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CLASSE DI SCIENZE

Ph.D. Thesis

Defects in Conformal Field Theories.

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My supervisor, Davide Gaiotto, deserves a special acknowledgment, not only for his support during the years of my Ph.D. Meeting him catapulted me into an entirely different world, in many senses. A new continent, where squirrels stroll around the cities instead of cats, the same shops and restaurants appear replicated on the main street, and winter teaches people to be courteous. A new physics institute, whose almost scary perfection is thankfully mitigated by the considerable amount of entropy that a large group of physicists generates. And new fields of research, more abstract, scattered with fundamental questions and filled with a myriad of technical details. As for the questions, I think I was never able to answer even one of those that Davide posed during these years. However, those questions helped me finding my own, less ambitious, problems to solve, and even the sketch of a research program. The interaction with Davide has something magic, or perhaps esoteric. When explaining the terms of a problem to him, I would typically first understand from a comment that I had been thinking about it all wrong, and then witness the development of a completely different approach, in real time, which would culminate in some final observation, much past the point where I stopped following the details. Those comments were often not just the keys to the solution of a specific issue, but the germs of new – and undoubtedly deeper – projects than those I was pursuing. Some of his remarks became suddenly crucial months after being made, when my knowledge had matured enough to understand them. Some others I try to carry in mind, hoping that sooner or later those seeds will sprout.

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A Stefano,
sebbene non sia abbastanza.

A Viola,
chissà che un giorno non ne sia incuriosita.

We know our friends by their defects
rather than by their merits.
W. Somerset Maugham

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Introduction

This thesis explores some features of extended probes in conformal field theories (CFTs). Therefore, it seems appropriate to spend a few words justifying why it is interesting to study conformal field theories in the first place.

In theoretical physics, symmetries are often used as a tool to simplify hard problems. They always reduce the room for dynamics, and therefore make life easier. This quest for symmetries, even in cases in which the physical system under inquiry does not possess them, is especially necessary in studying quantum field theories (QFTs). Indeed, in a generic situation, our analytic control over the dynamics of a QFT is extremely limited, it is essentially reduced to perturbation theory around some point in parameter space in which the theory is exactly solvable. And again, in a generic case, this means a weakly coupled regime. This is a shame, because QFT does account for natural phenomena like confinement - we know this, for instance, from lattice simulations - but our analytic control over, say, quantum chromodynamics (QCD) is not sufficient to achieve even a qualitative understanding of it. Of course, the most interesting models are the ones that achieve a subtle balance: they are symmetric enough to allow for analytic control, and at the same time they exhibit at least some of the phenomena present in less symmetric (and more physical) systems. It is mandatory to cite at least one very successful class of examples: the study of supersymmetric field theories has shed light on a number of phenomena which are part of the natural world, but can be handled with much more control in this arena. Duality, RG flows [1] and confinement [2] are three of the most important instances of this fact.

Supersymmetry and conformal symmetry have something in common: they are the only known extensions of the Lorentz (or Euclidean) group that leads to non-trivial quantum (or statistical) field theories. But conformal field theories are far more constrained, and the room for dynamics is very limited: all those physical phenomena for which the presence of a scale is crucial cannot be modeled using a CFT. However, there is another possible reason to study very symmetric theories, and that is if those theories are actually realized in nature. This is what happens with conformal field theories. It is somewhat surprising that quantum field theories with a large amount of symmetry, instead of being just an idealization, are actually rich of experimental consequences. The answer to the puzzle is well known: they sit at the endpoints of Renormalization group flows. Looking at larger and larger distances in any system described by a field theory, if something interesting remains, it is nearly always a conformal field theory. In particular, they provide an accurate explanation for the existence and the characteristics of critical phenomena in condensed matter physics, whose understanding has been one of the major achievements in physics in the last 50 years. Even more generally, an approximate conformal symmetry emerges every time a large separation exists between the ultraviolet and the infrared scales. Furthermore, developing analytical tools for handling CFTs would enlarge the set of points in parameter space around which it is possible to set up a perturbative expansion. CFTs are certainly a natural starting point, since knowledge of the spectrum of a CFT is equivalent to the knowledge of all the possible deformations that can trigger an RG flow, and in fact the choice of a deformation specifies the flow completely - even if it can be very difficult to follow it in practice. Besides the direct phenomenological implications, the interest in conformal field theories is also fueled by the revolutionary notion that they provide a window on regimes of quantum gravity inaccessible to perturbation theory [3]. And finally, on the abstract side, we would like to mention that if we are ever to solve exactly an interacting theory in dimensions higher than two, the first example is going to be a conformal field theory. In fact, there is a natural candidate, the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. Many ingredients conspire to make this result possible in principle. As we shall review in chapter 1, conformality implies that the complexity of the theory is encoded in the knowledge of a (infinite) set of numbers. Perturbation theory and AdS/CFT make it possible to compute these numbers both at weak and at strong coupling. Finally, the very special nature of the background, in which the dual string theory lives, endows the worldsheet theory with infinitely many conserved charges, thus allowing to extend to any finite coupling the knowledge of the fundamental data [4, 5]. In sum, conformal field theories are interesting from both a phenomenological and an abstract perspectives.

In this work, we shall study conformal field theories in the presence of defects. In the broadest sense, a defect is any modification of a system localized on a submanifold, and it is the standard tool to probe it. Local operators are the prototypical example. One can probe the system with extended defects, which can be put in the vacuum and break part of the conformal symmetry of the homogeneous system. Conformal defects are those extended modifications that preserve the subgroup of the conformal group whose action does not displace them. We shall primarily be concerned with CFTs in flat space, endowed with defects supported on a plane or a sphere, and we will often call this setup a *defect CFT*. One may ask if the RG flow grants to defect CFTs the same universality properties of the homogeneous theory. Certainly, as we flow to the infrared, we do not expect the modification to affect the critical properties of the theory in the bulk, and both the bulk and the defect are expected to be described by a scale invariant theory at the bottom of the flow. On the other hand, full conformal invariance is a stronger constraint. For unitary and Poincaré invariant quantum field theories, at least in $d = 2, 4$, scale invariance implies conformal invariance [6, 7]. It is not known if a similar result holds for scale invariant defects in a CFT as well - see [8] for some recent work in the case of a boundary - although it is natural to expect so. We will not have anything to say about this issue, except that in chapter 3 we will check conformal invariance on the lattice for a simple example of defect in a non-supersymmetric 3d CFT. Besides this technical aspect, we would like to stress that in practice conformal defects appear in a variety of situations of phenomenological and theoretical interest. The prototypical example is provided by boundaries and interfaces, whose conformal data have already been characterized in general dimensions in [9, 10]. Two-dimensional boundary and interface CFTs are as usual under far better control - see [11, 12, 13, 14, 15, 16] and many others - but higher dimensional conformal boundaries have been also extensively studied both in statistical and in high energy physics. With the only purpose of illustration, let us mention a few examples. Surface critical exponents of the ϕ^4 theory have been computed numerically [17] and perturbatively [18], and recently CFT predictions for correlation functions were tested on the lattice using the 3d Ising model with a spherical boundary [19]. In high energy physics, boundaries and interfaces can be engineered holographically [20, 21, 22, 23], while the systematic exploration of superconformal boundary conditions in $\mathcal{N} = 4$ SYM was carried out in [24]. Recently, the study of the CFT data associated to the D3-D5 brane system was initiated in [25], while the spectrum of defect operators on the NS5-like interface was considered in [26]. Defects of higher codimensions are equally common in high and low energy physics. Wilson and 't Hooft operators are pre-eminent examples [27], but surface operators also play an important role in gauge theories: in fact, order [28] and disorder [29] two-dimensional operators, analogous to Wilson and 't Hooft lines respectively, can be constructed in four dimensions. Again, lower dimensional defects also arise at brane intersections - see for instance [30]. Low energy examples of defects include vortices [31], higher dimensional descriptions of theories with long range interactions [32], all sort of impurity problems [33, 34, 35, 36, 37, 38] etc. In this work, we will add one more example to the list, defining and studying on the lattice a magnetic-like impurity which can be embedded in any spin system possessing a $\mathbb{Z}_2$ symmetry [39]. Finally, we will apply the defect CFT language to the study of entanglement [40].

In studying either homogeneous or defect CFTs, a first step consists in exploiting the constraints that conformal symmetry imposes on correlation functions. Unfortunately, it can be extremely hard to go beyond that. For instance, it is elementary to understand the presence of critical exponents - that is the power law decay of correlations at large distances in a condensed matter system - but conformal symmetry alone does not give any hint on how to actually compute the critical exponents. A strongly coupled CFT can be completely inaccessible as soon as quantities which are not fixed by symmetry are considered. One very important additional tool is the so called Operator Product Expansion (OPE): this is the statement that excitations triggered by two local operators, seen from large distances, are indistinguishable from a sum over excitations produced by a single operators placed, say, in the middle point between the two original ones. Conformal symmetry severely constrains the coefficient functions in this expansion, and the combination of these two ingredients leads to powerful conditions on the spectrum of operators. The attempt of extracting information from these constraints goes under the name of *conformal bootstrap*. In two dimensions, the conformal symmetry is locally promoted to an infinite dimensional algebra, and the combination of its representation theory and the conformal bootstrap allows to solve some strongly coupled CFTs exactly.[1] Although we shall have something to say about two dimensional theories in section 1.5, our main focus is on higher dimensional defect CFTs. The *conformal bootstrap* approach in higher dimensions has seen tremendous progress in recent years - see [41, 42, 43, 44, 45, 46, 47] and many others - and this has in turn fueled a renewed interest for the abstract study of CFTs.

This work is organized as follows. In chapter 1, we start from a quick review of the basics, *i.e.,* the structure of the conformal group and the OPE. We then put together a toolbox for analyzing a generic defect CFT - this is based on [48]. We develop a convenient language for solving the symmetry constraints on correlation functions of local operators in the presence of a conformal defect. In the special case of a two-point function of scalar primaries, we describe the constraints implied by convergence and associativity of the OPE. These, as usual, take the form of a crossing equation, which states that operators in the correlator can be fused in different orders. Depending on the order, different conformal families appear in the OPE, a conformal family being a set of operators which form a representation of the conformal group. The contribution of a conformal family to the correlator is called a conformal block, and we accumulate some results on the conformal block decomposition of this two-point function. These results are essential in the development of the bootstrap program for defect CFTs, which has recently moved the first steps in dimensions

[1] In fact, the progress in the study of $\mathcal{N} = 4$ SYM we alluded to is largely due to the secret two dimensional nature of the problem, due to its dual string theory description. However, the key word here is *integrability*, rather than conformal symmetry.

higher than two [49, 50, 51]. Finally we analyze the Ward identities of the stress-tensor, which exhibit additional contact terms when a conformal defect is present. This leads to the definition of a set of operators that always exist on top of a conformal defect in a local theory. These operators control the response of the defect to small deformations.

In the rest of the work, we apply various methods - perturbation theory, conformal bootstrap, lattice computations, AdS/CFT - to the study of specific examples of defect CFTs. Chapter 2 is dedicated to conformal boundaries and interfaces and is mostly based on [51]. We first briefly review the main existing techniques available in performing conformal bootstrap analyses, one of which, the *method of determinants* [43], is suitable for the purpose of studying non translational invariant problems. We apply this technique to the analysis of three conformal boundary conditions for $3d$ spin systems, thus extracting predictions for the low lying spectrum and OPE coefficients. We then describe a conformal interface between a theory of free bosons and the theory that lives at the Wilson-Fisher fixed point. This is an example of a renormalization group domain wall, whose study we tackle mainly employing the ϵ-expansion.

The rest of the thesis is dedicated to two examples of defects of codimension two. In chapter 3, which is based on [39], we define a twist line defect in the critical $3d$ Ising model, and we investigate its properties using Monte Carlo simulations. In this model the twist line defect is the boundary of a surface of frustrated links or, in a dual description, the Wilson line of the $\mathbb{Z}_2$ gauge theory. We test the hypothesis that the twist line defect flows to a conformal defect at criticality and evaluate numerically the low-lying spectrum of operators living on the defect, as well as mixed correlation functions of local operators in the bulk and on the defect.

In the last two chapters, we study one of the most popular measures of entanglement in QFT, a class of quantities named *Rényi entropies*. In this context, we are interested in quantifying the amount of entanglement between a region in space and its complement. The boundary between the two regions is a codimension two surface in spacetime, called the *entangling surface*. The Rényi entropy can be thought of as the expectation value of a defect placed on top of the entangling surface. In chapter 4, which is based on [40] we review some basic notions about entanglement in quantum mechanics and quantum field theory, then we make precise the relation between the Rényi entropies and defect CFT, by use of the so-called *replica trick*. We show how to exploit conformal symmetry to study the dependence of Rényi and entanglement entropy on the shape of the entangling surface. Along the way, we obtain results valid for any conformal defect. This work allows to relate a set of conjectures present in the literature, and express them in terms of the basic CFT data associated to the Rényi entropy defect. We then compute said CFT data in the case of a four dimensional free boson, finding agreement with the conjecture. Finally, in chapter 5 we extract the same quantities in the holographic dual of Einstein gravity. In $d = 4$, this computation has recently been performed in [52], where the conjecture was found to fail. The generalization of the procedure to arbitrary number of dimensions involves a deeper understanding of the consequences of conformal symmetry, which we provide in this chapter. The conjecture is found to fail in dimensions $3 \leq d \leq 6$, but the disagreement is, somewhat surprisingly, less than 2% in all cases. At the end of the chapter we comment on this fact and on the questions that it raises. We conclude the thesis with a brief discussion, mainly oriented to the many possible future directions.

Chapter 1

Defect Conformal field theories

We begin this chapter by reviewing some general features of conformal invariant quantum and statistical field theories on $\mathbb{R}^d$. In section 1.1, we define the conformal group, its action on operators and the Operator Product Expansion.[1]. We also describe the constraints imposed by symmetry on correlation functions, but we limit the discussion to scalar operators and we avoid the use of the embedding formalism. We close the section with a description of the CFT data associated to a defect. Section 1.2 is devoted to a review of the embedding formalism and to its application to the defect set-up. In particular, the rules for projecting correlators to physical space are explained, with a few examples. In section 1.3, we derive the tensor structures that can appear in correlation functions of spinning operators. The reader who is uninterested in the details can safely skip them, and directly apply the rules explained in the previous section to obtain the correlators in physical space. The crossing symmetry constraint for a scalar two-point function is treated in section 1.4. Finally, section 1.5 is dedicated to the Ward identities obeyed by the stress-tensor in the presence of a defect. We also derive constraints imposed by those identities on the CFT data, and illustrate the procedure in a few examples. Some technical details and some useful material are collected in the Appendices.

1.1 Conformal symmetry, the Operator Product Expansion and the CFT data

The conformal group in Euclidean space is defined as the group of transformations which preserve angles. In a number of dimensions $d > 2$, it is an extension of the Euclidean group $ISO(d)$ by dilations and the so called special conformal transformations:

$$x'^\mu = x^\mu + a^\mu, \qquad \text{translation} \tag{1.1}$$

$$x'^\mu = R^\mu{}_\nu \, x^\nu, \qquad \text{rotation} \tag{1.2}$$

$$x'^\mu = \lambda \, x^\mu, \qquad \text{dilatation} \tag{1.3}$$

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}, \qquad \text{special conformal transformation.} \tag{1.4}$$

Special conformal transformations look locally like a combination of the other ones. This provides an intuitive argument, related to locality, for why a theory at an RG fixed point usually possesses not only scale invariance but this larger symmetry: a deformation that looks locally like a dilatation cannot affect a scale invariant Hamiltonian with local interactions [55]. However, a more rigorous approach emphasizes the role of unitarity in

[1] In this introductory part we mainly follow the treatment of [53] (see also[54])

the enhancement of scale invariance to full conformal invariance [6, 7]. Special conformal transformations can also be expressed ad the combination of two inversions and a translation:

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu, \qquad \text{special conformal transformation.} \tag{1.5}$$

Although it is not necessary for a CFT to be invariant under inversion, all parity invariant CFTs also possess this discrete symmetry, and we shall restrict ourselves to this class, unless otherwise stated. From the action of the conformal group on spacetime coordinates, it is easy to derive the algebra. We denote with D the dilatation generator and with K_μ the generator of special conformal transformations. The non-vanishing commutators are

$$[D, P_\mu] = \mathrm{i}P_\mu, \tag{1.6a}$$
$$[D, K_\mu] = -\mathrm{i}K_\mu, \tag{1.6b}$$
$$[K_\mu, P_\nu] = 2\mathrm{i}(\delta_{\mu\nu}D - J_{\mu\nu}), \tag{1.6c}$$
$$[J_{\mu\nu}, K_\rho] = -\mathrm{i}(\delta_{\mu\rho}K_\nu - \delta_{\nu\rho}K_\mu) \tag{1.6d}$$
$$[J_{\mu\nu}, P_\rho] = -\mathrm{i}(\delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu) \tag{1.6e}$$
$$[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(J_{\mu\sigma}\delta_{\nu\rho} + J_{\nu\rho}\delta_{\mu\sigma} - J_{\mu\rho}\delta_{\nu\sigma} - J_{\nu\sigma}\delta_{\mu\rho}). \tag{1.6f}$$

In disguise, this is the algebra $so(d+1,1)$. To see this, it is sufficient to redefine the following generators:

$$J_{0,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right), \qquad J_{0,d+1} = D, \qquad J_{\mu,d+1} = \frac{1}{2}\left(P_\mu + K_\mu\right). \tag{1.7}$$

With these position, the algebra takes the form of the pseudo-orthogonal group, the coordinate labeled 0 being the time-like one. This observation is at the root of the *embedding formalism*, which greatly simplifies the analysis of the constraints imposed by symmetry on correlation functions. We will come back to this in section 1.2. For now, we keep working in the d physical dimensions and we define the action of the conformal group on local operators. Following [56] we use the convention $\phi'(x') = (1 - \mathrm{i}\epsilon G)\phi(x)$ for the action of a generator G. The little group of the origin of space consists of rotations, dilatations and special conformal transformations. Since D and $J_{\mu\nu}$ commute, fields are labeled by their spin J and, assuming that the dilatation generator can be diagonalized, by the eigenvalue under D, which we call the scale dimension Δ. As for the special conformal transformations, looking at (1.6b) one sees that they commute with the dilatations only when they annihilate the operator[2]. Fields which are invariant under special conformal transformations at the origin are called *quasi-primaries*:

$$K_\mu O(0) = 0 \quad \overset{\text{def}}{\Leftrightarrow} \quad O(x) \text{ is a quasi-primary.} \tag{1.8}$$

When $d > 2$, there is no such a thing as a *primary* operator, which is instead a highest weight state of the two-dimensional Virasoro algebra. Therefore, we shall often refer to operators satisfying (1.8) as primaries. Now (1.6a) and (1.6b) say that K_μ and P_μ are ladder operators, and we can construct an irreducible representation of the conformal algebra by acting with the momentum generator on a primary field, that is by taking

[2] We note here that the discussion at pag. 101 of [56] is somewhat imprecise. From the fact that the dilatation operator is diagonal, it is argued that the special conformal transformations must act trivially at the origin, since matrices commute with numbers and (1.6b) must be satisfied. In fact, the correct statement is that, when acting on a non primary field (see below), the action of K_μ is non-diagonal and produces a linear combination of fields with scale dimensions differing by one. For instance, it immediately follows from (1.6c) that if O is a scalar primary, $-\mathrm{i}b^\mu K_\mu\big(\partial_\nu O(0)\big) = \partial_\nu O(0) + 2\Delta b_\nu O(0)$.

derivatives. All operators obtained this way are called descendants, and the collection of a primary and of its descendants is called a conformal family. As usual, the action of the generators on fields at a generic point can be induced by conjugating the one at the origin with translations. For instance, the action of a special conformal transformation can be induced by translating (1.8):

$$-\,\mathrm{i}b^\mu K_\mu O(x) = \big(-2(b\cdot x)\Delta + 2\mathrm{i}b^\mu x^\nu S_{\mu\nu} - 2(b\cdot x)x^\nu\partial_\nu + x^2 b\cdot\partial\big)O(x), \tag{1.9}$$

where $S_{\mu\nu}$ is the spin matrix, and we are still suppressing internal indices. The finite form can be obtained by noticing that the last two terms in (1.9) form the Lie derivative of a scalar field with respect to the infinitesimal generator of the transformation (1.4): they are absorbed in the spatial part of the finite transformation of the field. The finite counterparts of the other addends in (1.9) are of course a point dependent multiplicative factor and a rotation. Satisfaction of the abelian composition law of special conformal transformations fixes the quadratic parts in the parameter b^μ. To write the final form, let us recall that conformal transformations preserve angles. Equivalently, they form the subgroup of the group of diffeomorphisms which preserves the metric up to a local scale factor[3]:

$$g'_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}. \tag{1.10}$$

It is immediate then to see that, when $g_{\mu\nu} = \delta_{\mu\nu}$, given an element g of the group one can construct in every point a matrix belonging to the fundamental of $O(d)$:

$$\mathcal{R}^g_{\mu\nu}(x) = \Omega(x)\frac{\partial x'^\mu}{\partial x^\nu}, \qquad \mathcal{R}^g_{\mu\nu}\mathcal{R}^g_{\mu\lambda} = \delta_{\nu\lambda}. \tag{1.11}$$

Now, for special conformal transformations $\Omega(x) = (1 + 2b\cdot x + b^2x^2)$. The $\mathcal{R}$ matrix then provides a representation of the finite counterpart of the rotation (1.9), and one finally obtains:

$$O'^i(x') = \Omega(x)^\Delta\, D^i{}_j(\mathcal{R}(x))O^j(x), \tag{1.12}$$

where $D^i{}_j$ belongs to the appropriate representation of $O(d)$. This rule is trivially compatible with translations ($\Omega(x) = 1$, $\mathcal{R}(x) = 1$), dilatations ($x' = \lambda x$, $\Omega(x) = \lambda^{-1}$, $\mathcal{R}(x) = 1$) and rotations ($x' = Rx$, $\Omega(x) = 1$, $\mathcal{R} = R$), so it summarizes the action of the conformal group on a quasiprimary field.

Let us emphasize, for later convenience, that inversions are characterized by

$$x'^\mu = \frac{x^\mu}{x^2}, \qquad \Omega(x) = x^2, \qquad \mathcal{R}_{\mu\nu}(x) = I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \tag{1.13}$$

One virtue of conformal field theories is that the behaviour of the observables is greatly constrained by a combination of kinematic restrictions coming from the extended bosonic symmetry and of a ultraviolet property inherent to the structure of quantum field theory.

[3] It is worth noticing that, despite this group theoretical characterization, the relation between the action of conformal transformations and diffeomorphisms in a field theory is slightly more elaborate. The definition of a CFT on flat space does *not* include a dynamical metric: the theory is not diffeomorphism invariant. This makes the statement of conformal symmetry non trivial. Vice versa, a general covariant theory on curved space is not necessarily conformal invariant when restricted to flat space: freezing the metric and changing coordinates is not a symmetry, unless the fields change appropriately. A diffeomorphism invariant theory which is also Weyl invariant - that is invariant under a gauge transformation which acts as (1.10) and leaves the coordinates untouched - turns out to be a CFT when restricted to flat space. This is because a Weyl transformation can be used to undo the effect of a coordinate transformation on the metric, if the coordinate transformation belongs to the conformal group. This last statement, which may be true only locally on non trivial spacetimes, is of course global when $g = \delta$.

The latter is captured by the idea of the Operator Product Expansion ([57], see also chap. 20 of [58]). On one hand, operators of a quantum field theory are probes of the system, their role being the one of creating local excitations acting on the vacuum, whose effect at different points is measured by correlation functions. If the set of operators is sufficient for describing all the excitations of the system, when two operators are brought to short distance the combined excitation should be reproduced by a sum of operators placed, say, at the middle point, with coefficients depending on the mutual distance. The nearer the operators, the better the approximation. On the other hand, probing arbitrarily short distances in a QFT gives rise to divergences, therefore the coefficient functions in the OPE will in general blow up as the separation is brought to zero. Per se, this property is not rich of information when the operators in a correlation function are put at generic positions. A priori, there is no reason to expect that the OPE would converge in a finite region - nevertheless, in a recent set of works [59, 60, 61] evidence was put forward for this to be the case in ϕ^4 and Yang-Mills theory. But if the theory is conformal, small and large separations are related by a symmetry transformation, therefore if the description given by the OPE is accurate in a limit, it must be accurate also elsewhere. In other words, the OPE of two operators converges inside correlation functions, with a radius of convergence given by the distance from the nearest other operator. A rigorous proof of this statement makes use of the state-operator correspondence [62], which provides a map between states in the Hilbert space of the system quantized on a sphere S^{d-1} and local operators. This map exists because the dilatation generator can be employed as the Hamiltonian, in the so-called *radial quantization*: a state on the sphere can be evolved backwards to the center of the sphere, where it is replaced by a local operator acting on the vacuum. Convergence of the OPE then follows from standard properties of Hilbert spaces.

The conformal group also fixes the form of the OPE up to two sets of numbers: the OPE coefficients and the scale dimensions of the operators in the OPE. Knowing these numbers, which are referred to as the *CFT data*, allows to compute all correlation functions, that is, coincides with being able to solve the theory. Indeed, by repeatedly fusing pairs of operators any correlator can be reduced to a set of two-point functions of identical operators[4], whose functional form is fixed in terms of the scale dimension of the operator involved. Still, one needs to compute the CFT data. This is a formidable task, which has been completed, for non-trivial cases, only in two dimensions, where the algebra of the conformal group possesses infinitely many generators [63].

Let us see how conformal invariance fixes the OPE of two primaries up to a discrete set of numbers (the CFT data). Its scale invariant form can be schematically written[5]

$$O_1(x_1)O_2(x_2) \sim c_{123}\ |x_{12}^\mu|^{\Delta_3-\Delta_1-\Delta_2}\ O_3(x_2) + \dots \tag{1.14}$$

where we considered scalar external operators for simplicity and we focused on the contribution of scalars to the OPE, which is sufficient for our purposes. Since K_μ acts removing a derivative from descendants, the coefficients of operators belonging to the same conformal family are mixed when a special conformal transformation is applied on both sides, and for the OPE to be invariant under special conformal transformations they cannot be independent. We see what we anticipated: knowledge of the scale dimensions of primaries and of one OPE coefficient for every conformal family is sufficient to determine the OPE, which can be re-expressed as

$$O_1(x_1)O_2(x_2) \sim c_{123}\ |x_{12}^\mu|^{\Delta_3-\Delta_1-\Delta_2}\ \widehat{\mathcal{D}}O_3(x_2), \tag{1.15}$$

[4] We are here restricting all considerations to theories whose dilatation operator can be diagonalized.

[5] We gather our set of conventions in appendix 1.A.

where $\widehat{\mathcal{D}}$ is a scale invariant differential operator whose form is fixed by conformal invariance.

We now turn to the analysis of the effect of conformal symmetry on observables. We first notice that the two-point function of primaries is fixed up to a single coefficient. It is easy to see that the two fields need to have the same scale dimension, for the correlator not to vanish. The correlator also vanishes if the two operators do not have equal spin. Precisely, for a field transforming in the representation D of the rotation group, one has

$$\langle O^i(x_1) O^j(x_2) \rangle = \frac{D^{ij}\big(I(x_{12})\big)}{(x_{12}^2)^\Delta}. \tag{1.16}$$

The meaning of the numerator is as follows: $D(I(x_{12}))$ is the appropriate matrix representation of the element of $O(d)$ associated to an inversion centered in x_{12}, defined as in (1.11) and (1.13). An arbitrary multiplicative factor on the right hand side would not spoil conformal invariance: this amounts to the choice of a normalization, which is usually unity, but may be different in special cases, like the stress-tensor, or the displacement operator that we introduce in subsection 1.5.1. One can use the two-point functions to build a metric in operator space. Indeed, consider applying an inversion only to one operator, sending it to infinity. Using (1.12) one gets

$$\langle O^i(0)\, O'^j(\infty) \rangle = g^{ij}. \tag{1.17}$$

Here g^{ij} is the invariant tensor for the representation D: $D(R)^T\, g\, D(R) = g$, and the prime on the field at infinity indicates that it is evaluated in the transformed frame, that is in the patch that covers the north pole on the compactification of $\mathbb{R}^d$. Eq. (1.17) defines the *Zamolodchikov metric*. In radial quantization, the action of a primary operator at the origin or at infinity corresponds to the creation of the relative in or out-state respectively. We see that positivity of the two-point functions is required by unitarity[6].

The three point function of primary operators is also completely determined by conformal symmetry. Indeed, one can use an inversion to send to infinity one of the operators, and then use translations, scale transformations and rotations to bring the others in any position in space: only the value of the correlator in one reference configuration is undetermined. This number is meaningful once the normalization of the operators is fixed, and must be related to the conformal data. For simplicity, we consider only a three point function of scalars $\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle$ (see [53] for the general case). One can use (1.15), noticing that by orthogonality of the Zamolodchikov metric only the conformal family of O_3 in the OPE of O_1 and O_2 contributes to the correlator. When the operator O_3 is sent to infinity, moreover, only the contribution of the primary matters, the two-point functions of the kind $\langle (\partial^n O_3)\, O_3 \rangle$ being subleading in the limit of large separation. The general three-point function is then obtained by applying the inversion again and using the transformation properties (1.12) of the operators:

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1}\, x_{13}^{\Delta_1+\Delta_3-\Delta_2}}, \tag{1.18}$$

where the constant is seen to be equal to the OPE coefficient in eq. (1.14).

