. **??**, that the unitarity of irreducible representations can be established, under certain assumptions, also for Lie groups.

Corollary A corollary of the above property is that all representations of finite groups are either irreducible or *fully reducible*. Indeed all representations are unitary, but we have seen in sec 1.1.8.1 that a unitary representation, if reducible, is fully reducible.

1.2.2 Schur's Lemma

1.2.2.1 First Schur's lemma

If D and D' are two *irreducible* representations of a group G and we have

$$D'(g)A = AD(g)\,, \ \forall g \in G \tag{1.2.73}$$

for some operator A (called an "inter-twiner"), then there are only two possibilities:

(i) $A = 0$;
(ii) $\dim G = \dim G'$ and G and G' are equivalent. A is then the change of basis expliciting their equivalence.

Hint of a proof For a proof, see e.g. Hammermesh, sec 3.14. We give here only some brief indications. Suppose that $\dim G > \dim G'$, and let $\{\psi_i\}$ be a set of basis vectors for the carrier space og D. If Eq. (1.2.73) holds for $A \neq 0$, it means that D admits an *invariant subspace* of dimension $\dim G'$, whose basis vectors ϕ_m are expressed by $\phi_m = A_m^{\ i}\psi_i$ in terms of the basis of D. Indeed, under any group action, we have $A\psi \overset{g}{\mapsto} AD(g)\psi$, but since $A\psi$ span an invariant subspace, we also have $A\psi \overset{g}{\mapsto} D'(g)A\psi$. *Irreducible* representations admit only trivial invariant subspaces: the null space $\{\vec{0}\}$, corresponding to the case 1), or the entire space, corresponding to the case 2).

1.2.2.2 Schur's lemma

Let D be an *irreducible* representation of a group G. A matrix A such that it commutes with *all* the representatives $D(g)$ of the elements of G can only be proportional to the identity matrix:

$$AD(g) = D(g)A\,, \ \forall g \in G \ \Rightarrow \ A \propto \mathbf{1}\,. \tag{1.2.74}$$

Hint of a proof Consider the eigenvalue equation for A, $A\psi = \lambda\psi$. Since A commutes with $D(g)$, every eigenvector of A is also an eigenvector of $D(g)$, for all $g \in G$. That is, an eigenspace of A is an invariant subspace of the representation D. D being irreducible, the only invariant subspaces are the trivial ones, i.e. the null space or the entire space, in which case A has a single eigenvalue, so that Eq. (1.2.74) follows.

1.2.2.3 Schur's lemma as an irreducibility criterion

Schur's lemma gives a very useful irreducibility criterion. Given a representation D of dimension d , we take a generic $d \times d$ matrix A and impose that it should commute with all the group representatives $D(g)$. If we find that this requirement implies that A is proportional to $\mathbf{1}$, then D is irreducible.

1.2.3 Orthogonality property of matrix elements

Let us denote, here and in the following, by D_μ the various irreducible representations of a group G, and by d_μ their dimensions.

There is a very important *orthogonality* relation between the matrix elements of group represetatives in irreducible representations, namely:

$$\frac{1}{|G|}\sum_{g\in G}[D_\mu(g)]^i{}_j[D_\mu(g^{-1}]^k{}_l = \frac{1}{d_\mu}\delta_{\mu\nu}\,\delta^i_l\,\delta^k_j\,. \tag{1.2.75}$$

If the representation is *unitary*, this can be written as

$$\frac{1}{|G|}\sum_{g\in G}[D_\mu(g)]^i{}_j[D_\mu(g)^\dagger]^k{}_l = \frac{1}{d_\mu}\delta_{\mu\nu}\,\delta^i_l\,\delta^k_j\,. \tag{1.2.76}$$

Hint of a proof For a proof of the orthogonality relation Eq. (1.2.75) see [Wu-ki-tung?] [put the following in the "technical appendix?]. The proof is based on Schur's lemma Eq. (1.2.74). One considers first the case $\mu = \nu$. It is not difficult to show that the matrix

$$A = \sum_{g\in G} D_\mu(g)BD_\mu(g^{-1}) \tag{1.2.77}$$

obtained by summing over al "conjugates" of any given matrix B satisfies

$$D_\mu(g)A = AD_\mu(g)\,,\ \forall g \in G\,. \tag{1.2.78}$$

This is shown by making use of the invariance property Eq. (1.2.72) of the sum over group elements. Schur's lemma appplied to Eq. (1.2.80) implies that $A = \lambda\mathbf{1}$. Choosing B as a matrix with a single element, say $B^l{}_m$ different from zero, so as to reconstruct in Eq. (1.2.79) the l.h.s. of the orthogonality relation Eq. (1.2.75), and tracing one finds that the corresponding eigenvalue of A is $\delta^l{}_m|G|/d_\mu$. Thus one obtains Eq. (1.2.75) in the case $\mu = \nu$. In the case $\mu \neq \nu$, one can show (again using only the invariance property of the sum over group elements) that

$$A = \sum_{g\in G} D_\mu(g)BD_\nu(g^{-1}) \tag{1.2.79}$$

with B generic satisfies

$$D_\mu(g)A = AD_\nu(g)\,,\ \forall g \in G\,. \tag{1.2.80}$$

From the first Schur's lemma, Eq. (1.2.73), it follows that $A = 0$. Choosing B as above, Eq. (1.2.79) reduces thus to the case $\mu \neq \nu$ of the orthogonality relation Eq. (1.2.75).

An important consequence The matrix elements $[D_\mu(g)]^i{}_j$ at *fixed* μ, i, j can be seen as entries of a $|G|$-dimensional vector (indicized by $g \in G$). Eq. (1.2.75) says that the $\sum_\mu (d_\mu)^2$ such vectors (recall that $i, j = 1, \ldots, d_\mu$) are mutually orthogonal. The maximal number of mutually orthogonal vectors in a $|G|$-dimensional space is $|G|$, so we must have

$$\sum_\mu d_\mu^2 \leq |G| \,. \tag{1.2.81}$$

We will see in the sequel that Eq. (1.2.81) actually holds with the equality sign.

1.2.4 Orthogonality properties of characters

By tracing the orthogonality relation Eq. (1.2.75) for matrix elements one obtains the following orthogonality relation for characters. We denote by $\chi^\mu(g)$ the character of the element g in the irrep D_μ, and we have then

$$\frac{1}{|G|} \sum_{g \in G} \chi^\mu(g) \chi^\nu(g^{-1}) = \delta_{\mu\nu} \,. \tag{1.2.82}$$

For unitary representation, the above can be written as

$$\frac{1}{|G|} \sum_{g \in G} \chi^\mu(g) \chi^\nu(g)^* = \delta_{\mu\nu} \,. \tag{1.2.83}$$

Characters depend only on the conjugacy classses. We will denote by χ_i^μ the character in the irrep D_μ of elements belonging to the conjucgacy class $\mathcal{C}_i$. We will also indicate as n_i the number of elements in the conjugacy class $\mathcal{C}_i$. Furthermore, let us denote by r the number of irrepses (i.e., we have $\mu = 1, \ldots, r$) and by k the number of conjugacy classes (i.e., we have $i = 1, \ldots, k$). The orthogonality of characters can be rewritten as follows (we consider the case of unitary representations, Eq. (1.2.83))

$$\frac{1}{|G|} \sum_i n_i \, \chi_i^\mu \, (\chi_i^\nu)^* = \delta_{\mu\nu} \,. \tag{1.2.84}$$

We can view $\sqrt{n_i}\chi_i^\mu$ as a set of r vectors labeled by μ, living in a k-dimensional space. The relation Eq. (1.2.84) states that these r vectors are mutually orthogonal, so r must not exceed the dimensionality k of the space:

$$r \leq k \,, \tag{1.2.85}$$

i.e., the number of irreducible representations is not bigger than the number of conjugacy classes. We will see in the following that actually Eq. (1.2.85) holds with the equality sign.

1.2.5 Characters and the decomposistion of representations

A generic, fully reducible, representation D of a group G decomposes as a direct sum of irreducible representations as in Eq. (**??**)

$$D = \oplus_\mu a_\mu D_\mu \ . \tag{1.2.86}$$

The characters $\chi(g)$ of D are then simply expressed in terms of the characters $\chi^\mu(g)$ of the irreducible representations D_μ. Indeed, the characters do not depend on the choice of basis, so we can always put the matrices $D(g)$ in the block-diagonal form of Eq. (1.1.31). By taking the trace, we get then

$$\chi(g) = \sum_\mu a_\mu \chi^\mu(g) \ , \tag{1.2.87}$$

or equivalently, labeling the characters with the conjugacy classes label m as in sec. **??**,

$$\chi_m = \sum_\mu a_\mu \chi^\mu_m \ . \tag{1.2.88}$$

Notice that in the literature the term "character" is often reserved to characters χ^μ_m in *irreducible* representations. The characters χ_m in a reducible representation being instead referred to as compound characters.

Using the orthogonality relation Eq. (**??**) we can extract from Eq. (1.2.88) an expression in terms of characters for the multeplicity a_μ of the irrep D_μ in the decomposition of D. Indeed, multiplying Eq. (1.2.88) by $(\chi^\mu_m)^*$ and summing over the conjigacy classes with weight N_m (the number of element in the class) we obtain

$$\sum_m N_m (\chi^\mu_m)^* \chi_m = \sum_\nu a_\nu \sum_m N_m (\chi^\mu_m)^* \chi^\nu_m = |G| a_\mu \ , \tag{1.2.89}$$

so that

$$a_\mu = \frac{1}{|G|} \sum_m N_m (\chi^\mu_m)^* \chi_m \ . \tag{1.2.90}$$

An irreducibility criterion Let us start again from Eq. (1.2.88). Let us multiply by the conjugate equation and sum over the classes with weight N_m, getting

$$\sum_\mu N_m (\chi_m)^* \chi_m = \sum_{\mu,\nu} a_\mu a_\nu \sum_m N_m (\chi^\mu_m)^* \chi^\nu_m \tag{1.2.91}$$

and thus, using agin the orthogonality of irreducible characters,

$$\sum_\mu N_m (\chi_m)^* \chi_m = |G| \sum_\mu a_\mu^2 \ . \tag{1.2.92}$$

If the representation D is irreducible, namely is equivalent to some D_μ for a certain μ, then only that specifc $a_\mu = 1$, all the others being zero. Thus,

$$D \text{ is irreducible} \Leftrightarrow \sum_\mu N_m (\chi_m)^* \chi_m = |G| \ . \tag{1.2.93}$$

This is a very useful irreducibility criterion.

1.2.6 The regular representation

The regular representation $\mathcal{R}$ of a finite group G is a faithful representation of dimension $|G|$. Its matrices $\mathcal{R}(g)$ are exactly the $|G| \times |G|$ matrix representatives (defined via Eq. (**??**)) of the *regular permutations* of $S_{|G|}$ that we encountered n the discussion of Cayley's theorem in sec **??**, and that describe the rows of the multiplication table of the group G. These matrices contain a signle 1 is each row and column, other elements bein zero. Moreover, while $\mathcal{R}(e) = \mathbf{1}$, all other matrices have no non-zero entry on the diagonal. Explicitely, we have

$$[R(g_j)]^i{}_k = \delta^i_k \Leftrightarrow g_j g_i = g_k \ . \tag{1.2.94}$$

Notice that the regular representation acts on a vector space obtained formally by taking as basis the group elements themselves.

In the regular representation, all matrices except $\mathcal{R}(e) = \mathbf{1}$ have vanishing diagonals, so

$$\begin{aligned} \chi(e) &= |G|\,; \\ \chi(g_i) &= 0 \quad \text{if } g_i \neq e\,. \end{aligned} \tag{1.2.95}$$

Let us consider the decomposition of the regular representation into irreducible representations,

$$\mathcal{R} = \oplus_\mu a_\mu D_\mu \ , \tag{1.2.96}$$

implying for the characters that

$$\chi_m = \sum_\mu a_\mu \chi^\mu_m \ . \tag{1.2.97}$$

The general expression Eq. (1.2.90) of the multiplicities a_μ simplifies in the case of the regular representation because of the properties Eq. (1.2.95):

$$a_\mu = \frac{1}{|G|}\sum_m N_m (\chi^\mu_m)^* \chi_m = \frac{1}{|G|} N_0 (\chi^\mu_0)^* \chi_0 = \frac{1}{|G|} d_\mu\, |G| = d_\mu \ . \tag{1.2.98}$$

Indeed only the characters of the identity class (which we conventionally label by $m = 0$) contribute to the sum; mooreover the characters of the identity, see Eq. (1.1.23), alway give the dimension of the representation:

$$\chi_0 = |G|\,, \quad \chi^\mu_0 = d_\mu \ . \tag{1.2.99}$$

On the other hand, Eq. (1.1.4)), for the conjugacy class of the identity gives

$$\chi_0 = \sum_\mu a_\mu \chi^\mu_0 \ \Rightarrow |G| = \sum_\mu a_\mu d_\mu\,. \tag{1.2.100}$$

Comparing Eq. (1.2.98) and Eq. (1.2.100) we find that

$$|G| = \sum_\mu d^2_\mu\,. \tag{1.2.101}$$

So the inequality Eq. (**??**) we derived in sec. **??** is in fact an equality, giving a very important *constraint* on the irreducible representations and their dimensions.

For instance, for an *Abelian* group G, since all irreducible representations have dimension 1, as we saw in sec. 1.2.2, the regular representation contains exactly once every irreducible representation:

$$\mathcal{R} = \oplus_\mu D_\mu \ . \tag{1.2.102}$$

Therefore, it is possible by a change of basis to *diagonalize* the regular representation matrices; the $|G|$ eigenvalues of each $\mathcal{R}(g)$ will give the representatives in the G irreducible of the element g. Moreover we have from Eq. (1.2.101) that

$$|G| = \sum_\mu = \text{number of irrepses} \qquad (G \text{ Abelian})\,. \tag{1.2.103}$$

Example The regular representation of the cyclic group $\mathbb{Z}_n$ is the $n \times n$ representation obtained if $\mathbb{Z}_n$ is realized as the subgroup of cyclic permutations in S_n, and the permutations are seen as matrices according to Eq. (**??**); see for instance Eq. (1.1.12) for the $\mathbb{Z}_2$ case and Eq. (1.1.17) for the $\mathbb{Z}_3$ case. Indeed the regular representation of the generator a is given, according to Eq. (1.2.94), as follows:

$$a a^i = a^{i+1} \text{ (with } a^n = a^0 \equiv e) \;\Rightarrow\; [\mathcal{R}(a)]^i{}_j = \delta_{i+1,j} \;, \tag{1.2.104}$$

namely (NB E" AL CONTRARIO!!!)

$$\mathcal{R}(a) = \begin{pmatrix} 0 & 0 & \dots & \dots & 1 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \vdots \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & 1 & 0 \end{pmatrix} . \tag{1.2.105}$$

Diagonalizing $\mathcal{R}(a)$ we obtain all irreducible representatives of the generator a. It is elementary to find the eigenvalues:

$$\det\left(\mathcal{R}(a) - \lambda \mathbf{1}\right) = 0 \;\Rightarrow\; \lambda = \exp\left(\frac{2\pi \mathrm{i} \mu}{n}\right) \;, \quad (\mu = 0, \dots n-1) \;. \tag{1.2.106}$$

The representative of the powers of a are of course the powers of the representatives of a, so altogether we can explcitely describe all the *irreducible* representations of $\mathbb{Z}_n$ (and their characters, which coincide with the 1×1 matrix elements) :

$$D_\mu(a^m) = \chi^\mu_m = \exp\left(\frac{2\pi \mathrm{i} \mu m}{n}\right) \;, \quad (\mu, m = 0, \dots n-1) \;. \tag{1.2.107}$$

1.2.7 The group algebra

...

1.2.7.1 Left ideals and projection operators

...

1.2.7.2 Idempotents

...

1.2.8 Completeness relations

Beside the orthogonality relations between characters for different irreducible representations, the characters satisfy also "completeness" relations, namely orthogonality relations between characters of different conjugacy classes. We do not prove such relations, for a proof see, e.g., Hamermesh, chap. ? or Wu-Ki-Tung, ?. The proof requires the introduction of some new concepts, as the *group algebra* or the algebra of the conjugacy classes, and makes use also of the properties of the regular representation. [maybe add it to the technical appendix in future?]