Since all the generators of the conformal group have been used to fix the position of three operators, correlation functions with higher number of points are not fully determined by conformal invariance. They depend on *cross-ratios*, that are conformal invariant

[6] We assume here that complex fields have been split in real and imaginary parts. Otherwise complex conjugation must be applied where appropriate.

combinations of coordinates. The four-point function of scalars, for instance, can be written

$$\begin{aligned}&\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle\\&\quad=(x_{12}^2)^{-\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{-\frac{1}{2}(\Delta_3+\Delta_4)}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}F(u,v),\end{aligned} \tag{1.19}$$

where the cross-ratios are

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}. \tag{1.20}$$

The function $F(u,v)$ is fixed once all the CFT data are known, since one can substitute, say, the products O_1O_2 and O_3O_4 with their OPE and reduce the four-point function to a sum of two-point functions. However, nothing prevents from choosing the channel (14)(23) for this decomposition. In the region in which both representations of the function $F(u,v)$ converge, they must of course be equal. This constraint, called *crossing symmetry*, is the starting point of the idea of the *conformal bootstrap*, which we shall describe in more detail in chapter 2.

1.1.1 CFT data in the presence of a defect

This thesis is concerned with the study of conformal defects. We will always consider spherical or flat defects, whose dimension we denote as p, embedded in $\mathbb{R}^d$. Such extended objects preserve an $SO(p+1,1)\times SO(q)$ subgroup of the full conformal group. To fix ideas, in what follows we shall mostly stick to the picture of a flat defect, but we explicitly discuss the spherical case in subsection 1.2.3. The discussion is easily adapted to defect CFTs in conformally flat spaces.

Besides the symmetry breaking pattern, a specific conformal defect is defined by the CFT data that completely specify the correlation functions of local operators. The fusion of primary operators in the bulk is clearly unaffected by the defect, and controlled by the usual bulk Operator Product Expansion (OPE), of form displayed in eq. (1.14). However, knowledge of the set of scale dimensions Δ_i and coefficients c_{ijk} is not sufficient to compute the correlation functions. The defect possesses local excitations, the defect operators $\widehat{O}_i$, whose conformal weights $\widehat{\Delta}_i$ are not related by symmetry to the bulk ones. When a bulk excitation is brought close to the extended operator, it becomes indistinguishable from a defect excitation, and the process is captured by a new OPE, with new bulk-to-defect OPE coefficients:

$$O(x^a,x^i)\sim b_{O\widehat{O}}\left|x^i\right|^{\widehat{\Delta}-\Delta}\widehat{O}(x^a)+\dots \tag{1.21}$$

Here we denoted by x^a the coordinates parallel to the flat extended operator, and by x^i the orthogonal ones. The defect OPE (1.21) converges within correlation functions like its bulk counterpart. Furthermore, the contribution of descendants of a defect primary $\widehat{O}$, i.e. its derivatives in directions parallel to the defect, is fixed by conformal symmetry. Precisely as the bulk OPE coefficients c_{ijk} appear in the three-point functions of a homogeneous CFT, the coefficients $b_{O\widehat{O}}$ determine the two-point function of a bulk and a defect operator. The existence of this non-trivial coupling between primaries of different scale dimensions is the trademark of a non trivial defect CFT. Its functional form is easily written down in the scalar case:

$$\langle O(x)\widehat{O}(0)\rangle=b_{O\widehat{O}}\left|x^i\right|^{\widehat{\Delta}-\Delta}|x^\mu|^{-2\widehat{\Delta}}, \tag{1.22}$$

while it is one of the task of this work to treat the case of spinning primaries. Among the bulk-to-defect OPE coefficients, the one of the identity plays a special rôle, as it allows bulk operators to acquire an expectation value:

$$\langle O(x)\rangle = a_O \left|x^i\right|^{-\Delta_1} . \tag{1.23}$$

Here, following the literature, we employ the notation a_O for $b_{O\widehat{\mathbf{1}}}$.

Finally, defect operators can be fused as well, so that one last OPE exists, which can be written just by adding hats to eq. (1.14). When considering correlation functions of defect operators, we are faced with an ordinary conformal field theory in p dimensions, with some specific features. The rotational symmetry around the defect is a global symmetry from the point of view of the defect theory, so that operators fall into representations of the $so(q)$ algebra. Moreover, unless there is a decoupled sector on the defect, there is no conserved defect stress-tensor. Indeed, energy is expected to be exchanged with the bulk, so that only the global stress-tensor is conserved. From a more formal point of view, if a separate defect current existed, the associated charge would translate the defect operators that couple to it without affecting bulk insertions. But such a symmetry is compatible with eq. (1.22) only if b_{12} vanishes. Besides the defect stress-tensor, which encodes the response of the defect to a change in its intrinsic geometry, other local operators might be present, which are associated to variations of the extrinsic geometry. Finally, there is always one protected defect primary for every bulk conserved current which is broken by the defect. In particular, the breaking of translational invariance induces a primary which is present in any local theory: the *displacement* operator. This primary appears as a delta function contribution to the divergence of the stress-tensor - see eq. (1.149) - that is, it measures its discontinuity across the defect. The displacement vanishes for trivial and topological defects. More generally, since the normalization of the displacement operator is fixed in terms of the stress-tensor, the coefficient of its two-point function is a physical quantity[7], which is parametrically small when a continuous family of defects connected to the trivial one is considered.

These remarks conclude the overview of the CFT data attached to a generic defect[8]. Any correlation function of local operators can be reduced to a sum of bulk-to-defect couplings by repeatedly fusing bulk and defect operators. Yet the coefficients a_i, b_{ij}, c_{ijk}, $\hat{c}_{ijk}$ are not independent: they obey crossing symmetry constraints, one instance of which we will consider in section 1.4.

1.2 Tensors as polynomials and CFTs on the light-cone

Due to the non-linear action of the special conformal generators on flat space coordinates, the constraints imposed by the conformal group on correlation functions can be annoyingly complicated, especially in the case of operators carrying spin. Luckily, the action of all generators can be linearized by embedding the physical space in a bigger one [66]. The physical d dimensional space is embedded in the light cone[9] of $\mathbb{R}^{d+1,1}$, and the linear action is given by the usual Lorentz transformations in $(d+2)$ dimensions. Spinning operators in $(d+2)$ dimensions have more degrees of freedom than their lower dimensional counterparts, so this redundancy needs to be resolved by a gauge choice. The machinery to do this was elucidated in [67] (see also [68]), where an algorithmic way of constructing conformally

[7] In fact, analogously to the central charge, the Zamolodchikov norm of the displacement is the coefficient of an anomaly in four dimensions [40], see section 4.5. We will not treat the problem of defect anomalies in this work. For recent work on the subject, see also [64, 65].

[8] Interfaces make a small exception, for they require the set of data to be slightly enlarged [51], see chapter 2.

[9] We focus on Euclidean signature, the extension being obvious.

covariant correlators of symmetric traceless tensors (STT) was set up. In this section, we will first review the main properties of the embedding space formalism, and then adapt it to the presence of a defect.

Let us start by recalling a first useful device to deal with the index structures. In this chapter, we will be concerned with operators that are symmetric, traceless tensors (STT).[10] Such tensors can be encoded in polynomials by using an auxiliary vector z^μ:

$$F_{\mu_1\dots\mu_J}(x) \to F_J(x,z) \equiv z^{\mu_1}\dots z^{\mu_J} F_{\mu_1\dots\mu_J}(x), \qquad z^2=0. \tag{1.24}$$

The null condition on the auxiliary vectors is there to enforce tracelessness of the tensor F. The correspondence is one to one, and the index structure can be recovered employing the Todorov differential operator [71]:

$$D_\mu = \left(\frac{d-2}{2} + z\cdot\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z^\mu} - \frac{1}{2} z_\mu \frac{\partial^2}{\partial z\cdot\partial z}. \tag{1.25}$$

Notice that this operator is interior to the condition $z\cdot z=0$. For example we can free one index by applying (1.25) once

$$F_{\mu_1\mu_2\dots\mu_J} z^{\mu_2}\dots z^{\mu_J} = \frac{D_{\mu_1} F_J(x,z)}{J\big(\frac{d}{2}+J-2\big)} \tag{1.26}$$

or we can free all indices by applying J times

$$F_{\mu_1\mu_2\dots\mu_J} = \frac{D_{\mu_1}\dots D_{\mu_J} F_J(x,z)}{J!\big(\frac{d-2}{2}\big)_J}. \tag{1.27}$$

We are now ready for a lightning review of the embedding formalism. A simple observation motivates it: the conformal algebra in d dimensions coincides with the group of rotations in $(d+2)$ dimensions. In fact, it is possible to embed the former space into the latter in such a way that the generators of $SO(d+1,1)$ act on the embedded $\mathbb{R}^d$ precisely as conformal transformations [66, 67, 68]. The advantage is manifest, since Lorentz transformations act linearly on points. The embedding works as follows. Points in physical space are mapped to the light-cone of the $(d+2)$-dimensional Minkowski space; note that the light-cone is an invariant subspace under the action of the Lorentz group. We shall find useful to pick light-cone coordinates in $\mathbb{R}^{d+1,1}$:

$$P\cdot P = \eta_{AB}P^A P^B = -P^+P^- + \delta_{\mu\nu}P^\mu P^\nu. \tag{1.28}$$

We still need to get rid of a dimension, and this can be obtained by declaring the light-cone to be projective, that is, by identifying points up to a rescaling: $P\sim\lambda P$, $\lambda\in\mathbb{R}^+$. This gauge freedom can then be fixed by choosing a section such that the induced metric be the Euclidean one. To this end, a point in $x\in\mathbb{R}^d$ is mapped to a null point P_x in $\mathbb{R}^{d+1,1}$ in the so called Poincaré section:

$$x \to P_x^M = (P^+,P^-,P^\mu) = (1,x^2,x^\mu)\,. \tag{1.29}$$

A generic element $g\in SO(d+1,1)$ does not fix the section (1.29), but one can define an action $\tilde g$ on the section by rescaling back the point: writing

$$gP_x^M = g(x)(1,x'^2,x'^\mu) \tag{1.30}$$

[10]The embedding space machinery was set up for antisymmetric tensors in [69] and for fields living in AdS in [70].

then we have $\tilde{g}x = x'$. It turns out that $\tilde{g}$ is precisely a conformal transformation.

Any field, $F^{\mu_1,\dots\mu_J}(x)$, of spin J in physical space can be obtained from a field, $F^{M_1,\dots M_J}(P)$, by restricting the latter to live on P_x defined in (1.29). The two operators are simply related by a pull-back:

$$F_{\mu_1\dots\mu_J}(x) = \frac{\partial P_x^{M_1}}{\partial x^{\mu_1}} \dots \frac{\partial P_x^{M_J}}{\partial x^{\mu_J}} F_{M_1\dots M_J}(P_x), \tag{1.31}$$

where, on the Poincaré section:

$$\frac{\partial P_x^M}{\partial x^\nu} = (0, 2x_\nu, \delta_\nu^\mu). \tag{1.32}$$

We further impose the following conditions on $F^{M_1,\dots M_J}(P)$:

- it is homogeneous of degree $-\Delta$, *i.e.* $F_{M_1\dots M_J}(\lambda P) = \lambda^{-\Delta} F_{M_1\dots M_J}(P),\ \lambda > 0$,
- it is transverse, $P_{M_1} F^{M_1,\dots M_J}(P) = 0$.

This ensures that F_J projects to a primary operator in physical space (see [68] for a derivation).

Symmetric traceless tensors in embedding space can be again easily encoded in polynomials:

$$F_{M_1\dots M_J}(P) \to F_J(P,Z) \equiv Z^{M_1} \dots Z^{M_1} F_{M_1\dots M_J}(P), \qquad Z^2 = 0 \tag{1.33}$$

We can impose further that $Z \cdot P = 0$ since this condition preserves the transversality of the tensor. Eqs. (1.33) and (1.24) agree, once using eq. (1.31), if

$$Z = (0, 2x \cdot z, z^\mu) \tag{1.34}$$

In particular, this satisfies $Z \cdot P = Z^2 = 0$, if $z^2 = 0$. Therefore, a correlation function in embedding space depends on a set of pairs (P_n, Z_n). The rules to project it down to physical space can be summarized as follows:

$$Z_m \cdot Z_n \to z_m \cdot z_n, \qquad -2P_m \cdot P_n \to x_{mn}^2, \qquad P_m \cdot Z_n \to z_n \cdot x_{mn}, \tag{1.35}$$

where $x_{mn}^\mu \equiv (x_m - x_n)^\mu$. Once in physical space, one can free the indices using the Todorov operator (1.25).

When computing the allowed structure for correlation functions of spinning operators, it is convenient in practice to reverse the logic: one starts by writing polynomials in the variables Z_n, and constrains their coefficients such that the polynomials obey the required properties. For this purpose it is convenient to rephrase the transversality condition:

$$F_J(P, Z + \alpha P) = F_J(P, Z)\,, \qquad (\forall\, \alpha). \tag{1.36}$$

One more simplification follows from defining identically transverse tensors that can be used as building blocks: their contractions automatically provide the right polynomials. When the vacuum is invariant under the full conformal group - i.e. no defect is present - only one tensor is required:

$$C_{MN} \equiv Z_M P_N - Z_N P_M\ . \tag{1.37}$$

For example, the structures appearing in the two and three point functions can be written in terms of C_{MN}

$$C_{m\,MN}C_n^{MN} = H_{mn} = -2\big[(Z_m\cdot Z_n)(P_m\cdot P_n) - (P_m\cdot Z_n)(P_n\cdot Z_m)\big]\,, \tag{1.38}$$

$$V_{m,nl} = \frac{C_{m,MN}P_n^M P_l^N}{P_m\cdot P_n P_m\cdot P_l} = \frac{(Z_m\cdot P_n)(P_l\cdot P_m) - (Z_m\cdot P_l)(P_n\cdot P_m)}{(P_n\cdot P_l)}. \tag{1.39}$$

These are transverse by construction. The same logic can be applied to higher point correlation functions.

Let us finally comment on the case in which one of the operators in a correlation function is a conserved tensor. The conservation condition in physical space is of course

$$\partial^\mu D_\mu T(x,z) = 0 \tag{1.40}$$

where we have used the Todorov operator (1.25) to open one index. As it is well known, this condition is only consistent with the conformal algebra if T has dimension $\Delta = d-2+J$. Furthermore, in a unitary theory the reverse is true as well: a tensor with the above scale dimension is conserved. Conservation of the tensor also implies constraints on correlation functions, which are again easier to analyze in embedding space. To this end, eq. (1.40) must be uplifted to the light-cone. This is easily done:

$$\partial^M D_M \widetilde{T}(X,Z) = 0. \tag{1.41}$$

Here D_M has the same expression as (1.25), but with z replaced by Z. The tensor $\widetilde{T}(P,Z)$ is obtained by uplifting $T(x,z)$ and imposing $Z^2 = Z\cdot P = 0$. It is important to impose $Z\cdot P = 0$, because $\partial^M D_M$ does not preserve this condition - for more details see the discussion in section 5 of [67].

In the remaining of the section, we introduce the necessary modifications to the formalism in order to deal with situations in which the conformal group is broken by the presence of a defect in the vacuum.

1.2.1 Defect CFTs on the light-cone

We now place a p-dimensional defect $\mathcal{O}_\mathcal{D}$ in the vacuum of the CFT, by which we mean that correlation functions are measured in the presence of this extended operator, whose expectation value is divided out. That is, a correlator with n bulk insertions and m defect insertions is defined as follows:

$$\langle O_1(x_1)\ldots O_n(x_n)\widehat{O}_1(x_1^a)\ldots\widehat{O}_m(x_m^a)\rangle \equiv \frac{1}{\langle O_\mathcal{D}\rangle_0}\langle O_1(x_1)\ldots O_n(x_n)\widehat{O}_1(x_1^a)\ldots\widehat{O}_m(x_m^a)\,O_\mathcal{D}\rangle_0\,, \tag{1.42}$$

where the subscript 0 denotes expectation values taken in the conformal invariant vacuum. As mentioned in the introduction, the reader can keep in mind the example of a flat defect: in subsection 1.2.3 we show explicitly how to deal with a spherical defect. Let us denote with q the codimension of the defect, so that $p+q=d$. The generators which belong to an $so(p+1,1)\times so(q)$ sub-algebra of $so(d+1,1)$ still annihilate the vacuum. In the picture in which the defect is flat these are just conformal transformations on the defect and rotations around it. In the special case of a codimension one defect, *i.e.* a CFT with a boundary or an interface, the embedding formalism has been set up in [49].

Defect operators carry both $SO(q)$ and $SO(p)$ quantum numbers. We call them transverse and parallel spin, and denote them with s and j respectively. Clearly, one can still encode spinning defect operators into polynomials. This time, two auxiliary variables w^i

and z^a are required, associated with transverse and parallel spin respectively. Again we restrict ourselves to symmetric traceless representations of both $SO(q)$ and $SO(p)$, thus imposing $w^i w_i = 0$ and $z^a z_a = 0$. When recovering tensors from the polynomials, one needs to remove the two kind of polarization vectors by use of the appropriate Todorov operators, that is, respectively,

$$D_a = \left(\frac{p-2}{2} + z^b \frac{\partial}{\partial z^b}\right)\frac{\partial}{\partial z^a} - \frac{1}{2} z_a \frac{\partial^2}{\partial z^b \partial z_b}, \tag{1.43}$$

$$D_i = \left(\frac{q-2}{2} + w^j \frac{\partial}{\partial w^j}\right)\frac{\partial}{\partial w^i} - \frac{1}{2} w_i \frac{\partial^2}{\partial w^j \partial w_j}. \tag{1.44}$$

From the point of view of the defect theory, the transverse spin is the charge under an internal symmetry, but of course both symmetries arise from the Euclidean group in the ambient space. In the correlator of a bulk and a defect operator, the allowed tensor structures couple indeed both transverse and parallel spins to the bulk Lorentz indices, in every way that preserves the stability group of the defect.

In the embedding space, let us split the coordinates in the two sets which are acted upon respectively by $SO(p+1,1)$ and $SO(q)$. We loosely call the first set "parallel" directions and denote them $A, B, \dots$, while "orthogonal" directions are labelled $I, J, \dots$:

$$M = (A, I), \qquad A = 1, \dots, p+2\,, \quad I = 1, \dots, q\,. \tag{1.45}$$

Since the symmetry is still linearly realized in embedding space, scalar quantities are simply built out of two scalar products instead of one:

$$P \bullet Q = P^A \eta_{AB} Q^B \qquad\qquad P \circ Q = P^I \delta_{IJ} Q^J. \tag{1.46}$$

There is of course also the possibility of using the Levi-Civita tensor density, which is relevant for correlators of parity odd primaries. We comment on this in subsection 1.3.4.[11] Bulk insertions still obey the conditions P^2, $Z^2\,, Z \cdot P = 0$. This implies that, for a single insertion, only a subset of scalar products is independent, since

$$P \bullet P = -P \circ P\,, \quad Z \bullet Z = -Z \circ Z\,, \quad Z \bullet P = -Z \circ P. \tag{1.47}$$

Under the symmetry breaking pattern, the transverse tensor (1.37) breaks up in three pieces: C^{AB}, C^{IJ}, C^{AJ}. It turns out that only the last one,

$$C^{A\,I} = P^A Z^I - P^I Z^A, \tag{1.48}$$

is necessary when dealing with bulk insertions. Indeed, the other structures can be written as linear combinations of C^{AI}:

$$C_{AB} Q^A R^B = \frac{P \bullet R}{P \circ G} C_{AI} Q^A G^I - \frac{P \bullet Q}{P \circ G} C_{AI} R^A G^I, \tag{1.49}$$

$$C^{IJ} Q^I R^J = \frac{P \circ Q}{P \bullet G} C_{AI} G^A R^I - \frac{P \circ R}{P \bullet G} C_{AI} G^A Q^I, \tag{1.50}$$

[11] There is at least another method to set up the formalism, which was suggested to us by Joao Penedones. The defect is specified by a q-form $V_{M_1 \dots M_q}$. Allowed tensor structures are produced via contraction of V, or of the Hodge dual, with the position and polarization vectors. One can take $V^{M_1,\dots,M_q} \propto n_1^{[M_1} \dots n_q^{M_q]}$, n_I^M being vectors normal to the embedded defect. This approach is equivalent to the one chosen in this work. Indeed, in the coordinate system (1.45) $n_I^M = \delta_I^M$ and $V_{M_1 \dots M_q} \propto \delta_{M_1}^{I_1} \dots \delta_{M_q}^{I_q} \epsilon_{I_1 \dots I_q}$, ϵ being the Levi-Civita symbol. Furthermore, a product of an even number of copies of V can be expressed in terms of the the orthogonal scalar product $\circ$, thus proving the equivalence.

for generic vectors Q, R and G - in particular, one can always choose $G^M = P^M$. The tensor (1.48) also obeys the following identity:

$$C^{AI}C_{BI}C^{BJ} = \frac{1}{2}(C^{BI}C_{BI})C^{AJ}, \tag{1.51}$$

so we never need to concatenate more than two of these structures.

Defect operators live on a $(p+1)$-dimensional light-cone within of the full $(d+1)$-dimensional one, and again they are encoded into polynomials of the two variables W^I and Z^A. They are subject to the usual transversality rule, so that parallel indices satisfy $Z\bullet Z = Z\bullet P = 0$. In particular, the polarization Z^A should appear in correlation functions only through the structure (1.37), restricted to the parallel indices.

Let us finally briefly comment on the issue of conservation in the presence of a defect. One possibility to study consequences of conservation on a correlator is to project the embedding space expression to physical space, open the indices with the Todorov operator and then take a derivative. This is completely harmless, but sometimes slightly inconvenient. The other possibility is to work directly in the embedding space. In this case, conservation corresponds to

$$\partial^M D_M \overline{T}(X, Z) = 0. \tag{1.52}$$

We use a different symbol with respect to (1.41), because $\overline{T}(X, Z)$ is obtained from $T(X, Z)$ imposing $Z \cdot P,\ Z \cdot Z = 0$ but in a specific way, namely, using eqs. (1.47) to replace everywhere

$$Z\bullet P \to -Z\circ P, \qquad Z\bullet Z \to -Z\circ Z. \tag{1.53}$$

The reason for this is that the operator $\partial^M D_M$ is not interior to the conditions $Z\bullet P = -Z\circ P$ and $Z\bullet Z = -Z\circ Z$ and, for any expression g, gives different results when applied to the l.h.s. or r.h.s. of

$$g(Z\circ Z, Z\circ P, \dots) = g(-Z\bullet Z, -Z\bullet P, \dots); \tag{1.54}$$

we thus have to make a choice. The easiest way to establish the correct one is to notice that $Z\circ Z$ and $Z\circ P$ are mapped to $z\circ z$ and $z\circ x$ in physical space when the defect is flat - see subsection 1.2.2 - and that $\partial^M D_M$ acting on $Z\circ Z$ and $Z\circ P$ gives the same result as $\partial^\mu D_\mu$ acting on $z\circ z$ and $z\circ x$. Clearly, there is no ambiguity in physical space. The correct prescription is thus to use $Z\circ Z$ and $Z\circ P$ instead of $Z\bullet P$ and $Z\bullet Z$, *i.e.* to apply eq. (1.53). Since the embedding formalism is insensitive to whether the defect is flat or spherical - see subsection 1.2.3 - the same prescription works in the latter case as well.

1.2.2 Projection to physical space: flat defect

The embedding space is a useful tool, but sometimes one is also interested in the result in physical space. A simple rule to project down is to pick the polynomial expressions in the embedding space, project to physical space using the Poincaré section (1.29) and then open the indices with the appropriate Todorov operator. A defect $\mathcal{O}_\mathcal{D}$ extended on a flat sub-manifold $\mathcal{D}$ is embedded in the Poincaré section as follows:

$$P^M \in \mathcal{D}: \quad P^A = (1, x^2, x^a), \quad P^I = 0. \tag{1.55}$$

Polarization vectors of bulk operators are evaluated according to (1.34), and a similar rule holds for defect operators:

$$W^I = w^i, \quad Z^A = (0, 2x^a z_a, z^a). \tag{1.56}$$

The projection to real space of orthogonal scalar products $(P{\circ}Q)$ is trivial, while for generic vectors $P_m,\ P_n$ and polarizations $Z_m,\ Z_n$ the rule is

$$-2P_m{\bullet}P_n = |x^a_{mn}|^2 + |x^i_m|^2 + |x^i_n|^2, \qquad P_m{\bullet}Z_n = x^a_{mn} z^a_n - z^i_n x^i_n. \tag{1.57}$$

Let us now present a few examples. They can be obtained from the embedding space correlators derived in section 1.3. Let us first consider the generic two-point function of a defect operator. This is a trivial example: we report it because it corresponds to a choice of normalization. The correlator in embedding space appears in eq. (1.78). Projection to physical space yields

$$\langle \widehat{O}^{i_1\dots i_s}_{\widehat{\Delta}}(x^a_1)\widehat{O}^{j_1\dots j_s}_{\widehat{\Delta}}(x^a_2)\rangle = \frac{\mathcal{P}^{i_1\dots i_s;j_1\dots j_s}}{(x^2_{12})^{\widehat{\Delta}}}, \tag{1.58}$$

where $\mathcal{P}$ is the projector onto symmetric and traceless tensors, as defined in [67] in terms of the Todorov operators (1.44):

$$\mathcal{P}^{i_1\dots i_s;j_1\dots j_s} \equiv \frac{1}{s!\left(\frac{q}{2}-1\right)_s} D_{i_1}\dots D_{i_s} w_{j_1}\dots w_{j_s}. \tag{1.59}$$

Let us consider next the one-point function of a $J=2$ bulk primary $O_{\Delta,2}(x)$:

$$\langle O^{ab}_{\Delta}(x)\rangle = \frac{q-1}{d}\frac{a_O}{|x^i|^{\Delta}}\delta^{ab}, \qquad \langle O^{ij}_{\Delta}(x)\rangle = -\frac{a_O}{|x^i|^{\Delta}}\left(\frac{p+1}{d}\delta^{ij} - n^i n^j\right), \tag{1.60}$$

where we introduced the versor $n^i \equiv x^i/|x^i|$ and we wrote a_O for $a_{O_{\Delta,2}}$ for simplicity. Notice that this correlator is compatible with conservation, so that in particular the stress-tensor can acquire expectation value.

Among defect operators, a special role is played by a primary of transverse spin $s=1$ and null parallel spin. This is the displacement operator, which we describe in some detail in section 1.5. Hence we choose to write here the correlator of a defect primary $\widehat{O}_{\widehat{\Delta},0,1}$ with these quantum numbers with bulk primaries of spin $J=1$ and $J=2$. The two-point function with the vector reads

$$\langle O^a_{\Delta}(x_1)\widehat{O}^i_{\widehat{\Delta}}(x^b_2)\rangle = \frac{-b_{O\widehat{O}}}{(x^2_{12})^{\widehat{\Delta}}\,|x^i_1|^{\Delta-\widehat{\Delta}}}\frac{x^i_1 x^a_{12}}{x^2_{12}}, \qquad \langle O^j_{\Delta}(x_1)\widehat{O}^i_{\widehat{\Delta}}(x^b_2)\rangle = \frac{-b_{O\widehat{O}}}{(x^2_{12})^{\widehat{\Delta}}\,|x^i_1|^{\Delta-\widehat{\Delta}}}\frac{x^i_1 x^j_1}{x^2_{12}}. \tag{1.61}$$

The correlator with a rank-two tensor $O_{\Delta,2}$ is a bit more lengthy, but straightforward to obtain:[12]

$$
\begin{aligned}
\langle O^{ab}_{\Delta}(x_1)\widehat{O}^i_{\widehat{\Delta}}(x_2^b)\rangle &= \frac{n_1^i/d}{(x_{12}^2)^{\widehat{\Delta}}\left|x_1^i\right|^{\Delta-\widehat{\Delta}}}\left\{b^1_{O\widehat{O}}\left(\frac{4d\left|x_1^i\right|^2 x_{12}^a x_{12}^b}{x_{12}^4}-\delta^{ab}\right)+b^2_{O\widehat{O}}(q-1)\delta^{ab}\right\}\\
\langle O^{ja}_{\Delta}(x_1)\widehat{O}^i_{\widehat{\Delta}}(x_2^b)\rangle &= \frac{1}{(x_{12}^2)^{\widehat{\Delta}}\left|x_1\right|^{\Delta-\widehat{\Delta}}}\frac{-x_{12}^a\left|x_1^i\right|}{x_{12}^2}\left\{2b^1_{O\widehat{O}}\,n_1^i n_1^j\left(1-\frac{2\left|x_1^i\right|^2}{x_{12}^2}\right)+b^3_{O\widehat{O}}(\delta^{ij}-n_1^i n_1^j)\right\},\\
\langle O^{jk}_{\Delta}(x_1)\widehat{O}^i_{\widehat{\Delta}}(x_2^b)\rangle &= \frac{1}{(x_{12}^2)^{\widehat{\Delta}}\left|x_1^i\right|^{\Delta-\widehat{\Delta}}}\left\{b^1_{O\widehat{O}}\,n_1^i\left[n_1^j n_1^k\frac{\left(x_{12}^2-2\left|x_1^i\right|^2\right)^2}{x_{12}^4}-\frac{1}{d}\delta^{jk}\right]\right.\\
&\quad+b^2_{O\widehat{O}}\,n_1^i\left(n_1^j n_1^k-\frac{p+1}{d}\delta^{jk}\right)\\
&\quad\left.+b^3_{O\widehat{O}}\left(1-\frac{2\left|x_1^i\right|^2}{x_{12}^2}\right)\left(\frac{(\delta^{ik}n_1^j+\delta^{ij}n_1^k)}{2}-n_1^i n_1^j n_1^k\right)\right\}. \qquad (1.63)
\end{aligned}
$$

1.2.3 Projection to physical space: spherical defect

A spherical defect is conformally equivalent to a flat one, therefore correlators in the presence of the former can be obtained from the homologous ones via a special conformal transformation - or simply an inversion. On the light-cone, such a transformation changes the embedding of the defect into the Poincaré section. On the other hand, the rules explained in subsection 1.2.1 only care about the $so(p+1,1)\times so(q)$ symmetry pattern, which stays unchanged.[13] As a consequence, correlators in embedding space encode both the flat and the spherical cases, the only difference lying in the choice of parallel and orthogonal coordinates, which we now describe. Without loss of generality, we consider a spherical p-dimensional defect of unit radius. Then, we abandon the light-cone coordinates in embedding space and use Cartesian coordinates instead:

$$P_x^M=(P^0,P^1,\dots,P^d,P^{d+1})=\left(\frac{x^2+1}{2},x^\mu,\frac{1-x^2}{2}\right), \qquad (1.64)$$

the first entry being the time-like one. The relation between light-cone and Cartesian coordinates is $P^\pm=P^0\pm P^{d+1}$. We parametrize the p-sphere with stereographic coordinates σ^a and center it at the origin of the d-dimensional space. A point on the defect turns out to be embedded as follows:

$$P^M\in\mathcal{D}:\quad P^M(\sigma)=\left(1,\frac{2\sigma^a}{\sigma^2+1},\frac{1-\sigma^2}{\sigma^2+1},\underbrace{0,\dots,0}_{\text{q times}}\right)=\frac{2}{\sigma^2+1}\left(\frac{\sigma^2+1}{2},\sigma^a,\frac{1-\sigma^2}{2},0,\dots,0\right). \qquad (1.65)$$

By comparing the second equality with the embedding of the flat defect eq. (1.55), we see that the two defects are related by a rotation in the plane (P^{p+1},P^{d+1}), up to the conformal factor needed to bring us back to the Poincaré section. The action of the two

[12] The notation used in the following section is slightly different from this one. The coefficients $b_{n_1\dots n_4}$ in (1.86) correspond to

$$b_{1,2,0,0}=b^1_{O\widehat{O}}\,,\quad b_{1,0,0,1}=b^2_{O\widehat{O}}\,,\quad b_{0,1,1,0}=b^3_{O\widehat{O}}. \qquad (1.62)$$

[13] The embedding of the stability subgroup into $SO(d+1,1)$ gets conjugated by the same special conformal transformation.

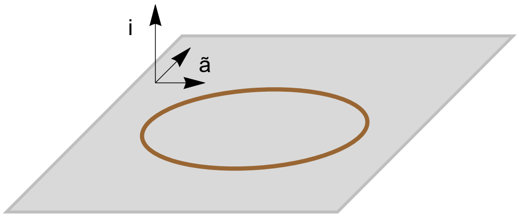

Figure 1.1: The picture illustrates the choice of coordinates in this subsection: the spherical defect is drawn in brown and is placed on the plane spanned by the coordinates $x^{\tilde{a}}$.

factors of the stability subgroup is clear from eq. (1.65), so that parallel and orthogonal coordinates are as follows:

$$P^A = (P^0, P^1, \dots, P^{p+1}) \,, \qquad P^I = (P^{p+2}, \dots, P^{d+1}). \tag{1.66}$$

This is already enough to project the one-point functions to real space. It is sufficient to plug this choice of indices into eqs. (1.79), (1.80) and evaluate the correlator on the Poincaré section (1.64). Before giving an example, we introduce one more bit of notation: we use a tilde for the $(p+1)$ directions in which the p-sphere is embedded - see figure 1.1 Correspondingly, the index i now only runs over the $(q-1)$ directions of the orthogonal subspace:

$$x^\mu = (x^{\tilde{a}}, x^i) \,, \qquad \tilde{a} = 1, \dots, p+1 \,, \quad i = 1, \dots, q-1. \tag{1.67}$$

We also use the radial coordinate $r = \left|x^{\tilde{a}}\right|$. The defect is placed in $r = 1$. Let us now consider the expectation value of a spin two primary. The projection to physical space is done using (1.64) and the polarization vector (1.34) in cartesian coordinates reads

$$Z = (x \cdot z, z^\mu, -x \cdot z). \tag{1.68}$$

Combining eqs. (1.64) and (1.68) we get the following projections:

$$\begin{gathered} P_m \circ P_n = x^i_m x^i_n + \frac{(1-x_m^2)(1-x_n^2)}{4}, \qquad P_m \circ Z_n = x^i_m z^i_n + \frac{(x_m^2 - 1)}{2} \, z_n \cdot x_n, \\ Z_m \circ Z_n = z^i_m z^i_n + x_m \cdot z_m \, x_n \cdot z_n, \end{gathered} \tag{1.69}$$

and similarly for the parallel scalar products. Let us now consider the one-point function of the stress-tensor. The embedding space formula is eq. (1.79). We project to physical space via eq. (1.69), we remove the polarization vectors with (1.25), and we obtain

$$\langle T_{ij} \rangle = a_T \frac{\frac{d+1-q}{4d}(r^4 + 2r^2(|x^i|^2 - 1) + (1 + |x^i|^2)^2)\delta_{ij} - r^2 x_i x_j}{\left(\frac{(x^2-1)^2}{4} + |x^i|^2\right)^{d/2+1}}, \tag{1.70a}$$

$$\langle T_{\tilde{a}i} \rangle = a_T x_{\tilde{a}} x_i \frac{1 - r^2 + |x^i|^2}{2\left(\frac{(x^2-1)^2}{4} + |x^i|^2\right)^{d/2+1}} \tag{1.70b}$$

$$\langle T_{\tilde{a}\tilde{b}} \rangle = a_T \frac{\frac{1-q}{4d}(r^4 + 2r^2(|x^i|^2 - 1) + (1 + |x^i|^2)^2)\delta_{\tilde{a}\tilde{b}} + r^2 x_{\tilde{a}} x_{\tilde{b}}}{\left(\frac{(x^2-1)^2}{4} + |x^i|^2\right)^{d/2+1}}. \tag{1.70c}$$

As for the defect primaries, their parallel indices can be pulled-back to physical space by means of the Jacobian $\partial P^A(\sigma)/\partial\sigma^a$ of the map (1.65) - which is the rule for any spinning operator of a CFT on a sphere. Then, the auxiliary variable Z^A is as usual determined in terms of its real space counterpart. With the choice of stereographic coordinates we get

$$Z^A = z^b\,\frac{\partial P^A(\sigma)}{\partial\sigma^b} = \left(0\,,\ 2\frac{z^a}{\sigma^2+1} - 4\frac{z^b\sigma_b\,x^a}{(\sigma^2+1)^2}\,,\ -8\frac{z^b\sigma_b}{(\sigma^2+1)^2}\right). \tag{1.71}$$

The $so(q)$ global symmetry now rotates vectors in the normal bundle to the sphere, and it is natural to choose the following basis of orthonormal vectors n_I^μ:

$$n_i^\mu = \delta_i^\mu\,, \qquad n_r^\mu = -\delta_{\tilde a}^\mu\,\frac{x^{\tilde a}}{r} = -\delta_{\tilde a}^\mu\,x^{\tilde a}. \tag{1.72}$$

We chose the radial vector to point inward. The reason for this will be clear in a moment. Notice that there is an apparent clash of notation, between the index $I=(i,r)$ of n_I^μ and the one in eq. (1.66). The clash is, indeed, only apparent. To see this, let us consider the projection to physical space of a defect primary $\widehat{O}^I(P(\sigma))$. The index has to be pulled back along the Poincaré section eq. (1.64):

$$\widehat{O}_\mu(\sigma) = \frac{\partial P_x^I}{\partial x^\mu}\widehat{O}_I(P(\sigma)). \tag{1.73}$$

We know from the case of a trivial defect that $\widehat{O}_\mu(\sigma)$ transforms as a vector under the $so(q)$ factor of the conformal algebra. Furthermore, it is easy to verify that this operator has only components normal to the p-sphere. Therefore we can further recover an operator $\widehat{O}_I(\sigma)$ in physical space via contraction with n_I^μ. However, we now notice that

$$n_J^\mu\frac{\partial P_x^I}{\partial x^\mu}\bigg|_{P_x=P(\sigma)} = \delta_J^I\,, \tag{1.74}$$

which trivializes the projection with the identification of the $(d+1)$-th direction on the light-cone with the radial one in physical space. The last equation also provides the rule for projecting down the transverse polarization vectors:

$$W^I = (w^i, w^r)\,. \tag{1.75}$$

The choice of an inward pointing radial vector in eq. (1.72) has been made in order to avoid a minus sign in the last entry of eq. (1.75). At this point, we are able to write down the expression in physical space of any correlation function. For instance, any bulk-to-defect two-point function involving a defect operator with $j=0$ is obtained by making the following substitutions in (1.86):

$$\begin{aligned} &P_1\circ W = x_1^i w^i + \frac{1-x_1^2}{2}w^r, \quad P_1\bullet P_2 = x_1^{\tilde a}x_2^{\tilde a} - \frac{1+x_1^2}{2},\\ &P_2\bullet Z_1 = z^{\tilde a}x_2^{\tilde a} - z\cdot x_1\,, \ \ W_2\circ Z_1 = w^i z^i - x_1\cdot z\,w^r. \end{aligned} \tag{1.76}$$

where we labelled the physical point corresponding to P_2 by means of its $p+1$ cartesian coordinates $x_2^{\tilde a}$, see eq. (1.67). Indeed, while the parametrization (1.65) makes it clear that operators on a spherical defect obey the same rules as operators in a CFT on a sphere, formulae may look nicer with this different choice. Removing the polarization vectors is straightforward. The bulk polarizations can be removed using eq. (1.25) as in the case of the one-point function, while the defect polarizations are removed by means of eq.s (1.43) and (1.44). As a simple example, let us present the correlator of a bulk scalar primary and the displacement operator:

$$
\begin{aligned}
\langle O_{\Delta,0}(x_1)\mathrm{D}_i(x_2)\rangle &= b_{\mathrm{OD}}\frac{x_{1,i}}{\left(\frac{(1-x_1^2)^2}{4}+|x_1^i|^2\right)^{\frac{\Delta_1-p}{2}}\left(x_1^{\tilde a}x_2^{\tilde a}-\frac{1+x_1^2}{2}\right)^{p+1}}\,,\\
\langle O_{\Delta,0}(x_1)\mathrm{D}_r(x_2)\rangle &= b_{\mathrm{OD}}\frac{(1-x_1^2)/2}{\left(\frac{(1-x_1^2)^2}{4}+|x_1^i|^2\right)^{\frac{\Delta_1-p}{2}}\left(x_1^{\tilde a}x_2^{\tilde a}-\frac{1+x_1^2}{2}\right)^{p+1}}\,.
\end{aligned}
\tag{1.77}
$$

1.3 Correlation functions in a Defect CFT

In the previous section we established the rules of the game: we now would like to play and construct the tensor structures appearing in correlation functions of a defect CFT. It makes sense to start from the correlators which are fixed by symmetry up to numerical coefficients - the latter being the CFT data associated to the defect.

Correlation functions with only defect insertions obey the constraints of a p dimensional CFT with a global symmetry. We will not have anything to say about this, besides the choice of normalization of a defect primary charged under the global symmetry:

$$
\langle \widehat{O}_{\widehat{\Delta},0,s}(P_1,W_1)\widehat{O}_{\widehat{\Delta},0,s}(P_2,W_2)\rangle = \frac{(W_1\circ W_2)^s}{(-2P_1\bullet P_2)^{\Delta}}. \tag{1.78}
$$

As mentioned in the introduction, the distinguishing feature of a conformal defect is the presence of bulk-to-defect couplings. The prototype of such interaction is the two-point function of a bulk and a defect operator, eq. (1.82). This correlator is fixed by conformal invariance, up to a finite number of coefficients. The set of two-point functions of a bulk primary with all defect primaries fixes its defect OPE. The simplest among such couplings is the expectation value of the bulk operator itself, aka the coupling with the identity on the defect. This has appeared in various places, and we derive it in the next subsection as a simple warm up. The first correlator which includes dependence on cross-ratios is the two-point function of bulk primaries. In section 1.3.3 we provide the elementary building blocks for the tensor structures in this case, and we briefly comment on the choice of cross-ratios. Finally, subsection 1.3.4 is dedicated to parity odd structures.

1.3.1 One-point function

The structure of the one-point function of a primary in the presence of a defect is easily constructed by means of the transverse tensor structure C^{AI}. Scale invariance implies that the one point function has the form

$$
\langle O_{\Delta,J}(P,Z)\rangle = a_O\frac{Q_J(P,Z)}{(P\circ P)^{\frac{\Delta}{2}}} \tag{1.79}
$$

where $Q_J(P,Z)$ is a homogeneous polynomial - whose normalization we will fix shortly - of degree J in Z. It must moreover have degree zero in P and be transverse, *i.e.*, it must satisfy $Q_J(P,Z+\ \alpha P)=\ Q_J(P,Z)$. The unique function with the aforementioned properties is

$$
Q_J = \left(\frac{C^{AI}C_{AI}}{2P\circ P}\right)^{\frac{J}{2}} = \left(\frac{(P\circ Z)^2}{P\circ P}-Z\circ Z\right)^{\frac{J}{2}}. \tag{1.80}
$$

The proof just follows from the fact that we are only allowed to use the C^{AI} building block, together with the identity (1.51). Clearly, this implies that only even spin operators acquire an expectation value in a parity preserving theory (but pseudo-tensors make an exception, see subsection 1.3.4). Furthermore, when the codimension is one the polynomial becomes trivial, which means that only scalars have non-vanishing one-point functions [49]. As a final remark, let us notice that the structure of the one-point function is compatible with conservation. Indeed, the condition

$$\partial_M D^M \frac{Q_J}{(P\circ P)^{\frac{\Delta}{2}}} = J(q+J-3)(d-\Delta+J-2)\,\frac{P\circ Z\ Q_{J-2}}{2(P\circ P)^{\frac{\Delta+2}{2}}} = 0 \tag{1.81}$$

is satisfied if $\Delta = d-2+J$.

1.3.2 Bulk-to-defect two-point function

A bulk-to-defect two-point function is a function of five variables:

$$\langle O_{\Delta,J}(P_1,Z_1)\rangle\, \widehat{O}_{\widehat{\Delta},j,s}(P_2,Z_2,W_2). \tag{1.82}$$

As explained in subsection 1.2.1, Z_2 should appear in correlation functions only through the building block

$$C_2^{AB} = P_2^A Z_2^B - P_2^B Z_2^A. \tag{1.83}$$

This elementary structure can be contracted with P_1, or with Z_1 through the building block C_1^{AI} defined in (1.48), while the other options lead to "pure gauge" terms, i.e., terms that vanish upon enforcing $P^2 = Z^2 = P\cdot Z = 0$. In turn, terms of type C_1^{AI} can be linked in chains of contractions, but luckily these chains do not become too long, thanks to the identity (1.51). When writing all the contractions which involve C_2^{AB}, one can remain loyal to the rule of exclusively using C_1^{AI} in order to introduce the vector Z_1 in the game. However, one soon realizes that all the structures factorize. The factor that contains Z_2 is always given by

$$Q_{BD}^0 = \frac{C_1^{AB} C_{2,AB}}{2P_1\bullet P_2} = \frac{P_1\bullet P_2\ Z_1\bullet Z_2 - P_2\bullet Z_1\ Z_2\bullet P_1}{P_1\bullet P_2}. \tag{1.84}$$

We conclude that the most general tensor structure is given by the product of (1.84) with other factors built out of at most two copies of C_1^{AI}, contracted with P_1, P_2 and W_2. The independent ones among these remaining structures are

$$\begin{aligned} Q_{BD}^1 &= \frac{P_1\circ W_2}{(P_1\circ P_1)^{1/2}}, \qquad Q_{BD}^2 = \frac{P_1\circ Z_1\, P_1\bullet P_2 - P_2\bullet Z_1\, P_1\circ P_1}{(P_1\circ P_1)^{1/2}(P_1\bullet P_2)}, \\ Q_{BD}^3 &= \frac{W_2\circ Z_1\, P_1\circ P_1 - P_1\circ W_2\, P_1\circ Z_1}{P_1\circ P_1}, \qquad Q_{BD}^4 = Q_2(P_1,Z_1), \end{aligned} \tag{1.85}$$

where Q_2 was defined in eq. (1.80). Thus, a generic bulk-to-defect two point function (1.82) is given by

$$\langle O_{\Delta,J}(P_1,Z_1)\widehat{O}_{\widehat{\Delta},j,s}(P_2,Z_2,W_2)\rangle = (Q_{BD}^0)^j \sum_{\{n_i\}} b_{n_1\ldots n_4} \frac{\prod_{k=1}^4 (Q_{BD}^k)^{n_k}}{(-2P_1\bullet P_2)^{\widehat{\Delta}}(P_1\circ P_1)^{\frac{\Delta-\widehat{\Delta}}{2}}}\ , \tag{1.86}$$

where the sum runs over integers n_i satisfying the condition $n_1+n_3 = s$ and $n_2+n_3+2n_4 = J-j$. The number of structures is given by

$$N_{s,j;J} = \sum_{k=0}^{\mathrm{Min}(s,J-j)} \left(1+\left\lfloor \frac{J-k-j}{2}\right\rfloor\right). \tag{1.87}$$

Notice that this only makes sense for $J \geq j$, which is easily understood from the leading order OPE.

The $N_{s,j;J}$ structures do not correspond to as many independent coefficients when one of the primaries is conserved. Let us consider, as an example, the correlator of an $s = 1$, $j = 0$ defect primary and a conserved $J = 2$ bulk primary. In this case the general results (1.85), (1.86) give:

$$\langle O_{\Delta,2}(P_1, Z_1)\widehat{O}_{\widehat{\Delta},0,1}(P_2, Z_2, W_2)\rangle = \frac{b_{1,2,0,0}Q^1_{BD}(Q^2_{BD})^2 + b_{0,1,1,0}Q^2_{BD}Q^3_{BD} + b_{1,0,0,1}Q^1_{BD}Q^4_{BD}}{(-2P_1\bullet P_2)^{\widehat{\Delta}}(P_1\circ P_1)^{\frac{\Delta-\widehat{\Delta}}{2}}}. \tag{1.88}$$

The projection to real euclidean space can be found in (1.63). By imposing conservation in the form of eq. (1.52), that is

$$\partial_M D^M_{Z_1} \langle O_{\Delta,2}(P_1, Z_1)\widehat{O}_{\widehat{\Delta},0,1}(P_2, Z_2, W_2)\rangle = 0, \tag{1.89}$$

we get the following constraint:

$$\begin{aligned} &2b_{1,2,0,0}(\widehat{\Delta} + d(p - \widehat{\Delta})) + (q-1)\left(b_{0,1,1,0}d - 2b_{1,0,0,1}\widehat{\Delta}\right) = 0, \\ &b_{0,1,1,0}d(\widehat{\Delta} - p) - 2b_{1,0,0,1}(p+1) - 2b_{1,2,0,0} = 0, \end{aligned} \tag{1.90}$$

where we have used $\Delta = d$. For generic values of $\widehat{\Delta}$, p and d this implies that there is just one independent coefficient. However, if $\widehat{\Delta} = p+1$ the rank decreases and two independent coefficients remain. This happens when $\widehat{O}$ is the displacement operator - see section 1.5.

1.3.3 Two-point function of bulk primaries

In this section we analyze the structure of two-point functions of operators with spin in the bulk. The main novelty compared to the bulk-to-defect two-point function is that conformal symmetry is not powerful enough to fix completely the dependence on the positions. There are two cross-ratios, which we may choose as follows:

$$\xi = -\frac{2P_1 \cdot P_2}{(P_1\circ P_1)^{\frac{1}{2}}(P_2\circ P_2)^{\frac{1}{2}}}, \qquad \cos\phi = \frac{P_1\circ P_2}{(P_1\circ P_1)^{\frac{1}{2}}(P_2\circ P_2)^{\frac{1}{2}}}. \tag{1.91}$$

Let us pause to make a few comments on this choice. The first cross-ratio vanishes in the bulk OPE limit and diverges in the defect OPE one. The angle ϕ is defined in fig. 1.2. This angle is not defined in the codimension one case, where the number of cross-ratios reduces to one.[14] The cross-ratios (1.91) are especially suitable for describing the bulk OPE in Lorentzian signature, where $\xi \to 0$ while $\cos\phi$ may remain constant. On the contrary, since $\cos\phi$ goes to one in the Euclidean OPE limit, it is not a useful variable in this case, and may be traded for instance for the following one:

$$\zeta = \frac{1-\cos\phi}{\xi}. \tag{1.92}$$

Indeed, in the Euclidean OPE limit the two points approach each other along some direction v, or in other words $x_2 = x_1 + \epsilon\, v$. In the small ϵ limit the cross ratios (1.91) behave as

[14] Notice that $\xi = 4\xi_{[10]}$, the latter being the cross ratio defined in [10]. Their convention is motivated by the natural appearance of this factor in specific examples of boundary CFTs, as it is easy to see using the method of images. The same simplifications seem not to occur in examples with greater codimension q.

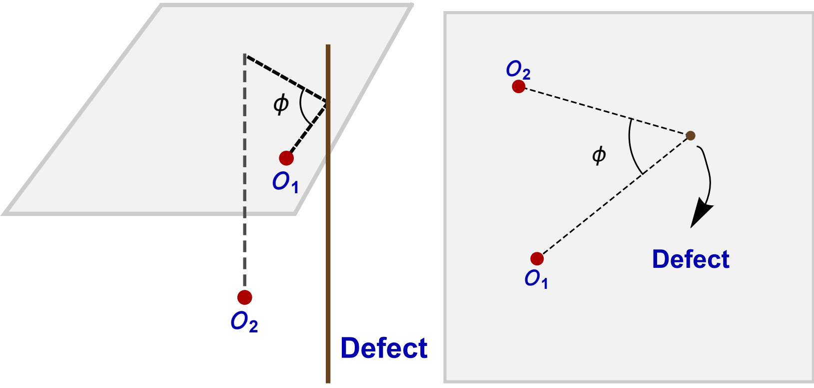

Figure 1.2: The angle ϕ is formed by the projections of the vectors P_1 and P_2 onto the q-dimensional space orthogonal to the defect. The left figure gives a perspective view of this angle, while the one on the right gives the top view. The defect is represented here by a brown line (or brown point in the top view).

$\xi \approx O(\epsilon^2)$ and $\cos\phi \approx 1 + O(\epsilon^2)$, so that (1.92) stays fixed in the limit. This choice is by no means unique. We leave a thorough analysis for a future work, in which the choice of the most appropriate cross-ratio will be further discussed, along the lines of [72].

Finally, in the defect OPE limit, it is convenient to substitute ξ with the following cross-ratio:

$$\chi = -\frac{2P_1 \bullet P_2}{(P_1 \circ P_1)^{\frac{1}{2}}(P_2 \circ P_2)^{\frac{1}{2}}}, \tag{1.93}$$

which has the property of being invariant under $SO(q)$ transformations applied only to one of the two coordinates. The same applies to $\cos\phi$, now with respect to $SO(p+1,1)$ transformations. This property is useful in solving the defect Casimir equation (1.101).

Let us now turn to the allowed structures in the correlation function. Recall that the polarization vectors Z_1 and Z_2 appear through the transverse structures (1.48). These can be concatenated and contracted with P_1 and P_2. In the end, one can write the two-point function as follows:

$$\langle O_{\Delta_1,J_1}(P_1,Z_1) O_{\Delta_2,J_2}(P_2,Z_2)\rangle = \sum_{\{n_i\}} \frac{\prod_{k=1}^{8} (Q^k_{BB})^{n_k} f_{n_1\ldots n_8}(\chi,\phi)}{(P_1 \circ P_1)^{\frac{\Delta_1}{2}} (P_2 \circ P_2)^{\frac{\Delta_2}{2}}}. \tag{1.94}$$

The indices n_i are subject to the constraints $n_1 + n_2 + n_5 + n_6 + 2n_7 = J_1$ and $n_3 + n_4 + n_5 + n_6 + 2n_8 = J_2$, which impose that the two point function is homogeneous of degrees J_1 and J_2 in the polarization vectors Z_1 and Z_2. Here are the building blocks Q^k_{BB}:

$$\begin{aligned}
&Q^1_{BB} = \frac{C_1^{AI} P_{1A} P_{2I}}{(P_1 \circ P_1)\,(P_2 \circ P_2)^{1/2}}, \quad Q^2_{BB} = \frac{C_1^{AI} P_{2A} P_{2I}}{(P_2 \circ P_2)\,(P_1 \circ P_1)^{1/2}}, \quad Q^3_{BB} = \frac{C_2^{AI} P_{1A} P_{2I}}{(P_2 \circ P_2)\,(P_1 \circ P_1)^{1/2}}, \\
&Q^4_{BB} = \frac{C_2^{AI} P_{1A} P_{1I}}{(P_2 \circ P_2)^{1/2} (P_1 \circ P_1)}, \quad Q^5_{BB} = \frac{C_1^{AI} C_2^{BI} P_{1A} P_{2B}}{(P_2 \circ P_2)(P_1 \circ P_1)}, \quad Q^6_{BB} = \frac{C_1^{AI} C_2^{AJ} P_{2I} P_{2J}}{(P_2 \circ P_2)^{3/2} (P_1 \circ P_1)^{1/2}}, \\
&Q^7_{BB} = Q_2(P_1, Z_1), \quad Q^8_{BB} = Q_2(P_2, Z_2).
\end{aligned} \tag{1.95}$$

The number of allowed structures is

$$\sum_{k=0}^{\mathrm{Min}(J_1,J_2)} (k+1)\left(J_1 - k + 1 + \lfloor \tfrac{J_1-k}{2} \rfloor\right)\left(J_2 - k + 1 + \lfloor \tfrac{J_2-k}{2} \rfloor\right). \tag{1.96}$$

Again, in the case of conserved operators the number of independent coefficients is smaller. The procedure used to obtain the relevant constraints in the previous subsection can be applied here as well.

1.3.4 Parity odd correlators

Let us finally comment on correlation functions of parity odd primaries. The strategy to construct them follows the same lines as in a homogeneous CFT: one has to consider all the allowed additional structures involving the ϵ-tensor [67]. The main difference here is that the part of the residual symmetry group connected to the identity does not relate parity transformations applied to parallel and orthogonal coordinates. As a consequence, it is possible to use two more ϵ-tensors.

The simplest possible example is provided by the one-point function of a bulk pseudo vector field in a CFT with a codimension $q = 2$ defect. Since it is a bulk operator, it transforms according to irreducible representations of $O(d+1,1)$, so we have to use the total ϵ-tensor:

$$\langle O_{\Delta,1}(P,Z)\rangle = a_O \frac{\epsilon_{01\dots p+1\, IJ} Z^I P^J}{(P\circ P)^{\frac{\Delta+1}{2}}} . \tag{1.97}$$

where the first $p+2$ coordinates are fixed by the defect.

Defect operators carry separate parallel and orthogonal parity quantum numbers. Correlation functions involving primaries which are odd under one or the other require the use of the ϵ-tensors $\epsilon_{AB\dots}$ and $\epsilon_{IJ\dots}$ respectively. The orthogonal ϵ-tensor has q indices that can be contracted with the vectors P_1, Z_1 and W_2. Therefore there are solutions only up to $q = 3$. When $q = 2$, the only transverse contractions are $\epsilon_{IJ} P_1^I W_2^J$ and $\epsilon_{IJ} P_1^I Z_1^J$, while for $q = 3$, $\epsilon_{IJK} P_1^I Z_1^J W_2^K$ is the only possible parity odd structure. For instance, the two-point function of a defect operator odd under inversion of an orthogonal coordinate and a bulk vector, with $q = 3$, reads

$$\langle O_{\Delta,1}(P_1,Z_1)\widehat{O}_{\widehat{\Delta},0,1}(P_2,W_2)\rangle = b_{O\widehat{O}} \frac{\epsilon_{IJK} W_2^I Z_1^J P_1^K}{(P_1\circ P_1)^{\frac{\Delta-\widehat{\Delta}+1}{2}}(-2P_1\bullet P_2)^{\widehat{\Delta}}} . \tag{1.98}$$

On the other hand, the parallel ϵ-tensor has $p+2$ indices that can be contracted with four vectors P_1, P_2, Z_1 and Z_2, which implies that there are solutions up to $p = 2$ and the correlators can be constructed exactly in the same way as was done above. In particular, when $p = 1$ only contractions of the kind $\epsilon_{ABC} P_1^A P_2^B Z_n^C$, $n = 1, 2$ are allowed.

1.4 Scalar two-point function and the conformal blocks

This section is devoted to the simplest crossing equation which contains information about the bulk-to-defect couplings. Let us consider the correlator between two scalar bulk operators $O_{1,2}$ of dimensions $\Delta_{1,2}$, which we write again in terms of the cross-ratios defined in subsection 1.3.3:

$$\langle O_1(P_1)O_2(P_2)\rangle = \frac{f_{12}(\xi,\phi)}{(P_1\circ P_1)^{\frac{\Delta_1}{2}}(P_2\circ P_2)^{\frac{\Delta_2}{2}}} . \tag{1.99}$$

The function $f_{12}(\xi,\phi)$ can be decomposed into two complete sets of conformal blocks[15] by plugging either the bulk or the defect OPE (1.237), (1.239) in the l.h.s. of eq. (1.99):

$$f_{12}(\xi,\phi) = \xi^{-(\Delta_1+\Delta_2)/2} \sum_k c_{12k}\, a_k\, f_{\Delta_k,J}(\xi,\phi) = \sum_{\widehat{O}} b_{1\widehat{O}}\, b_{2\widehat{O}}\, \widehat{f}_{\widehat{\Delta},0,s}(\xi,\phi)\,. \tag{1.100}$$

Equality of the last two expressions is an instance of the crossing symmetry constraint. The bulk conformal blocks $f_{\Delta_k,J}$ are eigenfunctions of the quadratic Casimir of the full conformal group $SO(d+1,1)$, while the defect conformal blocks $\widehat{f}_{\widehat{\Delta},j,s}$ are eigenfunctions of the quadratic Casimir of the stability subgroup $SO(p+1,1)\times SO(q)$. In what follows we study the solutions of the two associated Casimir equations. We refer to appendix (1.B.2) for a derivation of the equations themselves.