The completeness relation for characters reads (for unitary representations)

$$\frac{1}{|G|}\sum_\mu \chi_i^\mu\,(\chi_j^\mu)^* = \frac{1}{n_i}\delta_{ij}\,. \tag{1.2.108}$$

We can view χ_i^μ as a set of k vectors labeled by i (k being the number of conjugacy classes) living in a r-dimensional space. Eq. (1.2.108) states that these vectors are mutually orthogonal. Therefore their number, k, must not exceed the dimensionaly r of the space:

$$k \le r\,, \tag{1.2.109}$$

namely the number of conjugacy classes is not bigger than the number of irrepses. Since from the orthogonality of characters we had already proved in Eq. (1.2.85) that $r \le k$, we conclude that actually

$$k = r\,, \tag{1.2.110}$$

namely, for a discrete group, the number of irreducible representations and of conjugacy classes coincides.

The completeness relation can be extended to compact Lie groups, in which case it reads

$$\sum_\mu \chi^\mu(g)\,\chi_j^\mu(g')^* = \delta_G(g,g')\,, \tag{1.2.111}$$

where $\delta_G(g,g')$is the conjigation-invariant delta-function on the group.

Example: Fourier basis Let us consider the Lie group SO(2) $\sim$ U(1). Its irreducible representations were given in Eq. (1.1.67). They are labeled by an integer that we now call $\mu \in \mathbb{Z}$, and they are all uni-dimensional, and we have

$$D_\mu(\theta) = \chi^\mu(\theta) = \mathrm{e}^{\mathrm{i}\mu\,\theta}\,. \tag{1.2.112}$$

The orthogonality and completeness relations Eq. (1.3.207) and Eq. (1.2.111) become explicitely

$$\begin{aligned} \frac{1}{2\pi}\int d\theta\,\chi^\mu(\theta)\,(\chi^\nu(\theta))^* &= \frac{1}{2\pi}\int d\theta\,\mathrm{e}^{\mathrm{i}\mu\theta}\mathrm{e}^{-\mathrm{i}\nu\theta} = \delta_{\mu\nu}\,,\\ \sum_\mu \chi^\mu(\theta)\,(\chi^\mu(\theta'))^* &= \sum_\mu \mathrm{e}^{\mathrm{i}\mu\theta}\mathrm{e}^{-\mathrm{i}\mu\theta'} = \delta_{\mathrm{P}}(\theta-\theta')\,, \end{aligned} \tag{1.2.113}$$

where δ_P is the periodic delta-function. Thus, the characters of U(1) are nothing else than the basis functions of the Fourier expansion, an the orthogonality and completeness relations for these characters are nothing else but the usual orthogonality and completeness of the Fourier basis.

1.2.9 The character table and its properties

...

Example: S_3 ...

1.2.10 Irreducible vectors and projection operators

The elements of the carrier space V of an irreducible representation D of a group G are called *irreducible vectors* transforming in the representation D. Suppose that a set of d_μ vectors[3] belonging to V

$$|\mu, i\rangle\,, \quad (\mu = 1, \ldots d_\mu) \tag{1.2.114}$$

form a basis for (one of the a_μ copies of) the irreducible representation D_μ contained in D, namely are a set of irreducible vectors for D_μ. This means that for every operator $D(g)$ of the representation D we have

$$D(g)|\mu, i\rangle = |\mu, j\rangle [D_\mu(g)]^j{}_i\ . \tag{1.2.115}$$

An important result is that, as a consequence of the orthogonality of the matrix elements of irreducible representations, two sets of *irreducible vectors* in V for *inequivalent representations* are *mutually orthogonal*:

$$\langle \nu, j|\mu, i\rangle = 0 \quad \text{if } \mu \neq \nu\ . \tag{1.2.116}$$

Indeed we have, using the unitarity of the representation D to insert an identity operator written in the form

$$\mathbf{1} = \frac{1}{G} \sum_{g \in G} D^\dagger(g) D(g)\,, \tag{1.2.117}$$

we have

$$\begin{aligned}\langle \nu, j|\mu, i\rangle &= \frac{1}{G} \sum_{g \in G} \langle \nu, j|\mu, i D^\dagger(g) D(g)|\mu, i\rangle \\ &= \frac{1}{G} \sum_{g \in G} [D^\dagger_\nu(g)]^j{}_k [D_\mu(g)]^l{}_i \langle \nu, k|\mu, l\rangle = \frac{1}{d_\mu} \delta_{\mu\nu} \delta_{ij} \langle \nu, k|\mu, k\rangle\,,\end{aligned} \tag{1.2.118}$$

where we used Eq. (1.2.115) and the orthogonality property Eq. (**??**).

When instead $\mu = \nu$, there are two possibilities:

i) The subspaces generated by the two sets of irreducible vectors may *not* overlap, so that they are orthogonal. This means that in the decomposition of D the irrep D_μ appears more that once (i.e., $\alpha > 1$) and the two bases are acted by two different copies of D_μ.
ii) If the two subspaces overlap, they are acted upon by the same copy of D_μ, and they are related by a univaty basis transformation.

We well from now adopt the following notation:

$$|\mu, \alpha, i\rangle\,, \quad \alpha = 1, \ldots a_\mu\ , \quad i = 1, \ldots d_\mu\ , \tag{1.2.119}$$

for sets of irreducible vectors for the α-th copy of the irrep D_μ inside $D = \oplus a_\mu D_\mu$.

We would like to be able to find out explicitely in V sets of irreducible vectors for the irreducible components of D, thus singling out explicitely the invariant subspaces in V. In other words, we woud like to be able to project explicitely the vectors $|x\rangle$ in V onto the subspaces V_μ. To this purpose, consider the operators

$$P^j_{\mu i} = \frac{d_\mu}{|G|} \sum_{g \in G} [D^{-1}_\mu(g)]^j{}_i\, D(g)\ , \tag{1.2.120}$$

[3] The Dirac notation by bra and kets is quite handy (and familiar expecially in applications to Quantum mechanics), so we will often use it in the following.

with $i, j = 1, \ldots d_\mu$. For any *fixed value* of j, if we apply the above operators to *any vector* $|x\rangle \in V$ we obtain a set of d_μ vectors (labeled by i)

$$P^j_{\mu i}|x\rangle \tag{1.2.121}$$

which transform in the irreducible representation D_μ. Indeed, applying any $h \in G$ to any of the above vectors we find, by using Eq. (1.2.120),

$$\begin{aligned} D(h)P^j_{\mu i}|x\rangle &= \frac{d_\mu}{|G|}\sum_{g\in G}[D^{-1}_\mu(g)]^j{}_i\, D(h)D(g)|x\rangle = \frac{d_\mu}{|G|}\sum_{hg\in G}[D^{-1}_\mu(g)]^j{}_i\, D(hg)|x\rangle \\ &= \frac{d_\mu}{|G|}\sum_{hg\in G}[D^{-1}_\mu(gh)]^j{}_k[D(h)]^k{}_i\, D(hg)|x\rangle = P^j_{\mu i}|x\rangle[D(h)]^k{}_i\ . \end{aligned} \tag{1.2.122}$$

We have used the homomorphic properties of D and D_μ, and the *invariance* of the sum over group elements, see sec. 1.2.1: $\sum_{g\in G} = \sum_{hg\in G}$. The discussion generalizes thus, as we argued in sec. 1.3.2, to compact Lie groups as well. One can see that the vectors $P^j_{\mu i}|x\rangle$ are mutually orthogonal, but not normalized.

Example Reconsider once more the 2-dimensional representation of $\mathbb{Z}_2$ of Eq. (1.1.12). It decomposes into irrepses as in Eq. (1.1.35), and we understood this decomposition in the first example of sec. 1.1.6 as due to the existence of invariant subspaces corresponding to definite symmetry requisites (in that example, being an even or an odd function). Now we could rephrase the procedure as follows. Sart from any state $|\psi\rangle \in V$, where V is the 2-dimensional carrier space of D. Vectors spanning the invariant subspaces for the trivial representation D_μ $|mu = 0, 1)$ are obtained by applying the operators Eq. (1.2.120), namely (all indices i, j have any 1 value so can be omitted)

$$\begin{aligned} P_0 &= \frac{1}{2}\left((D(e) + D(a)\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ P_1 &= \frac{1}{2}\left((D(e) - D(a)\right) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \end{aligned} \tag{1.2.123}$$

where the explicit matrices $D(g)$ were given in Eq. (1.1.12). Applying these operators, for instance, to the basis vector $\binom{|\psi\rangle_1=}{10}$ we get

$$\begin{aligned} P_0|\psi_1\rangle &= \frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle)\,, \\ P_1|\psi_1\rangle &= \frac{1}{2}(|\psi_1\rangle - |\psi_2\rangle)\,. \end{aligned} \tag{1.2.124}$$

The operators $P^j_{\mu i}$ can be thought of as *generalized projection operators*. Their on an irreducible basis is the following:

$$P^j_{\mu i}|\nu, \alpha, k\rangle = \delta_{\mu\nu}\delta_{jk}|\nu, \alpha, i\rangle\ . \tag{1.2.125}$$

Prove this as an exercise, using the orthogonality property Eq. (**??**).

Since $P^j_{\mu i}|x\rangle$ is, for any $|x\rangle$, an irreducible set, applying Eq. (1.2.125) to it we obtain

$$P^j_{\mu i}P^l_{\nu k} = \delta_{\mu\nu}\delta_{jk}P^l_{\mu i} \tag{1.2.126}$$

Inverting the definition Eq. (1.2.120) of the P's, one finds that the operators $D(g)$ of the representation D can be decomposed on the operators $P^j_{\mu i}$, the coefficients being the matrix elements of the irrepses:

$$\forall g \in G\,,\ D(g) = \sum_{\mu,i,j} [D_\mu(g)]^i{}_j \, P^j_{\mu i}\ . \tag{1.2.127}$$

Out of the $P^j_{\mu i}$ we can construct *projection operators*

$$\begin{aligned} P_{\mu i} &\equiv P^{j=i}_{\mu i}\,, \\ P_\mu &\equiv \sum_i P_{\mu i}\,, \end{aligned} \tag{1.2.128}$$

which satisfy ...

1.2.11 Irreducible representations of S_n and Young tableaux

...

1.2.12 Direct product of representations and its decomposition

...

1.2.13 Irreducible tensor operators and Wigner-Eckhart theorem

...

1.3 Representations of Lie groups

...

Can be obtained starting from the representations of the corresponding Lie algebrae; we will pursue this approach in the next Chapter.

Also tensor methods ...

1.3.1 Introductory examples

Let us consider the simplest Lie groups, namely SO(2) and $(\mathbb{R}, +)$, both uni-dimensional and abelian (i.e., with the same, trivial Lie algebra). The group manifold of SO(2) is the circle S_1, so it is a compact group, while that of $(\mathbb{R}, +)$ is obviously $\mathbb{R}$, so we have a non-compact group.

Let us now consider the *irreducible representations*, starting from the rotation group. In any representation D_μ, we must have

$$\begin{aligned} D(\theta_2)D(\theta_1) &= D(\theta_2 + \theta_1)\,, \\ D(\theta + 2\pi) &= D(\theta)\,, \end{aligned} \tag{1.3.129}$$

the first requirement sbeing simply the homomorphicity property, the second stemming from the global requirement Eq. (**??**). It is easy to see that we can repeat for any representation exactly the same analysis in terms of infinitesimal generators we did above, so that in the end one can write

$$D_\mu(\theta) = \mathrm{e}^{\mathrm{i}\,\theta\, J_\mu}\,, \tag{1.3.130}$$

having denoted as J_μ the infinitesimal rotation generator in the representation D_μ. If the representations D_μ have to be *unitary* (which is the only possibility since this group is a *compact* Lie group, see sec. 1.3.2), it follows from Eq. (1.3.130) that J_μ has to be an *Hermitean* operator.

Since the groups is Abelian, all its irreducible representations have dimension 1. The Hermitean operator J_μ is therefore a *real* number μ. Let us denote by $|\mu\rangle$ the single basis vector, so that we can write

$$J|\mu\rangle = \mu|\mu\rangle \tag{1.3.131}$$

to express the fact that the irrep D_μ is precisely that irrep in which the abstract generator J of the group is represented by the number μ. It follows then from Eq. (1.3.130) that the representative af a generic element in the irrep D_μ is given by

$$D_\mu(\theta) = \mathrm{e}^{\mathrm{i}\,\theta\,\mu}\,. \tag{1.3.132}$$

However, we still have to impose the periodicity condition on θ (the second line of Eq. (1.3.129)). This condition reads

$$D_\mu(\theta + 2\pi) = \mathrm{e}^{\mathrm{i}\,(\theta+2\pi)\,\mu} = \mathrm{e}^{\mathrm{i}\,\theta\,\mu} = D_\mu(\theta)\,, \tag{1.3.133}$$

from which it follows that

$$\mu \equiv m \in \mathbb{Z}\,, \tag{1.3.134}$$

i.e., that the angular momentum is quantized in any unitary irreducible representation of the rotation group SO(2):

$$J|m\rangle = m|m\rangle\,, \quad m \in \mathbb{Z}\,. \tag{1.3.135}$$

If we realize the group as a group of transformations φ_θ on the space S_1, see Eq. (**??**), we search (irreducible, unitary) representations of the group as particular subspaces of the space of *functions* over S_1. Since the action $D(\theta')$ of teh group element $\varphi_{\theta'}$ on a function ψ is defined as usual by the requirement that

$$D(\theta')\psi(\theta) = \psi(-\theta' + \theta)\,, \tag{1.3.136}$$

by considering an infinitesimal element $D(d\theta) = \mathbf{1} + \mathrm{i}\,d\theta\,J$ we find

$$(\mathbf{1} + \mathrm{i}\,d\theta\,J)\psi(\theta) = \psi(\theta) - d\theta\,\frac{d\psi(\theta)}{d\theta}\,, \tag{1.3.137}$$

so that on functions $\psi(\theta)$ the infinitesimal generator si realized as

$$J = \mathrm{i}\partial_\theta\,. \tag{1.3.138}$$

An irrep D_m acts on a space spanned by a function $\psi_m(\theta)$ characterized (see Eq. (**??**)) by

$$J\psi_m(\theta) = m\psi_m(\theta) \;\Leftrightarrow \mathrm{i}\partial_\theta\psi_m(\theta) = m\psi_m(\theta)\,, \quad (m \in \mathbb{Z})\,, \tag{1.3.139}$$

where we used Eq. (**??**). Eq. **??**, with the condition that $\psi_m(0) = 1$, gives immediately

$$\psi_m(\theta) = \mathrm{e}^{-im\theta}\,. \tag{1.3.140}$$

We may also consider the irreducible representations D_m as being obtained by the action of the rotation group on some invariant 1-dimensional subspace of the space of functions $\psi(r,\theta)$ on the two-dimensional plane. Here $\mathbf{x} = \binom{x_1}{x_2}$ and we introduced polar coordinates by $x^1 = r\cos\theta$ and $x^2 = r\sin\theta$. Since rotations $R(\theta)$ leave the radial distance invariant, we recover again a description in which these trasnformations act (for each fixed radius r) on an S_1 parametrized by an angle θ. We have

$$(\mathbf{1} + \mathrm{i}\, d\theta\, J)\, \psi\,(\mathbf{x}) = \psi\left(R^{-1}(d\theta)\mathbf{x}\right) = \psi\,(r, \theta - d\theta) = \psi(r,\theta) - d\theta \frac{d\psi(\theta)}{d\theta} \;\;, \tag{1.3.141}$$

so that[4]

$$J = \mathrm{i}\partial_\theta \;= -\mathrm{i}(x^1\partial_2 - x^2\partial_1)\,. \tag{1.3.142}$$

Thus, J is nothing else but the angular momentum operator for rotations around an axis orthogonal to the x^1, x^2 plane. The condition Eq. (1.3.131) translates then, using the explicit expression Eq. (1.3.142) of the infinitesimal generator of functions, into the differential equation

$$\mathrm{i}\,\partial_\theta\psi(r,\theta) = m\,\psi(r,\theta)\,, \tag{1.3.143}$$

with the general solution

$$\psi(r,\theta) = \varphi(r)\,\mathrm{e}^{-\mathrm{i}\,m\,\theta}\,. \tag{1.3.144}$$

A far as irreducible representations are concerned, we arrive, in exactly the same way as in the SO(2) case, to the conclusion that in any irreducible representation D_p we have

$$D_p(x) = \mathrm{e}^{\mathrm{i}\,x\,P} \tag{1.3.145}$$

with P being represented as an *Hermitean* operator if we *require* that the representation be *unitary*. We will therefore label by $|p\rangle$ the basis vector of an irreducible representation D_p where P has *real* eigenvalue p:

$$P|p\rangle = p|p\rangle \quad p \in \mathbb{R}\,. \tag{1.3.146}$$

Differently from rotations, no periodicity has to be imposed on x, so that no quantization arises on the representation label p.