1.4.1 Defect channel Casimir equation

The sum in the rightmost side of eq. (1.100) runs over the defect primaries that appear in both the defect OPE of O_1 and O_2. As we remarked in subsection 1.3.2, only defect scalars ($j=0$) have a chance of being present. Since the stability subgroup is a direct product, the defect channel Casimir equation factorizes correspondingly, so that each defect conformal block satisfies separately the following eigenvalues equations:

$$\begin{aligned}
(\mathcal{L}^2 + \widehat{C}_{\widehat{\Delta},0}) \frac{\widehat{f}_{\widehat{\Delta},0,s}(\xi,\phi)}{(P_1\circ P_1)^{\frac{\Delta_1}{2}}(P_2\circ P_2)^{\frac{\Delta_2}{2}}} &= 0,\\
(\mathcal{S}^2 + \widehat{C}_{0,s}) \frac{\widehat{f}_{\widehat{\Delta},0,s}(\xi,\phi)}{(P_1\circ P_1)^{\frac{\Delta_1}{2}}(P_2\circ P_2)^{\frac{\Delta_2}{2}}} &= 0,
\end{aligned} \tag{1.101}$$

where $\widehat{C}_{\widehat{\Delta},s} = \widehat{\Delta}(\widehat{\Delta}-p) + s(s+q-2)$ and the differential operators $\mathcal{L} \equiv \frac{1}{2}(\mathcal{J}_{AB})^2$, $\mathcal{S} \equiv \frac{1}{2}\,(\mathcal{J}_{IJ})^2$ are defined through the generators

$$\mathcal{J}_{MN} = P_M \frac{\partial}{\partial P^N} - P_N \frac{\partial}{\partial P^M}. \tag{1.102}$$

In eq. (1.101), the operators $\mathcal{L}^2$ and $\mathcal{S}^2$ act only on one of the points, say P_1. Eqs. (1.101) immediately translate into differential equations for $\widehat{f}_{\widehat{\Delta},0,s}(\xi,\phi)$:

$$\begin{aligned}
\mathcal{D}^{L^2}_{\text{def}}\, \widehat{f}_{\widehat{\Delta},0,s}(\xi,\phi) &= 0,\\
\mathcal{D}^{S^2}_{\text{def}}\, \widehat{f}_{\widehat{\Delta},0,s}(\xi,\phi) &= 0.
\end{aligned} \tag{1.103}$$

The differential operators are most conveniently expressed by trading ξ for the variable χ, which was defined in (1.93):

$$\begin{aligned}
\mathcal{D}^{S^2}_{\text{def}} &\equiv 4\cos\phi(1-\cos\phi)\frac{\partial^2}{\partial\cos\phi^2} + 2(1-q\cos\phi)\frac{\partial}{\partial\cos\phi} + \widehat{C}_{0,s},\\
\mathcal{D}^{L^2}_{\text{def}} &\equiv (4-\chi^2)\frac{\partial^2}{\partial\chi^2} - (p+1)\chi\frac{\partial}{\partial\chi} + \widehat{C}_{\Delta,0}.
\end{aligned} \tag{1.104}$$

[15] See [73, 74] for an extensive discussion of conformal blocks in CFT, and [10, 49] for the case of a boundary.

The complete solution of the system (1.103) is then:

$$\widehat{f}_{\widehat{\Delta},0,s}(\chi,\phi) = \alpha_{s;q}\,\chi^{-\widehat{\Delta}}\,{}_2F_1\left(\frac{q}{2}+\frac{s}{2}-1,-\frac{s}{2};\frac{q}{2}-\frac{1}{2};\sin^2\phi\right) \times {}_2F_1\left(\frac{\widehat{\Delta}}{2}+\frac{1}{2},\frac{\widehat{\Delta}}{2};\widehat{\Delta}+1-\frac{p}{2};\frac{4}{\chi^2}\right), \tag{1.105}$$

where $\alpha_{s;q} = 2^{-s}\frac{\Gamma(q+s-2)}{\Gamma\left(\frac{q}{2}+s-1\right)}\frac{\Gamma\left(\frac{q}{2}-1\right)}{\Gamma(q-2)}$. The normalization has been chosen such that, given a leading contribution to the defect OPE of the kind:

$$O_1(x) = b_{1\widehat{O}}|x^i|^{-\Delta_1+\widehat{\Delta}-s}\,x_{i_1}...x_{i_s}\widehat{O}^{i_1...i_s}(\hat{x}) + ..., \tag{1.106}$$

the contribution of $\widehat{O}$ to the two-point function (1.99) is as shown in eq. (1.100). Also, recall that the normalization of the defect-defect correlator is fixed by (1.78) (see also eq. (1.58)). Finally, note that the transverse factor in the conformal block (1.105) correctly reduces to a Gegenbauer polynomial for integer s:

$${}_2F_1\left(\frac{q}{2}+\frac{s}{2}-1,-\frac{s}{2};\frac{q}{2}-\frac{1}{2};\sin^2\phi\right) = \frac{\Gamma(s+1)\Gamma(q-2)}{\Gamma(q+s-2)}\,C_s^{\left(\frac{q}{2}-1\right)}(\cos\phi)\,. \tag{1.107}$$

1.4.2 Bulk channel Casimir equation

The sum over bulk operators in the second equality of eq. (1.100) runs over all primaries admitted in $O\times O$ with non-vanishing one point functions. In particular, as follows from subsection 1.3.1, the sum can be restricted to even spins J. Each bulk conformal block is an eigenfunction of the $SO(d+1,1)$ Casimir operator $\mathcal{J}^2$ with eigenvalue $C_{\Delta_k,J} = \Delta_k(\Delta_k-d)+J(J+d-2)$:

$$(\mathcal{J}^2 + C_{\Delta_k,J})\frac{\xi^{-(\Delta_1+\Delta_2)/2}f_{\Delta_k,J}(\xi,\phi)}{(P_1\circ P_1)^{\frac{\Delta_1}{2}}(P_2\circ P_2)^{\frac{\Delta_2}{2}}} = 0, \tag{1.108}$$

where $\mathcal{J}^2 \equiv \frac{1}{2}(\mathcal{J}^{(1)}_{MN}+\mathcal{J}^{(2)}_{MN})^2$ and $\mathcal{J}_{MN}$ is defined in (1.102). The differential equation for $f_{\Delta_k,J}(\xi,\phi)$ which follows from eq. (1.108),

$$\mathcal{D}_{\text{bulk}}f_{\Delta_k,J}(\xi,\phi) = 0, \tag{1.109}$$

contains the differential operator

$$\begin{aligned}\mathcal{D}_{\text{bulk}} \equiv\; & 2\xi^2\left(2+\xi\cos\phi+2\cos^2\phi\right)\frac{\partial^2}{\partial\xi^2} + 2\sin^2\phi\left(2\sin^2\phi-\xi\cos\phi\right)\frac{\partial^2}{\partial\cos\phi^2} \\ & - 4\xi\sin^2\phi\,(\xi+2\cos\phi)\frac{\partial^2}{\partial\xi\partial\cos\phi} + 2\xi\left[2(1+\cos^2\phi)-(2d-\xi\cos\phi)\right]\frac{\partial}{\partial\xi} \\ & +\left[2\xi(q-2+\cos^2\phi)-4\cos\phi\sin^2\phi\right]\frac{\partial}{\partial\cos\phi} - \left[\Delta_{12}^2\cos\phi\left(\cos\phi+\frac{\xi}{2}\right)-\Delta_{12}^2+2C_{\Delta_k,J}\right],\end{aligned} \tag{1.110}$$

where $\Delta_{12} = \Delta_1-\Delta_2$. We will not be able to solve this differential equation in closed form in the most general case. In the next subsection, we provide a recurrence relation for the light-cone expansion of the conformal block. Then, in subsection (1.4.2), we will consider a special case, in which the Casimir equation can be mapped to a different one, well studied

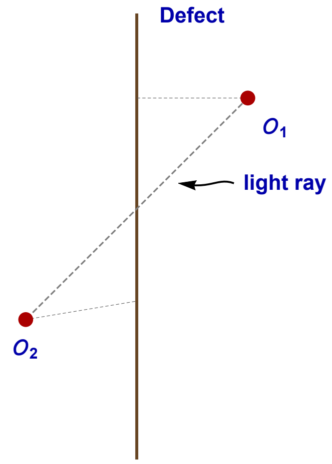

Figure 1.3: The light-cone limit relevant for the bulk Casimir equation: the defect is time-like and the operators are space-like separated from it.

in the literature. Finally, we leave to appendix 1.B.3 a solution for the scalar block in terms of an infinite series. For now, let us point out that the equation can be solved when the codimension is one ($q = 1$). In this case, only scalars acquire an expectation values ($J = 0$), and (1.109) reduces to an hypergeometric equation in ξ. The solution with the correct asymptotics is [10]

$$f_{\Delta_k,0}(\xi) = \xi^{\frac{\Delta_k}{2}}\, {}_2F_1\left(\frac{\Delta_k + \Delta_{12}}{2}, \frac{\Delta_k - \Delta_{12}}{2}, \Delta_k + 1 - \frac{d}{2}; -\frac{\xi}{4}\right). \tag{1.111}$$

We fix the normalization of the conformal blocks in the next subsection, while discussing the collinear block - see eq. (1.116).

Lightcone expansion

It is a well understood fact that the nature of the light cone limit in the Lorentzian OPE limit is different from the Euclidean one. For instance, the operators that dominate in each limit are not the same: while in the Lorentzian the operators with lowest twist contribute the most (with the twist $\tau = \Delta_k - J$ being defined as the dimension minus the spin), in the Euclidean it is the ones with lowest dimension. This is easily seen by considering the leading order OPE

$$O_1(x_1)O_2(x_2) = \sum_k \frac{c_{12k}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_k}{2}}} \left[\frac{x_{12\mu_1}\dots x_{12\mu_J}}{(x_{12}^2)^{\frac{J}{2}}}\, O_k^{\mu_1\dots\mu_J}(x_2) + \text{descendants}\right]. \tag{1.112}$$

It follows directly from the OPE that ξ controls the twist of the operator being exchanged in the OPE limit. The conformal block resums the contribution of a conformal family. It follows from the observation above that it is possible to organize every such family into operators with the same twist, which contribute at the same order in the light-cone limit. In said limit, the operators become light-like separated without colliding - see fig. 1.3. In terms of the cross-ratios, this corresponds to sending $\xi \to 0$ while holding ϕ fixed. Then, from a direct analysis of the solution of the Casimir equation for the first few orders in small ξ, one is led to the following ansatz:

$$f_{\Delta_k,J}(\xi,\phi)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}c_{m,k}g_{\tau+2m,J-2k}(\xi,\phi),\qquad g_{\tau,J}(\xi,\phi)=\xi^{\tau/2}\tilde{g}_{\tau,J}(\phi),\tag{1.113}$$

where $g_{\tau,J}(\xi,\phi)$ includes the contribution of operators with twist τ and spin J, and we determine it below. Notice that, given a primary operator $O_{\mu_1,\ldots,\mu_J}(x_1)$ it is possible to construct different descendants with the same twist but different spin. For example, given a primary operator with spin J and twist τ we can create a descendant operator with twist $\tau+2$ either by acting with P^2 or by contracting one of the indices of the primary operator with P^μ. In formula (1.113) this degeneracy is labelled by k. For a given m, the number of descendants with different spin is $m+1$.

In order to constrain the functions $g_{\tau,J}(\xi,\phi)$ and the coefficients $c_{m,k}$, it is convenient to divide the Casimir differential operator in two parts [72], one that keeps the degree of ξ and other that does not, $\mathcal{D}_{\text{bulk}}=\mathcal{D}_0+\mathcal{D}_1+2C_{\Delta_k,J}$:

$$\begin{aligned}\mathcal{D}_0\equiv{}&4\xi^2(1+\cos^2\phi)\frac{\partial^2}{\partial\xi^2}+4\sin^4\phi\frac{\partial^2}{\partial\cos\phi^2}-8\xi\sin^2\phi\cos\phi\frac{\partial^2}{\partial\xi\partial\cos\phi}-4\xi(d-1-\cos^2\phi)\frac{\partial}{\partial\xi}\\&-4\cos\phi\sin^2\phi\frac{\partial}{\partial\cos\phi}-\Delta_{12}^2(\cos^2\phi-1),\end{aligned}\tag{1.114}$$

$$\begin{aligned}\mathcal{D}_1\equiv{}&2\xi^3\cos\phi\frac{\partial^2}{\partial\xi^2}-2\xi\sin^2\phi\cos\phi\frac{\partial^2}{\partial\cos\phi^2}-4\xi^2\sin^2\phi\frac{\partial^2}{\partial\xi\partial\cos\phi}+2\xi^2\cos\cos\phi\frac{\partial}{\partial\xi}\\&+2\xi(d-p-2+\cos^2\phi)\frac{\partial}{\partial\cos\phi}-\frac{\xi}{2}\Delta_{12}^2\cos\phi.\end{aligned}\tag{1.115}$$

By inspection of the Casimir equation, one can conclude that $g_{\tau,J}(\xi,\phi)$ is an eigenfunction of the differential operator $\mathcal{D}_0$ with eigenvalue $2C_{\tau,J}=(J+\tau)(J+\tau-d)+J(d+J-2)$. More precisely, for $m=0$ this property coincides with the leading order of the Casimir equation. At all orders, we simply check that the equation can be solved by the ansatz (1.113) and this choice of functions. Clearly, the solution is unique once the asymptotic behavior has been chosen. The eigenvalue equation fixes $g_{\tau,J}(\xi,\phi)$ to be

$$g_{\tau,J}(\xi,\phi)=\xi^{\frac{\tau}{2}}\sin^J\phi\,{}_2F_1\left(\frac{2J+\tau+\Delta_{12}}{4},\frac{2J+\tau-\Delta_{12}}{4},\frac{2J+\tau+1}{2},\sin^2\phi\right).\tag{1.116}$$

We chose the solution in such a way that $g_{\tau,J}(\xi,\phi)\sim\xi^{\frac{\tau}{2}}\sin^J\phi$ in the limit $\phi\to0$, which is the asymptotics required by the Euclidean OPE limit. The normalization of $g_{\tau,J}$ has been fixed so that, given the leading order OPE eq. (1.112), and the normalization of the one-point function eq. (1.79), the conformal block (1.113) contributes to the two-point function as in eq. (1.100), once $c_{0,0}=1$.[16]

The action of $\mathcal{D}_1$ on $g_{\tau,J}(\xi,\phi)$ can be expressed as a combination of the these building

[16]This matching is most easily done by noticing that in the light cone limit x^μ in eq. (1.112) becomes null. Then, comparing with the one point function (1.79), we can just specify $z=x$. This immediately leads to

$$\langle O_1(x_1)O_2(x_2)\rangle\sim\frac{c_{12k}a_O}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_k+J)}}\frac{((x_1\circ x_{12})^2-x_{12}\circ x_{12}\;x_1\circ x_1)^{\frac{J}{2}}}{(x_1\circ x_1)^{\frac{\Delta_k+J}{2}}}.\tag{1.117}$$

We can compare with the collinear block by taking the limit $x_{12}\to0$, which corresponds to $\sin\phi\to0$.

blocks with one more unit of twist

$$\mathcal{D}_1 g_{\tau,J}(\xi,\phi) = b^+_{\tau,J}\, g_{\tau+1,J}(\xi,\phi) + b^-_{\tau,J}\, g_{\tau+1,J-2}(\xi,\phi), \tag{1.118}$$
$$b^-_{\tau,J} = 8J(q+J-3), \quad b^+_{\tau,J} = \frac{(J+\tau-1)\left(4J^2+4J\tau+\tau^2-\Delta_{12}^2\right)(q-J-\tau-2)}{2(2J+\tau-1)(2J+\tau+1)}.$$

Thus, the Casimir equation can be translated into a recurrence relation that the coefficients $c_{n,k}$ must satisfy:

$$2(C_{\tau+2n,J-2k} - C_{\tau,J})c_{m,k} = b^+_{\tau+2n-2,J-2k}c_{n-1,k} + b^-_{\tau+2n-2,J-2k+2}c_{n-1,k-1}. \tag{1.119}$$

All coefficients $c_{n,k}$ are determined once we impose the initial conditions $c_{n,k} = 0$ for $n < j$ and $c_{0,0} = 1$. In particular, in the case of an exchanged scalar primary, the general term of the recursion can be recovered from eq. (1.263). Finally, notice that this solution can be straightforwardly applied to the conformal block for the four-point function without a defect. In that case we would be solving the Casimir equation in the limit of small u and fixed v - see appendix A of [75].

Defects of codimension two and the four-point function

The four-point function of scalar primaries in a homogeneous CFT has been extensively studied in the literature. Many results are available for the conformal blocks in this case [73, 74, 75, 76, 77, 78, 79]. In the special case of a codimension two defect - with equal external dimensions - all those results apply in fact to our problem. Indeed, let us consider the change of variables

$$u = -\xi e^{-i\phi}, \qquad v = e^{-2i\phi}; \tag{1.120a}$$
$$\xi = -\frac{u}{\sqrt{v}}, \qquad \cos\phi = \frac{1+v}{2\sqrt{v}}. \tag{1.120b}$$

Let us denote by $\mathcal{D}_{\text{CFT}}$ the Casimir differential operator for the four-point function without defect, and let us choose pairwise equal external dimensions $\widetilde{\Delta}_{12} = \widetilde{\Delta}_{34} = 0$, so that they do not appear in $\mathcal{D}_{\text{CFT}}$ - see for instance eqs. (2.10) and (2.11) in [74]. We obtain the following relation with the operator (1.110), when $\Delta_1 = \Delta_2$:

$$\mathcal{D}_{\text{def}}[f_{\Delta_k,J}(u,v)] = \left(\mathcal{D}_{\text{CFT}} f_{\Delta_k,J}(u,v) + \mathcal{D}_q f_{\Delta_k,J}(u,v)\right) = 0, \tag{1.121}$$

where

$$\mathcal{D}_q = \frac{4u(q-2)}{1-v}\left(u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}\right). \tag{1.122}$$

Hence, in the special case of $q = 2$, the functions $f_{\Delta_k,J}(u,v)$ solve the Casimir equation for the four point function with pairwise equal external operators, $f^{4\text{pt}}_{\Delta_k,J}(u,v)$:

$$f_{\Delta_k,J}(u,v) = f^{4\text{pt}}_{\Delta_k,J}(u,v). \tag{1.123}$$

To confirm that (1.123) is an equality between conformal blocks, we need to match the asymptotic behavior.[17] In fact, we can check that (1.123) holds explicitly at leading order in the lighcone expansion (1.113):

$$f_{\Delta_k,J}(\xi,\phi) = g_{\tau,J}(\xi,\phi) + \mathcal{O}(\xi), \tag{1.124}$$

[17] Conformal blocks of the four-point function with internal quantum numbers Δ' and J', such that $C_{\Delta_k,J} = C_{\Delta',J'}$, solve eq. (1.110) as well. The asymptotics single out the unique physical solution (1.123), with $\Delta' = \Delta_k$ and $J' = J$.

where $g_{\tau,J}(\xi,\phi)$ is defined in (1.116). An identity between hypergeometric functions

$$e^{-i(a+m)\phi}{}_2F_1\left(a+m, m+\frac{a+b}{2}, a+b+2m; 2i\sin\phi e^{-i\phi}\right) \\ = {}_2F_1\left(\frac{a+m}{2}, \frac{b+m}{2}, \frac{a+b+1}{2}+m; \sin^2\phi\right). \quad (1.125)$$

allows to rewrite (1.116) in terms of u and v

$$g_{\tau,J}(\xi,\phi) = \mathrm{i}^{-\Delta}u^{\frac{\tau}{2}}\left(\frac{1-v}{2}\right)^J {}_2F_1\left(\frac{\tau}{2}+J, \frac{\tau}{2}+J, \tau+2J; 1-v\right). \quad (1.126)$$

Here we have assumed once again $\Delta_1 = \Delta_2$. The result agrees with the collinear block for the four-point function - see for instance eq. (118) in [75]. Hence, all formulae for the conformal blocks of a homogeneous CFT provide as many results for the bulk channel blocks of a defect CFT with codimension two. It would be interesting to have a geometric understanding of this fact. For now, let us just notice that the mapping (1.120) has a chance of having some kinematic meaning only when the defect is placed in a space with Lorentzian signature.

An interesting check of eq. (1.123) is related to the reality property of $f^{\text{4pt}}_{\Delta_k,J}$. If ξ and ϕ are real, this block should be real when evaluated on the map (1.120). Certainly, $\left(f^{\text{4pt}}_{\Delta_k,J}(u,v)\right)^*$ and $f^{\text{4pt}}_{\Delta_k,J}(u^*,v^*)$ coincide, since they satisfy the same Casimir equation with the same boundary conditions. Then one notices that, through eq. (1.120), the complex conjugation acts on u, v as

$$^* : (u,v) \longrightarrow \left(\frac{u}{v}, \frac{1}{v}\right). \quad (1.127)$$

But this is easily recognized as the effect of exchanging points x_1 and x_2.[18] Conformal blocks for the exchange of a primary with even spin are invariant under this crossing, when $\widetilde{\Delta}_{12} = \widetilde{\Delta}_{34} = 0$. We conclude that $f^{\text{4pt}}_{\Delta_k,J}(u^*,v^*) = f^{\text{4pt}}_{\Delta_k,J}(u,v)$ precisely in the case of interest.

1.5 Ward identities and the displacement operator

This section is devoted to the Ward identities involving the stress-tensor, in the presence of a flat defect, both in the Poincaré (or rather, Euclidean) invariant and in the fully conformal cases. Throughout the section, we ignore the issue of defect anomalies. We derive the Ward identities in the next subsection. These involve a number of defect operators, one of which plays a pre-eminent role: the displacement operator defined in eq. (1.150). In subsection 1.5.2, we focus on the displacement operator and its properties, and derive some constraints on its appearance in the defect OPEs of a generic theory. In subsection 1.5.3, we take a look at two dimensional interfaces, and prove unitarity bounds for the Zamolodchikov norm C_{D} of the displacement. Finally, in subsection 1.5.4, we consider examples of free defect theories, in which specific identities can be given to the operators defined in subsection 1.5.1, in terms of the elementary fields.

[18] Recall that in the standard notation $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

1.5.1 The Ward identities for diffeomorphisms and Weyl transformations

Let us consider a defect quantum field theory defined on a manifold $\mathcal{M}$ and coupled to a background metric g. We define the embedding of the defect sub-manifold $\mathcal{D}$ through

$$x^\mu = X^\mu(\sigma^a), \tag{1.128}$$

where the coordinates σ^a, $a = 1, \dots, p$ provide a parametrization of $\mathcal{D}$. We assume that a vacuum energy functional is defined through a functional integral of the form

$$W[g, X] = \log \int [D\phi] \exp \{-S[g, X, \phi, \partial\phi, \dots]\}\,. \tag{1.129}$$

We ask the action functional to be invariant under diffeomorphisms on $\mathcal{M}$, once the embedding function is allowed to change accordingly:

$$S[g + \delta_\xi g,\, X + \delta_\xi X,\, \phi + \delta_\xi \phi,\, \partial\phi + \delta_\xi \partial\phi, \dots] = S[g, X, \phi, \partial\phi, \dots], \tag{1.130}$$
$$\delta_\xi X^\mu = \xi^\mu, \tag{1.131}$$
$$\delta_\xi g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \tag{1.132}$$

This can be achieved by coupling the defect degrees of freedom to local geometric quantities. The induced metric γ_{ab}, the q-tuple of normal unit vectors n^μ_A, the extrinsic curvatures K^A_{ab} and the spin connection on the normal bundle μ_{aIJ} are defined in eqs. (1.266), (1.268), (1.270) and (1.271) respectively. Further available building blocks are given by components of the bulk Riemann tensor evaluated at the defect, etc. All of these quantities transform as tensors under a diffeomorphism in $\mathcal{M}$, up to a local rotation in the normal bundle - see in particular eqs. (1.284). We find therefore convenient to ask for symmetry under local rotations in the normal bundle. Then the defect action may be constructed as the integral of a scalar function over $\mathcal{D}$. Let us emphasize that the set of counterterms needed to make the functional integral finite should be diffeomorphism invariant as well. Of course, we also assume the action to be invariant under reparametrizations of the defect.

In this section, we shall first follow the procedure employed in [9], which consists in defining many defect operators, as the response of the partition function to the variations of each defect geometric quantity. Later we comment on the relation with a quicker approach, in which the only operators involved are the stress-tensor and the variation of the partition function with respect to the embedding coordinate $X(\sigma)$.

The Ward identities that follow from diffeomorphism invariance are obtained by the standard procedure. Given a set of bulk local operators O_i, let us also define the abbreviation

$$\mathcal{X} = O_1(x_1, z_1) \dots O_n(x_n, z_n). \tag{1.133}$$

The following equation holds:

$$- \delta_\xi \langle \mathcal{X} \rangle - \int_{\mathcal{M}} \nabla_\mu \xi_\nu \langle T^{\mu\nu}\, \mathcal{X} \rangle + \int_{\mathcal{D}} \Big\langle \Big(\frac{1}{2} B^{ab} \delta_\xi \gamma_{ab} + {\eta_\mu}^a \delta_\xi e^\mu_a + {\lambda_\mu}^I \delta_\xi n^\mu_I$$
$$+ \frac{1}{2} C^{ab}_I \delta_\xi K^I_{ab} + D_\mu\, \delta_\xi X^\mu + j^{aIJ} \delta_\xi \mu_{aIJ} + \dots \Big) \mathcal{X} \Big\rangle = 0. \tag{1.134}$$

Here we made use of the standard definition of the stress-tensor:

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}; \tag{1.135}$$

It is important that this definition is adopted only for points in the bulk. The operators B^{ab}, ${\eta_\mu}^a$, ${\lambda_\mu}^I$, C_I^{ab}, D_μ, j^{aIJ} are local in local theories, and are defined by *minus* the variation of the action with respect to the relevant quantities, in analogy with (1.135). In particular, D_μ is obtained by the variation with respect to X^μ, keeping fixed the intrinsic and extrinsic geometry of the defect. Finally, the dots stand for higher dimension geometric quantities, that we disregard for simplicity. Let us note that eq. (1.134) holds for other symmetries of the vacuum functional, provided the variations under diffeomorphisms $\delta_\xi g_{\mu\nu}, \delta_\xi \gamma_{ab}, \ldots$ are appropriately replaced; we will use this fact repeatedly below.

Taking into account eqs. (1.284), the relation (1.134) simplifies to

$$-\delta_\xi \langle \mathcal{X} \rangle - \int_\mathcal{M} \nabla_\mu \xi_\nu \langle T^{\mu\nu} \mathcal{X} \rangle + \int_\mathcal{D} \langle \left(\partial_a \xi^\mu {\eta_\mu}^a + n_I^\mu \partial_\mu \xi^\nu {\lambda_\nu}^I + \xi^\mu D_\mu + \ldots \right) \mathcal{X} \rangle = 0. \quad (1.136)$$

In deriving this, we used the invariance of the path integral under local rotations, which implies the following Ward identity, valid in the absence of defect operators:

$$\langle \left(\lambda^{\mu[I} n_\mu^{J]} + \frac{1}{2} C^{ab[I} K_{ab}^{J]} + \nabla_a j^{aIJ} \right) \mathcal{X} \rangle = 0; \quad (1.137)$$

this can be seen by plugging the variations in eq. (1.272) into the analogue of eq. (1.134).