1.3.2 The Haar measure

A crucial role in many properties of finite group representations is played by the *invariance*, Eq. (1.2.72), of the sum over group elements under left or right group multiplication. For Lie groups, the sum over group elements must be replaced by *integration* over the group manifold. It is extremely important to find an *integration measure* that is *invariant* under group multiplication. It turns out that this is possible; let us discuss some points of the construction of the invariant measure for a Lie group.

Let us use for the integration over a Lie group G of dimension d the notation

$$\int_G Dg \equiv \int_G d^d\alpha\,\mu(\alpha)\;, \tag{1.3.147}$$

where the σ are explicit coordinates on G, and $\mu(\alpha)$ a measure function.

[4] Notice that also the generator J of Eq. (**??**) acting matricially on $\binom{x_1}{x_2}$ can be expressed as a differential operator $\mathrm{i}(x^1\partial_2 - x^2\partial_1)$. Indeed

$$\mathrm{i}(x^1\partial_2 - x^2\partial_1)\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \mathrm{i}\begin{pmatrix} -x^2 \\ x^1 \end{pmatrix} = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\,.$$

The measure is *left-invariant* (and we denote it as $D_L g$), or *right-invariant* (let it be $D_R g$) when

$$\int_G D_L g\, f(g) = \int_G D_L g\, f(hg) \quad \Leftrightarrow \quad D_L g = D_L(hg) \ ;$$
$$\int_G D_R g\, f(g) = \int_G D_R g\, f(gh) \quad \Leftrightarrow \quad D_R g = D_R(hg) \ . \tag{1.3.148}$$

In terms of the measure density $\mu_{L,R}(\alpha)$, the above invariance requrement reads

$$\mu_L(\alpha) d^d\alpha = \mu_L(\varphi(\beta,\alpha)) d^d\varphi \ ,$$
$$\mu_R(\alpha) d^d\alpha = \mu_R(\varphi(\alpha,\beta)) d^d\varphi \ , \tag{1.3.149}$$

where $\{\alpha\}$ are the coordinates of the group element g, $\{\beta\}$ those of h and $\varphi : G \times G \to G$ is the bi-continuous, differentiable map which represents the group product, see sec. **??**. Thus $\varphi^\mu(\beta,\alpha)$ are the coordinates of hg, and $\varphi^\mu(\alpha,\beta)$ those of gh. Eq. (1.3.149) requires that

$$\frac{\mu_L(\alpha)}{\mu_L(\varphi(\beta,\alpha))} = \det\left(\frac{\partial\varphi(\beta,\alpha)}{\partial\alpha}\right) \ , \tag{1.3.150}$$

where on the r.h.s appears the *Jacobian* determinant of the transformation from α to φ. Mutatis mutandis, the same holds for the right-invariant measure:

$$\frac{\mu_R(\alpha)}{\mu_R(\varphi(\alpha,\beta))} = \det\left(\frac{\partial\varphi^\mu(\alpha,\beta)}{\partial\alpha^\nu}\right) \ . \tag{1.3.151}$$

A general solution of the conditions Eq. (1.3.150) or Eq. (1.3.151) can be obtained by expressing $\mu(\alpha)$ in terms of the particular constant value it assumes at the origin $\alpha = 0$, by setting

$$\mu_L(\alpha) = \mu_L(0)\left[\det\left(\frac{\partial\varphi(\alpha,\beta)}{\partial\beta}\right)\Big|_{\beta=0}\right]^{-1} \ ,$$
$$\mu_R(\alpha) = \mu_R(0)\left[\det\left(\frac{\partial\varphi(\beta,\alpha)}{\partial\beta}\right)\Big|_{\beta=0}\right]^{-1} \ . \tag{1.3.152}$$

The constants $\mu_{L,R}(0)$ can be adjusted so that the volume of the group G computed with the invariant metric, if it turns out to be finite, is normalized to 1.

1.3.2.1 Some examples

It is clear that the simplest solution to the condition Eq. (1.3.150) is obtained when the coordinates $\varphi^\mu(\beta,\alpha)$ are *linear* functions of the coordinates α. Such a situation is encountered, for instance, for the SO(2) group described in sec. **??**, and again in sec. 1.3.1; see Eq. (**??**):

$$\varphi(\theta_1,\theta_2) = \theta_1 + \theta_2 \ . \tag{1.3.153}$$

When $\varphi^\mu(\beta,\alpha)$ is linear in α, the Jacobian is trivial and a trivial constant measure $\mu(\alpha) = \text{const}$ is an invariant measure. In the SO(2) example, we obtain a measure both left- and right-invariant, by setting simply[5], for $g = \exp(\theta J)$,

$$Dg = \frac{d\theta}{2\pi} \ , \tag{1.3.154}$$

[5] The constant $1/2\pi$ is chosen so that the volume of the group computed with the invariant measure is normalized to $\int Dg = 1$.

so that, if $h = \exp(\theta' J)$,

$$D(hg) = \frac{d(\theta' + \theta)}{2\pi} = Dg \ . \tag{1.3.155}$$

If we realize isomorphically the rotation group as U(1), i.e., if we use the irreducible representation where $J = \mathrm{i}$, the group elements are simply given by the uni-modular numbers $g = \mathrm{e}^{\mathrm{i}\theta}$ and the product law is simply the usual product. We could use directly g as coordinate, in which case the invariant measure is

$$Dg = \frac{1}{2\pi\mathrm{i}} \frac{dg}{g} \ , \tag{1.3.156}$$

which agrees with Eq. (1.3.154) under the change of variable $g = \mathrm{e}^{\mathrm{i}\theta}$. Indeed,

$$D(hg) = \frac{1}{2\pi\mathrm{i}} \frac{d(hg)}{hg} = \frac{1}{2\pi\mathrm{i}} \frac{hdg}{hg} = \frac{1}{2\pi\mathrm{i}} \frac{dg}{g} = \frac{1}{2\pi\mathrm{i}} \frac{dg}{g} \ . \tag{1.3.157}$$

The U(1) group is *locally* isomorphic to the group $\mathrm{GL}(1,\mathbb{R})$, i.e. $\mathbb{R} \setminus \{0\}$ with the usual product. An invariant measure on $\mathrm{GL}(1,\mathbb{R})$ is given, following the same reasoning as in Eq. (1.3.157), by

$$Dg = \frac{dg}{g} \ , \tag{1.3.158}$$

which can be written simply as dx in terms of the exponential parametrization $g = \mathrm{e}^x$. We did not include any normalization factor in Eq. (1.3.158) since the invariant volume of the group is infinite (the group is non-compact).

The measure Eq. (1.3.158) generalizes to the matrix groups $\mathrm{GL}(n,\mathbb{R})$, parametrized by the entries g^i_j of the matrices $g \in \mathrm{GL}(n,\mathbb{R})$. Since the product law is realized as in Eq. (**??**), we have the Jacobian matrix for left translations

$$\frac{\partial (h^i{}_m g^m_j}{\partial g^p{}_q} = h^i{}_m \delta^m{}_p \delta^q{}_j = h^i{}_p \delta^q{}_j \tag{1.3.159}$$

and the Jacobian determinant

$$\det_{(ij)(pq)} \frac{\partial (h^i{}_m g^m_j}{\partial g^p{}_q} = \det_{(ij)(pq)} (h^i{}_p \delta^q{}_j) = \det_{ip} h \det_{jq} \mathbf{1} = \det h \ , \tag{1.3.160}$$

where we indicated explicitely the matrix indices over which the determinants are to be understood, where unclear. The same Jacobian would be found for right translations. The invariant measure is therefore given by

$$Dg = \frac{\prod_{i,j} dg^i{}_j}{\det g} \ , \tag{1.3.161}$$

so that indeed we have (for left translations, for instance)

$$D(hg) = \frac{\prod_{i,j} dhg^i{}_j}{\det(hg)} = \frac{\det h \prod_{i,j} dg^i{}_j}{\det h \det g} = Dg \ . \tag{1.3.162}$$

For the special linear groups $\mathrm{Sl}(n,\mathbb{R})$, the n^2 entries $g^i{}_j$ of the matrices give an over-parametrization, as the constraint $\det g = 1$ must hold (so that the actual dimension of the group is $n^2 - 1$). Nevertheless, in the coordinates g^i_j the invariant measure would simply the trivial one, $\prod_{i,j} dg^i{}_j$, since the determinant in Eq. (1.3.161) is one. We can therefore write the measure as

$$Dg = \delta(\det g - 1) \prod_{i,j} dg^i{}_j \ , \tag{1.3.163}$$

from which the actual expression in terms of $n^2 - 1$ parameters can be obtained by expressing, for instance, one of the entries in terms of the remaining ones, or by other convenient changes of variables.

1.3.2.2 Invariant measure for SU(2)*, first way.*

A reasoning similar to the one used above for SL$(n, \mathbb{R})$ can help to establish the invariant measure for the SU(2) group. Recall (see Eq. (**??**)) that the special unitary 2×2 matrices g can be parametrized in terms of 4 real parameters x^a $(a = 0, 1, 2, 3)$, as

$$g = x^0 \mathbf{1} + \mathrm{i} x^i \sigma_i \ , \tag{1.3.164}$$

with the constraint

$$\sum_a (x_a)^2 = 1 \tag{1.3.165}$$

corresponding to the requirement that $\det g = 1$. A group translation (e.g., a left one) $g \to hg$ gives a matrix parametrized by some new numbers x'_a, depending linearly on the x_a (similarly to Eq. (1.3.159)) and subject to the constraint that $\sum_a (x'_a)^2 = 1$. Since the norm of the vector x_a is preserved, the linear transformation must actually be an *orthogonal* transformation:

$$(x')^a = L^a{}_b x^b \ , \qquad \det L = 1 \ , \tag{1.3.166}$$

whose Jacobian determinant is 1. The invariant measure on the group manifold, which according to Eq. (1.3.165) is evidently the three-sphere S_3 of unit radius, can be simply written, in the coordinates x^a of the R^4 space in which S_3 is embedded, as

$$Dg = \delta(\sum_a (x_a)^2 - 1) \prod_a dx^a \ . \tag{1.3.167}$$

Namely, the sphere has the natural measure induced by its immersion in $\mathbb{R}^4$. By changing to suitable polar coordinates in $\mathbb{R}^4$, the constraint can be explicitely solved and the measure easily written down. In particular, let us consider the parametrization suggested by the exponentiation of the su(2) Lie algebra, Eq. (**??**), which we reopeat here for convenience:

$$g = \cos\frac{\psi}{2}\mathbf{1} + \mathrm{i}\sin\frac{\psi}{2}\hat{\alpha}^i\sigma_i \ . \tag{1.3.168}$$

where $\psi \in [0, 2\pi]$ and the versor $\hat{\alpha}$ individuates a point on S^2 and can be therefore parametrized by the usual polar angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ in $\mathbf{R}^3$:

$$\hat{\alpha}^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ . \tag{1.3.169}$$

Comparing with Eq. (1.3.164), we see that the exponential parametrization Eq. (1.3.168) corresponds to the following choice of polar coordinates:

$$x^0 = r\cos\frac{\psi}{2} \ ,$$

$$\begin{aligned}
x^1 &= r\sin\frac{\psi}{2}\sin\theta\cos\phi\ ,\\
x^2 &= r\sin\frac{\psi}{2}\sin\theta\sin\phi\ ,\\
x^3 &= r\sin\frac{\psi}{2}\cos\theta\ .
\end{aligned} \tag{1.3.170}$$

Of course, in these coordinates the sphere constraint Eq. (1.3.165) corresponds simply to $r = 1$. By straightforward computation (do it as an exercise) the Jacobian determinant of the above coordinate change is given by

$$\left|\frac{\partial(x^0,x^1,x^2,x^3)}{\partial(r,\psi,\theta,\phi)}\right| = r^3\frac{1}{2}\sin^2\frac{\psi}{2}\sin\theta\ . \tag{1.3.171}$$

The invariant measure for SU(2) in the parametrization Eq. (1.3.168) is thus given by

$$Dg = \frac{1}{2\pi^2}\sin^2\frac{\psi}{2}d\frac{\psi}{2}\ \sin\theta d\theta\, d\phi = \frac{1}{2\pi^2}\sin^2\frac{\psi}{2}d\frac{\psi}{2}\, d\Omega_2\ , \tag{1.3.172}$$

where $d\Omega_2$ is the usual volume element for the sphere S_2. The constant in front has been chosen so that the volume is normalized to 1[6]:

$$\int_{\mathrm{SU}(2)} Dg = \frac{1}{2\pi^2}\int_0^{2\pi}\sin^2\frac{\psi}{2}d\frac{\psi}{2}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi = 1\ . \tag{1.3.173}$$

1.3.2.3 A general expression in Lie-algebraic terms

We have discussed in sec. **??** the adjoint action of the elements of a Lie group G over its Lie algebra $\mathbb{G}$. Let's parametrize the elements of a Lie group G via the exponential map:

$$g(\alpha) = \mathrm{e}^{\alpha^a J_a}\ , \qquad (a = 1,\dots,\dim\mathbb{G})\ , \tag{1.3.174}$$

in terms of a basis of generators J_a. Let us consider the expression

$$g^{-1}(\alpha)\frac{\partial g(\alpha)}{\partial\alpha^a}\ . \tag{1.3.175}$$

By direct computation (do it as an exercise) one can show that

$$g^{-1}(\alpha)\frac{\partial g(\alpha)}{\partial\alpha^a} = \left(J_a + \frac{1}{2!}\left[J_a,\alpha\cdot J\right] + \frac{1}{3!}\left[\left[J_a,\alpha\cdot J\right],\alpha\cdot J\right] + \dots\right)\ , \tag{1.3.176}$$

i.e., it belongs to the Lie algebra $\mathbb{G}$. It can be therefore expanded on the basis the generators $\{J\}$, defining

$$g^{-1}(\alpha)\frac{\partial g(\alpha)}{\partial\alpha^a} \equiv \left[\mathcal{A}(\alpha)\right]_a^{\ b}(\alpha)\, J_b\ . \tag{1.3.177}$$

The expression

$$\mu(\alpha) = \det\mathcal{A}(\alpha)\ , \tag{1.3.178}$$

[6] In general, the volume of the unit S_n sphere embedded into R^{n+1} is given by $2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2}$. Prove this formula for exercise (Hint: write a gaussian integral in polar coordinates: $\int d^{n+1}x\mathrm{e}^{-x^2} = \int_0^\infty r^n dr d\Omega_n$, where $d\Omega_n$ is the volume element for S_n, and evaluate the two sides.

where A is the matrix of components $A_a{}^b$ defined in Eq. (1.3.177), has the right properties to represent a left-invariant measure on the group G. Let us first notice that this measure does not depend on the explicit coordinate choice. Changing variables to any other local coordinates $\{\beta\}$, we have

$$[\mathcal{A}(\beta)]_a{}^b J_b = g^{-1}\frac{\partial g}{\partial \beta^a} = g^{-1}\frac{\partial g}{\partial \alpha^c}\frac{\partial \alpha^c}{\partial \beta^b} = \frac{\partial \alpha^c}{\partial \beta^b}[\mathcal{A}(\alpha)]_c{}^b J_b \ , \tag{1.3.179}$$

i.e., we have, in matrix notation,

$$\mathcal{A}(\beta) = \frac{\partial \alpha}{\partial \beta}\mathcal{A}(\alpha) \ . \tag{1.3.180}$$