The components of D_μ parallel to the defect are further related to the other local operators defined above. The link is provided by the Ward identities associated to reparametrizations of the defect $\sigma'^a = \sigma^a + \zeta^a$, under which

$$\delta_\zeta X^\mu = -e_a^\mu \zeta^a, \quad (1.138)$$

which in turn induces variations of all other quantities which are summarized in eq. (1.287). In the absence of operators localized at the defect, the following identity is then easily obtained:

$$\begin{aligned} \langle D_a \mathcal{X} \rangle = \langle \Big(& \nabla^b B_{ab} - K_{ab}^I {\eta_I}^b + \nabla_b {\eta_a}^b + \Gamma^\mu_{ab} {\eta_\mu}^b + \mu_{aIJ} \lambda^{IJ} - \partial_a n_I^\mu {\lambda_\mu}^I \\ & + \frac{1}{2} \left(-\nabla_a K_{bc}^I + 2 \nabla_c K_{ab}^I + 2 K_{ab}^I \nabla_c \right) C_A^{bc} + \left(2 K_{bc}^I {K^c}_a{}^J - \frac{1}{2} R_{abIJ} + R_{a[IJ]b} \right) j^{bIJ} \Big) \mathcal{X} \rangle. \end{aligned} \quad (1.139)$$

We can work out the consequences of Weyl invariance on correlation functions in the presence of a defect in the same way. A Weyl transformation acts on the metric as

$$\delta_\sigma g_{\mu\nu} = 2\sigma(x)\, g_{\mu\nu} \quad (1.140)$$

and induces the variations given in eq. (1.289). Therefore, the Ward identity for Weyl transformations reads

$$-\delta_\sigma \langle \mathcal{X} \rangle + \int_\mathcal{M} \sigma \langle T^\mu_\mu \mathcal{X} \rangle + \int_\mathcal{D} \sigma \langle (B^a_a - n_I^\mu {\lambda_\mu}^I + \frac{1}{2} K^I_{ab} C_I^{ab}) \mathcal{X} \rangle - \frac{1}{2} \int_\mathcal{D} \partial_\mu \sigma\, n_I^\mu \langle C_a^{aI} \mathcal{X} \rangle = 0. \quad (1.141)$$

As usual, the transformation law of correlation functions in a conformal field theory can be inferred from the ones above, due to the fact that conformal transformations are a subgroup of diffeo×Weyl transformations, for manifolds which possess conformal killing vectors. The variation of the partition function under conformal transformations vanishes in the absence of the defect, but this is not true any more in our set-up. Let us consider a conformal killing vector $\hat{\xi}$, for which

$$\nabla_{(\mu} \hat{\xi}_{\nu)} = -\frac{\nabla_\rho \hat{\xi}^\rho}{d} g_{\mu\nu} \equiv -\hat{\sigma} g_{\mu\nu} \ . \quad (1.142)$$

Effecting a diffeomorphism of parameter $\hat{\xi}^\mu$ and a compensating Weyl rescaling of parameter $\hat{\sigma}$ corresponds to a conformal transformation that leaves the metric invariant. However, no Weyl transformation can undo the action of the diffeomorphism on the embedding function X^μ, hence the non vanishing variation of the partition function. Let us decompose the diffeomorphism in his tangent and normal components:

$$\hat{\xi}^\mu(X) = e^\mu_a \hat{\xi}^a + n^\mu_I \hat{\xi}^I . \tag{1.143}$$

A diffeomorphism tangent to the defect can be undone by a reparametrization: in the end, we are left with the variation induced by the components of the conformal killing vector which do not preserve the defect. In order to see this, one starts from the Ward identities associated to the composition of the diffeomorphism $\hat{\xi}$ - eq. (1.136) - with the compensating Weyl transformation - eq. (1.141). Accordingly, the stress-tensor cancels out in the sum. Then, one eliminates $\hat{\xi}^a D_a$ by use of the Ward identity for reparametrizations, eq. (1.139). Finally, a straightforward but lengthy manipulation shows that all remaining terms involving the parallel components ξ^a correspond to transverse rotations and can be reabsorbed using eq. (1.137). The resulting formula is[19]

$$\begin{aligned} -(\delta_{\hat{\xi}} + \delta_{\hat{\sigma}}) \langle \mathcal{X} \rangle = & -\int_\mathcal{D} \hat{\xi}^I \langle D_I \, \mathcal{X} \rangle + \int_\mathcal{D} \hat{\xi}_I K^I_{ab} \langle B^{ab} \mathcal{X} \rangle + \int_\mathcal{D} \left(\hat{\xi}^J \Gamma^\nu_{IJ} + e^{\nu a} \nabla_a \hat{\xi}_I \right) \langle \lambda_\nu{}^I \mathcal{X} \rangle \\ & + \int_D \left(-\nabla_b \hat{\xi}^I n^\mu_I + \hat{\xi}_I K^I_{ab} e^{a\mu} + \hat{\xi}^I \Gamma^\mu_{Ib} \right) \langle \eta_\mu{}^b \mathcal{X} \rangle \\ & + \frac{1}{2} \int_\mathcal{D} \left(- \nabla_a \nabla_b \hat{\xi}^I + K_{ac}{}^I K_b{}^{cJ} \hat{\xi}_J + \hat{\xi}^J R_{Jab}{}^I \right) \langle C^{ab}_I \mathcal{X} \rangle \\ & + \int_\mathcal{D} \left(2 K^{[I}_{ab} \nabla^b \hat{\xi}^{J]} - \frac{1}{2} \hat{\xi}^K R_{KaIJ} + \hat{\xi}^K R_{K[IJ]a} \right) \langle j^{aIJ} \mathcal{X} \rangle . \end{aligned} \tag{1.144}$$

Eq. (1.144) says that the role of a certain subset of defect operators contained in the OPE of the stress-tensor is to implement a conformal transformation which does not fix the defect.

From now on, we would like to focus on a flat defect in flat space, and recover explicitly the contact terms of the stress-tensor with the defect, via the unintegrated form of equations (1.136) and (1.141). Before doing it, it is maybe worth stressing that eq. (1.141) contains already all the information about the trace of the stress-tensor in the case of scale but not conformal invariant theories. We can choose the usual cartesian coordinates $\mu = (a, i)$, pick an adapted basis of normal vector fields ($I \to i$), and place the defect in $X^i = 0$. The Ward identities for diffeomorphism invariance read

$$\begin{aligned} \langle \partial_\mu T^{\mu\nu}(x) \, \mathcal{X} \rangle = & -\delta_\mathcal{D}(x) \delta^\nu_i \langle \left(D^i(x^a) - \partial_a \eta^{ia}(x^a) \right) \mathcal{X} \rangle - \delta_\mathcal{D}(x) \delta^\nu_a \langle \partial_b B^{ab}(x^a) \, \mathcal{X} \rangle \\ & + \partial_i \delta_\mathcal{D}(x) \langle \lambda^{\nu i}(x^a) \, \mathcal{X} \rangle + \text{contact terms associated with } \mathcal{X}, \end{aligned} \tag{1.145}$$

where $\delta_\mathcal{D}(x)$ is a delta function with support on the defect, in particular $\delta_\mathcal{D}(x) = \delta^q(x^j)$ in this case. Eq. (1.139) reduces to

$$\langle \left(D_a - \partial_b \eta_a{}^b \right) \mathcal{X} \rangle = \langle \partial_b B^b_a \, \mathcal{X} \rangle . \tag{1.146}$$

The physical meaning of these equations is transparent: they regulate the exchange of energy and momentum between the bulk and the defect. In particular, when the defect

[19] For instance, if $\mathcal{X}$ contains a scalar field O, then $\delta_\xi O = -\xi^\mu \partial_\mu O$ and $\delta_{\hat{\sigma}} O = -\hat{\sigma} \Delta O$.

is decoupled both D_a and $\eta_\mu{}^a$ generically vanish: eq. (1.146) tells us that in these cases the defect stress-tensor is separately conserved, as expected. Notice also that C^I_{ab} does not appear in eqs. (1.145) and (1.146). This is expected, since a coupling with the extrinsic curvatures is never required in order to reformulate the theory in a diffeomorphism invariant way. On the contrary, this coupling can be necessary to enforce Weyl invariance - as can be foreseen by comparing the third line of eq. (1.289) with the transformation law of the derivative of a bulk primary operator. It is therefore not surprising that C^I_{ab} does appear in the Ward identities associated with the trace of the stress-tensor in a CFT:

$$\langle T^\mu_\mu(x)\,\mathcal{X}\rangle = -\delta_{\mathcal{D}}(x)\,\langle\big(B^a_a(x^a)-\lambda_i{}^i(x^a)\big)\mathcal{X}\rangle - \frac{1}{2}\partial_i\delta_{\mathcal{D}}(x)\,\langle C_a^{a\,i}(x^a)\,\mathcal{X}\rangle$$
$$+\ \text{contact terms associated with } \mathcal{X}. \quad (1.147)$$

One can summarize (1.145) and (1.147) by adding contact terms to the definition of the stress-tensor itself. In other words, let us define

$$T^{\mu\nu}_{\text{tot}} = T^{\mu\nu}+\delta_{\mathcal{D}}(x)\left(\delta^\mu_a\delta^\nu_b B^{ab} - 2\delta_a^{(\mu}\delta_i^{\nu)}\left(\lambda^{ai}+\frac{1}{2}\partial_b C^{ab\,i}\right) - \delta_i^{(\mu}\delta_j^{\nu)}\lambda^{ij}\right)+\frac{1}{2}\partial_i\delta_{\mathcal{D}}(x)\delta^\mu_a\delta^\nu_b C^{ab\,i}. \quad (1.148)$$

In terms of T_{tot}, the Ward identities take a simpler form:

$$\partial_\mu T^{\mu a}_{\text{tot}} = 0, \qquad \partial_\mu T^{\mu i}_{\text{tot}} = -\delta_{\mathcal{D}}(x)\mathrm{D}^i + \partial_k\delta_{\mathcal{D}}(x)\lambda^{[ik]}, \qquad (T_{\text{tot}})^\mu_\mu = 0, \quad (1.149)$$

where we defined the *displacement operator*:

$$\mathrm{D}_i = D_i - \partial_a\eta_i{}^a + \partial_a\lambda^a{}_i + \frac{1}{2}\partial_a\partial_b C^{ab}_i. \quad (1.150)$$

The form (1.149) of the Ward identities - which clearly still has to be interpreted as an operatorial equation - make it manifest that one can construct globally conserved currents for the space-time symmetries preserved by the defect.[20]

By using eqs. (1.275) to (1.280), it is easy to see that T_{tot} and D_i are defined by the total variations with respect to the bulk metric and the embedding function respectively. This provides a compact way of writing the consequences of diffeomorphism and Weyl invariance; on the other hand, it obscures the complete set of possible contact terms of the stress-tensor, which from the CFT point of view correspond to the presence of specific singularities in its defect OPE.

Finally, the flat space version of eq. (1.144) is the following:

$$(\delta_{\xi_{\text{c}}} + \delta_{\hat{\sigma}})\,\langle\mathcal{X}\rangle = \int_{\mathcal{D}} \hat{\xi}^i\,\langle \mathrm{D}_i\,\mathcal{X}\rangle\,. \quad (1.151)$$

We see that the displacement operator completely encodes the effect of conformal transformations on correlation functions in the presence of a flat defect. Furthermore, eq. (1.149) fixes both the scale dimension $\Delta_D = p+1$ and the normalization of this operator. It follows that its Zamolodchikov norm C_{D}, defined by

$$\langle \mathrm{D}_i(x)\mathrm{D}_j(0)\rangle = C_{\mathrm{D}}\,\frac{\delta_{ij}}{(x^2)^{p+1}}, \quad (1.152)$$

is a piece of CFT data.

[20]Eq. (1.137) implies that the second addend in $\partial_\mu T^{\mu i}_{\text{tot}}$ is a total derivative. This allows to define a conserved current for rotations in transverse directions, which in general contains a contribution from internal degrees of freedom of the defect.

1.5.2 Constraints on CFT data

The integrated Ward identity (1.151) provides information on the coupling of the displacement to bulk operators, analogously to what happens for the appearance of the stress-tensor in the bulk OPE. Indeed, when $\mathcal{X}$ comprises only one operator, both sides in eq. (1.151) are free of cross-ratios, and only depend on the CFT data. This generates constraints, of which we consider three examples, namely the defect OPE of a scalar, vector, and 2-index tensor primaries.

The easiest way of proceeding is to lift eq. (1.151) to the projective light-cone. On the l.h.s., we just need to apply the appropriate generator of the Lorentz group in $d+2$ dimensions. On the r.h.s., the corresponding killing vector is contracted with the displacement, and one should make sense of the integration by defining the correct measure on the light-cone. The issue of integration has be settled in [80], whose results we borrow. We are going to assume that the operator D_i is a primary, and check that the functional form after integration matches the left hand side.

Let us start from a scalar operator $O_{\Delta,0}(P)$. Under the Lorentz group, the change of its one-point function coincides with the coordinate transformation, so the generator is the same as the killing vector:

$$\mathcal{J}_{AI} = 2\, P_{[A} \frac{\partial}{\partial P^{I]}}. \tag{1.153}$$

We can then write eq. (1.151) as follows:

$$P_A \frac{\partial}{\partial P^I} \langle O_{\Delta,0}(P) \rangle = - \int D^p Q\, Q_A \langle O_{\Delta,0}(P) \mathrm{D}_I(Q) \rangle \,. \tag{1.154}$$

We just need to plug in the form of the correlators involved, that is:

$$\langle O_{\Delta,0}(P_1) \mathrm{D}(P_2, Z_2) \rangle = b_{O\mathrm{D}} \frac{Z_2 \circ P_1}{(P_1 \circ P_1)^{(\Delta-p)/2} (-2 P_1 \bullet P_2)^{(p+1)}}, \tag{1.155}$$
$$\langle O_{\Delta,0}(P_1) \rangle = \frac{a_O}{(P_1 \circ P_1)^{\Delta/2}}, \tag{1.156}$$

The integrand has dimension p, so that the integral is well defined and can be computed along the lines of [80]. We obtain the following scaling relation:

$$\Delta\, a_O = \left(\frac{\pi}{4}\right)^{\frac{p}{2}} \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}(p+1)\right)}\, b_{O\mathrm{D}}. \tag{1.157}$$

This equation states that the coefficient of the identity and the one of the displacement in the defect OPE of a scalar operator are linearly related. It is important to notice that, since the displacement is not canonically normalized, the coefficient appearing in the defect OPE is $b_{O\mathrm{D}}/C_{\mathrm{D}}$. Analogous considerations apply in the rest of this subsection. Eq. (1.157) is the generalization of a result found by Cardy in the codimension one case [81], where the normal component of the stress-tensor plays the role of the displacement [9, 51]. In that case, the group of transverse rotations is trivial and the displacement does not carry indices, but the result can be obtained by directly plugging $p = d-1$ in eq. (1.157).

Odd spin primaries do not acquire a one-point function in parity invariant theories - unless they are pseudo-tensors, a case which we do not consider here. The left hand side of eq. (1.151) vanishes, while the right hand side for a spin one primary $O_{\Delta,1}(P, Z)$ can be computed using[21]

$$\begin{aligned}\langle O_{\Delta,1}(P_1,Z_1)\mathrm{D}(P_2,Z_2)\rangle = &\frac{1}{(P_1\circ P_1)^{(\Delta-p-1)/2}(-2P_1\bullet P_2)^{p+1}}\\ &\times\left(b^1_{O\mathrm{D}}\frac{P_1\circ Z_2(P_1\bullet P_2\,P_1\circ Z_1 - P_1\circ P_1\,P_2\bullet Z_1)}{P_1\circ P_1\,P_1\bullet P_2}\right.\\ &\left.+b^2_{O\mathrm{D}}\frac{P_1\circ P_1\,Z_1\circ Z_2 - P_1\circ Z_2\,P_1\circ Z_1}{P_1\circ P_1}\right).\end{aligned} \tag{1.158}$$

The r.h.s. of eq. (1.151) does not vanish:

$$\int D^pQ\,Q_A\,\langle O_{\Delta,1}(P,Z)\mathrm{D}_I(Q)\rangle \neq 0, \tag{1.159}$$

therefore we conclude that $b^1_{O\mathrm{D}} = b^2_{O\mathrm{D}} = 0$, that is, spin one primaries do not couple with the displacement. This conclusion holds true in the codimension one case, in which only the structure proportional to $b^1_{O\mathrm{D}}$ survives, and still does not vanish upon integration.

Let us turn to the spin 2 case. The action of a conformal transformation on the one-point function of a tensor is lifted simply as a Lorentz transformation involving both the coordinate of the field insertion and the auxiliary vector, so that eq. (1.151) reads in this case

$$2\left(P_{[A}\frac{\partial}{\partial P^{I]}} + Z_{[A}\frac{\partial}{\partial Z^{I]}}\right)\langle O_{\Delta,2}(P,Z)\rangle = -\int D^pQ\,Q_A\,\langle O_{\Delta,2}(P,Z)\mathrm{D}_I(Q)\rangle\,. \tag{1.160}$$

The one and two-point functions involved are written in eqs. (1.79), (1.88) respectively. Eq. (1.160) translates into the following relations:[22]

$$b^2_{O\mathrm{D}} = \frac{1}{p+1}\left(\frac{\Delta}{2}b^3_{O\mathrm{D}} - b^1_{O\mathrm{D}}\right), \tag{1.161a}$$

$$b^3_{O\mathrm{D}} = 2^{p+2}\pi^{-\frac{p+1}{2}}\Gamma\left(\frac{p+3}{2}\right)a_O. \tag{1.161b}$$

Let us add a few comments. If the codimension is one, the structures multiplying $b^2_{O\mathrm{D}}$ and $b^3_{O\mathrm{D}}$ vanish, and so does the integral of the remaining one: this allows in particular the displacement to couple with the stress-tensor, which is expected since the first is the defect limit of a component of the second. Comparing the relations (1.161) with the ones which follow from conservation (1.90), one sees that compatibility is ensured by $\Delta = d$. Finally, when $O_{\mu\nu} = T_{\mu\nu}$ is the stress-tensor, another linear combination of the parameters in its correlator with the displacement can be seen to be equal to C_D, defined in eq. (1.152). Thus the correlator is completely specified in terms of the latter, and of the coefficient of the one-point function a_T. To see this, let us look back at eq. (1.145), and specifically let us choose the free index in a direction orthogonal to the defect. By comparison with eq. (1.150) we see that the δ-function contribution on the defect is given by the displacement plus descendants. The Ward-identity fixes the scale dimension of all the operators involved, so the displacement is certainly orthogonal to $\partial_a\lambda^a{}_i$ and $\partial_a\partial_b C^{ab}_i$. We can therefore write the following equality:

$$\langle\partial_\mu T^{\mu i}(x)\,\mathrm{D}^j(0)\rangle = -\delta_\mathcal{D}\,\langle\mathrm{D}^i(x^a)\mathrm{D}^j(0)\rangle\,. \tag{1.162}$$

[21] The relation with the conventions of section 3 is $b_{1100} = b^1_{O\mathrm{D}}, b_{0010} = b^2_{O\mathrm{D}}$

[22] The relation with the conventions of section 3 is $b_{1200} = b^1_{O\mathrm{D}}, b_{0110} = b^3_{O\mathrm{D}}, b_{1001} = b^2_{O\mathrm{D}}$.

This equation is easily lifted to the projective light-cone by use of the Todorov operator:

$$\frac{1}{d}\,\partial_{P_1}\cdot D_{Z_1}\,\langle T(P_1,Z_1)\mathrm{D}(P_2,Z_2)\rangle = -\frac{C_{\mathrm{D}}\,\delta^q(P_1^J)\;Z_1\circ Z_2}{(-2P_1\bullet P_2)^{p+1}}. \tag{1.163}$$

Notice that, although on the Poincaré section

$$-\,2P_1\bullet P_2 = |(x_1-x_2)^a|^2 + \left|x_1^i\right|^2, \tag{1.164}$$

eq. (1.163) is correct, thanks to the δ-function which kills the dependence on the transverse coordinates. Upon integration against a test function, eq. (1.163) provides the following relation:

$$b_{T\mathrm{D}}^2 = \frac{1}{p}\left((d+q-2)\frac{b_{T\mathrm{D}}^3}{2} + q\frac{C_{\mathrm{D}}}{\Omega_{q-1}}\right), \tag{1.165}$$

$\Omega_{q-1} = 2\pi^{q/2}/\Gamma(q/2)$ being the volume of S_{q-1}. As promised, this relation, together with eqs. (1.161), fixes the bulk-to-defect coupling of the displacement with the stress-tensor in terms of the norm of the displacement (C_{D}) and the coefficient of the one-point function of the stress-tensor (a_T).

1.5.3 Displacement and reflection in two dimensional CFTs

The story of two dimensional defect conformal field theories is rich, and dates back to the essential work of Cardy [11, 12] and Cardy and Lewellen [13]. Here we would like to comment on the role of the displacement operator in this case. It is not difficult to see that C_{D} is intimately related with the *reflection coefficient* introduced in [82]. A straightforward analysis of the defect operators present in the defect OPE of the stress-tensor also allows to prove unitarity bounds for this coefficient. This analysis is greatly simplified by some easy consequences of holomorphy. We put a CFT$_1$ on the upper half and a CFT$_2$ on the lower half of the complex plane, parametrized by $z = x + \mathrm{i}y$. Since both components of the stress-tensor of either theory, $T(z)$ and $\bar{T}(\bar{z})$, are purely (anti)holomorphic, their defect OPE cannot be singular. It follows at once by dimensional analysis that the only defect operator surviving among the ones appearing in eqs. (1.145), (1.147) is the displacement operator. This means in particular that the component T_{xy} is continuous across the defect, or in other words

$$T^1(x) - \bar{T}^1(x) = T^2(x) - \bar{T}^2(x), \quad x\in\mathbb{R}. \tag{1.166}$$

In fact, in any dimension an interface CFT can be mapped to a boundary CFT by folding the system. In $2d$, mapping the CFT$_2$ on the upper half plane corresponds to exchanging holomorphic and anti-holomorphic fields, so that the interface is equivalent to a boundary condition for the theory CFT$_1\times\overline{\mathrm{CFT}_2}$.[23] The condition (1.166) is simply Cardy's condition for the stress-tensor of the folded theory $T^1+\bar{T}^2$. Holomorphy and translational and scale invariance fix all correlators of T^1 and T^2 up to constants. The gluing condition (1.166)

[23]Consequences of the folding trick are in fact not completely understood - see [83] for some comments in the case of topological defects - especially in non-rational theories, which lack a nice factorization of holomorphic and anti-holomorphic parts. However, related subtleties are not relevant for the considerations below. We thank Jurgen Fuchs for correspondence on this point.

relates many of the constants:

$$\langle T^1(z)T^2(z')\rangle = \frac{a/2}{(z-z')^4} \tag{1.167}$$

$$\langle \bar{T}^1(\bar{z})\bar{T}^2(\bar{z}')\rangle = \frac{(a+\bar{c}_1-c_1)/2}{(\bar{z}-\bar{z}')^4} = \frac{(a+\bar{c}_2-c_2)/2}{(\bar{z}-\bar{z}')^4} \tag{1.168}$$

$$\langle T^1(z)\bar{T}^1(\bar{z}')\rangle = \frac{(c_1-b)/2}{(z-\bar{z}')^4} \tag{1.169}$$

$$\langle T^2(z)\bar{T}^2(\bar{z}')\rangle = \frac{(c_2-b)/2}{(z-\bar{z}')^4}. \tag{1.170}$$

$$\tag{1.171}$$

This imposes the condition

$$c_1 - \bar{c}_1 = c_2 - \bar{c}_2, \tag{1.172}$$

but we will assume

$$c_1 = \bar{c}_1, \quad c_2 = \bar{c}_2 \tag{1.173}$$

in the following. Also, the parameters a and b are related by the remaining correlator:

$$\langle T^1(z)\bar{T}^2(\bar{z}')\rangle = \frac{(a-b)/2}{(z-\bar{z}')^4}. \tag{1.174}$$

Here the minus sign is dictated by translational invariance along the real direction. However, this correlator doesn't fall off when the fields are far away on the opposite sides of the interface, therefore the boundary condition in this direction prompts us to fix

$$a = b. \tag{1.175}$$

The displacement operator is (up to a sign)

$$\mathrm{D}(x) = (T^1 + \bar{T}^1 - T^2 - \bar{T}^2)(x) = 2(T^1 - T^2)(x) = 2(\bar{T}^1 - \bar{T}^2)(x). \tag{1.176}$$

Since there are four operators of dimension 2 and one gluing condition, there are other two dimension 2 operators on the interface. One of them might be taken to be $T = T^1 + \bar{T}^2$, the boundary stress-tensor for the folded theory, which is also, up to a factor 2, the displacement for the folded theory. Notice that, unless $c_1 = c_2$, this operator is not orthogonal to D. The coefficient of the displacement two-point function is

$$C_{\mathrm{D}} = 2(c_1 + c_2 - 2a) \geq 0, \tag{1.177}$$

where the inequality holds in a unitary theory. We can actually obtain a stronger lower bound and an upper bound for C_{D} by considering the matrix of two-point functions of all the dimension 2 fields at our disposal. We go to the folded picture and denote:

$$\tilde{T}^2(z) = \bar{T}^2(\bar{z}'), \quad \bar{z} = z', \ \Im z' < 0. \tag{1.178}$$

Then one can verify that, with respect to the total stress tensor,

$$T(z) = T^1(z) + \tilde{T}^2(z), \tag{1.179}$$

the following two fields are primaries[24] (in the notation of [82])

$$W(z) = c_2 T^1(z) - c_1 \tilde{T}^2(z), \tag{1.180}$$

$$\bar{W}(\bar{z}) = c_2 \bar{T}^1(\bar{z}) - c_1 \bar{\tilde{T}}^2(\bar{z}). \tag{1.181}$$

[24] In this section primaries are intended to be Virasoro primaries, as customary in 2d CFT.

Being (anti)holomorphic, these fields have non-singular defect OPEs, and the coefficients of the two point functions of the boundary operators compute the overlap of the corresponding states in radial quantization. The matrix of inner products should have positive eigenvalues in a unitary theory:

$$G = \frac{1}{2}\begin{pmatrix} c_1+c_2 & 0 & 0 \\ 0 & c_1c_2(c_1+c_2) & (c_1+c_2)(c_1c_2-a(c_1+c_2)) \\ 0 & (c_1+c_2)(c_1c_2-a(c_1+c_2)) & c_1c_2(c_1+c_2) \end{pmatrix}. \tag{1.182}$$

The eigenvalues of G are

$$\lambda_1 = \frac{1}{2}(c_1+c_2), \qquad \lambda_2 = \frac{a}{2}(c_1+c_2)^2, \qquad \lambda_3 = -\frac{1}{2}(a(c_1+c_2)-2c_1c_2). \tag{1.183}$$

From positivity of λ_2 and λ_3 it follows that

$$2\,\frac{(c_1-c_2)^2}{c_1+c_2} \leq C_{\mathrm{D}} \leq 2\,(c_1+c_2). \tag{1.184}$$

Notice that the upper bound is saturated by the case of a boundary condition, for which $a=0$ in eq. (1.167).

As already mentioned, a general definition of reflection and transmission coefficients in $2d$ CFT was put forward in [82]. It is easy to see that the reflection coefficient $\mathcal{R}$ can be expressed in terms of the coefficient of the two-point function of the displacement operator:

$$\mathcal{R} = \frac{C_{\mathrm{D}}}{2(c_1+c_2)}. \tag{1.185}$$

We find therefore that, in a unitary theory, reflection is less than unity, as it should, and transparency is bounded by the square of the difference of the central charges:

$$\left(\frac{c_1-c_2}{c_1+c_2}\right)^2 \leq \mathcal{R} \leq 1. \tag{1.186}$$

1.5.4 Examples

When dealing with strongly coupled CFTs, usually the conformal data are all that matters, and the explicit expression of renormalized operators in terms of elementary fields is inaccessible. On the contrary, when perturbation theory makes sense, it is useful to consider the free field composite operators as starting point. Here we give a few free theory examples of the defect operators which appear in eqs. (1.134)-(1.150) above. Dealing with free theories, we never need to worry about running couplings, so that conformal invariance of the defects that we consider simply follows from dimensional analysis. Most of the examples have already been considered elsewhere. In what follows, we will sometimes employ the notation $\sigma = \{\sigma^a\}$, along with the usual $x_\parallel = \{x^a\}$ for the parallel coordinates and $x_\perp = \{x^i\}$ for the orthogonal ones.

Minimal coupling to a free scalar Let us start by considering a single free scalar field ϕ in a flat d dimensional Euclidean space. Rather than the usual QFT normalization of the field ϕ we will use the CFT one,[25] in which the two-point function is

$$G_0(x,y) \equiv \langle \phi(x)\phi(y)\rangle_0 = \frac{1}{|x-y|^{2\Delta}}\ . \tag{1.188}$$

[25] In the CFT normalization, the free action is

$$\frac{1}{(d-2)\Omega_{d-1}}\int_{\mathcal{M}} d^dx\, \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi\ , \tag{1.187}$$

where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of S_{d-1}.

We place in the vacuum the following planar p-dimensional defect:

$$\mathcal{O}_{\mathcal{D}} = \exp\left(\lambda \int_{\mathcal{D}} d^p\sigma \phi(\sigma)\right) . \tag{1.189}$$

Notice that $\mathcal{O}_{\mathcal{D}}$ is the extended operator itself, according to the notation of eq. (1.42). In what follows, we sometimes alternatively use the defect action $S_{\mathcal{D}} = -\log \mathcal{O}_{\mathcal{D}}$. The defect is conformal when p equals the dimension Δ of the scalar:

$$p = \Delta = \frac{d}{2} - 1 , \quad q \equiv d - p = \frac{d}{2} + 1 ; \tag{1.190}$$

the scalar Wilson line in four dimensions is one example [27].

The equation of motion reads

$$\Box\phi = -\lambda\,(d-2)\Omega_{d-1}\,\delta_{\mathcal{D}} . \tag{1.191}$$

In the presence of the defect, the one-point function of ϕ can be computed in various ways. For instance, one can solve the equation of motion (1.191), or use the properties of free exponentials to get

$$\langle\phi(x)\rangle = \frac{\langle\phi(x)\mathcal{O}_{\mathcal{D}}\rangle_0}{\langle\mathcal{O}_{\mathcal{D}}\rangle_0} = \lambda \int d^p\sigma\, \langle\phi(x)\phi(\sigma)\rangle_0. \tag{1.192}$$

Either way,

$$\langle\phi(x)\rangle = \frac{a_\phi}{|x_\perp|^\Delta} , \quad \text{with} \quad a_\phi = 2\lambda\frac{\Omega_{d-1}}{\Omega_{q-1}} . \tag{1.193}$$

For completeness, let us also write down the two-point function:

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle &= \frac{\langle\phi(x)\phi(x)\mathcal{O}_{\mathcal{D}}\rangle_0}{\langle\mathcal{O}_{\mathcal{D}}\rangle_0} = \langle\phi(x)\phi(x)\rangle_0 + \langle\phi(x)\rangle\langle\phi(y)\rangle \\ &= \frac{1}{|x-y|^{2\Delta}} + \frac{a_\phi^2}{|x_\perp|^\Delta|y_\perp|^\Delta} = \frac{1}{|x_\perp|^\Delta|y_\perp|^\Delta}\left(\frac{1}{\xi^\Delta} + a_\phi^2\right) .\end{aligned} \tag{1.194}$$

The form of this correlator means in particular that the defect OPE of the fundamental field is only mildly deformed from the case of a trivial defect: the identity appears, but the other couplings remain untouched.