Defining $\mu(\beta) = \det \mathcal{A}(\beta)$ in accordance with Eq. (**??**), gives thus

$$\mu(\beta) = \det\left(\frac{\partial \alpha}{\partial \beta}\right)\mu(\alpha) \ , \tag{1.3.181}$$

so that the volume element is indeed invariant:

$$\mu(\beta)d^d\beta = \mu(\alpha)\det\left(\frac{\partial \alpha}{\partial \beta}\right)\det\left(\frac{\partial \beta}{\partial \alpha}\right)d^d\alpha = \mu(\alpha)d^d\alpha \ , \tag{1.3.182}$$

having taken into account the Jacobian determinant. Now, having chosen local coordinates α at the point g, and for a fixed $h \in G$, let us define local coordinates at the left-tranlated point hg by declaring

$$(hg)(\alpha) = h\,g(\alpha) \ . \tag{1.3.183}$$

With this choice of coordinates, we have

$$(hg)^{-1}\frac{\partial}{\partial \alpha^a}(hg) = g^{-1}h^{-1}h\frac{\partial g}{\partial \alpha^a} = g^{-1}\frac{\partial g}{\partial \alpha^a} \ . \tag{1.3.184}$$

It follows then from the definition Eq. (1.3.178) that

$$\mu_{hg}(\alpha) = \mu_g(\alpha) \ , \tag{1.3.185}$$

where we denoted explicitely the point to which the measure is referred. We have shown above that the measure $\mu_{hg}(\alpha)d^d\alpha$ does not depend explicit on a specific choice of coordinates, such as Eq. (1.3.183). Therefore we have shown that the measure $Dg = \mu_g(\alpha)d^d\alpha$ is left-invariant:

$$D(hg) = Dg \ . \tag{1.3.186}$$

Similarly, we can define a *right-invariant* measure by choosing

$$\tilde{\mu}(\alpha) = \det \tilde{\mathcal{A}}(\alpha) \ , \tag{1.3.187}$$

where the matrix $\mathcal{A}(\alpha)$ is defined by

$$\frac{\partial g(\alpha)}{\partial \alpha^a}g^{-1}(\alpha) \equiv [\tilde{\mathcal{A}}(\alpha)]_a{}^b(\alpha)\,J_b \ . \tag{1.3.188}$$

Indeed, one can show that $\frac{\partial g(\alpha)}{\partial \alpha^a}g^{-1}(\alpha)$ belongs to the Lie algebra, analogously to what we did in Eq. (1.3.176) for $g^{-1}(\alpha)\frac{\partial g(\alpha)}{\partial \alpha^a}$.

The expression of the invariant measures Eq. (1.3.178), Eq. (1.3.187) given in this section must agree with the expressions given in Eq. (1.3.152). Let us show this explicitely, as a sort of exercise focusing, for instance, on the left-invariant measure. Using the exponential parametrization of Eq. (1.3.174), the function $\varphi(\beta, \alpha)$ that describes the group product is implicitely defined by

$$e^{\beta\cdot J} e^{\alpha\cdot J} = e^{\varphi(\beta,\alpha)\cdot J} \ . \tag{1.3.189}$$

Notice that $\varphi(\beta, \alpha = 0) = \beta$. Let us consider the derivative w.r.t. α^a, evaluated at $\alpha = 0$, of this relation. From the r.h.s., we have

$$\left.\frac{\partial e^{\varphi\cdot J}}{\partial \alpha^a}\right|_{\alpha=0} = \left.\frac{\partial \varphi^c}{\partial \alpha^a}\right|_{\alpha=0} \left.\frac{\partial e^{\varphi\cdot J}}{\partial \varphi^c}\right|_{\alpha=0} \ . \tag{1.3.190}$$

Using Eq. (**??**), we rewrite this as follows:

$$\left.\frac{\partial e^{\varphi\cdot J}}{\partial \alpha^a}\right|_{\alpha=0} = \left.\frac{\partial \varphi^c}{\partial \alpha^a}\right|_{\alpha=0} \left.\left(e^{\varphi\cdot J}\, [\mathcal{A}(\varphi)]_c{}^b J_b\right)\right|_{\alpha=0} = \left.\frac{\partial \varphi^c}{\partial \alpha^a}\right|_{\alpha=0} e^{\beta\cdot J} [\mathcal{A}(\beta)]_c{}^b J_b \ . \tag{1.3.191}$$

From the l.h.s. of Eq. (1.3.189) we get instead

$$\left.\frac{\partial (e^{\beta\cdot J} e^{\alpha\cdot J})}{\partial \alpha^a}\right|_{\alpha=0} = e^{\beta\cdot J} \left.\frac{\partial (e^{\alpha\cdot J})}{\partial \alpha^a}\right|_{\alpha=0} = e^{\beta\cdot J} \left.\left(e^{\alpha\cdot J} J_a\right)\right|_{\alpha=0} \ , \tag{1.3.192}$$

where we have used the fact that $\partial e^{\alpha\cdot J}/\partial \alpha^a = J_a$, as it follows from Eq. (1.3.176), which is nothing else than the deinition of infinitesimal generators. Comparing Eq. (1.3.191) and Eq. (1.3.192) we see that

$$\left.\frac{\partial \varphi^c(\beta, \alpha)}{\partial \alpha^a}\right|_{\alpha=0} [\mathcal{A}(\beta)]_c{}^b = \delta_a{}^b, \tag{1.3.193}$$

so that, in matrix notation,

$$\left.\frac{\partial \varphi(\beta, \alpha)}{\partial \alpha}\right|_{\alpha=0} = [\mathcal{A}(\beta)]^{-1} \ . \tag{1.3.194}$$

Therefore, the expressions Eq. (1.3.178) and Eq. (1.3.152) of the invariant measure coincide:

$$\mu_L(\beta) = \det \mathcal{A}(\beta) = \left[\det \left.\left(\frac{\partial \varphi(\beta, \alpha)}{\partial \alpha}\right)\right|_{\alpha=0}\right]^{-1} \ . \tag{1.3.195}$$

Haar measure for compact Lie groups It is possible to proof that, for a *compact* Lie group, the left- and right-invariant measures defined above coincide (see [**?**] for the proof):

$$D_L g = D_R g \equiv Dg \ , \tag{1.3.196}$$

and this measure Dg, enjoying both left- and right-invariance, is called the*Haar measure* on the group. The volume of the group G computed with the Haar measure is finite:

$$\mathrm{Vol}(G) = \int_G Dg < \infty \ . \tag{1.3.197}$$

The integration over a compact Lie group with the Haar measure has essentially the same properties as the discrete sum over the elements of a finite group. Thus, many results about representation theory which we will derive for finite groups extend naturally to compact Lie groups upon replacing all sums over group elements with invariant integrations.

In particular, all representations of a compact Lie group are unitary and, if reducible, fully reducible.

1.3.2.4 Invariant measure for SU(2)*, second way.*

Let us apply the recipe given in the previous subsection to the group SU(2). Let us choose the generators

$$J_i = \frac{\mathrm{i}}{2}\sigma^i \ , \tag{1.3.198}$$

where σ^i are the Pauli matrices. A generic group element g can be parametrized as

$$g(\alpha) = \mathrm{e}^{\alpha^i J_i} = \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha^i\sigma_i} = \cos\frac{|\vec{\alpha}|}{2}\mathbf{1} + \mathrm{i}\cos\frac{|\vec{\alpha}|}{2}\hat{\alpha}^i\sigma_i \ , \tag{1.3.199}$$

see Eq. (**??**) in sec. **??**, where $\hat{\alpha}^i = \alpha^i/|\vec{\alpha}|$. We can therefore compute explicitely[7]

$$\frac{\partial g(\alpha)}{\partial\alpha^i} = -\frac{1}{2}\sin\frac{|\vec{\alpha}|}{2}\hat{\alpha}^i\mathbf{1} + \frac{\mathrm{i}}{2}\cos\frac{|\vec{\alpha}|}{2}\hat{\alpha}^i\hat{\alpha}^j\sigma_j + \mathrm{i}\sin\frac{|\vec{\alpha}|}{2}\frac{\delta^{ij}-\hat{\alpha}^i\hat{\alpha}^j}{|\vec{\alpha}|}\sigma_j \ , \tag{1.3.200}$$

finding, after some algebra,

$$g^{-1}(\alpha)\frac{\partial g(\alpha)}{\partial\alpha^i} = \left(\hat{\alpha}^i\hat{\alpha}^j + \frac{\sin|\vec{\alpha}|}{|\vec{\alpha}|}\left(\delta^{ij}-\hat{\alpha}^i\hat{\alpha}^j\right) - 2\frac{\sin^2\frac{|\vec{\alpha}|}{2}}{|\vec{\alpha}|}\epsilon^{ijk}\hat{\alpha}^k\right)\frac{\mathrm{i}}{2}\sigma_j \ . \tag{1.3.201}$$

We see that indeed $g^{-1}\frac{\partial g}{\partial\alpha^i}$ belongs to the Lie algebra, as it can be expanded on the generators $J_i = \frac{\mathrm{i}}{2}\sigma_i$. According to the definition Eq. (1.3.177), we have

$$\mathcal{A}_i{}^j = \frac{\alpha^i\alpha^j}{|\vec{\alpha}|^2} + \frac{\sin|\vec{\alpha}|}{|\vec{\alpha}|}\left(\delta^{ij} - \frac{\alpha^i\alpha^j}{|\vec{\alpha}|^2}\right) - 2\frac{\sin^2\frac{|\vec{\alpha}|}{2}}{|\vec{\alpha}|^2}\epsilon^{ijk}\alpha^k \ . \tag{1.3.202}$$

By direct computation, one finds

$$\det\mathcal{A} = 4\frac{\sin^2\frac{|\vec{\alpha}|}{2}}{|\vec{\alpha}|^2} \ . \tag{1.3.203}$$

We can therefore write the left-invariant measure on SU(2) in terms of the coordinates α^i as

$$Dg \propto \frac{\sin^2\frac{|\vec{\alpha}|}{2}}{|\vec{\alpha}|^2}d\alpha^1 d\alpha^2 d\alpha^3 \ . \tag{1.3.204}$$

Changing to polar coordinates in the α space, and calling then $\psi = |\vec{\alpha}|$ as in sec. 1.3.2.4 above, we have

$$Dg \propto \frac{\sin^2\frac{|\vec{\alpha}|}{2}}{|\vec{\alpha}|^2}|\vec{\alpha}|^2 d|\vec{\alpha}|d\Omega_2 = \sin^2\frac{\psi}{2}d\psi d\Omega \ , \tag{1.3.205}$$

in agreement with the measure we found in Eq. (1.3.172). ...

1.3.3 Orthogonality properties

The proof of the orthogonality properties of matrix elements of irreducible group representations established in sec. **??** extends to the case of Lie groups for which we can find an invariant measure. If we label by D_μ the irreducible representations of the Lie group G, we will have

$$\int Dg\,[D_\mu(g)]^i{}_j[D_\mu(g^{-1})]^k{}_l = \frac{1}{d_\mu}\delta_{\mu\nu}\,\delta^i_l\,\delta^k_j \,. \tag{1.3.206}$$

[7] Recall that $|\vec{\alpha}| = \sqrt{\sum_i(\alpha^i)^2}$, so that $\partial_i|\vec{\alpha}| = \alpha^i/|\vec{\alpha}| = \hat{\alpha}^i$ and $\partial_i\hat{\alpha}^j = \partial_i(\alpha^i/|\vec{\alpha}|) = \ldots = (\delta_i^j - \hat{\alpha}^i\hat{\alpha}^j)/|\vec{\alpha}|$.

Similarly, also the characters $\chi_\mu(g)$ in the irreducible representations are orthogonal with respect to the Haar measure:

$$\int Dg\chi^\mu(g)\chi^\nu(g)^* = \delta_{\mu\nu}\,. \tag{1.3.207}$$

1.3.4 Multi-valued representations

Consider a non-simply connected group $\hat{G}$, so that its fundamental group $\Pi_1(\hat{G})$ is a non-trivial group. We can can construct *multi-valued* representations of this group, as follows. If D is a representation of $\hat{G}$, we obtain a multi-valued representation D_Γ of $\hat{G}$ by associating to every $\hat{g} \in \hat{G}$ several images:

$$D_\Gamma \;:\; \hat{g} \in \hat{G} \mapsto D(\hat{g})\Gamma(\gamma)\,,\ \forall\gamma \in \Pi_1(\hat{G})\,, \tag{1.3.208}$$

where Γ is a representation of $\Pi_1(\hat{G})$. The idea is that one requires only that D_Γ is an homomorphism for small group elements, i.e. elements close to the identity. Consider now a loop $\gamma(t)$ based in an element $\hat{g} \in \hat{G}$. We assign continously to every element $\gamma(t)$ on the path a representative $D_\Gamma(\gamma(t))$. Given the continuity of the product law on $\hat{G}$, it follows that $D_\Gamma(\gamma(t))$ changes smoothly under continuous deformations of the path γ, so it can depend at most on the homotopy class of γ. If γ is trivial, we have $D_\Gamma(\gamma(1)) = D_\Gamma(\gamma(0)) = D_\Gamma(\hat{g})$. If γ is non-trivial, then we may have different images for $\hat{g} = \gamma(0) = \gamma(1)$:

$$D_\Gamma(\gamma(1)) = \Gamma(\gamma)D_\Gamma(\gamma(0)) \neq D_\Gamma(\gamma(0))\,. \tag{1.3.209}$$

It is clear that this behaviour is consistent only if it respects the product of loops, namely if it respects the group product in $\Pi_1(\hat{G})$. Thus we obtain Eq. (1.3.208).

As we discussed above, the non-simply-connected group $\hat{G}$ can in fact be seen as the factor group $G/\mathcal{D}$, where the discrete group $\mathcal{D}$ is isomorphic to $\Pi_1(\hat{G})$. The multi-valued representations D_Γ of $\hat{G}$, see Eq. (1.3.208), are ordinary representations of the cover group G. Indeed, consider a representation D of G. If in this representation the subgroup $\mathcal{D}$ is mapped to the identity, $D(\mathcal{D}) = D(e) = \mathbf{1}$, the representation D descends naturally to a representation of the factor group $\hat{G} = G/\mathcal{D}$. It corresponds in Eq. (1.3.208) to the case where Γ is the trivial representation of $\Pi_1(\hat{G})$. Otherwise, it induces a multi-valued representation of $\hat{G}$. Writing an element $g \in G$ as $g = \hat{g}\gamma$, with $\hat{g} \in \hat{G}$ and $\gamma \in \mathcal{D}$, we have

$$D(g) = D(\hat{g})D(\gamma)\,. \tag{1.3.210}$$

From the point of view of the group $\hat{G}$, D assigns to a given element $\hat{g} \in \hat{G}$ a number of different images:

$$D \;:\; \hat{\in}\hat{G} \mapsto D(\hat{g})D(\gamma)\,,\ \forall\gamma \in \mathcal{D}\,, \tag{1.3.211}$$

and we retrieve Eq. (1.3.208). The multi-valued representations still respect the $\hat{G}$ product, up to the ambiguity inherent to the definition Eq. (1.3.211): one has (check the steps for exercise)

$$D(\hat{g}_1)D(\gamma_1)\,D(\hat{g}_2)D(\gamma_2) = D(\hat{g}_1\hat{g}_2)D(\hat{g}_2^{-1}\gamma_1\hat{g}_2\,\gamma_2) \tag{1.3.212}$$

(recall that $\mathcal{D}$ is a normal subgroup so $\hat{g}_2^{-1}\gamma_1\hat{g}_2 \in \mathcal{D}$).

Example: multi-valued representations of SO(2). We have seen in Eq. (**??**) that the multiply-connected group SO(2), can be seen as the quotient of $(\mathbb{R}, +)$ by $\mathbb{Z}$. Indeed we have $\Pi_1(\mathrm{SO}(2)) = \mathbb{Z}$. In this framework, SO(2) is naturally parametrized by $a = \theta/2\pi \in [0, 1]$ (with periodic identification). Multi-valued representations of SO(2)

1.3.5 Tensor methods: irreducible representations of $\mathrm{GL}(n,\mathbb{R})$

...

1.3.6 Representations of the $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ groups

1.3.6.1 Finite-dimensional representations of the $\mathrm{su}(2)$ *algebra*

Let us use combine the generators $J_i = \frac{\mathrm{i}}{2}\sigma_i$ into the Cartan basis:

$$\begin{aligned} H = \mathrm{i}J_3 &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \\ J_\pm &= \mathrm{i}(J_1 \pm \mathrm{i}J_2) \ , \end{aligned} \tag{1.3.213}$$

so that J_+ and J_- are respectively the raising and lowering operators

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{1.3.214}$$

The commutation relations are

$$\begin{aligned} [H, J_\pm] &= \pm J_\pm \ , \\ [J_+, J_-] &= 2H \ . \end{aligned} \tag{1.3.215}$$

Let us notice that the quadratic Casimir operator reads

$$C = 2H^2 + J_+J_- + J_-J_+ \tag{1.3.216}$$

and can be rewritten, using the algebra Eq. (1.3.215), as either

$$C = 2H(H+1) + 2J_-J_+ \tag{1.3.217}$$

or

$$C = 2H(H-1) + 2J_+J_- \ . \tag{1.3.218}$$

In any irreducible representation of this algebra, the eigenvalue c of the Casimir is fixed. Within the representation, we can label the states by means of the eigenvalue of the H generator: we denote thus the states as

$$|c; m\rangle \ , \tag{1.3.219}$$

such that

$$H|c;m\rangle = m|c;m\rangle \ , \qquad C|c;m\rangle = c|c;m\rangle \ . \tag{1.3.220}$$

Let us suppose that these states are normalized to 1:

$$\langle c, m|c, m'\rangle = \delta_{m,m'} \ . \tag{1.3.221}$$

The raising and lowering operators do indeed raise and lower the H-eigenvalue:

$$HJ_\pm|c;m\rangle = ([H, J_\pm] + J_\pm H)\,|c;m\rangle = (\pm J_\pm + J_\pm m)\,|c;m\rangle = (m \pm 1)J_\pm|c;m\rangle \ , \tag{1.3.222}$$

so that

$$J_\pm|c;m\rangle \propto |c; m \pm 1\rangle \ . \tag{1.3.223}$$

The proportionality constant is fixed by the normalization requirement Eq. (1.3.221). The squared norm of $J_\pm|c;m\rangle$ is given by

$$\langle c;m|J_\mp J_\pm|c;m\rangle = \langle c;m|C/2 - H(H\pm 1)|c;m\rangle = (c/2 - m(m\pm 1)) \ , \tag{1.3.224}$$

where we used the relations (1.3.217),(1.3.218). Thus we have

$$J_\pm|c;m\rangle = \sqrt{c/2 - m(m\pm 1)}|c;m\pm 1\rangle \ . \tag{1.3.225}$$

Looking for a finite-dimensional representation, we have to assume that there exist a state (the *highest state*) $|c;j\rangle$ that is annihilated by the raising operator, otherwise we would have an infinity of states of increasing H-eigenvalue. We require thus that

$$J_+|c;j\rangle = 0 \tag{1.3.226}$$

for some appropriate value j. Using Eq. (1.3.217), this requirement fixes the Casimir eigenvalue in terms of j:

$$C|c;j\rangle = 2\left(H(H+1) + J_-J_+\right)|c;j\rangle = 2j(j+1)|c;j\rangle \Rightarrow \ c = 2j(j+1) \ . \tag{1.3.227}$$

At this point, we should label the states as $|2j(j+1);m\rangle$; for ease of use, let us simplify the notation as follows:

$$|2j(j+1);m\rangle \to |j;m\rangle \ . \tag{1.3.228}$$

The eigenvalues of H within the representation have also to be bounded from below, otherwise we would get an infinity of states. Thus, after having applied k times the lowering operator J_-, to the highest state $|j;j\rangle$ we must find a lowest state $|j;j-k\rangle$ such that

$$J_-|j;j-k\rangle = 0 \ . \tag{1.3.229}$$

The norm of the state $J_-|j;j-k\rangle$ which, according to Eq. (1.3.224) is given by

$$\langle j;j-k|J_+J_-|j;j-k\rangle = j(j+1) - (j-k)(j-k-1) = (2j-k)(k+1) \tag{1.3.230}$$

must therefore vanish, which requires

$$j = k/2 \ . \tag{1.3.231}$$

Thus, irreducible representations of su(2) are characterized by a *semi-integer spin* j. The irrep D_j of spin j, in which the Casimir takes the value $2j(j+1)$, has dimension

$$\dim D_j = 2j+1 \ . \tag{1.3.232}$$

Indeed, it is spanned by the $2j+1$ states of definite H-eigenvalue m

$$|j;m\rangle \ , \quad m = j, j-1, j-2, \ldots, -j+1, -j \ . \tag{1.3.233}$$

The matrix elements of the generators in the representation of spin j are given, according to the above eq.s (1.3.220) and (1.3.225), by

$$\begin{aligned}
\left[\mathcal{D}_j(H)\right]_{mm'} &= \langle j;m|H|j;m'\rangle = m'\langle j;m|j;m'\rangle = m'\,\delta_{m,m'} \ , \\
\left[\mathcal{D}(J_-)\right]_{mm'} &= \langle j;m|J_-|j;m'\rangle = N_{m'}\langle j;m|j;m'\rangle = N_{m'}\,\delta_{m,m'-1} \ , \\
\left[\mathcal{D}_j(J_+)\right]_{mm'} &= \langle j;m|J_+|j;m'\rangle = N_{m'+1}\langle j;m|j;m'\rangle = N_{m'+1}\,\delta_{m,m'+1} \ ,
\end{aligned} \tag{1.3.234}$$

where

$$N_m = \sqrt{j(j+1) - m(m-1)} = \sqrt{(j+m+1)(j-m)} \ . \tag{1.3.235}$$

For the basis J_i of generators, inverting the change Eq. (1.3.213) to the Cartan basis, we have

$$\begin{aligned}
[\mathcal{D}_j(J_3)]_{mm'} &= = -\mathrm{i}m'\,\delta_{m,m'} \ , \\
[\mathcal{D}(J_1)]_{mm'} &= -\frac{\mathrm{i}}{2}\left(N_{m'+1}\,\delta_{m,m'+1} + N_{m'}\,\delta_{m,m'-1}\right) \ , \\
[\mathcal{D}(J_2)]_{mm'} &= \frac{1}{2}\left(-N_{m'+1}\,\delta_{m,m'+1} + N_{m'}\,\delta_{m,m'-1}\right) \ .
\end{aligned} \tag{1.3.236}$$

As an example, let us check that for $j = 1/2$ we retrieve the fundamental bi-dimensional representation of su(2), in which the generators are given by $J_i = \frac{\mathrm{i}}{2}\sigma_i$. The casimir is $2j(j+1) = 3/2$. From Eq. (1.3.235), we find

$$\begin{aligned}
N_{\frac{1}{2}} &= \sqrt{\frac{3}{4} - \frac{1}{2}\left(\frac{1}{2} - 1\right)} = 1 \ , \\
N_{-\frac{1}{2}} &= 0 \ .
\end{aligned} \tag{1.3.237}$$

Notice that, according to Eq. (1.3.235), $N_{-j} = 0$ in the irrep of any spin j. Collecting in matrix notation the matrix elements (with $m = 1/2, -1/2$ corresponding to the first, second row, and $m' = 1/2, -1/2$ to the first, second column) given in Eq. (1.3.236), we find

$$\begin{aligned}
\mathcal{D}_{\frac{1}{2}}(J_3) &= -\mathrm{i}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = -\frac{\mathrm{i}}{2}\sigma_3 \ , \\
\mathcal{D}_{\frac{1}{2}}(J_1) &= -\frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{\mathrm{i}}{2}\sigma_1 \ , \\
\mathcal{D}_{\frac{1}{2}}(J_2) &= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\frac{\mathrm{i}}{2}\sigma_2 \ .
\end{aligned} \tag{1.3.238}$$

The next example is provided by the $j = 1$ representation, which is 3-dimensional. The Casimir is $2j(j+1) = 4$. The non-zero coefficients $N_m = \sqrt{2 - m(m-1)}$ are

$$N_1 = \sqrt{2} \ , \quad N_0 = \sqrt{2} \ . \tag{1.3.239}$$

The matrices cooresponding to the matrix elements Eq. (1.3.236), again with the convention that rows are labeled by decreasing m and coumns by decreasing m', are then the following:

$$\begin{aligned}
\mathcal{D}_1(J_3) &= -\mathrm{i}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \\
\mathcal{D}_1(J_1) &= -\frac{\mathrm{i}}{2}\begin{pmatrix} 0 & N_1 & 0 \\ N_1 & 0 & N_0 \\ 0 & N_0 & 0 \end{pmatrix} = -\frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \\
\mathcal{D}_1(J_2) &= \frac{1}{2}\begin{pmatrix} 0 & -N_1 & 0 \\ N_1 & 0 & -N_0 \\ 0 & N_0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ .
\end{aligned} \tag{1.3.240}$$

These matrices are equivalent to the usual so(3) generators L_i given in Eq.s (**??**,**??**). As we remarked after the afore-mentioned eq.s, these generate the *adjoint* representation of su(2), since $(L_i)_j{}^k = c_{ij}{}^k = \epsilon_{ijk}$, in accordance with the definition Eq. (**??**) of the adjoint generators.

The similarity transformation from the L_i to the $D_1(J_i)$ is the one that diagonalizes L_3 and reorders its eigenvalues $\pm\mathrm{i}, 0$ as in $D_1(J_3)$ above[8].

1.3.6.2 Finite-dimensional representations of the SU(2) *group*

As we discussed in general, the irreducible group representations are obtained by exponentiating the irreducible representations of the Lie algebra. The finite-dimensional SU(2) irrepses D_j are therefore labeled by a semi-integer spin j, have dimension $2j+1$ and are given by

$$D_j(g = \mathrm{e}^{\alpha^i J_i}) = \exp\left(\alpha^i \mathcal{D}_j(J_i)\right) \ , \tag{1.3.243}$$

where $\mathcal{D}_j(J_i)$ are the Lie algebra generators is the irrep of spin j described in the previous section 1.3.6.1, see Eq. (1.3.236).

The matrix exponential in the r.h.s. of Eq. (1.3.243) is defined as a power series; it is not immediate to resum explicitely this series and get explicit expressions, but some results can be achieved.

First of all, consider the $j = 1/2$ case, which is nothing else than the defining (or *fundamental* representation in which the Lie algebra elements are antihermitean, traceless matrices and the Lie group elements are unitary, unimodular matrices. In this case we already carried out several times the exponentiation (see, e.g., Eq. (1.3.168)): choosing $J_i = -\frac{\mathrm{i}}{2}\sigma_i$, we have

$$g(\alpha) \equiv \exp\left(\alpha^i J_i\right) = \cos|\alpha|\mathbf{1} - \mathrm{i}\sin|\alpha|\hat{\alpha}^i\sigma_i \ , \tag{1.3.244}$$

which we can write, passing to coordinates $\psi = |\alpha|$ and to polar coordinates (θ, ϕ) for the versor $\hat{\alpha}$, as in Eq. (1.3.169), as

$$g(\psi,\theta,\phi) = \begin{pmatrix} \cos\frac{\psi}{2} - \mathrm{i}\sin\frac{\psi}{2}\cos\theta & -\mathrm{i}\sin\frac{\psi}{2}\sin\theta\mathrm{e}^{-\mathrm{i}\phi} \\ -\mathrm{i}\sin\frac{\psi}{2}\sin\theta\mathrm{e}^{\mathrm{i}\phi} & \cos\frac{\psi}{2} + \mathrm{i}\sin\frac{\psi}{2}\cos\theta \end{pmatrix} \ . \tag{1.3.245}$$

The direct exponentiation of the $j = 1$ Lie algebra element $\alpha^i D_1(J_i)$ (corresponding,as we just saw, to the adjoint representation of su(2) and to the fundamental representation of so(3)), was carried out in sec. (**??**), eq.s (**??**-**??**).

Non-canonical parametrizations. Euler angles For higher j representations, the direct exponentiation of a generic Lie algebra element $\alpha^i D_j(J_i)$ becomes involved. It is convenient to make use of non-canonical parametrizations of the group elements (see [**?**], chap. 5, sec. VI for several

[8] Note however that in Eq. (**??**) we had chosen the L_i to satisfy the algebra $[L_i, L_j] = -\epsilon_{ijk}$, while in this section we chose $[J_i, J_j] = \epsilon_{ijk}$. The similarity can only be to a set of generators satifying exactly the same algebra, for instance $\mathcal{D}_1(J_1), -\mathcal{D}_1(J_2), \mathcal{D}_1(J_3)$, so we may take explicitely

$$T^{-1}L_{1,3}T = \mathcal{D}_1(J_{1,3}) \ , \qquad T^{-1}L_2T = -\mathcal{D}_1(J_2) \ , \tag{1.3.241}$$

where

$$T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{\mathrm{i}}{\sqrt{2}} & 0 & \frac{\mathrm{i}}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \ . \tag{1.3.242}$$

examples of such parametrizations) . For instance, a *generic* element of the group SU(2) can be parametrized as follows:

$$g(\alpha,\beta,\gamma) = \mathrm{e}^{\gamma J_3}\mathrm{e}^{\beta J_2}\mathrm{e}^{\alpha J_3}\ . \tag{1.3.246}$$

If J_i are geometrically interpreted as the so(3) generators[9], namely the generators of rotations in the 3-dimensional Euclidean space, the above parametrization corresponds to the parametrization of a generic rotation by means of *Euler angles*. More on this later.

Utilizing the Cmpbell-Hausdorff formulae, one finds indeed (check the first terms for exercise)

$$\mathrm{e}^{\gamma J_3}\mathrm{e}^{\beta J_2}\mathrm{e}^{\alpha J_3} = \mathrm{e}^{\left[\frac{1}{2}(\alpha-\gamma)+\beta+\ldots\right]J_1+(\beta+\ldots)J_2+(\alpha+\gamma+\ldots)J_3} = \mathrm{e}^{\alpha^i(\alpha,\beta,\gamma)J_i}\ . \tag{1.3.247}$$

The three independent functions $\alpha^i(\alpha\beta\gamma)$, connecting the canonical parametrization of a generic group element to its Euler angles parametrization, are in principle completely determined by the direct application of Cambell-Hausdorff relations, and depend thus only on the commutation relations of the generators. They are the same in whatever representation.