The (improved) stress-tensor for the free scalar field ϕ reads

$$T_{\mu\nu} = \frac{1}{(d-2)\Omega_{d-1}}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\partial\phi\cdot\partial\phi - \frac{1}{4}\frac{d-2}{d-1}\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box\right)\phi^2\right) . \tag{1.195}$$

Contact terms in the Ward identities of the stress-tensor follow from the definition (1.195):

$$\partial_\mu T^{\mu\nu} = \frac{1}{(d-2)\Omega_{d-1}}\partial^\nu\phi\,\Box\phi, \qquad T_\mu{}^\mu = \frac{1}{(d-2)\Omega_{d-1}}\frac{d-2}{2}\phi\Box\phi , \tag{1.196}$$

and from the use of the e.o.m. (1.191):

$$\langle\partial_\mu T^{\mu\nu}\mathcal{X}\rangle = -\lambda\,\langle\partial^\nu\phi\,\mathcal{X}\rangle\;\delta_{\mathcal{D}}(x), \qquad \langle T_\mu{}^\mu\mathcal{X}\rangle = -\lambda\,\frac{d-2}{2}\,\langle\phi\,\mathcal{X}\rangle\,\delta_{\mathcal{D}}(x) . \tag{1.197}$$

This is in agreement with the Ward identities (1.145), (1.147). Indeed, the defect in curved space-time becomes

$$O_{\mathcal{D}} = \lambda\int_{\mathcal{D}} d^p\sigma\sqrt{\gamma}\,\phi\left(X(\sigma)\right) , \tag{1.198}$$

and

$$B^{ab} \equiv \frac{2}{\sqrt{\gamma}} \frac{\delta O_{\mathcal{D}}}{\delta \gamma_{ab}}\bigg|_{\text{flat}} = \lambda\, \phi\, \delta_{ab}\ , \tag{1.199}$$

We also have, in the flat space limit,

$$D_a = \lambda\, \partial_a \phi\ , \quad \mathrm{D}_i = D_i = \lambda\, \partial_i \phi\ . \tag{1.200}$$

Eq.s (1.199) and (1.200) are consistent with the Ward identity for reparametrization symmetry, eq. (1.146). The norm C_{D} of the displacement follows from the two-point function

$$\langle \mathrm{D}_i(x_\parallel)\mathrm{D}_j(y_\parallel)\rangle = \lambda^2 \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\langle\phi(x)\phi(y)\rangle\bigg|_{x_\perp=y_\perp=0}\ , \qquad C_{\mathrm{D}} = 2\lambda^2\Delta. \tag{1.201}$$

The coupling of ϕ to the displacement is also easily computed:

$$\langle \mathrm{D}_i(y_\parallel)\phi(x)\rangle = \lambda\, \frac{\partial}{\partial y^i}\langle\phi(y)\phi(x)\rangle\bigg|_{y_\perp=0} = 2\Delta\lambda \frac{x_i}{((x-y)_\parallel^2 + x_\perp^2)^{\Delta+1}}\ , \tag{1.202}$$

and since

$$b_{\phi\mathrm{D}} = 2\Delta\lambda = \Delta\left(\frac{\pi}{4}\right)^{-\frac{p}{2}} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\, a_\phi\ , \tag{1.203}$$

this explicitly verifies the general relation (1.157) between the coefficient of the identity and the one of the displacement in the defect OPE of ϕ.

The one-point function of the stress tensor in presence of the defect can be computed by diagrammatic methods or by evaluating $T_{\mu\nu}$ on the classical solution, i.e., on the one-point function eq. (1.193). The result agrees with the expected form eq. (1.60) with

$$a_T = -\frac{(d-2)}{4(d-1)}\frac{a_\phi^2}{\Omega_{d-1}}\ . \tag{1.204}$$

Similarly, a direct computation shows that the coupling between the displacement operator and the stress-tensor is of the form (1.63), with $\Delta = d$, $\widehat{\Delta} = p+1$ and with the following defect OPE coefficients:

$$b^1_{\mathrm{D}T} = -\lambda^2\frac{d(d-2)}{2(d-1)}\,\pi^{-\frac{d+2}{4}}\Gamma\left(\frac{d+6}{4}\right), \quad b^2_{\mathrm{D}T} = -\lambda^2\frac{(d-2)^3}{16(d-1)}\,\pi^{-\frac{d+2}{4}}\Gamma\left(\frac{d-2}{4}\right),$$
$$b^3_{\mathrm{D}T} = -\lambda^2\frac{d(d-2)}{2(d-1)}\,\pi^{-\frac{d+2}{4}}\,\Gamma\left(\frac{d+2}{4}\right). \tag{1.205}$$

Equations (1.201) (1.204) (1.205) verify the constraints (1.161) and (1.165).

Minimal coupling to a p-form Let us consider an (Abelian) p-form A minimally coupled to a p-dimensional object $\mathcal{D}$. The action reads

$$\frac{1}{2}\int_{\mathcal{M}} F\wedge {}^*F - \lambda\int_{\mathcal{D}} A = \frac{1}{2(p+1)!}\int_{\mathcal{M}} d^dx\, F_{\mu_1\dots\mu_{p+1}}F^{\mu_1\dots\mu_{p+1}} - \lambda\int_{\mathcal{D}} d^px_\parallel A_{1\dots p}\ , \tag{1.206}$$

where $F = dA$. This system is scale invariant when

$$p = \frac{d}{2} - 1\ . \tag{1.207}$$

In fact, there is full conformal invariance, as can be checked from the trace of the stress-tensor eq. (1.209). This is a simple generalization of a charged particle in four dimensions, which corresponds to the case $p = 1$ in $d = 4$. The equations of motion read

$$\partial^\mu F_{\mu\, 1\ldots p} = -\lambda\, \delta_{\mathcal{D}} \ , \tag{1.208}$$

while $\partial^\mu F_{\mu\nu_1\ldots\nu_p} = 0$ for all other sets of indices $\nu_1, \ldots \nu_p$. The bulk stress-energy tensor for the p-form field is given by

$$T_{\mu\nu} = \frac{1}{p!}\left(F_{\mu\rho_1\ldots\rho_p} F_\nu{}^{\rho_1\ldots\rho_p} - \delta_{\mu\nu}\frac{1}{2(p+1)} F_{\rho_1\ldots\rho_{p+1}} F^{\rho_1\ldots\rho_{p+1}}\right) \ . \tag{1.209}$$

Direct computation of the one-point function yields the expected form eq. (1.60), with

$$a_T = -\left(\frac{\lambda}{\Omega_{q-1}}\right)^2 \ . \tag{1.210}$$

The trace of the stress tensor vanishes identically, therefore the only contact terms appear in its derivative:

$$\partial_\mu T^{\mu i} = -\lambda \delta_{\mathcal{D}}\, F_{i\,1\ldots p} \ , \qquad T_\mu{}^\mu(x) = 0 \ . \tag{1.211}$$

The defect action in eq. (1.206) is directly generalized to a curved case, since it is written as the integral of a form:

$$S_{\mathcal{D}} = -\lambda \int_{\mathcal{D}} A = -\frac{\lambda}{p!}\int_{\mathcal{D}} A_{\mu_1\ldots\mu_p} e^{\mu_1}_{a_1} \ldots e^{\mu_p}_{a_p} d\sigma^{a_1} \wedge \ldots d\sigma^{a_p} \ . \tag{1.212}$$

Since the induced metric γ_{ab}, the normal vectors n^μ_I and the extrinsic curvature K^I_{ab} do not appear, the operator B_{ab}, $\lambda_\mu{}^I$ and C^{ab}_I vanish, so that (1.211) is consistent with the Ward identity for Weyl rescalings, eq. (1.147).

We have instead non-trivial operators

$$\eta_\mu{}^a = -\frac{\delta S_{\mathcal{D}}}{\delta e^\mu_a} = \frac{\lambda}{(p-1)!} A_{\mu a_2\ldots a_p} \epsilon^{a a_2\ldots a_p} \tag{1.213}$$

and

$$D_\mu = -\frac{\delta S_{\mathcal{D}}}{\delta X^\mu} = \frac{\lambda}{p!}\partial_\mu A_{a_1\ldots a_p}\epsilon^{a_1\ldots a_p} \ . \tag{1.214}$$

These are not primary operators, since they are not gauge invariant. The gauge invariant combination is the one appearing in the Ward identities (1.145), which coincides with the displacement operator, i.e.

$$\mathrm{D}_i = D_i - \partial_a \eta_i{}^a = \lambda F_{i0\ldots p-1} \ . \tag{1.215}$$

This is in agreement with eq. (1.211).

Vector coupled with lower dimensional matter. We now go back to a free vector field in a generic dimension d, but this time we add degrees of freedom on a p dimensional subspace. If we choose these additional fields to possess a global $SO(q)_I$ symmetry, we can couple the two theories by a symmetry breaking term that only preserves the diagonal of $SO(q)_I \times SO(q)$, the latter being the usual transverse rotational symmetry. It is convenient to start from the Stuckelberg Lagrangian, so that we have enough fields at disposal to make the defect coupling gauge invariant. The redundant description contains a vector A_μ and a scalar B, and a symmetry is imposed under the following local transformations:

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \qquad B \to B + m\Lambda. \tag{1.216}$$

We couple the usual action with a defect in the following way:

$$S = \int_{\mathcal{M}} d^d x \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{m^2}{2} \left(A_\mu - \frac{1}{m} \partial_\mu B \right)^2 \right) + \frac{\lambda}{2} \int_{\mathcal{D}} d^p x \left((\partial_a \phi^i)^2 + 2\mu \left(A_i - \frac{1}{m} \partial_i B \right) \phi^i \right). \tag{1.217}$$

While at low energy the bulk theory contains a massive vector, at high energy the longitudinal degree of freedom decouples. We will come back later to the massless limit. The equations of motion are

$$\partial_\mu F^{\mu\nu} - m^2 (A^\nu - \partial^\nu B/m) = \delta_i^\nu \, \delta_{\mathcal{D}}(x) \, \lambda\mu \, \phi^i, \tag{1.218}$$

$$m^2 \partial^\mu (A_\mu - \partial_\mu B/m) = -\partial_i \delta_{\mathcal{D}}(x) \, \lambda\mu \, \phi^i, \tag{1.219}$$

$$\mu (A^i - \partial^i B/m) = \Box \phi^i. \tag{1.220}$$

Notice that the e.o.m. are compatible with the antisymmetry of the field strength, in the sense that $\partial_\mu \partial_\nu F^{\mu\nu} = 0$. The stress-tensor reads

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho + m^2 (A_\mu - \partial_\mu B/m)(A_\nu - \partial_\nu B/m) - \delta_{\mu\nu} \left(\frac{1}{4} F_{\rho\sigma}^2 + \frac{m^2}{2} (A_\mu - \partial_\mu B/m)^2 \right). \tag{1.221}$$

Let us consider its divergence:

$$\partial^\mu T_{\mu\nu} = \delta_{\mathcal{D}} \, \lambda\mu\phi^i F_{\nu i} - \partial_i \delta_{\mathcal{D}} \, \lambda\mu\phi^i \, (A_\nu - \partial_\nu B/m). \tag{1.222}$$

It is important to notice that the fields appearing in the contact terms preserve their full space-time dependence. In other words, upon integration by parts the derivative acting on the delta function in the second addend ends up acting on A and B. On the contrary, the Ward identities (1.145) were written assuming that defect operators only depend on the parallel coordinates. With this in mind, we can rewrite eq. (1.222) as follows:

$$\partial^\mu T_{\mu\nu} = \delta_{\mathcal{D}} \, \lambda\mu\phi^i \partial_\nu (A_i - \partial_i B/m) - \partial_i \delta_{\mathcal{D}} \, \lambda\mu\phi^i \, (A_\nu - \partial_\nu B/m), \tag{1.223}$$

where now all fields and their derivatives are evaluated at the defect and do not carry dependence on transverse coordinates any more.

When coupling the theory to a background metric, we also need to upgrade the $SO(q)_I$ symmetry to a local one, so that the bulk-to-boundary mixing term couples the rotations in the normal bundle to the internal symmetry. For the same reason, the background gauge field for the internal symmetry has to coincide with the spin connection $\mu_{a\,IJ}$ as defined in (1.271). Altogether, the defect action in curved space time - without any improvement, since we cannot achieve conformal invariance in the bulk anyway - is

$$S_{\mathcal{D}} = \frac{\lambda}{2} \int_{\mathcal{D}} \left(\left(D_a \phi^I \right)^2 + 2\mu \, (A_\mu - \partial_\mu B/m) \, n_I^\mu \phi^I \right), \qquad D_a \phi^I = \partial_a \phi^I + \mu_a{}^I{}_J \phi^J. \tag{1.224}$$

Let us identify the defect operators appearing in the contact terms coming from the action (1.224):

$$B_{ab} = \frac{\lambda}{2} \left(2\partial_a \phi^i \partial_b \phi^i - \left((\partial_c \phi^i)^2 + 2\mu \, (A_i - \partial_i B/m) \, \phi^i \right) \delta_{ab} \right), \tag{1.225}$$

$$D_\mu = -\lambda\mu\phi^i \, \partial_\mu (A_i - \partial_i B/m), \tag{1.226}$$

$$\lambda_\mu{}^i = -\lambda\mu \, (A_\mu - \partial_\mu B/m) \, \phi^i, \tag{1.227}$$

$$j_a{}^{ij} = -\lambda \partial_a \phi^{[i} \phi^{j]}. \tag{1.228}$$

On shell, one finds in particular

$$\partial^a B_{ab} = \lambda \Box \phi^i \partial_b \phi^i - \lambda\mu \partial_b \left(\left(A_i - \partial_i B/m \right) \phi^i \right) = D_b, \tag{1.229}$$

$$\partial^a {j_a}^{ij} = -\lambda\mu \left(A^{[i} - \partial^{[i} B/m \right) \phi^{j]} = \lambda^{[ij]}, \tag{1.230}$$

in agreement with eqs. (1.146) and (1.137). Plugging this into (1.145), we find perfect agreement with eq. (1.223).

Let us now look for a fixed point of the action (1.217). In order to have a non singular massless limit on the defect, μ needs to vanish with m. This means that the transverse degrees of freedom of the photon decouple from the lower dimensional matter, and we end up with the coupling of a bulk with a defect scalar field:

$$S_{\text{massless}} = \int_{\mathcal{M}} d^d x \, \frac{1}{2} \left(\partial_\mu B \right)^2 + \frac{\lambda}{2} \int_{\mathcal{D}} d^p x \left((\partial_a \phi^i)^2 - 2\kappa \partial_i B \phi^i \right), \quad \kappa = \frac{\mu}{m}. \tag{1.231}$$

All couplings are dimensionless for a codimension two defect. In this case one can achieve Weyl invariance by improving both the bulk and the defect action with the usual coupling to the Ricci scalar, and furthermore by supplementing the defect action with a linear coupling to the extrinsic curvatures:

$$S_{\mathcal{D}} = \frac{\lambda}{2} \int_{\mathcal{D}} d^p x \sqrt{\gamma} \left(\left(D_a \phi^I \right)^2 + \frac{p-2}{4(p-1)} \widehat{R} \, (\phi^I)^2 - 2\kappa \partial_I B \phi^I + \kappa B \phi_I K^I \right). \tag{1.232}$$

By means of the massless limit of eq. (1.219), and repeating the step which led us from (1.222) to (1.223), we obtain the trace of the stress-tensor in flat space:

$$T_\mu{}^\mu = \frac{d-2}{2} \lambda\kappa \left(-\delta_{\mathcal{D}} \, \partial_i B \phi^i + \partial_i \delta_{\mathcal{D}} \, B \phi^i \right). \tag{1.233}$$

Comparing with the operators B_{ab}, ${\lambda_\mu}^I$ and C^I_{ab} extracted from $S_{\mathcal{D}}$, we find that these are the right contact terms for a conformal defect when $p = d - 2$, as expected.

1.6 Summary and outlook

In this chapter, we have used the embedding formalism to analyze correlation functions in a generic CFT endowed with a conformal defect. The tensor structures appearing in a two-point function of symmetric traceless (bulk or defect) primaries have been classified. We also started the exploration of the crossing constraints for defects of generic codimension, by deriving the conformal blocks for the scalar two-point function in the defect channel and setting up the light-cone expansion for the bulk channel. Moreover, for codimension two defects, we found that the bulk channel blocks for identical external scalars are equal to the blocks of the four-point function of a homogeneous CFT (i.e. without defects), up to a change of variables. This means in particular that the bulk blocks are now known in closed form for codimension two defects in even dimensional CFTs, and of course all the results available in odd dimensions apply as well (see [79] for a recent progress in $d = 3$.). Finally, we described the possible protected defect operators appearing in the OPE of the stress-tensor with the defect, and derived constraints on the CFT data starting from the appropriate Ward identities. In the two dimensional case, we pointed out that the Zamolodchikov norm of the displacement operator contains the same information as the reflection coefficient defined in [82], and we derived unitarity bounds for the latter.

These results can be extended in multiple directions. It should be possible to find an algorithmic way of constructing correlation functions involving an arbitrary number of primaries. Also, the number of independent structures depend on the dimension of space and of the defect, so one may inquire the case of low dimensional CFTs. We did not consider mixed symmetry tensors or spinors. The latter might be done generalizing the results contained in [69] and [68] respectively. Much in the same way, we expect that many of the techniques that have been developed for the four-point function could be bent to the purpose of computing conformal blocks for a scalar two-point function in the presence of a defect. This is work in progress [84]. It is also worth mentioning that the simplest bootstrap equation, the one we considered, does not exhaust the constraints coming from crossing symmetry. Indeed, it is not difficult to realize that all correlation functions automatically obey crossing only when the three-point functions $\langle O_1 O_2 \widehat{O}\rangle$ do.

The results presented here are sufficient to set up a bootstrap analysis of codimension two defects. Upon completion of some of the formal developments outlined above, it will be possible to do the same for generic flat extended operators. On the contrary, everything was ready for bootstrapping boundaries and interfaces already with the works [9, 10]. Two-point functions of bulk operators can be bootstrapped with the method of determinants [43], while the linear functional method [41] meets the obstruction of lack of positivity in the bulk channel.[26] On the contrary, four-point functions of defect operators are clearly amenable to the latter technique [50]. Finally, it is also possible to study the bootstrap equations analytically in the presence of defects, and obtain asymptotic information about the spectrum of defect primaries [85]. In the next chapter, we undertake the bootstrap analysis of the two-point function for codimension one defects in spin systems.

Appendix

1.A Notations and conventions

$\mathcal{M}$ is the d-dimensional space-time with coordinates x^μ, $\mu = 1, ..., d$. $\mathcal{M} = \mathbb{R}^d$ throughout the thesis, except for section 1.5. The coordinates in the embedding space are P^M, with $M = (+, -, \mu)$, except in subsection 1.2.3, where we use Cartesian coordinates $M = (0, \mu, d+1)$, with $P^\pm = P^0 \pm P^{d+1}$.

The defect is a p dimensional sub manifold $\mathcal{D}$ with coordinates σ^a, $a = 1, ..., p$, embedded into $\mathcal{M}$ by $x^\mu = X^\mu(\sigma)$. We denote the codimension with q: $p + q = d$. $\mathcal{D}$ is a plane or a sphere throughout the thesis, except in section 1.5. For a flat defect in $\mathbb{R}^d$, we use an adapted system of coordinates $x^\mu = (x^a, x^i)$, where the defect $\mathcal{D}$ lies at $x^i = 0$. In the embedding light-cone, the defect is a $(p+2)$-dimensional sub-manifold, and we use an adapted system of coordinates $P^M = (P^A, P^I)$, where the defect lies at $P^I = 0$. In subsection 1.2.3 and in section 1.5, we also denote by $I, J = 1, ..., q$ an index in the normal bundle to $\mathcal{D}$.

The stability group of the defect is $SO(p+1,1) \times SO(q)$, where q is the codimension. Bulk operators considered in this chapter are rank-J symmetric, traceless, tensors of $SO(d)$. Defect operators are rank-j symmetric, traceless, tensors of $SO(p)$ and rank-s symmetric, traceless, tensors of their flavour symmetry group $SO(q)$. We denote them as follows:

$$O_\Delta^{\mu_1,\dots\mu_J}(x^\mu) \quad \text{and} \quad \widehat{O}_{\widehat{\Delta}}^{a_1,\dots a_j; i_1,\dots i_s}(x^a). \tag{1.234}$$

When several operators are present, we distinguish them with a further label n (and not i). Operators uplifted to the embedding space are encoded in homogeneus polynomials of

[26] See for instance the introduction of [79] for a comparison between the two techniques.

an auxiliary variable Z (or Z and W for defect operators), so we use notations like

$$O_{\Delta,J}(P,Z) \quad \text{and} \quad \widehat{O}_{\widehat{\Delta},j,s}(P,Z,W) \tag{1.235}$$

In certain cases, like with the stress tensor, the labels Δ, J (or $\widehat{\Delta}, j, s$) are redundant and we omit them.

We denote the conformal blocks as follows:

$$\begin{array}{ll} f_{\Delta,J} & \text{two-point function, bulk channel,} \\ \widehat{f}_{\widehat{\Delta},j,s} & \text{two-point function, defect channel,} \\ f^{4pt}_{\Delta,J} & \text{four-point function without defect.} \end{array} \tag{1.236}$$

1.B OPE channels of scalar primaries and the conformal blocks

This appendix collects some facts related to bulk scalar primaries. In particular, we describe the most general defect OPE of a free scalar in 1.B.1, and we give a formula for the scalar block in the bulk-channel, eq. (1.263). All other results are not new.

1.B.1 Bulk and defect OPEs

The regular OPE in the bulk can be expressed as

$$O_1(x_1)O_2(x_2) = \sum_k \frac{c_{12k}}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_k+J)}} \mathcal{C}^{(J)}(x_{12},\partial_2)_{\mu_1\dots\mu_J} O_k^{\mu_1\dots\mu_J}(x_2), \tag{1.237}$$

where c_{12k} are the structure constants appearing in the three-points functions in the absence of a defect and $\mathcal{C}^{(J)}$ are differential operators fixed by $SO(d+1,1)$ symmetry [86]. For the exchange of a scalar, one can derive the following expression [53]:

$$\begin{aligned} \mathcal{C}^{(0)}(s,\partial_2) &= \frac{1}{B(a,b)} \int_0^1 \mathrm{d}\alpha\ \alpha^{a-1}(1-\alpha)^{b-1} e^{\alpha s\cdot\partial_2} \\ &\times \sum_{m=0} \frac{1}{m!} \frac{1}{(\Delta_k+1-d/2)_m} [-\frac{1}{4} s^2 \alpha(1-\alpha)\partial_2^2]^m, \end{aligned} \tag{1.238}$$

where $(k)_m = \Gamma(k+m)/\Gamma(k)$, $B(a,b)$ is the Euler Beta function, $a = (\Delta_k+\Delta_{12})/2$, $b = (\Delta_k - \Delta_{12})/2$ and we assume $[s,\partial]=0$.

Similarly, the defect OPE can be written as follows:[27]

$$O(x) = \sum_{\widehat{O}} \frac{b_{O\widehat{O}}}{|x^i|^{\Delta-\widehat{\Delta}}} \mathcal{C}_{\widehat{O}}\left(|x^i|^2\partial_\parallel^2\right) \frac{x_{i_1}\dots x_{i_s}}{|x^i|^s} \widehat{O}^{i_1,\dots i_s}(x^a), \tag{1.239}$$

where $\partial_\parallel^2 \equiv \partial_a\partial^a$. The explicit form of the differential operator $\mathcal{C}_{\widehat{O}}$ is fixed by the $SO(p+1,1)\times SO(q)$ symmetry:

$$\mathcal{C}_{\widehat{O}}\left(|x^i|^2\partial_\parallel^2\right) = \sum_{m=0} c_m \left(|x^i|^2\partial_\parallel^2\right)^m, \quad c_m = \frac{1}{m!} \frac{(-4)^{-m}}{\left(\widehat{\Delta}+1-\frac{p}{2}\right)_m}. \tag{1.240}$$

[27] With respects to eq. (1.86) we have $b_{s,0,0,0} = b_{O\widehat{O}}$.

Defect OPE of a free scalar

The spectrum of defect primaries can be complicated, no matter how simple the bulk theory is. This simply follows from the fact that new degrees of freedom with intricate dynamics might be present on the defect. However, bulk-to-defect couplings are constrained by the properties of the bulk. In a free theory, a differential operator exists that annihilates all correlators of the elementary field, and as a consequence it annihilates all its OPEs [87]. Consider for instance the theory of a free boson φ. Let us focus on the contribution of a defect primary to the OPE (1.239). When applying the Laplace operator $\partial^2 = \partial_\perp^2 + \partial_\parallel^2$ to the right hand side, we can disregard the parallel part, which acts on the primary to give descendants. Denoting the transverse twist $\widehat{\tau} \equiv \widehat{\Delta} - s$, we then get

$$0 \sim b_{O\widehat{O}}\,(\Delta - \widehat{\tau})\,(\Delta - \widehat{\tau} + 2 - q - 2s)\,\frac{x_{i_1}\dots x_{i_s}}{|x^i|^{\Delta-\widehat{\tau}+2}}\,\widehat{O}^{i_1,\dots i_s}(x^a) + \dots \tag{1.241}$$

where of course $\Delta = (d-2)/2$. It follows that $b_{O\widehat{O}} = 0$ unless

$$\widehat{\tau} = \Delta, \quad \text{or} \quad \widehat{\tau} = \Delta + 2 - q - 2s. \tag{1.242}$$

We see that primaries with protected dimension are generically induced on the defect. The first equation isolates a tower of primaries that we may denote $\widehat{\partial_\perp^s \varphi}$. The second equation has solutions only for finitely many values of the transverse spin s in a unitary theory, precisely $s \leq (4-q)/2$. The inequality is saturated for a defect scalar at the unitarity bound, and in particular, at $s = 1$ this is possible only for codimension two defects. This is precisely the case of the last example discussed in subsection 1.5.4. Finally, the identity can appear in the defect OPE of φ only when $q = d/2 + 1$, which matches the first of the examples in the same subsection.

1.B.2 The Casimir equations for the two-point function

For the sake of completeness, we give here a derivation of the Casimir equations that appear in section 1.4. In a defect CFT, there are two ways of doing radial quantization: we can define a state on a sphere Σ centered in some point in the bulk, or on a sphere $\widehat{\Sigma}$ centered in a point on the defect. In the first case, in the absence of insertions and when the radius of the sphere is smaller than the distance from the defect, the state on the sphere is the conformal invariant vacuum $|0\rangle$. The partition function in the presence of the defect is given by the superposition of this vacuum with a state $\langle\mathcal{D}|$ defined by evaluating the path integral on the exterior of Σ - see fig. 1.B.1. If we center the quantization on the defect, the sphere $\widehat{\Sigma}$ is decorated by its intersection with the defect - see fig. 1.B.2 - and a new vacuum $|\widehat{\mathcal{D}}\rangle$ is thus defined. Of course, we can write a two-point function in the two schemes:

$$\begin{aligned}\langle O_1(x_1)O_2(x_2)\rangle &= \langle\mathcal{D}|\,O_1(x_1)O_2(x_2)\,|0\rangle\\ &= \langle\widehat{\mathcal{D}}|\,O_1(x_1)O_2(x_2)\,|\widehat{\mathcal{D}}\rangle\,.\end{aligned} \tag{1.243}$$

The conformal block decompositions in the bulk and defect channels correspond to the insertion of a complete set of states in the two lines of eq. (1.243) respectively:

$$\langle O_1(x_1)O_2(x_2)\rangle = \sum_{\alpha=O,P_\mu O,\dots} \langle\mathcal{D}|\alpha\rangle\,\frac{1}{\langle\alpha|\alpha\rangle}\,\langle\alpha|\,O_1(x_1)O_2(x_2)\,|0\rangle + \text{other families}, \tag{1.244}$$

$$= \sum_{\widehat{\alpha}=\widehat{O},P_a\widehat{O},\dots} \langle\widehat{\mathcal{D}}|\,O_1(x_1)\,|\widehat{\alpha}\rangle\,\frac{1}{\langle\widehat{\alpha}|\widehat{\alpha}\rangle}\,\langle\widehat{\alpha}|\,O_2(x_2)\,|\widehat{\mathcal{D}}\rangle + \text{other families.} \tag{1.245}$$

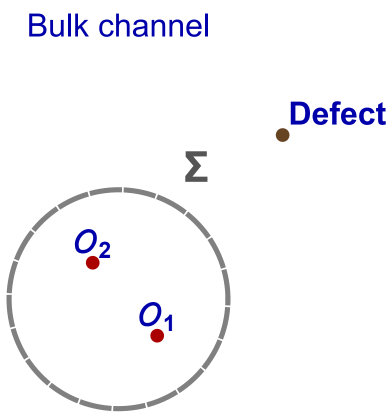

Figure 1.B.1: One way to do radial quantization is to quantize around a bulk point. We can always choose a sphere which does not intersect the defect and such that both insertions lie in its interior.

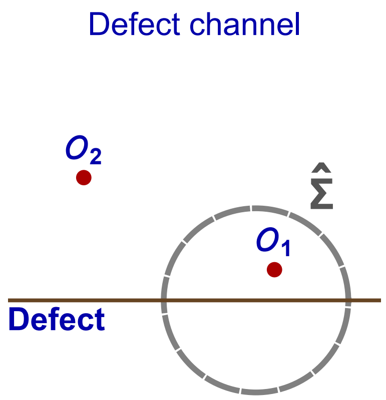

Figure 1.B.2: The theory can also be quantized around a point on the defect. This is usually associated with the defect OPE.

We will focus on a single conformal family from here on. The generators of the full conformal group J_{MN} act on scalar primaries as follows

$$\mathrm{i}[J_{MN}, O_i(x_i)] = \mathcal{J}^{(i)}_{MN} O_i(x_i), \qquad \mathcal{J}_{MN} = \left(P_M \frac{\partial}{\partial P^N} - P_N \frac{\partial}{\partial P^M} \right). \tag{1.246}$$

Bulk and defect vacua are invariant under a different set of generators:

$$J_{MN} \left|0\right\rangle = 0, \quad J_{AB} \left|\widehat{\mathcal{D}}\right\rangle = J_{IJ} \left|\widehat{\mathcal{D}}\right\rangle = 0. \tag{1.247}$$

If we define the Casimir operators of $SO(d+1,1)$, $SO(p+1,1)$ and $SO(q)$ respectively:

$$J^2 = \frac{1}{2} J_{MN} J^{MN}, \quad L^2 = \frac{1}{2} J_{AB} J^{AB}, \quad S^2 = \frac{1}{2} J_{IJ} J^{IJ}, \tag{1.248}$$

their action on any state belonging to the family of O or $\widehat{O}$ is given by

$$J^2 \left|\alpha\right\rangle = C_{\Delta,J} \left|\alpha\right\rangle, \quad L^2 \left|\widehat{\alpha}\right\rangle = \widehat{C}_{\widehat{\Delta},0} \left|\widehat{\alpha}\right\rangle, \quad S^2 \left|\widehat{\alpha}\right\rangle = \widehat{C}_{0,s} \left|\widehat{\alpha}\right\rangle, \tag{1.249}$$

with $C_{\Delta,J} = \Delta(\Delta - d) + J(J + d - 2)$ and $\widehat{C}_{\widehat{\Delta},s} = \widehat{\Delta}(\widehat{\Delta} - p) + s(s + q - 2)$. The Casimir equations are now easily obtained. For the bulk channel, we apply the operator J^2 in eq. (1.244), we use alternatively eq. (1.246) and eqs. (1.247), (1.249), and we obtain

$$\begin{aligned}
\sum_{\alpha=O,P_\mu O,\dots} \langle \mathcal{D}|\alpha\rangle \frac{1}{\langle\alpha|\alpha\rangle} \langle\alpha| \frac{1}{2}[J_{MN}, [J^{MN}, O_1(x_1)O_2(x_2)]] |0\rangle \\
= -\frac{1}{2}\left(\mathcal{J}^{(1)}_{MN} + \mathcal{J}^{(2)}_{MN}\right)^2 \langle O_1(x_1)O_2(x_2)\rangle_O \\
= C_{\Delta,J}\, \langle O_1(x_1)O_2(x_2)\rangle_O\,.
\end{aligned} \tag{1.250}$$

Here the subscript means restriction to the contribution of a single conformal family:

$$\langle O_1(x_1)O_2(x_2)\rangle_O = \sum_{\alpha=O,P_\mu O,\dots} \langle \mathcal{D}|\alpha\rangle \frac{1}{\langle\alpha|\alpha\rangle} \langle\alpha| O_1(x_1)O_2(x_2) |0\rangle\,. \tag{1.251}$$

The Casimir equation in the defect channel is obtained similarly. In this case the Casimir operator is only made to act on one of the operators. For instance, starting from

$$\sum_{\widehat{\alpha}=\widehat{O},P_a\widehat{O},\dots} \langle\widehat{\mathcal{D}}| \frac{1}{2}[J_{AB}[J^{AB}, O_1(x_1)]] |\widehat{\alpha}\rangle \frac{1}{\langle\widehat{\alpha}|\widehat{\alpha}\rangle} \langle\widehat{\alpha}| O_2(x_2) |\widehat{\mathcal{D}}\rangle\,, \tag{1.252}$$

one immediately obtains the first of the eqs. (1.101). The second one can be derived analogously.

1.B.3 The bulk-channel block for an exchanged scalar

Here, we obtain a formula for the scalar conformal block starting from the OPE (1.237). The result follows from manipulating the r.h.s. of the following equation:

$$a_k \xi^{-(\Delta_1+\Delta_2)/2} f_{\Delta_k,0}(\xi,\theta) = \frac{|x_1^i|^{\Delta_1}|x_2^i|^{\Delta_2}}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_k)}} \mathcal{C}^{(0)}(x_{12},\partial_2)\langle O_k(x_2)\rangle, \tag{1.253}$$

where the differential operator $\mathcal{C}^{(0)}$ is defined in (1.238) and

$$\langle O_k(x_2)\rangle = \frac{a_k}{|x_2^i|^{\Delta_k}}. \tag{1.254}$$

For ease of notation, in this appendix we use $s^\mu = x_{12}^\mu$. Let us compute the expression $\mathcal{C}^{(0)}(s,\partial_2)|x_2^i|^{-\Delta_k}$. Since $[s,\partial_2] = 0$ by definition, $e^{\alpha s\cdot\partial_2}$ acts as a translation:

$$e^{\alpha s\cdot\partial_2}|x_2^i|^{-\Delta_k} = |x_2^i + \alpha s^i|^{-\Delta_k}, \tag{1.255}$$

Furthermore:

$$(\partial_{2\perp}^2)^m |x_2^i|^{-\Delta_k} = 4^m \left(\frac{\Delta_k}{2}\right)_m \left(\frac{\Delta_k}{2} + 1 - \frac{q}{2}\right)_m |x_2^i|^{-(\Delta_k+2m)}, \tag{1.256}$$

where $\partial_{2\perp} \equiv \partial_2^i \partial_2^i$. Hence

$$\begin{aligned}
\mathcal{C}^{(0)}(s,\partial_2)|x_2^i|^{-\Delta_k} &= \frac{1}{B(a,b)}\int_0^1 \mathrm{d}\alpha\ \alpha^{a-1}(1-\alpha)^{b-1} \\
&\times \sum_{m=0} \frac{1}{m!}\frac{\left(\frac{\Delta_k}{2}\right)_m \left(\frac{\Delta_k}{2}+1-\frac{q}{2}\right)_m}{(\Delta_k+1-d/2)_m}\frac{[-s^2\alpha(1-\alpha)]^m}{|x_2^i|^{(\Delta_k+2m)}} \\
&= \frac{1}{B(a,b)}\int_0^1 \mathrm{d}\alpha\ \alpha^{a+m-1}(1-\alpha)^{b+m-1} \\
&\times \sum_{m=0} \frac{1}{m!}\frac{\left(\frac{\Delta_k}{2}\right)_m \left(\frac{\Delta_k}{2}+1-\frac{q}{2}\right)_m}{(\Delta_k+1-d/2)_m}\frac{(-s^2)^m|x_2^i|^{-(\Delta_k+2m)}}{(1+\alpha^2\frac{|s^i|^2}{|x_2^i|^2}+2\alpha\frac{s^i x_2^i}{|x_2^i|})^{(\Delta_k+2m)}}.
\end{aligned} \tag{1.257}$$

Recall that $a=(\Delta_k+\Delta_{12})/2$, $b=(\Delta_k-\Delta_{12})/2$. Using now

$$\int_0^1 \mathrm{d}\alpha\ \alpha^{w-1}(1-\alpha)^{r-1}(1-\alpha x')^{-\rho}(1-\alpha y')^{-\sigma} = B(w,r)F_1(w,\rho,\sigma,w+r;x';y'), \tag{1.258}$$

we find

$$\begin{aligned}
\mathcal{C}^{(0)}(s,\partial_2)|x_2^i|^{-\Delta_k} &= \frac{1}{B(a,b)}\int_0^1 \mathrm{d}\alpha\ \alpha^{a+m-1}(1-\alpha)^{b+m-1} \\
&\times \sum_{m=0} \frac{1}{m!}\frac{\left(\frac{\Delta_k}{2}\right)_m \left(\frac{\Delta_k}{2}+1-\frac{q}{2}\right)_m}{(\Delta_k+1-d/2)_m} B(a+m,b+m) \\
&\times \frac{(-s^2)^m}{|x_2^i|^{(\Delta_k+2m)}} F_1\left(a+m, m+\frac{\Delta_k}{2}, m+\frac{\Delta_k}{2}, 2m+\Delta_k; x', y'\right),
\end{aligned} \tag{1.259}$$

where x', y' are solutions of

$$\begin{cases} x'+y' = -2\dfrac{s^i x_2^i}{|x_2^i|^2}, \\ x'y' = \dfrac{|s^i|^2}{|x_2^i|^2}. \end{cases} \tag{1.260}$$

The Appell function in (1.259) is secretly a hypergeometric:

$$F_1(w,\rho,\sigma,\rho+\sigma;x';y') = (1-y')^w {}_2F_1\left(w,\rho,\rho+\sigma;\frac{x+y'}{1-y'}\right). \tag{1.261}$$

This formula also allows to reconstruct the explicit dependence on the cross-ratios:

$$\begin{aligned}
f_{\Delta,0}(\xi,\phi) = \frac{1}{B(a,b)}\xi^{\frac{\Delta_k}{2}} \sum_m \frac{1}{m!}\frac{(\frac{\Delta_k}{2})_m(\frac{\Delta_k}{2}+1-\frac{q}{2})_m}{(\Delta_k+1-\frac{d}{2})_m}(-\xi e^{-i\phi})^m \times \\
B(m+a,m+b)e^{-ia\phi}{}_2F_1\left(m+a, m+\frac{\Delta_k}{2}, \Delta_k+2m; 2i\sin\phi e^{-i\phi}\right).
\end{aligned} \tag{1.262}$$

(1.262) can be finally recast in a form which agrees with the light-cone expansion (1.113) by means of the quadratic transformation (1.125):

$$\begin{aligned}
f_{\Delta_k,0}(\xi,\phi) = \frac{1}{B(a,b)}\xi^{\frac{\Delta_k}{2}} \sum_m \frac{1}{m!}\frac{(\frac{\Delta_k}{2})_m(\frac{\Delta_k}{2}+1-\frac{q}{2})_m}{(\Delta_k+1-\frac{d}{2})_m}(-\xi)^m \\
\times B(m+a,m+b){}_2F_1\left(\frac{m+a}{2}, \frac{m+b}{2}, \frac{\Delta_k+1}{2}+m; \sin^2\phi\right).
\end{aligned} \tag{1.263}$$

As a special case, (1.262) reduces to a nice closed form when $\phi = 0$:

$$f_{\Delta_k,0}(\xi,0) = \xi^{\frac{\Delta_k}{2}}\,{}_3F_2\left(1-\frac{q}{2}+\frac{\Delta_k}{2}, a, b; \frac{1}{2}+\frac{\Delta_k}{2}, 1-\frac{d}{2}+\Delta_k; -\frac{\xi}{4}\right), \tag{1.264}$$

and reproduces the boundary CFT result [10, 49] when $q = 1$:

$$f_{\Delta_k,0}(\xi) = \xi^{\frac{\Delta_k}{2}}\,{}_2F_1\left(a, b, \Delta_k+1-\frac{d}{2}; -\frac{\xi}{4}\right). \tag{1.265}$$

1.C Differential geometry of sub-manifolds

In this appendix we collect some definitions and elementary results in differential geometry of sub-manifolds which are useful in section 1.5. We refer, for instance, to [88] and [89] for more details.

The induced metric on a sub-manifold $\mathcal{D}$ is defined as follows:

$$\gamma_{ab} = e^\mu_a\, e^\nu_b\, g_{\mu\nu}, \qquad e^\mu_a = \frac{\partial X^\mu}{\partial \sigma^a}\ . \tag{1.266}$$

We raise and lower latin indices with γ, so that the inverse induced metric can be written as

$$\gamma^{ab} = e^a_\mu e^b_\nu g^{\mu\nu}\ . \tag{1.267}$$

We choose a set of q unit vector fields n^μ_I normal to the defect through the following conditions:

$$n^I_\mu e^\mu_a = 0, \qquad n^I_\mu n^{\mu J} = \delta^{IJ}\ . \tag{1.268}$$

When restricted to the defect, the bulk metric enjoys the following decomposition, which is just the completeness relation for the basis in the tangent bundle of $\mathcal{M}$:

$$g_{\mu\nu}(X) = e_\mu{}^a e_{\nu a} + n^I_\mu n^I_\nu\ . \tag{1.269}$$

The extrinsic curvatures are defined as follows:

$$\nabla_a e_b{}^\mu = \partial_a e_b{}^\mu - \hat\Gamma^c_{ab} e_c{}^\mu + \Gamma^\mu_{\lambda\sigma} e_a{}^\lambda e_b{}^\sigma = n^\mu_I K^I_{ab}\ , \qquad K^I_{ab} = K^I_{(ab)}\ . \tag{1.270}$$

We denoted with a hat the Christoffel symbols of the induced metric. The Weingarten decomposition holds:

$$\nabla_a n^\mu_I = \partial_a n^\mu_I + \Gamma^\mu_{\lambda\nu} e_a{}^\lambda\, n_I{}^\nu - \mu_a{}^J{}_I\, n^\mu_J = -K^I_{ab} e^{\mu b}\ , \qquad \mu_{aIJ} = \mu_{a[IJ]}\ . \tag{1.271}$$

μ_{aIJ} is sometimes called the torsion tensor, and provides a spin connection for the normal bundle. In this way ∇_a, which we defined to act on both flat and curved indices, is covariant both under diffeomorphisms in $\mathcal{D}$ and local orthogonal transformations in the normal bundle. In other terms, an infinitesimal local rotation in the normal bundle, parametrized by an antisymmetric matrix ω, acts as

$$\delta_\omega n^\mu_I = \omega_{IJ} n^{\mu J}\ , \quad \delta_\omega K_{Iab} = \omega_{IJ} K^J_{ab}\ , \quad \delta_\omega \mu_{aIJ} = -\nabla_a \omega_{IJ}\ . \tag{1.272}$$

The bulk and defect Riemann tensors and the extrinsic curvatures are related by the Gauss-Codazzi equations:

$$\hat R_{abcd} = K^I_{ac} K^I_{bd} - K^I_{ad} K^I_{bc} + R_{abcd}\ , \tag{1.273}$$

$$\nabla_c K^I_{ab} - \nabla_b K^I_{ac} = R_a{}^I{}_{bc}\ . \tag{1.274}$$

Here we defined $R_{abcd} \equiv R_{\mu\nu\rho\sigma} e^\mu_a e^\nu_b e^\rho_c e^\sigma_d$, and similarly in the second line, where the contraction with a normal vector appears.

Hereafter, we report some notions in variational calculus on sub-manifolds, which are needed in section 1.5. The variations of the normal vectors, of the extrinsic curvatures and of the torsion tensor are induced, through their definition in eq.s (1.266-1.271), from the variations of the metric $g_{\mu\nu}$ (and of its Christoffel connection) evaluated on $\mathcal{D}$ and of the tangent vectors e^μ_a. From eq. (1.266) we find

$$\delta\gamma_{ab} = 2e_{\mu(a}\delta e^\mu_{b)} + e^\mu_a e^\nu_b \delta g_{\mu\nu} \ . \tag{1.275}$$

Decomposing the variation of the normal vectors on the basis given by e^μ_a and n^μ_I and considering the variation of eq.s (1.268) one finds[28]

$$\delta n^\mu_I = \left(-\frac{1}{2} n^\rho_I n^\sigma_J \delta g_{\rho\sigma} + \omega_{IJ}\right) n^{\mu J} - \left(n^\rho_I e^\sigma_a \delta g_{\rho\sigma} + n_{I\rho}\delta e^\rho_a\right) e^{a\mu} \ , \tag{1.277}$$

where the antisymmetric matrix ω_{IJ} is undetermined. The variation of the extrinsic curvatures can be derived from eq. (1.270), and it involves also the variation of the Christoffel connection:

$$\delta K^I_{ab} = \nabla_{(a}\delta e^\mu_{b)} n^I_\mu + n^I_\mu \Gamma^\mu_{\lambda\sigma}\delta e^\lambda_{(a} e^\sigma_{b)} + n^I_\mu \delta\Gamma^\mu_{\lambda\sigma} e^\lambda_a e^\sigma_b + n^\mu_J K^J_{ab}\delta n^I_\mu \ . \tag{1.278}$$

Finally, from eq. (1.271) we obtain the variation of the torsion tensor μ:

$$\delta\mu^{IJ}_a = -\nabla_a \delta n^{[I\mu} n^{J]}_\mu - n^{[J}_\mu \Gamma^\mu_{\lambda\nu}\delta e^\lambda_a n^{I]\nu} - n^{[J}_\mu \delta\Gamma^\mu_{\lambda\nu} e^\lambda_a n^{I]\nu} + K^{[I}_{ab} e^{b\mu}\delta n^{J]}_\mu \ . \tag{1.279}$$

From these formulæ we can obtain the expressions pertaining to the various symmetries by specializing the form of δe^μ_a, $\delta g_{\mu\nu}$ and $\delta\Gamma^\mu_{\nu\rho}$. Note that in general the total variation $\delta g_{\mu\nu}(X) = g'_{\mu\nu}(X+\delta X) - g_{\mu\nu}(X)$ can be decomposed into a variation in form, $\delta_g g_{\mu\nu}$, and a part due to the change in the argument:

$$\delta g_{\mu\nu} = \delta_g g_{\mu\nu} + \delta_X g_{\mu\nu} \ , \quad \text{with } \delta_X g_{\mu\nu} = \delta X^\lambda \partial_\lambda g_{\mu\nu} \ . \tag{1.280}$$

The same applies to the connection $\Gamma^\mu_{\nu\rho}$ (and to any bulk quantity evaluated on $\mathcal{D}$). The variation of the tangent vectors is instead only due to the change in the embedding function: $\delta e^\mu_a = \delta_X e^\mu_a$

Diffeomorphisms Under an infinitesimal diffeomorphism $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$, the embedding coordinates of the sub-manifold change by $\delta_\xi X^\mu = \xi^\mu$, so that

$$\delta_\xi e^\mu_a = \partial_a \xi^\mu \ . \tag{1.281}$$

The in form variations of the metric and the Christoffel symbols are

$$\begin{aligned} \delta_{\xi,g} g_{\mu\nu} &= -2\nabla_{(\mu}\xi_{\nu)} \ , \\ \delta_{\xi,g}\Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\tau}\left(\nabla_\nu \delta_g g_{\rho\tau} + \nabla_\rho \delta_g g_{\nu\tau} - \nabla_\tau \delta_g g_{\nu\rho}\right) = \xi^\sigma R_{\sigma(\nu\rho)}{}^\mu - \nabla_{(\nu}\nabla_{\rho)}\xi^\mu \ . \end{aligned} \tag{1.282}$$

[28] It is useful to write down also the variation of $n_{\mu I} \equiv g_{\mu\nu} n^\nu_I$:

$$\delta n_{\mu I} = \left(\frac{1}{2} n^\rho_I n^\sigma_J \delta g_{\rho\sigma} + \omega_{IJ}\right) n^J_\mu - n_{I\rho}\delta e^\rho_a e^a_\mu \ . \tag{1.276}$$

The total variations of the same quantities follow the usual law:

$$\begin{aligned}\delta_\xi g_{\mu\nu} &= -2\partial_{(\mu}\xi^\lambda g_{\nu)\lambda} \ , \\ \delta_\xi \Gamma^\mu_{\nu\rho} &= \partial_\sigma \xi^\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \xi^\sigma \Gamma^\mu_{\sigma\rho} - \partial_\rho \xi^\sigma \Gamma^\mu_{\sigma\nu} - \partial^2_{\nu\rho}\xi^\mu \ .\end{aligned} \tag{1.283}$$

Inserting this into eq.s (1.275-1.279) we find in the end

$$\begin{aligned}\delta_\xi \gamma_{ab} &= 0 \ , \\ \delta_\xi n^\mu_I &= n^\lambda_I \partial_\lambda \xi^\mu + \omega'_{IJ} n^{\mu J} \ , \\ \delta_\xi K^I_{ab} &= \omega'_{IJ} K^J_{ab} \ , \\ \delta_\xi \mu_{aIJ} &= -\nabla_a \omega'_{IJ} \ ,\end{aligned} \tag{1.284}$$

with ω'_{IJ} an arbitrary antisymmetric matrix.[29] Up to transverse rotations, these are just the expected tensorial transformations: n^μ_I is a vector, the other quantities are scalars.

Re-parametrizations Under an infinitesimal redefinition $\sigma^a \to \sigma^a + \zeta^a(\sigma)$ of the coordinates on $\mathcal{D}$, the embedding functions change by $\delta_\zeta X^\mu = -e^\mu_a \zeta^a$. The bulk metric varies just because of this shift in its argument:

$$\delta_\zeta g_{\mu\nu} = -\zeta^a e^\lambda_a \partial_\lambda g_{\mu\nu} \ . \tag{1.285}$$

The various quantities describing the geometry of $\mathcal{D}$ vary in form according to their tensor structure in the indices $a, b \ldots$, so in particular

$$\delta_\zeta e^\mu_a = -\partial_a \zeta^b \eta^\mu_b - \zeta^b \partial_b e^\mu_a = -\nabla_a \zeta^b e^\mu_b - K^I_{ab} \zeta^b n^\mu_I + \Gamma^\mu_{b\lambda} e^\lambda_a \zeta^b \ , \tag{1.286}$$

where in the second step we introduced covariant derivatives and made use of eq. 1.270. Inserting eq.s (1.285),1.286) into eq.s (1.275-1.279) we find then

$$\begin{aligned}\delta_\zeta \gamma_{ab} &= -2\nabla_{(a}\zeta_{b)} \ , \\ \delta_\zeta n^\mu_I &= K^I_{ab} e^{\mu a} \zeta^b + \Gamma^\mu_{\lambda b} n^\lambda_I \zeta^b + \omega''_{IJ} n^{\mu J} \ , \\ \delta_\zeta K^I_{ab} &= -2\nabla_{(a}\zeta^c K^I_{b)c} - \zeta^c \nabla_c K^I_{ab} + \omega''_{IJ} K^J_{ab} \ , \\ \delta_\zeta \mu_{aIJ} &= K_a^{[Ib} K^{J]}_{bc} \zeta^c - \frac{1}{2}\zeta^c R_{caIJ} - \zeta^c R_{c[IaJ]} - \nabla_a \omega''_{IJ} \ ,\end{aligned} \tag{1.287}$$

where ω''_{IJ} is an arbitrary antisymmetric matrix.[30]

Weyl rescalings If we perform an arbitrary infinitesimal rescaling of the metric

$$\delta_\sigma g_{\mu\nu}(X) = 2\sigma(X) g_{\mu\nu}(X) \ , \tag{1.288}$$

which does not affect the position X and the tangent vectors e^μ_a, from eqs. (1.275-1.279) we find

$$\begin{aligned}\delta_\sigma \gamma_{ab} &= 2\sigma \gamma_{ab} \ , \\ \delta_\sigma n^\mu_I &= -\sigma n^\mu_I + \omega_{IJ} n^{\mu J} \ , \\ \delta_\sigma K^I_{ab} &= \sigma K^I_{ab} - \gamma_{ab} n^{\mu I} \partial_\mu \sigma + \omega^{IJ} K_{Jab} \ , \\ \delta_\sigma \mu_{aIJ} &= -\nabla_a \omega_{IJ} \ .\end{aligned} \tag{1.289}$$

[29] With respect to eq. (1.277) we have $\omega'_{IJ} = -\partial_\lambda \xi^\rho n^\lambda_{[I} n_{J]\rho} + \omega_{IJ}$.

[30] With respect to eq. (1.277) we have $\omega''_{IJ} = +\zeta^c \Gamma^\lambda_{c\rho} n_{\lambda[I} n^\rho_{J]} + \omega_{IJ}$.

Chapter 2

Boundaries and interfaces in spin systems

This chapter is dedicated to an exploration of (some of) the conformal boundaries and interfaces which can be coupled to the 3d Ising model, and more generally to systems possessing an $O(N)$ symmetry. In particular, we consider the ordinary, special and extraordinary transitions, which characterize the critical behavior of the Ising model with a boundary, as we review in section 2.3. The conformal bootstrap is the main player of this part of the chapter. Therefore, we start in the next section with a brief review of the basic ideas that lead to a large amount of progress in this topic in the last few years. For a recent review and a set of lectures on the conformal bootstrap, see [90, 91]. In section 2.4, we tackle the study of an interface between free theory and an interacting spin system. Although we exhibit a solution of the bootstrap equations describing this situation, we mainly employ the ϵ-expansion in this study. Along the way, we prove some general results for interfaces between theories which live "close-by" from the RG point of view, valid at leading order in conformal perturbation theory.

2.1 Crossing symmetry and the conformal bootstrap

Let us start by briefly illustrating the crossing symmetry constraints in the homogeneous case. Consider the four-point function of scalars as represented in eq. (1.19), that we report here:

$$\begin{aligned}\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle \\ = (x_{12}^2)^{-\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{-\frac{1}{2}(\Delta_3+\Delta_4)}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}F(u,v),\end{aligned} \tag{2.1}$$

where

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}. \tag{2.2}$$

The function $F(u,v)$ can be decomposed in conformal blocks in the s-channel or in the t-channel, by which we mean, in a sum over primaries exchanged in the fusion (12)(34) and (14)(23). The result is the following crossing equation:

$$v^{\frac{1}{2}(\Delta_2+\Delta_3)}\sum_k c_{12k}c_{34k}\, f^{\text{4pt}}_{\Delta_k,J_k}(u,v)=u^{\frac{1}{2}(\Delta_3+\Delta_4)}\sum_k c_{14k}c_{23k}\, f^{\text{4pt}}_{\Delta_k,J_k}(v,u). \tag{2.3}$$

Notice that, even if not explicitly shown, the conformal blocks $f^{\mathrm{4pt}}_{\Delta,J}(u,v)$ depend, besides on the spin and scale dimension of the exchanged primary O, also on the scale dimension of the external operators.

Eq. (2.3) can be thought of as an infinite set of equations, one for each values of the continuous parameters u and v, whose unknowns are the OPE coefficients and the scale dimensions of the external and exchanged primaries. Recently, two methods have been devised in order to extract information from this kind of equations. Although their degree of development is not the same, they are in some sense complementary. We refer to them as the *linear functional* and the *determinant* methods.

The linear functional method was introduced in [41]. In its simplest version, is concerned with a correlator of four identical scalars of dimension Δ_{ext}. In this case the same subset of operators appear on both sides of eq. (2.3), and one of them is the identity. Let us make a remark at this point: the conformal blocks have in general logarithmic singularities at the boundary of their region of convergence. Vice versa, the conformal block of the identity is a simple power law. Now consider the limit, say, $x_1 \to x_2$. The (12)(34) channel is dominated by the identity, and infinitely many conformal blocks are needed in the crossed channel to reproduce the power law singularity through a sum of logarithms. In other words, every OPE in a CFT has to contain infinitely many operators, which is the main challenge the bootstrap program has to face. Coming back to the linear functional method, if the scalars are identical only squared OPE coefficients appear in eq. (2.3). OPE coefficients can be chosen to be real in a unitary theory, and therefore one can rewrite the crossing equation as follows:

$$1 = \sum_{\Delta, J\neq 0} p_{\Delta,J} F_{\Delta,J}(u,v), \qquad p_{\Delta,J} > 0, \tag{2.4}$$

where we isolated the contribution of the identity and we set

$$F_{\Delta,J}(u,v) = \frac{v^{\Delta_{\mathrm{ext}}} f^{\mathrm{4pt}}_{\Delta_k,J}(u,v) - u^{\Delta_{\mathrm{ext}}} f^{\mathrm{4pt}}_{\Delta_k,J}(v,u)}{u^{\Delta_{\mathrm{ext}}} - v^{\Delta_{\mathrm{ext}}}}. \tag{2.5}$$

Now suppose that a linear functional Λ can be found such that for a putative spectrum $\Sigma = \{\Delta, J\}$

$$\Lambda[F_{\Delta,J}(u,v)] > 0, \qquad \Lambda[1] = -1, \qquad \forall\ (\Delta, J) \in \Sigma. \tag{2.6}$$

Then the equation (2.4) does not admit solution and the spectrum is ruled out. In practical applications, usually $\Lambda = \sum_{m,n} c_{m,n} \partial_u^m \partial_v^n \Big|_{u=v=1/4}$ [1] - but this is in no way mandatory, see for instance [92]. The typical strategy is as follows. One makes an assumption on the low lying spectrum, for instance one can look for CFTs such that the lowest scalar has dimension $\Delta_{\mathrm{min}} > \Delta_0$. Then well established numerical technology exists to search in the space of functionals Λ for one that respects the condition (2.6), where Σ does not contain scalars of dimension smaller than Δ_0 and the spectrum is discretized and truncated at some high value of Δ and J. For Δ_0 big enough, such a functional is found and one discovers that CFTs without a scalar lighter than Δ_0 do not exist[2]. The linear functional method has been employed in a vast number of works, it has been refined and applied to a variety of problems [94, 95, 96, 42, 44, 49, 97, 50, 98, 99, 100, 45, 101, 102, 103, 104].

[1]To be precise, for technical reasons it is convenient to make a change of variables $u = (a^2 - b)/4$, $v = \big((a-2)^2 - b\big)/4$, take derivatives with respect to a and b and evaluate them at $(a,b) = (1,0)$.

[2]Since it can be shown that heavy operators give exponentially suppressed contribution to the conformal block decomposition [93], the error coming from the truncation of the spectrum is under control.

Here, let us emphasize a couple of facts. By its nature, the linear functional method defines forbidden regions in the space of the CFT data. At the boundary of these regions, a spectrum which is crossing symmetric up to some maximum scale dimension can be extracted numerically [97]. It is not difficult to show that the four-point functions of local operators on the vacuum encode all of the constraints coming from crossing symmetry: however, one needs in principle all of them, and therefore the trial spectrum extracted from a specific correlator is not guaranteed to correspond to a unitary CFT. Sometimes it does, though [42], or maybe a set of minimal hypotheses on the spectrum can be put in place to lower the bound disregarding uninteresting solutions which stand in the way [50]. Another possibility is to consider more than one four-point function, so that further requirements on the spectrum can be made: for instance, internal symmetries differentiate the set of primaries appearing in different OPEs. This strategy was applied to the $3d$ Ising model in [45], providing strong evidence that the presence of $\mathbb{Z}_2$ symmetry and two relevant primaries defines only one theory.