This being the case, to derive the explicit relation between the two parametrizations it is convenient to work in the fundamentale $j = 1/2$ representation, where the exponentials both in the canonical and the Euler-angles expression of a group element are easily worked out. In the canonical parametrization we obtained, using the coordinates ψ,θ,ϕ, Eq. (1.3.245). For the parametrization in Eq. (1.3.246) we have obviously

$$\mathrm{e}^{\alpha J_3} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha\sigma_3} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} \end{pmatrix}\ , \tag{1.3.248}$$

and similarly for $\mathrm{e}^{\gamma J_3}$. We also have (it is just a particular case of Eq. (1.3.244))

$$\mathrm{e}^{\beta J_2} = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\ . \tag{1.3.249}$$

Altogether we find (check it)

$$\mathrm{e}^{\gamma J_3}\mathrm{e}^{\beta J_2}\mathrm{e}^{\alpha J_3} = \begin{pmatrix} \cos\frac{\beta}{2}\mathrm{e}^{-\mathrm{i}\frac{\alpha+\gamma}{2}} & -\sin\frac{\beta}{2}\mathrm{e}^{-\mathrm{i}\frac{\alpha-\gamma}{2}} \\ \sin\frac{\beta}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha-\gamma}{2}} & \cos\frac{\beta}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha+\gamma}{2}} \end{pmatrix}\ . \tag{1.3.250}$$

Comparing with the real and imaginary parts of the matrix elements in Eq. (1.3.245), one obtains following relations:

$$\begin{aligned} \phi &= \frac{\pi-\alpha+\gamma}{2}\ , \\ \sin\frac{\psi}{2}\sin\theta &= \sin\frac{\beta}{2} \\ \cos\frac{\psi}{2} &= \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}\ , \\ \sin\frac{\psi}{2}\cos\theta &= \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} \end{aligned} \tag{1.3.251}$$

(the last three relations are not independent: the square of the second second is obtained by summeing the squares of the third and fourth ones). The analyitic relation between the two

[9] That happens, as explained above, in the representation of spin $j = 1$.

parametrizations can then be described, for instance, as follows:

$$\begin{aligned}
\phi &= \frac{\pi - \alpha + \gamma}{2} \ , \\
\cos\frac{\psi}{2} &= \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} \ , \\
\tan\theta &= \tan\frac{\beta}{2}\frac{1}{\sin\frac{\alpha+\gamma}{2}} \ .
\end{aligned} \tag{1.3.252}$$

Using the non-canonical parametrization Eq. (**??**), the matrix representative of a group element $g(\alpha,\beta,\gamma)$ in a representation of spin j is given by

$$D_j(\alpha,\beta,\gamma) = D_j(\mathrm{e}^{\gamma J_3})D_j(\mathrm{e}^{\beta J_2})D_j(\mathrm{e}^{\alpha J_3}) = \mathrm{e}^{\gamma \mathcal{D}_j(J_3)}\mathrm{e}^{\beta \mathcal{D}_j(J_2)}\mathrm{e}^{\alpha \mathcal{D}_j(J_3)} \ . \tag{1.3.253}$$

Using the fact that $J_3 = -\mathrm{i}H$ is represented diagonally, see Eq. (1.3.236), can write the matrix elements of $D_j(\alpha,\beta,\gamma)$ as

$$[D_j(\alpha,\beta,\gamma)]_{mm'} = \mathrm{e}^{-\mathrm{i}\gamma m} d(\beta)^j_{m,m'} \mathrm{e}^{-\mathrm{i}\alpha m'} \ , \tag{1.3.254}$$

where

$$d(\beta)^j_{m,m'} \equiv \langle j;m|\mathrm{e}^{\beta J_2}|j;m'\rangle \ . \tag{1.3.255}$$

The functions $d(\beta)^j_{mm'}$ can be determined by showing that they satisfy the following two differential recursion relations:

$$\begin{aligned}
\left[\sin\beta\frac{\partial}{\partial\beta} + (m - \cos\beta m')\right] d(\beta)^j_{m,m'} &= -\sin\beta N_{m'+1} d(\beta)^j_{m,m'+1} \ , \\
\left[\sin\beta\frac{\partial}{\partial\beta} + -(m - \cos\beta m')\right] d(\beta)^j_{m,m'} &= \sin\beta N_{m'} d(\beta)^j_{m,m'-1} \ .
\end{aligned} \tag{1.3.256}$$

These relations can bederived as follows. On the one hand, we have

$$\langle j;m|J_3\mathrm{e}^{\beta J_2}|j;m'\rangle = -\mathrm{i}m\, d(\beta)^j_{m,m'} \ . \tag{1.3.257}$$

On the other hand, since

$$J_3\mathrm{e}^{\beta J_2} = \mathrm{e}^{\beta J_2}\mathrm{e}^{-\beta J_2}J_3\mathrm{e}^{\beta J_2} = \mathrm{e}^{\beta J_2}\left(\cos\beta J_3 - \sin\beta J_1\right) \tag{1.3.258}$$

(which is easily verified by direct computation, and corresponds to the fact that the generators J_i behave as vectors under rotations), we have

$$\begin{aligned}
\langle j;m|J_3\mathrm{e}^{\beta J_2}|j;m'\rangle &= \langle j;m|\mathrm{e}^{\beta J_2}\left(\cos\beta J_3 - \sin\beta J_1\right)|j;m'\rangle \\
&= -\mathrm{i}m\cos\beta d(\beta)^j_{m,m'} + \mathrm{i}\sin\beta\langle j;m|\mathrm{e}^{\beta J_2}\left(\pm J_2 + J_\pm\right)|j;m'\rangle \\
&= -\mathrm{i}m\cos\beta d(\beta)^j_{m,m'} \pm \mathrm{i}\sin\beta\frac{\partial}{\partial\beta}d(\beta)^j_{m,m'} + \mathrm{i}\sin\beta\langle j;m|\mathrm{e}^{\beta J_2}J_\pm|j;m'\rangle \ .
\end{aligned} \tag{1.3.259}$$

In the second step, we reexpressed $J-1$ in terms of J_2 and J_+ or J_-, using Eq. (1.3.213). From Eq. (1.3.259), the relations Eq. (1.3.256) follow straightforwardly.

Out of two recursive relations in Eq. (1.3.256) it is possible to derive a second-order differential equation obeyed by the $d(\beta)^j_{m,m'}$ functions (see [?], sec. 8.5):

$$\left[\frac{1}{\sin\beta}\frac{d}{d\beta}\sin\beta\frac{d}{d\beta} - \frac{1}{\sin^2(\beta)}\left((m')^2 + m^2 - 2mm'\cos\beta\right) + j(j+1)\right] d(\beta)^j_{m,m'} = 0 \ . \qquad (1.3.260)$$

This allows to relate the $d(\beta)^j_{m,m'}$ to the Jacobi polynomials $P^{\alpha,\beta}_l(x)$, by comparing Eq. (1.3.260) with the hypergeometric differential equation satisfied by the latter, see e.g. [?], eq. 8.964. One gets (see [?], sec. 8.5)

$$d(\beta)^j_{m,m'} = \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}}\left(\cos\frac{\beta}{2}\right)^{m+m'}\left(\sin\frac{\beta}{2}\right)^{m-m'} P^{m-m',m+m'}_{j-m}(\cos\beta) \ . \qquad (1.3.261)$$

Specializing to the case $m' = 0$ (which implies that j has to be integer: let us denote it as l to remark this fact), we have

$$d(\beta)^l_{m,0} = \frac{\sqrt{(l+m)!(l-m!}}{l!}\frac{1}{2^m}(\sin\beta)^m P^{m,m}_{l-m}(\cos\beta) \ . \qquad (1.3.262)$$

Eq. 8.961.4 of [?] states that

$$P^{m,m}_{l-m}(x) = 2^m\frac{l!}{(l+m)!}\frac{d^m}{dx^m}P^{0,0}_l(x) \ , \qquad (1.3.263)$$

and moreover, see 8.962.2 of [?], $P^{0,0}_l(x)$ are just the Legendre polynomials: $P^{0,0}_l(x) = P_l(x)$. Thus Eq. (1.3.264) is rewritten as

$$d(\beta)^l_{m,0} = \sqrt{\frac{(l-m)!}{(l+m)!}}(\sin\beta)^m\frac{d^m}{d\cos\beta^m}P_l(\beta) = \sqrt{\frac{(l-m)!}{(l+m)!}}P^m_l(\beta) \ , \qquad (1.3.264)$$

where $P^m_l(\beta) \equiv (\sin\beta)^m\frac{d^m}{d\cos\beta^m}P_l(\beta)$ are the so-called *associated Legendre functions.*

Consider the matrix elements of the group representatives $[D^l(\alpha,\beta,\gamma)]^m{}_0$. Using Eq. (1.3.254) and Eq. (1.3.264) we can relate them to *spherical harmonic* functions Y^m_l:

$$\begin{aligned}[D^l(\alpha,\beta,\gamma)]^m{}_0 &= \mathrm{e}^{-im\alpha}d(\beta)^l_{m,0} = \sqrt{\frac{(l-m)!}{(l+m)!}}\mathrm{e}^{-im\alpha}P^m_l(\beta) = \sqrt{\frac{4\pi}{2l+1}}(-1)^m N_{lm}\mathrm{e}^{-im\alpha}P^m_l(\beta) \\ &= \sqrt{\frac{4\pi}{2l+1}}\,Y^m_l(\beta,\pi-\alpha) \ , \end{aligned} \qquad (1.3.265)$$

where $N_{lm} = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}$ is the usual normalization factor of the spherical harmonic functions

$$Y^m_l(\theta,\phi) \equiv N_{lm}\mathrm{e}^{im\phi}\,P^m_l(\cos\theta) \ . \qquad (1.3.266)$$

1.3.6.3 Irreducible representations of SU(2) *by tensor methods*

1.3.6.4 Finite-dimensional representations of the SO(3) *group*

As we discussed in sec.s **??** and **??**, the Lie algebra of three-dimensional rotation group $SO(3)$ is isomorphic to that of SU(2), and the two groups are thus *locally* isomorphic. However, globally they differ, and $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$.

...

Spinors The rotation group in three dimensions, SO(3), is non simply-connected. We explicitely showed indeed, in Sec. **??**, that $SO(3) = SU(2)/\mathbb{Z}_2$. The group SU(2) is simply connected: as a manifold it is S^3, so it is the covering group of SO(3), which as a manifold is $S^3/\mathbb{Z}_2$, and has $\Pi_1(S^3/\mathbb{Z}_2) = \mathbb{Z}_2$. Faithful representations of SU(2), for instance the fundamental 2×2 on which SU(2) acts naturally, constitute *doubly valued* representations of SO(3). The elements of the carries spaces of such doubly-valued representations, i.e., the objects transforming in such representations, are called *spinors*. The electron wave function has a spinorial nature.

1.3.7 Representations of Euclidean and Poincaré groups

Let us consider now the Euclidean groups $\mathrm{Eucl}_d = \mathrm{O}(d) Ⓢ \mathbb{R}^d$, namely the isometry group of d-dimensional flat space $\mathbb{R}^d$, as well as their generalization to spaces with Minkowskian signature, $\mathrm{O}(p,q) Ⓢ \mathbb{R}^{p+q}$, a particular case of which is the Poincaré group $\mathrm{O}(1,3) Ⓢ \mathbb{R}^4$.

These group are semi-direct products of (pseudo)-rotation groups with translation groups. The translation group form an *invariant* subgroup, so these groups are not simple; since the invariant translation subgroup is also *Abelian*, they are not semi-simple either.

These (pseudo)-Euclidean groups are obviously of great relevance in Physics. It is therefore important to understand their irreducible representations. Being Lie groups, it is clear that we can proceed by studying the representations of the associated Lie algebrae, and then exponentiate. However, we will here pursue first the possibility, offered by the semi-direct product structure of the (pseudo)-Euclidean groups, to build their irreducible representations starting from representations of "smaller groups" related to the factor group w.r.t. the invariant translation group, namely to the rotations. This goes under the name of *induced representation* method, as it generalizes the induced representations discussed in sec. 1.1.9.

We proceed by studying first the simplest example of $\mathrm{ISO}(2) = \mathrm{SO}(2) Ⓢ \mathbb{R}^2$ (for simplicity, we leave out the inversions, i.e. we consider the component of Eucl_2 connected to the identity). Then we consider $\mathrm{ISO}(3) = \mathrm{SO}(3) Ⓢ \mathbb{R}^3$, which displays already the general structure of all $\mathrm{ISO}(d)$ groups. Finally, we move to the restricted Poincaré group $\mathrm{ISO}(1,3) = \mathrm{SO}(1,3) Ⓢ \mathbb{R}^4$, which follows a pattern similar to that of Euclidean groups, but where the Minkowskian signature implies several important differences.

1.3.7.1 Representations of ISO(2)

Let us recall the Lie algebra of ISO(2). Using as much as possible the conventions and notations of sec.s **??** and 1.3.1. Let us denote by $\mathbf{P} = (P_1, P_2)$ the generators of $\mathbb{R}^2$, and by J the generator of SO(2). The Lie algebra is summarized in the commutation relations

$$\begin{aligned} [P_1, P_2] &= 0 \ , \\ [J, P_i] &= \mathrm{i}\epsilon_{ij} P_j \ . \end{aligned} \tag{1.3.267}$$

The first relation simply states that the translation subgroup is abelian, while the second implies that it is an invariant subgroup, and that the generators $\mathbf{P}$ transform under the rotation subgroup as a two-dimensional vector. Recalling that the finite group elements (translations and rotations) are given by

$$T(\mathbf{b}) = \mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{P}} \ , \quad R(\theta) = \mathrm{e}^{\mathrm{i}\theta J} \ , \tag{1.3.268}$$

we can infer from the infinitesimal commutations Eq. (1.3.267) the "adjoint" action Eq. (**??**) of the rotations on the translations. Indeed, we have (using the properties of the exponential map)

$$R(\theta) T(\mathbf{b}) R^{-1}(\theta) = \mathrm{e}^{\mathrm{i}\theta J} \mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{P}} \mathrm{e}^{-\mathrm{i}\theta J} = \exp\left(\mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta J} (\mathbf{b}\cdot\mathbf{P}) \mathrm{e}^{-\mathrm{i}\theta J}\right) \ . \tag{1.3.269}$$

Moreover, (using a matrix notation where ϵ is the matrix of components ϵ_{ij}, and satisfies thus $\epsilon^2 = -\mathbf{1}$)

$$\begin{aligned} e^{i\theta J}\mathbf{P}e^{-i\theta J} &= \mathbf{P} + i\theta\,[J,\mathbf{P}] + \frac{(i\theta)^2}{2!}\,[J,[J,\mathbf{P}]] + \frac{(i\theta)^3}{3!}\,[J,[J,[J,\mathbf{P}]]]\ldots \\ &= \mathbf{P} - \theta\epsilon\mathbf{P} + \frac{\theta^2}{2!}\epsilon^2\mathbf{P} - \frac{\theta^3}{3!}\epsilon^3\mathbf{P} + \ldots \\ &= \left((1 - \frac{\theta^2}{2!} + \ldots)\mathbf{1} - (\theta - \frac{\theta^3}{3!} + \ldots)\epsilon\right)\mathbf{P} = (\cos\theta\mathbf{1} - \sin\theta\epsilon))\,\mathbf{P} \\ &= \mathcal{R}^{-1}(\theta)\mathbf{P}\ . \end{aligned} \tag{1.3.270}$$

Inserting Eq. (**??**) into Eq. (1.3.269), and taking into account the orthogonality of he matrix $\mathcal{R}(\theta)$, we find

$$R(\theta)T(\mathbf{b})R^{-1}(\theta) = \exp\left(i\mathbf{b}\cdot\mathcal{R}^{-1}(\theta)\mathbf{P}\right) = \exp\left(i(\mathcal{R}(\theta)\mathbf{b}\cdot\mathbf{P}\right) = T(\mathcal{R}(\theta)\mathbf{b})\ , \tag{1.3.271}$$

in accordance with Eq. (**??**).

The ISO(2) group admits a quadratic Casimir, given by

$$P^2 = \mathbf{P}\cdot\mathbf{P} = P_1^2 + P_2^2\ . \tag{1.3.272}$$

This is a Casimir operator because, beside commuting, obviously, with the Abelian generators P_i, it commutes also with the rotation generator J (check it starting from Eq. (1.3.267)) since rotation preserve the norm of vectors.

As we discussed in sec. 1.1.9, any representation[10] D_m of the factor group $\mathrm{SO}(2) = ISO(2)/\mathbb{R}^2$ induces a representation of the full ISO(2) group, by declaring that, for a generic element $T(\mathbf{b})R(\theta)$ we have

$$D_m(T(\mathbf{b})R(\theta)) = D_m(R(\theta))\ , \tag{1.3.273}$$

i.e., we represent trivially the translations.

What happens if we start with a non-trivial representation of the translation group? Recall that the group $\mathbb{R}^2$ is abelian, its irrepses are uni-dimensional and are labeled by the eigenvalues $\mathbf{p}$ of the generators $\mathbf{P}$ on the state vector[11]:

$$\mathbf{P}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle\ . \tag{1.3.275}$$

As the momentum $\mathbf{p}$ is not invariant under rotations, it cannot be constant for all the states in a representation of ISO(2). Its norm $p^2 = \mathbf{p}\cdot\mathbf{p}$ however, being the eigenvalue of the Casimir operator P^2, will be the same for all states. We fix therefore $p^2 \neq 0$ and start from a *reference*, standard momentum $\mathbf{p}_0$, for instance $\mathbf{p}_0 = (p, 0)$ and denote the corresponding state vector as $|p^2;\mathbf{p}_0\rangle$.

[10] Let us recall that the irreducible representations D_m of SO(2) are uni-dimensional, and we have $D_m(\theta) = \exp(im\theta)$, with $m \in \mathbb{Z}$ if the representation must be single-valued. On the state vector $|m\rangle$corresponding to the representation D_m the generator J satisfies $J|m\rangle = m|m\rangle$.

[11] Representing the generators $\mathbf{P}$ as the differential operator $-i\frac{\partial}{\partial\mathbf{x}}$ on $\mathbb{R}^2$, the state vector of this representation is identified with the plane wave of momentum $\mathbf{p}$:

$$|\mathbf{p}\rangle \leftrightarrow e^{i\mathbf{p}\cdot\mathbf{x}}\ . \tag{1.3.274}$$

Previously, we considered the particular situation $p^2 = 0$, in which case $\mathbf{p}_0 = \mathbf{0}$ *is* invariant under the entire SO(2). Then, every representation of SO(2) induces a representation of the full group.

For a non-zero p^2, we must begin by investigating whether a subgroup of rotations leaves $\mathbf{p}_0$ invariant. Such a subgroup would be called the *little group* of $\mathbf{p}_0$ and would play the same role that SO(2) plays in the $\mathbf{p}_0 = \mathbf{0}$ case: every irrep of the little group would induce a representation of the full group.

In this respect, ISO(2) is peculiar: for $\mathbf{p}_0 \neq \mathbf{0}$ the little group is trivial, as no two-dimensional rotation leaves the vector $\mathbf{p}_0$ invariant; this is not the case for higher-dimensional Euclidean groups. Since the little group is trivial, it has only the trivial representation, and we can forget about it.

We can instead start acting on $|p^2;\mathbf{p}_0\rangle$ with the rotations: the set of states

$$|p^2;\mathbf{p}\rangle = R(\theta)|p^2;\mathbf{p}_0\rangle \tag{1.3.276}$$

span by construction an irreducible invariant space for the group ISO(2), namely form the basis for an irreducible representation. Notice that the momentum $\mathbf{p}$ of the rotated state $R(\theta)|p^2;\mathbf{p}_0\rangle$ is just the rotated momentum of components $p^i = (\mathcal{R}(\theta))^i{}_j p^j$, justifying the notation in Eq. (1.3.276). Indeed, we have

$$\mathbf{P}R(\theta)|p^2;\mathbf{p}_0\rangle = R(\theta)R^{-1}(\theta)\mathbf{P}R(\theta)|p^2;\mathbf{p}_0\rangle = R(\theta)\mathcal{R}(\theta)\mathbf{P}|p^2;\mathbf{p}_0\rangle = \mathbf{p}R(\theta)|p^2;\mathbf{p}_0\rangle \ . \tag{1.3.277}$$

The representation of the group elenments on the states $|p^2;\mathbf{p}\rangle$ is simply described as follows. For translations,

$$T(\mathbf{b})\mathbf{P}|p^2;\mathbf{p}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{p}}|p^2;\mathbf{p}\rangle \ . \tag{1.3.278}$$

As far as rotations are concerned, we have

$$R(\phi)|p^2;\mathbf{p}\rangle = |p^2;\tilde{\mathbf{p}}\rangle \ , \tag{1.3.279}$$

where $\tilde{\mathbf{p}}$ is the rotated momentum

$$\tilde{p}^i = (\mathcal{R}(\phi))^i{}_j p^j \ . \tag{1.3.280}$$

1.3.7.2 Representations of ISO(3)*: helicity and the little group.*

In the case of $\mathrm{ISO}(3) = \mathrm{SO}(3)Ⓢ\mathbb{R}^3$, the Lie algebra reads

$$\begin{aligned} [P_i, P_j] &= 0 \ , \\ [J_i, P_j] &= \mathrm{i}\epsilon_{ijk}P_k \ , \\ [J_i, J_j] &= \mathrm{i}\epsilon_{ijk}J_k \ . \end{aligned} \tag{1.3.281}$$

The first line states that the translations are commutative, the second line that $\mathbf{P}$ behaves as a tri-vector under the rotations, an the third line is the well-known so(3) rotation algebra. There are two quadratic Casimir operators, namely

$$P^2 = \mathbf{P}\cdot\mathbf{P} \ , \qquad \mathbf{J}\cdot\mathbf{P} \ . \tag{1.3.282}$$

The operator P^2 commutes with all generators exactly for the same reasons as in the two-dimensonal case (see the discussion after Eq. (1.3.272) above). The operator $\mathbf{J}\cdot\mathbf{P}$ is clearly invariant under rotations (it is a scalar) and commutes with $\mathbf{P}$ because

$$\left[J_iP^i, P^j\right] = \left[J_i, P^j\right]P^i = \mathrm{i}\epsilon_{ijk}P^kP^i = 0 \ . \tag{1.3.283}$$

To find the irreducible representations of this group, we start again from a representation of the translation group, labeled by a tri-momentum. Between all the momenta corresponding to the same eigenvalue p^2 of the Casimir operator $\mathbf{P} \cdot \mathbf{P}$, let us select a standard momentum

$$\mathbf{p}_0 = (0, 0, p) \ . \tag{1.3.284}$$

If $p^2 = 0$ (i.e., translations are trivially represented), any representation D_j of SO(3) induces a representation of ISO(3), as discussed in sec. 1.1.9: given a generic element of ISO(3) written as the product of a translation and a rotation $T(\mathbf{b})R(\{\alpha\})$,where $\{\alpha\}$ are three parameters specifying the rotation[12], we set

$$D_j\left(\mathbf{b}, \{\alpha\}\right) = D_j\left(\{\alpha\}\right) \tag{1.3.285}$$

and we obtain an irreducible representation.

If $p^2 \neq 0$, the standard momentum $\mathbf{p}_0$, whose only non-zero component is $p_3 = p$, is left invariant by the rotations around the 3-rd axis, i.e. those generated by J_3, as it is clear from the algebra Eq. (1.3.281) and from geometric intuition. These rotation form an SO(2) subgroup, which constitutes the *little group* of $\mathbf{p}_0$. It is clear that the subspace corresponding to $\mathbf{p}_0$ is in principle degenerate: we still have to specify the value of J_3, i.e. we have to specify a representation of the little group. This representation will induce a representation of the full group.

We select therefore an irreducible representation of the little group SO(2) labeled by $m \in \mathbf{Z}$. The state vector of this representation can be labeled in terms of the eigenvalues $\{p^2, mp; \mathbf{p}_0\}$ of the Casimirs and of the maximal commuting set of operators given by $\mathbf{P}$, and satisfies

$$\begin{aligned} \mathbf{P}|p^2, m; \mathbf{p}_0\rangle &= \mathbf{p}_0|p^2, m; \mathbf{p}_0\rangle \ , \\ J_3|p^2, m; \mathbf{p}_0\rangle &= m|p^2, m; \mathbf{p}_0\rangle \ . \end{aligned} \tag{1.3.286}$$

The value of the two Casimirs on this state is indeed

$$\begin{aligned} P^2|p^2, m; \mathbf{p}_0\rangle &= p^2|p^2, m; \mathbf{p}_0\rangle \ , \\ \mathbf{J} \cdot \mathbf{P}|p^2, m; \mathbf{p}_0\rangle &= mp|p^2, m; \mathbf{p}_0\rangle \ . \end{aligned} \tag{1.3.287}$$

The proportionality constant $m = (mp)/p$ between the eigenvalues of the two Casimir operators is called *helicity*, and is the same for every state in the irreducible representation of ISO(3) we are constructing.

By acting on the reference state $|p^2, m; \mathbf{p}_0\rangle$ with rotations *outside* the little group we construct all the states $|p^2, m; \mathbf{p}\rangle$ spanning an irreducible invariant subspace for the full ISO(3) group:

$$|p^2, m; \mathbf{p}\rangle \equiv R(\alpha^1, \alpha^2, 0)|p^2, m; \mathbf{p}_0\rangle \ , \tag{1.3.288}$$

where $p^i = (\mathcal{R}(\alpha^1, \alpha^2, 0))^i{}_j p_0^j$.

How are the elements of ISO(3) represented on these states? The translation generators are obvious:

$$T(\mathbf{b})|p^2, m; \mathbf{p}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{p}}|p^2, m; \mathbf{p}\rangle \ . \tag{1.3.289}$$

For rotations we have instead

$$R(\beta^1, \beta^2, \beta^3)|p^2, m; \mathbf{p}\rangle = \mathrm{e}^{\mathrm{i}m\tilde{\beta}^3}|p^2, m; \tilde{\mathbf{p}}\rangle \ , \tag{1.3.290}$$

[12] We use the "exponential" parametrization as in sec. **??**, so that $R(\alpha^1, \alpha^2, \alpha^3)$ is a rotation of an angle $|\alpha|$ around an axis individuated by the versor $\hat{\alpha}$.

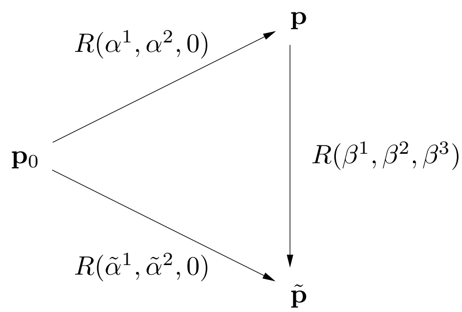

Figure 1.1. The transformation $R(\tilde{\alpha}^1, \tilde{\alpha}^2, 0)^{-1} R(\beta^1, \beta^2, \beta^3) R(\alpha^1, \alpha^2, 0)$ is in the litle group of $\mathbf{p}_0$.

where $\tilde{\mathbf{p}} = \mathcal{R}(\beta^1, \beta^2, \beta^3)\mathbf{p}$ is the rotated momentum, and the parameter $\tilde{\beta}^3$ is determined as follows. Both $\mathbf{p}$ and $\tilde{\mathbf{p}}$ can be obtained from the standard momentum $\mathbf{p}_0$ by rotations outside the little group:

$$\begin{aligned} \mathbf{p} &= \mathcal{R}(\alpha^1, \alpha^2, 0)\mathbf{p}_0 \ , \\ \tilde{\mathbf{p}} &= \mathcal{R}(\tilde{\alpha}^1, \tilde{\alpha}^2, 0)\mathbf{p}_0 \ . \end{aligned} \qquad (1.3.291)$$

It is then clear (see Fig. 1.1) that the product $R(\tilde{\alpha}^1, \tilde{\alpha}^2, 0)^{-1} R(\beta^1, \beta^2, \beta^3) R(\alpha^1, \alpha^2, 0)$ is in the little group, as it leaves $\mathbf{p}_0$ invariant, so we have

$$R(\tilde{\alpha}^1, \tilde{\alpha}^2, 0)^{-1} R(\beta^1, \beta^2, \beta^3) R(\alpha^1, \alpha^2, 0) = R(0, 0, \tilde{\beta}^3) \qquad (1.3.292)$$

for a parameter $\tilde{\beta}^3$ which is in fact determined from Eq. (1.3.292) itself.

1.3.7.3 Representations of ISO$(1,3)$

The Lie algebra of ISO$(1,3)$ is described by

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [J_{\mu\nu}, P_\rho] &= \mathrm{i}((\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \ , \\ [J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} \ . \end{aligned} \qquad (1.3.293)$$

The second line states that P_μ behaves as a four-vector, the third is the so$(1,3)$ Lie algebra of Eq. (**??**). A Casimir operator is clearly provided by

$$P^2 = \eta^{\mu\nu} P_\mu P_\nu = -P_0^2 + P_1^2 + P_2^2 + P_3^2 \ . \qquad (1.3.294)$$

A second Casimir operator can be easily constructed if we can form with the generators a second four-vector W^μ that commutes with P^μ: in this case,

$$W^2 = \eta^{\mu\nu} W_\mu W_\nu \qquad (1.3.295)$$

is certainly a Casimir operator, as it is invariant under the (pseudo)-rotations and commutes with the translations. Such a vector indeed exists, and is called the *Pauli-Lubanski vector*. It is given by

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \ . \qquad (1.3.296)$$

This vector is obviously orthogonal to the momentum generator:

$$W^{\mu}P_{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_{\sigma}P_{\mu} = 0 \tag{1.3.297}$$

because of the antisymmetry of the ϵ-symbol. For the same reason we find, using the algebra Eq. (1.3.293), that W^{μ} is translationally invariant:

$$[P_{\tau}, W^{\mu}] = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[P_{\tau}, J_{\nu\rho}]P_{\sigma} = \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}(\eta_{\rho\tau}P_{\nu} - \eta_{\nu\tau}P_{\rho})P_{\sigma} = 0 \ . \tag{1.3.298}$$

That W^{μ} behaves as a four-vector is clear by its definition Eq. (1.3.296) in terms of contractions of vectors and tensors; in any case, one can check as an exercise, using the Lie algebra Eq. (1.3.293), that

$$[J_{\mu\nu}, W_{\rho}] = \mathrm{i}\,(\eta_{\mu\rho}W_{\nu} - \eta_{\nu\rho}W_{\mu}) \ . \tag{1.3.299}$$

Finally, one can prove from the definition Eq. (1.3.296) (do it for exercise) that

$$[W^{\mu}, W^{\nu}] = \mathrm{i}\epsilon^{\mu\nu\rho\sigma}W_{\rho}P_{\sigma} \ . \tag{1.3.300}$$

We will now discuss the unitary irreducible representations of $\mathrm{ISO}(1,3)$, utilizing, as we did for the Euclidean groups, the method of induced representations. In doing so, it will become clear that the basis states of irreducible representations of the Poincaré group are physically interpreted as the quantum states of elementary particles.

We consider start from representations of the translation subgroup, i.e. from states of definite quadri-momentum. Differently from the Euclidean case, the Casimir operator P^2 is not semipositive definite, so we have more case to consider, according to the norm p^2 of the momentum:

(i) $p^2 = 0$ which implies $\mathbf{p} = \mathbf{0}$, i.e. *null momentum*;
(ii) $p^2 < 0$, in which case one talks of *time-like* momentum;
(iii) $p^2 = 0$, but $\mathbf{p} \neq \mathbf{0}$, in which case one talks of *light-like* momentum;
(iv) $p^2 > 0$, i.e. *space-like* momentum.

Let us now consider the various cases.

Null momentum In this case the translation subgroup is trivialy represented and, as described in sec. 1.1.9, every unitary irreducible $D(\Lambda)$ representation of the factor group $\mathrm{SO}(1,3)$ induces an unitary irreducible representation of $\mathrm{ISO}(1,3)$ in which every element $T\Lambda$ (T being a tanslation, Λ a pseudo-rotation) is represented by

$$D(T\Lambda) = D(\Lambda) \ . \tag{1.3.301}$$

Recall that the unitary irreducible representations of the *non-compact* factor group $\mathrm{SO}(1,3)$ are infinite-dimensional. The case of null momentum is not particularly relevant in Physics.

Time-like momentum In the case $p^2 = -M^2 < 0$, we can choose a reference momentum of the form

$$\hat{p}^{\mu} = (M, \mathbf{0}) \ . \tag{1.3.302}$$

Physically, this corresponds to a state at rest with rest energy, i.e. *mass* M.

On a state correponding to the reference momentum $\hat{p}^\mu$, the Pauli-Lubanski vector W^μ takes the following expression (we use the convention that $i, j, \ldots = 1, 2, 3$, and that $\epsilon^{0123} = \epsilon^{123}$):

$$\begin{aligned} W^0 &= \frac{1}{2}\epsilon^{0ijk} J_{ij}\hat{p}_k = 0 \ , \\ W^i &= -\frac{1}{2}\left(2\epsilon^{i0jk} J_{0i}\hat{p}_k + \epsilon^{ijk0} J_{jk}\hat{p}_0\right) = -\frac{1}{2}\epsilon^{ijk} J_{jk} M = -M J^i \ , \end{aligned} \tag{1.3.303}$$

where we introduced the usual notation $J^i = \frac{1}{2}\epsilon^{ijk} J_{jk}$, see Eq. (**??**), for the generators of the SO(3) subgroup of spatial rotations. The Casimir operator W^2 takes thus the value

$$W^2 = M^2 J^2 \ , \tag{1.3.304}$$

where $J^2 = J^i J_i$ is the Casimir of the SO(3) subgroup.

The SO(3) subgroup generated by J^i, on the other hand, is precisely the little group, i.e. the subgroup of the factor group $\mathrm{SO}(1,3)$ that leaves p^μ invariant. The space individuated by $\hat{p}^\mu$ is in principle degenerated, and organized in representations of the little group. Every unitary irreducible representation D_j of SO(3) (where $j \in \mathbb{Z} + \frac{1}{2}$ is the spin) induces then a representation of $\mathrm{ISO}(1,3)$. We start from the the basis states

$$| -M^2, j; \hat{p}^\mu, m\rangle \tag{1.3.305}$$

individuated, beside the standard momentum $\hat{p}^\mu = (M, \mathbf{0})$ corresponding to the casimir $p^2 = -M^2$, by the eigenvalue $j(j+1)$ of the SO(3) Casimir J^2 and by the 3-rd component m of the spin. Notice that, as it follows from Eq. (1.3.304), $j(j+1)$ is also ($1/M^2$ times) the value of the Casimir operator W^2, and thus is constant within the representation of $\mathrm{ISO}(1,3)$ we are constructing. For shortness, since they are constant in the representation we're seeking to build, we will drop the labels $-M^2$ and j from the states of Eq. (1.3.305) in the sequel.

Summarizing, we have

$$\begin{aligned} P^2|\hat{p}^\mu, m\rangle &= -M^2|\hat{p}^\mu, m\rangle \ , \\ W^2|\hat{p}^\mu, m\rangle &= M^2 J^2|\hat{p}^\mu, m\rangle = M^2 j(j+1)|\hat{p}^\mu, m\rangle \ , \\ P^\mu|\hat{p}^\mu, m\rangle &= \hat{p}^\mu|\hat{p}^\mu, m\rangle = (M, \mathbf{0})|\hat{p}^\mu, m\rangle \ , \\ J_3|\hat{p}^\mu, m\rangle &= m|\hat{p}^\mu, m\rangle \ , \qquad (m = -j, \ldots, j) \ . \end{aligned} \tag{1.3.306}$$

Acting on these states with transformations of $\mathrm{SO}(1,3)$ *outside* the little group SO(3) we construct the full irreducible invariant subspace corresponding to the representation induced by the representation of spin j of the little group: we set

$$|p^\mu, m\rangle = K(p^\mu)|\hat{p}^\mu, m\rangle \ , \tag{1.3.307}$$

where the momentum p^μ is a *generic* momentum with norm $p^2 = -M^2$, obtained by the action of the Lorentz matrix $(\mathcal{K}(p^\mu))^\mu{}_\nu$ on the reference momentum $\hat{p}^\mu$, namely $p^\mu = (\mathcal{K}(p^\mu))^\mu{}_\nu \hat{p}^\nu$. The transformation $K(p^\mu) = K(\nu, \alpha^1, \alpha^2)$ depends in fact from 3 parameters (6 are the parameters of $\mathrm{SO}(1,3)$ but 3 pertain to the little group) and is obtained by first boosting by a parameter ν in, say, the 3-d spatial direction, and then rotating the resulting 3-momentum in a generic spatial direction by a rotation around an axis in the $1-2$ plane:

$$\hat{p}^\mu = (M, \mathbf{0}) \xrightarrow{L_3(\nu)} (M\cosh\nu, 0, 0, M\sinh\nu) \xrightarrow{R(\alpha^1,\alpha^2,0)} p^\mu = (M\cosh\nu, \mathbf{p}) \ , \tag{1.3.308}$$

with $p^i = M \sinh\nu\,(\mathcal{R}(\alpha^1,\alpha^2,0))^i{}_3$.

On the states $|p^\mu, m\rangle$ of our representation, the translations are obviously represented simply by

$$T(\mathbf{b})|p^\mu, m\rangle = \mathrm{e}^{\mathrm{i}b^\mu p_\mu}|p^\mu, m\rangle\ . \tag{1.3.309}$$

For a generic Lorentz transformations Λ, we have

$$\Lambda|p^\mu, m\rangle = [D_j\,(R(\Lambda,p))]^m{}_{m'}\,|\tilde{p}^\mu, m\rangle\ , \tag{1.3.310}$$

where

$$\tilde{p}^\mu = \Lambda^\mu{}_\nu p^\nu \tag{1.3.311}$$

and the $SO(3)$ rotation $R(\Lambda,p)$ is the little group element determined, similarly to the case of ISO(3) described in Fig. 1.1, by

$$R(\Lambda,p) = K^{-1}(\tilde{p})\Lambda K(p)\ , \tag{1.3.312}$$

where $K(p)$, defined in Eq.s 1.3.307-1.3.308, maps the rest frame momentum $\hat{p}^\mu = (M,\mathbf{0})$ to the momentum p^μ. Notice that the state label m in $|p^\mu, m\rangle$ can be interpreted as the eigenvalue of the operator $\mathbf{J}\cdot\mathbf{P}/|mathbfp$, which is the rotation leaving invariant $\mathbf{p}$; in the intermediate situation of Eq. (1.3.308) in which $\mathbf{p}$ is along the 3-rd axis, it is the eigenvalue of J_3.

Physically, an irreducible unitary representation of the ISO(1,3) Poincaré group, characterized by Casimirs $P^2 = -M^2$ and $W^2 = M^2 j(j+1)$, corresponds to a *particle of rest mass M and spin j*.

What happens if, instead of the group ISO(1,3) we consider the full Poincaré group $\mathrm{O}(1,3)Ⓢ T_4$? In this case, one must take into account the discrete transformations contained in O(1,3) as opposed to SO(1,3), in particular the *space inversion* $S = \mathrm{diag}(1,-1-1-1)$. We do not discuss the details here (see for instance [**?**], Chapter 11), but the space invariance I_s changes sign to the spatial momentum $\mathbf{p}$ and, since $S^2 = \mathbf{1}$, it may have eigenvalues $\eta_s = \pm 1$.

The action of I_s on the states $|p^\mu, m\rangle = |(p_0,\mathbf{p}), m\rangle$ of a representation of spin j of ISO(1,3), see Eq.s **??**-1.3.307, turns out to be[13]

$$I_s|(p_0,\mathbf{p}), m\rangle = \eta_s \mathrm{e}^{\mp\mathrm{i}\pi j}|(p_0,-\mathbf{p}), -m\rangle\ . \tag{1.3.313}$$

Recall indeed (see the discussion after Eq. (**??**)) that the helicity m is the eigenvalue of $\mathbf{J}\cdot\mathbf{P}/|\mathbf{p}|$; since $\mathbf{P}$ chenges sign under I_s, while $\mathbf{J}$ is invariant, m changes sign. Notice that for integer spin, we can write simply

$$I_s|(p_0,\mathbf{p}), m\rangle = \eta_s(-1)^j|(p_0,-\mathbf{p}), -m\rangle\ . \tag{1.3.314}$$

Since in every representation of spin j both the eigenvalues m and $-m$ appear, the space spanned by the states $|(p_0,\mathbf{p}), m\rangle$ is invariant the action of I_s, and thus under the full Poincaré group. Thus, every irrep of ISO(1,3) labeled by the mass M and the spin j, and spanned by the states $|p^\mu, m\rangle$ leads to *two* irrepses of the Poincaré group, further distinguished by thei *parity* $\eta_s = \pm 1$ under space inversions, and spanned by the states

$$|p^\mu, m, \eta_s\rangle\ . \tag{1.3.315}$$

[13] The phase $\mathrm{e}^{\mp\mathrm{i}\pi j}$ has the upper (lower) sign when the azimuthal angle of $\mathbf{p}$ is less (more) that π; see [**?**], sec. 11.3.2 for a discussion)

Light-like momentum Physically, when $p^2 = 0$, $p_0 = |\mathbf{p}|$, so the velocity $|\mathbf{p}|/p_0$ is fixed to be 1 (or c, if we kept track of dimensional factors); this cannot be changed by a Lorentz rotation. So particles with such momenta, such as photons, always travel at the speed of light, and these momenta are aclled light-like.

In the class of light-like momenta we can pick up a standard momentum, for instance

$$\hat{p}^\mu = (1, 0, 0, 1) \ . \tag{1.3.316}$$

Usin the definition Eq. (1.3.296), we see that fixing the momentum to this $\hat{p}^\mu$ the componens of the Pauli-Lubanski vector take the following values:

$$\begin{aligned} W^0 &= \frac{1}{2}\epsilon^{0ij3} J_{ij}\hat{p}_3 = J_{12} = J^3 \ , \\ W^3 &= \frac{1}{2}\epsilon^{3ij0} J_{ij}\hat{p}_0 = J_{12} = J^3 \ , \\ W^1 &= \frac{1}{2}\left(\epsilon^{1ij0} J_{ij}\hat{p}_0 + \epsilon^{1\mu\nu3}\hat{p}_3\right) = J_{23} + J_{20} = J^1 + L^2 \ , \\ W^2 &= \frac{1}{2}\left(\epsilon^{2ij0} J_{ij}\hat{p}_0 + \epsilon^{2\mu\nu3}\hat{p}_3\right) = J_{31} - J_{10} = J^1 - L^2 \ , \end{aligned} \tag{1.3.317}$$

(recall that $\hat{p}_3 = \hat{p}^3 = 1$, but $\hat{p}_0 = -\hat{p}^0 = -1$), where we use the notation of sec. **??** for rotations and Lorentz boosts. As we saw in the case of time-like momenta, the independent components of the Pauli-Lubanski vector evaluated on $\hat{p}^\mu$ generate the little group of $\hat{p}^\mu$. The Lie algebra of the independent components W^1, W^2, J^3 is easily foudn to be

$$\begin{aligned} \left[W^1, W^2\right] &= 0 \ , \\ \left[J^3, W_i\right] &= \mathrm{i}\epsilon_{ij} W^j \ , \qquad (i, j = 1, 2) \ . \end{aligned} \tag{1.3.318}$$

This is isomorphic to the ISO(2) algebra, see Eq. (1.3.267) with W^i taking the place of P^i and J_3 that of J.

We have discussed the representations of ISO(2) in sec. 1.3.7.1. They are labeled by the eigenvalue w^2 of the Casimir operator $W^2 = (W^1)^2 + (W^2)^2$, and we must distinguish two cases. If $w^0 = 0$, the representations are uni-dimensional and are induced from that of the factor group SO(2) generated by J_3 and labeled by its eigenvalue m; the state vector can be denoted as $|w^2 = 0; m\rangle$ or simply as $|m\rangle$. If $w^2 > 0$, there's no little group, and the states are obtained by acting on a reference vector $|w^2; \overline{\mathbf{w}}\rangle$ with SO(2) rotations.

As usual, we can build an irreducible representation of ISO(1, 3), we can start from anyy of the irreducible representations of the little group ISO(2). However, it turns out that only the representations constructed out of the little group representations with $w^2 = 0$ are realized in Nature as elementary particles, so we will discuss in some detail only these.

Let us start therefore from the state

$$|p^2 = 0, w^2 = 0; \hat{p}^\mu, m\rangle \ , \tag{1.3.319}$$

which for simplicity (since the Casimirs are invariant within the representation) we well denote in the sequel as $|\hat{p}^\mu, m\rangle$. This state satisfies

$$\begin{aligned} P^2|\hat{p}^\mu, m\rangle &= W^2|\hat{p}^\mu, m\rangle = 0 \ , \\ P^\mu|\hat{p}^\mu, m\rangle &= \hat{p}^\mu|\hat{p}^\mu, m\rangle = (1, 0, 0, 1)|\hat{p}^\mu, m\rangle \ , \\ J_3|\hat{p}^\mu, m\rangle &= m|\hat{p}^\mu, m\rangle \ . \end{aligned} \tag{1.3.320}$$

Transformations $K(p)$ outside the little group modify the reference momentum $\hat{p}^\mu$ into a generic light-like momentum p^μ (such that $p^2 = 0$), and reconstruct the full set of basis vectors $|p^\mu, m\rangle$ of an irreducible representation of $\mathrm{ISO}(1,3)$:

$$|p^\mu, m\rangle = K(p)|\hat{p}^\mu, m\rangle \ . \tag{1.3.321}$$

Such a transformation consists in general of a Lorentz boost in the 3-rd direction, followed by a rotation in the plane $1-2$:

$$\hat{p}^\mu = (1,0,0,1) \stackrel{L_3(\nu)}{=\!\!\longrightarrow} (\mathrm{e}^\nu, 0, 0, \mathrm{e}^\nu) \stackrel{R(\alpha^1,\alpha^2,0)}{\longrightarrow} p^\mu = (\mathrm{e}^\nu, \mathbf{p}) \ , \tag{1.3.322}$$

with $p^i = \mathrm{e}^\nu\,(\mathcal{R}(\alpha^1,\alpha^2,0))^i{}_3$, so that $|\mathbf{p}|^2 = \mathrm{e}^\nu$. On the generic state $|p^\mu, m\rangle$ of our irreducible representation, the translations are simply represented as follows:

$$T(\mathbf{b})|p^\mu, m\rangle = \mathrm{e}^{\mathrm{i}b^\mu p_\mu}|p^\mu, m\rangle \ . \tag{1.3.323}$$

For a Lorentz rotation Λ, we have

$$\Lambda|p^\mu, m\rangle = \mathrm{e}^{\mathrm{i}m\alpha^3}|p^\mu, m\rangle \ , \tag{1.3.324}$$

where

$$\tilde{p}^\mu = \Lambda^\mu{}_\nu p^\nu \tag{1.3.325}$$

and the $\alpha^3 = \alpha^3(\Lambda, p)$ specifies the rotation angle around the 3-rd axis for the little group element $R(0,0,\alpha^3) = \exp(\mathrm{i}\alpha^3 J_3)$ determined, similarly to the case of ISO(3) described in Fig. 1.1, and to the case of time-like momenta discussed in Eq. (1.3.312), by

$$R(0,0,\alpha 3) = K^{-1}(\tilde{p})\Lambda K(p) \ . \tag{1.3.326}$$

The irreducible representations of $\mathrm{ISO}(1,3)$ corresponding to $P^2 = W^2 = 0$ and labeled by m corrispond, in physical terms, to elementary *massless* particles of *helicity* m. In nature, only cases with m integer (corresponding to single-valued representations of the SO(2) little group) or to m half-integer (corresponding to doubly-valued reps. of SO(2)) are realized ($m = \pm 1$: photon, $\ldots$; $m = \pm 1/2$: neutrino).

Notice the difference between the helicity m of massless particles and the spin j, with third component m, of massive particles. Massless states with a given helicity are *not* mixed by *continuous* Poincaré representations, i.e. representations in $\mathrm{ISO}(1,3)$, see Eq. (1.3.324): they correspond to a specific irreducible representations. Massive states with different 3-rd component m of the spin are instead rotated into each other within a representation of spin j, see Eq. (1.3.311).

Let us now consider the full Poincaré group, i.e., let us include the effect of space inversions. In the light-like case, space inversion acts on the states $|p^\mu, m\rangle = |(p_0, \mathbf{p}), m\rangle$ of Eq. (1.3.321) changing sign to both $\mathbf{p}$ and the helicity $m = \mathbf{J}\cdot\mathbf{p}/|\mathbf{p}|$:

$$I_s|(p_0, \mathbf{p}), m\rangle = \eta_s \mathrm{e}^{\mp\mathrm{i}\pi|m|}|(p_0, -\mathbf{p}), -m\rangle \ . \tag{1.3.327}$$

Fixing $\eta_s = 1$, we see...

Space-like momentum ... hysically not so relevant (tachyons), pseudo-orthogonal little groups -¿ infinite-dimensiona reps ...