The linear functional method can be clearly rephrased as follows. By Taylor expanding the function defined in (2.5) around a point, the crossing symmetry constraint (2.4) becomes a system of linear equations in the $\{p_{\Delta,\ell}\}$, whose coefficients depend on the spectrum Σ and whose number is determined by the maximum order included in the Taylor expansion. The rest of the procedure provides a way to investigate the existence of solutions of this equation with the positivity constraint on the $\{p_{\Delta,\ell}\}$. The determinant method, proposed by Gliozzi in [43], diverges from the linear functional procedure in this second part. Consider the system defined by one inhomogeneous equation

$$\sum_{\Delta,J\neq 0} p_{\Delta,J}\, F_{\Delta,J}\left(\frac{1}{4},\frac{1}{4}\right) = 1, \tag{2.7}$$

and N homogeneous ones:

$$\sum_{\Delta,J\neq 0} p_{\Delta,J}\; \partial_u^m \partial_v^n F_{\Delta,J}\big|_{u=v=1/4} = 0, \qquad m+n\leq N. \tag{2.8}$$

Fix also a truncation of the spectrum containing M primary operators, such that the linear system has M unknowns. Choosing $N > M$, the homogeneous system admits a non-trivial solution if and only if all the $\binom{N}{M}$ minors of the system vanish. This condition provides a set of non-linear equations in the M unknown scale dimensions, that can be solved numerically to determine the spectrum. Solving the linear system then provides the OPE coefficients. It is important to notice that positivity of the $\{p_{\Delta,J}\}$ is not required, and the output of the method is the CFT data instead of bounds on it. On the other hand, the procedure is less systematic than with the positive linear functional. Indeed, in practice one chooses the number of operators at each spin level, and the numerical search of a solution for the spectrum of anomalous dimensions requires a starting point in the space of Δs. This means that this method is especially suitable if one is interested in studying a specific CFT, for which some information about the low-lying spectrum, even very approximate, can be found by other means. Furthermore, while the linear functional method can be modified in order to allow for a search over continuous trial spectra of scale dimensions, it is not clear how to do this with Gliozzi's procedure. The method of determinants has been used both in the study of positive and non-positive crossing equations. The latter situation presents itself often - it is enough to consider a correlator of non identical primaries - but non-unitary theories, like the Lee-Yang model [105], or theories placed on non-trivial manifolds [106] have been tackled the most. The rest of the chapter is dedicated to an example of this kind: we shall analyze the crossing equation for the two-point function in the presence of a codimension one defect, for which one of the channel is not positive. We shall therefore bootstrap the spectrum using Gliozzi's method. What follows reproduces part of the content of [51].

2.2 Defect CFTs and the method of determinants

The conformal bootstrap was first applied to the boundary setup in [49], while the twist line defect defined in [39] - see chapter 3 - was tackled in [50]. Both papers are concerned with the $3d$ Ising model, and both used the linear functional method. In the latter, four-point functions of defect operators were considered, while the former focused on two-point functions of bulk operators. Correlators of defect operators are blind to bulk-to-defect couplings, but correlators of bulk primaries do not satisfy in general the positivity constraints required by the linear functional method, and *ad hoc* assumptions were made in [49], motivated by computations in $2d$ and in ϵ-expansion. Here we concentrate on the two-point function of bulk scalar primaries, using the method of determinants, which can be safely applied to this case. Since our main interest is again the $3d$ Ising model, we compare our results for the special and the extraordinary transitions with those of [49]. We also find approximate solutions to the crossing equations corresponding to the ordinary transition, which cannot be studied with the linear functional. In the latter case we extended the analysis to the $O(N)$ models with $N = 0, 2, 3$, where a comparison can be made with two-loop calculations. The main results are summarized in the tables 2.1 and 2.2. Finally, we initiate the study of an example of RG domain wall, an interface between two CFTs connected by the renormalization group, which is obtained by turning on a relevant deformation on half of the space and flowing to the IR. Specifically, we study the flow triggered by the $(\phi^2)^2$ coupling in a bosonic theory. We give a first order description in ϵ-expansion which applies to models with $O(N)$ symmetry and can be easily generalized to other perturbation interfaces. We then focus on the Ising model when looking for a numerical solution to the crossing equations in $3d$.

In the rest of this section we briefly recapitulate some of the material presented in chapter 1, and we explain the method of determinants. Section 2.3 is devoted to the study of the boundary CFTs associated to the 3d Ising and other spin systems. We define and study the domain wall in section 2.4. Finally, we draw our conclusions in section 2.5. Appendix 2.A contains some details of the ϵ-expansion computations.

Correlation functions of excitations living on a defect are the same as in an ordinary $(d-1)$-dimensional CFT, and are completely characterized by the spectrum of scale dimensions $(\widehat{\Delta}_l)$ and the coefficients of three-point functions $(\widehat{c}_{lmn})$. While no conserved stress-tensor is expected to exist on the defect, a protected scalar operator of dimension d is always present: the displacement operator, which we call $\mathrm{D}(x^a)$, measures the breaking of translational invariance, and is defined by the Ward identity for the stress-tensor:

$$\partial_\mu T^{\mu d}(x) = -\mathrm{D}(x^a)\,\delta(x^d). \tag{2.9}$$

Here we denoted by latin indices the directions along the defect, which is placed at $x^d = 0$, while Greek letters run from 1 to d. Similarly, for every bulk current whose conservation is violated by the defect, a protected defect operator exists.

In the bulk, there is of course the usual OPE. For scalar primaries,

$$O_1(x)O_2(y) = \frac{\delta_{12}}{(x-y)^{2\Delta_1}} + \sum_k c_{12k}\mathcal{C}[x-y,\partial_y]O_k(y)\,, \tag{2.10}$$

where $\mathcal{C}[x-y,\partial_y]$ are determined by conformal invariance, and we isolated the contribution of the identity. One can also fuse a local operator with the defect. The bulk-to-defect OPE for a scalar primary can be written

$$O_1(x) = \frac{a_1}{|2x^d|^{\Delta_1}} + \sum_l b_{1l}\mathcal{D}[x^d,\partial_a]\widehat{O}_l(x^b)\,, \tag{2.11}$$

where we denoted defect operators with a hat. Again, the differential operators $\mathcal{D}[x^d,\partial_a]$ are fixed by conformal invariance. Similar OPEs can be written for bulk tensors. The c_{12k}'s in eq. (2.10) are the coefficients of three-point functions without the defect, while b_l is the coefficient of the correlator $\langle O(x)\widehat{O}_l(y^a)\rangle$, otherwise fixed by conformal symmetry. Even if, for the sake of simplicity, some abuse of notation is present[3], all OPE coefficients refer to canonically normalized operators, with one exception: the normalization of the displacement operator is fixed by eq. (2.9). While scalars have one-point functions proportional to a_O, the coefficient of the identity in the bulk-to-defect OPE, tensors do not acquire an expectation value in the presence of a codimension one defect.

The easiest crossing equation involves the OPEs (2.10) and (2.11). Consider the two-point function $\langle O_1(x)O_2(x')\rangle$. One can decompose it into the bulk channel by plugging in eq. (2.10): a sum over one-point functions is obtained, that is, a sum over the coefficients $c_{12k}a_k$ multiplying some known functions of the kinematic variables. Or, one can substitute both operators with their Defect OPE, and in this case the sum involves the quantities $b_{1l}b_{2l}$. Let us introduce the conformal invariant combination[4]

$$\xi = \frac{(x-x')^2}{4x^d x'^d}. \tag{2.12}$$

This cross-ratio is conveniently positive when both points are chosen in the half-plane $x^d > 0$. This is not the case when considering bulk operators on opposite sides of an interface. Moreover, in this setup the bulk OPE is not defined. The issue is solved by folding the system and treating it as a boundary CFT: the folding trick provides us with a trivial OPE, fixed by the absence of local interactions between the two primaries. We shall have more to say on this point in section 2.4. For now, we just point out that the natural cross-ratio is the one constructed from a point and the mirror image of the second one, and it is again positive. We assume $\xi \geq 0$ in the rest of this section.

Conformal symmetry justifies the following parametrization:

$$\langle O_1(x)O_2(x')\rangle = \frac{1}{(2x^d)^{\Delta_1}(2x'^d)^{\Delta_2}}\, f_{12}(\xi). \tag{2.13}$$

Then the crossing equation can be written as a double decomposition of the function $f_{12}(\xi)$:

$$f_{12}(\xi) = \xi^{-(\Delta_1+\Delta_2)/2}\left(\delta_{12} + \sum_k c_{12k}\, a_k\, f(\Delta_{12},\Delta_k;\xi)\right) = a_1 a_2 + \sum_l b_{1l}\, b_{2l}\, \widehat{f}(\widehat{\Delta}_l;\xi)\,, \tag{2.14}$$

where [10] - see also section 1.4,

$$f(\Delta_{12},\Delta,\xi) = \xi^{\Delta/2}\,{}_2F_1\left(\frac{1}{2}(\Delta_1-\Delta_2+\Delta),\frac{1}{2}(\Delta_2-\Delta_1+\Delta);\Delta+1-\frac{d}{2},-\xi\right), \tag{2.15a}$$

$$\widehat{f}(\Delta,\xi) = \xi^{-\Delta}\,{}_2F_1\left(\Delta,\Delta+1-\frac{d}{2};2\Delta+2-d;-\frac{1}{\xi}\right). \tag{2.15b}$$

[3]For instance, the coefficient $b_{\phi^2 D}$ in free theory appears in the two point function $\langle \frac{\phi^2}{\sqrt{2N}} D\rangle$.

[4]In this chapter we use for the cross-ratio the normalization which is customary for boundaries, see footnote 14 in chapter 1.

We slightly changed notation with respect to section 1.4, in accordance with [49], removing quantum numbers that become trivial when the codimension is one, and including explicitly the dependence on Δ_{12}. It is worth noticing that the conformal blocks of the boundary channel in $d = 3$ can be expressed as elementary algebraic functions, namely,

$$\widehat{f}(\Delta, \xi)|_{d=3} = \frac{1}{2\sqrt{\xi}} \left(\frac{4}{1+\xi}\right)^{\Delta-\frac{1}{2}} \left[1 + \sqrt{\frac{\xi}{1+\xi}}\right]^{-2(\Delta-1)} . \tag{2.16}$$

This is of course of great help in numerical calculations.

Going back to eq. (2.14), we rewrite it isolating the identity:

$$-\sum_k c_{12k}\, a_k\, f(\Delta_{12}, \Delta_k; \xi) + \xi^{(\Delta_1+\Delta_2)/2} \left(a_1 a_2 + \sum_l b_{1l}\, b_{2l}\, \widehat{f}(\widehat{\Delta}_l; \xi)\right) = \delta_{12}\,. \tag{2.17}$$

As outlined in section 2.1 in the case of a four-point function, the strategy described in [43] consists in trading one functional equation for infinitely many linear equations: one for each coefficient of the Taylor expansion around, say, $\xi = 1$. Then we truncate both the Taylor expansions, keeping only the first M derivatives, and the spectrum, keeping the first N operators in total from the two channels. The bulk identity is excluded from the count. We denote this truncation with a triple (n_{bulk}, n_{bdy}, s), the three numbers counting respectively bulk and boundary operators of non vanishing dimension, and the presence ($s = 1$) or absence ($s = 0$) of the boundary identity. We obtain this way a finite system, at the price of introducing a systematic error, coming from the disregarded higher order derivatives and heavier operators:

$$\begin{aligned}
&-\sum_k^{n_{bulk}} p_k\, f^k\Big|_{\xi=1} + \sum_l^{n_{bdy}} q_l\, \widehat{f}^l\Big|_{\xi=1} + a_1 a_2 = \delta_{12}, && n_{bulk} + n_{bdy} + s = N \\
&-\sum_k^{n_{bulk}} p_k\, \partial_\xi^n f^k\Big|_{\xi=1} + \partial_\xi^n \left(\xi^{(\Delta_1+\Delta_2)/2} \sum_l^{n_{bdy}} q_l \widehat{f}^l + a_1 a_2\right)\Bigg|_{\xi=1} = 0, && n = 1, \dots, M,
\end{aligned} \tag{2.18}$$

where we used a shorthand notation for the OPE coefficients $p_k = c_{12k} a_k$, $q_l = b_{1l} b_{2l}$. Let us focus for definiteness on the case of two identical external scalars, $\delta_{12} = 1$. The p_k's, q_l's and a_1^2 are the unknowns of a linear system whose coefficients depend nonlinearly on the bulk and defect spectra. Choosing $M \geq N$, the homogeneous system, i.e. the second line in (2.18), admits a non-trivial solution if and only if all the $\binom{M}{N}$ minors of the system vanish. This condition provides a set of non-linear equations in the N unknown scale dimensions. When this set admits a (numerical) solution we say that the the two-point function under study is truncable. In such a case, inserting the obtained (approximate) spectrum in the complete linear system (2.18), we get the OPE coefficients.

Notice that every consistent CFT data is in particular a solution to this crossing equation. Therefore, some input has to be provided: here we are implicitly assuming that the external dimensions are known, and in fact this is going to be the strategy when we try to isolate the $3d$ Ising model. One does not expect to find an exact solution for a generic truncation: heavier defect and bulk operators become more and more important when moving respectively towards the bulk ($\xi \to 0$) or the defect ($\xi \to \infty$), therefore we expect a good truncation to require N to grow with M. In practice, in this work we usually choose $M = N + 1$, and we find that the space of solutions to the system of nonlinear equations has in general non-zero dimension. By fixing the free parameters with the best known values of the lowest lying bulk primaries, we give predictions for the low lying defect spectrum and for heavier primaries.

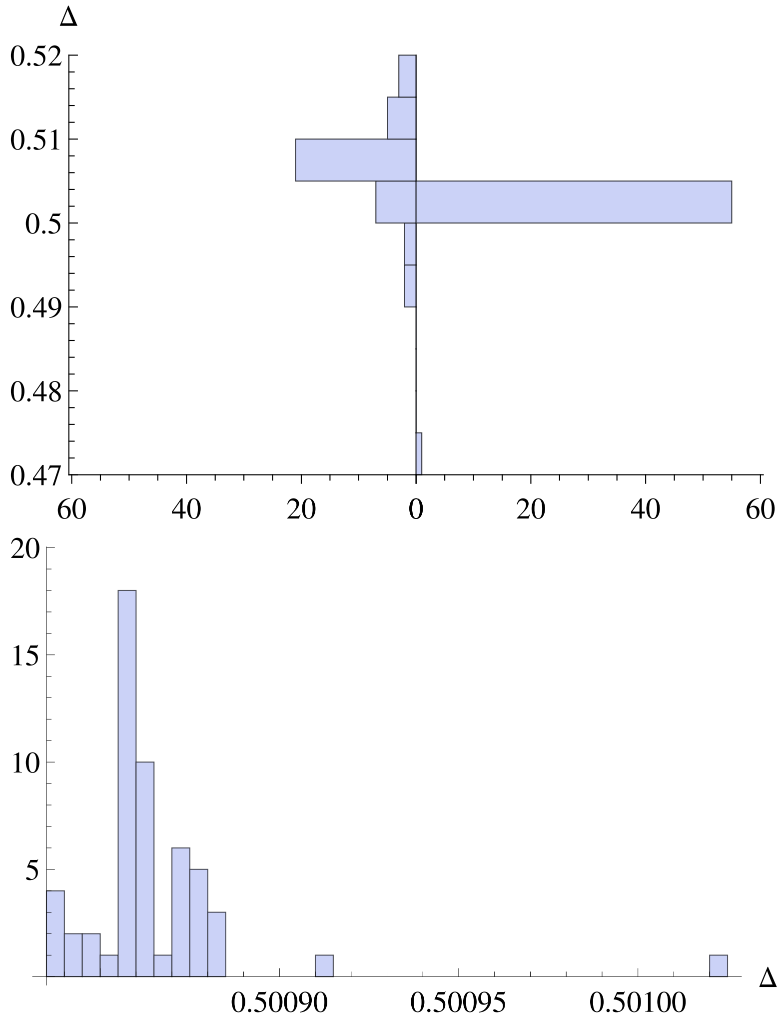

Figure 2.1: Top panel: paired histograms of the solutions of two different truncations of the crossing equations for the ordinary transition of the 2d Ising model. Left: histogram for the scale dimensions of the first boundary operator in the (2,1,0) truncation. The exact result is at $\widehat{\Delta} = \frac{1}{2}$. Right: the corresponding histogram for the (4,3,0) truncation. Bottom panel: a more detailed view of the latter histogram.

As a general rule, a finite truncation of the crossing symmetry equations is a good approximation of a given CFT if the missing operators can be consistently put at $\Delta = \infty$ or at zero coupling. When a trial spectrum has been found, one can check its stability by adding one operator and one derivative. It turns out in most cases that the scaling dimension of the new operator acts as a free parameter which can vary in a fixed range. We use the solution for predictions only if it does not depend very strongly on this parameter. This gives a way of controlling the systematic error, albeit not an algorithmic one. Let us also observe that the general agreement with the results of the epsilon expansion suggests that the error is rather small, at least for what concerns the boundary case. Another important check comes from the Ward identity associated with the displacement operator, which, as we shall see, yields non-trivial relations among the CFT data. These relations

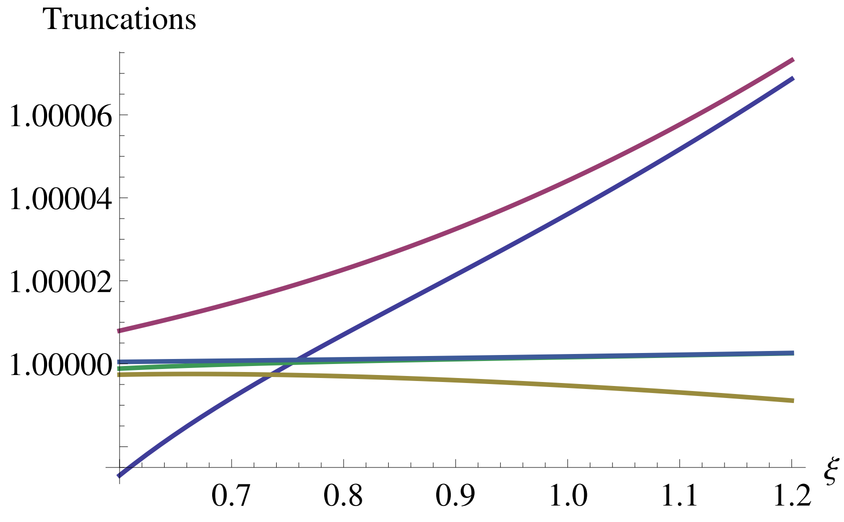

Figure 2.2: The left-hand-side of the sum rule (2.17) for various truncations $(n_{bulk}, n_{bdy}, 0)$ of the two-point function of the 2d Ising model in the ordinary transition. Only in the $n_{bulk} \to \infty$, $n_{bdy} \to \infty$ limit the sum rule is saturated.

are perfectly verified by the numerical solutions, as described in the next section.

Another parameter to be considered in order to check the quality of a given truncation is the spread of the solutions. As soon as the number M of equations exceeds the number of unknowns, the system is over-determined and can be split in consistent subsystems, each of them giving in principle a different solution. The spread of these solution gives a rough estimate of the error. In the cases where the exact solution is known the narrower is the spread the closer is the solution to its exact value. This is the case for instance of the four-point function of the free scalar massless theory in any dimension [43]. On the contrary large spreads are associated to large systematic errors due to too rough approximations of the crossing equations. A clear illustration of this behavior can be found in the ordinary transition of the 2d Ising model, where the exact two-point function is known [56]. Assuming we already know the bulk spectrum, we can start considering the truncation (2,1,0) to evaluate the scale dimensions of the first surface operator. We have to look at the zeros of 3×3 determinants. Taking for instance 8 derivatives we have 56 equations whose solutions are plotted in the histogram of of fig. 2.1. Their large spread is associated with a rather rough approximation of the sum rule (2.17) as fig. 2.2 shows. The same figure points out also that the truncation (4,3,0) is much better. In this case the unknowns are the dimensions of the three surface operators. The consistent subsystems are made of sets of three 7×7 determinants. With 8 derivatives we have again 56 possible solutions. Their spread is drastically reduced and the mean value is closer to the exact one, as fig. 2.1 shows. We anticipate that all the solutions considered in the next section have a microscopic spread (see e.g. fig. 2.1 and fig. 2.3).

2.3 The boundary bootstrap and the $3d$ Ising and $O(N)$ models

In this section we shall consider the boundary conformal field theories (BCFTs) associated with the Ising model and other magnetic systems. Specifically, the IR properties of the surface transitions in these systems are controlled by RG fixed points, which of course are described by just as many defect CFTs. We denote with $\sigma(x)$ the scalar field (i.e. the order parameter of the theory) and with $\widehat{\sigma}$ the corresponding surface operator. The surface Hamiltonian associated with a flat $d-1$ dimensional boundary of a semi-infinite system can be written in terms of the three relevant surface operators (see for instance [55])

$$H = \int d^{d-1}x \left(c\widehat{\sigma}^2 + h_1\widehat{\sigma} + h_2\partial_z\widehat{\sigma} \right) . \tag{2.19}$$

Here $z \equiv x^d$ is the coordinate orthogonal to the boundary. This Hamiltonian has three fixed points for which the $\mathbb{Z}_2$ symmetry is not explicitly broken:

$$O: \quad h_1 = h_2 = 0,\, c = +\infty\,; \tag{2.20}$$
$$E: \quad h_1 = h_2 = 0,\, c = -\infty\,; \tag{2.21}$$
$$S: \quad h_1 = h_2 = c = 0\,. \tag{2.22}$$

Near the first fixed point the configurations with $\widehat{\sigma} \neq 0$ are exponentially suppressed, then $\widehat{\sigma} = 0$ (i.e. Dirichlet boundary condition). This fixed point controls the ordinary transition. The only relevant surface operator in this phase is $\partial_z\widehat{\sigma}$. The fixed point with $c = -\infty$ favors the configurations with $\widehat{\sigma} \neq 0$: it is associated with the extraordinary transition, where the $\mathbb{Z}_2$ symmetry is spontaneously broken on the boundary and no relevant surface operator can couple with it; the lowest dimensional surface operator, besides the identity, is the displacement, whose scaling dimension is d. The fixed point with $c = 0$ controls the special transition, a multicritical phase with two relevant primaries. The even operator $\widehat{\sigma}^2$ is responsible for the flow of c to ∞ or $-\infty$ according to the initial sign, while the odd one, $\widehat{\sigma}$, is the symmetry breaking operator of this phase, characterized by the Neumann boundary condition $\partial_z\widehat{\sigma} = 0$. We omitted a classically marginal coupling, $\partial_z\widehat{\sigma}^2$, because it vanishes with both Neumann and Dirichlet boundary conditions, and it cannot be turned on in the extraordinary transition, where there is no local odd relevant excitation. We shall come back to this operator when considering the RG domain wall.

One important question to address within a BCFT is how to find the scale dimensions of the surface operators and their OPE coefficients in terms of the bulk data. This problem has been completely solved in 2d [13] thanks to the modular invariance. In $d > 2$ useful information can be extracted by the epsilon expansion and other perturbative methods. Recently the conformal bootstrap approach has been shown to be very promising [49]. Here we face this problem with the method of determinants.

We study the 2-point function $\langle\sigma(x)\sigma(y)\rangle$. The general criterion we use to classify the surface transition associated with a specific truncation (n_{bulk}, n_{bdy}, s) of the crossing symmetry equations (2.18) is based on three steps. First, we verify that the solution is compatible with a unitary theory by requiring the positivity of all the non-vanishing couplings b_a^2 $(a = 1, 2, \ldots, n_{bdy})$. Then we look at the sign of the couplings to the bulk blocks $a_k c_{\sigma\sigma k}$ $(k = 1, \ldots, n_{bulk})$. As in [49], we will assume that the ordinary transition is signaled by the presence of at least one negative coupling in the bulk channel. On the other hand, positivity of the couplings indicates the extraordinary or the special transition, depending on the presence or absence of the surface identity. We should point out that these assumptions have not been proven. However, the results of this work seem to confirm them, serving as a consistency check on the whole setup.

2.3.1 The ordinary transition

We start by considering what is perhaps the simplest successful truncation of eq. (2.18), corresponding to the fusion rules

$$\begin{aligned} &\sigma \times \sigma \sim 1 + \varepsilon + \varepsilon', && \text{bulk channel,} \\ &\sigma \sim \widehat{O}, && \text{boundary channel.} \end{aligned} \tag{2.23}$$

This truncation is denoted by the triple (2,1,0). The system (2.18) admits a solution if and only if the 3×3 determinants made with the derivatives of the conformal blocks associated with $\varepsilon, \varepsilon', \widehat{O}$ vanish. We assume that the scale dimensions of σ, ε and ε' are known ($\Delta_\sigma = \frac{1}{2} + \frac{\eta}{2}$; $\Delta_\varepsilon = 3 - 1/\nu$; $\Delta_{\varepsilon'} = 3 + \omega$, see table 2.1) and in this particular case the only unknown scale dimension is $\Delta_{\widehat{O}}$. Fig. 1 shows the values of few determinants of this kind. Clearly they all apparently vanish at the same point. In fact there is a microscopic spread of the solutions and we find $\Delta_{\widehat{O}} = 1.276(2)$. The solution of the complete linear system yields a negative $a_\varepsilon c_{\sigma\sigma\varepsilon}$, thus, according to the above criterion, we are faced with the ordinary transition of the 3d Ising model. Hence, $\widehat{O}$ has to be identified with $\partial_z \widehat{\sigma}$. A two-loop calculation in the 3d ϕ^4 model yields [107] $\Delta_{\partial_z \widehat{\sigma}} \simeq 1.26$ in good agreement with our result.

N	η	ν	ω
0	0.0314(32)	0.5874(2)	0.812(16)
1	0.03627(10)	0.63002(10)	0.832(6)
2	0.0380(4)	0.67155(27)	0.789(11)
3	0.0364(6)	0.7112(5)	0.782(13)

	$\Delta_{\partial_z \widehat{\sigma}}$		
N	2-loop	Monte Carlo	Bootstrap
0	1.33	–	1.332(6)
1	1.26	1.2751(6)	1.276(2)
2	1.211	1.219(2)	1.2342(9)
3	1.169	1.187(2)	1.198(1)

N	$a_\varepsilon c_{\sigma\sigma\varepsilon}$	$a_{\varepsilon'} c_{\sigma\sigma\varepsilon'}$	$b^2_{\widehat{\Delta}}$
0	−0.8447(34)	0.0366(17)	0.692(1)
1	−0.789(3)	0.042(1)	0.755(13)
2	−0.747(1)	0.0488(4)	0.80022(5)
3	−0.710(1)	0.0509(6)	0.8395(6)

Table 2.1: The first table collects the input parameters. The second one is a comparison between two-loop calculations [107], Monte Carlo simulations (reference [17] for $N = 1$ and reference [108] for $N > 1$) and our bootstrap results for the scaling dimension of the surface operator $\partial_z \widehat{\sigma}$ in the ordinary transition of 3d $O(N)$ models. The last three columns collect our results for the OPE coefficients. The critical indices η and ν for $N = 0, 1, 2, 3$ are taken respectively from references [109], [110], [111] and [112]. Those for ω from[113].

This solution admits a straightforward generalization to any 3d $O(N)$ model by simply replacing the critical indices with the appropriate values. Table 2.1 shows our results for $N = 0$ (the non-unitary self-avoiding walk model), $N = 1$ (Ising), $N = 2$ (XY model) and $N = 3$ (Heisenberg model), where we can compare our results with the two-loop calculation of [107].

2.3.2 The extraordinary transition

Such a transition is characterized by the non-vanishing contribution of the boundary identity to the two-point functions of $\mathbb{Z}_2$ odd operators. In this case the boundary surface is

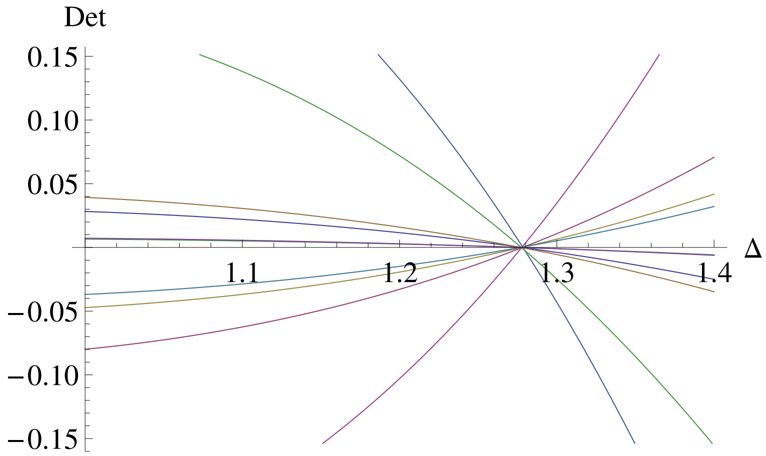

Figure 2.1: Plot of the 10 3×3 minors made with the first 5 derivatives of the conformal blocks associated with ε, ε' and $\widehat{O}$ as functions of $\Delta_{\widehat{O}}$. They all vanish approximately at he same point, selecting the allowed value of $\Delta_{\widehat{O}}$.

in an ordered phase, therefore the degrees of freedom described by $\mathbb{Z}_2$ odd operators are frozen. The first non-vanishing surface operator, besides the identity, is the displacement D with $\Delta_D = 3$. As a consequence, the most relevant contribution to the boundary channel is known and the crossing equations can be exploited to obtain information on the bulk channel.

Actually adding the boundary identity to the truncation requires adding more bulk operators as well. We found a first stable solution of the type (4,1,1). This time the scaling dimensions of the two needed bulk scalars ε'' and ε''' cannot be used as input parameters because, once fixed Δ_σ, Δ_ε and $\Delta_{\varepsilon'}$[5], we get a solution only if

$$N = 1\,, \qquad \Delta_{\varepsilon''} = 7.316(14)\,, \qquad \Delta_{\varepsilon'''} = 13.05(4)^6. \tag{2.24}$$

The other parameters of the solution are

$$\begin{aligned} a_\varepsilon c_{\sigma\sigma\varepsilon} &= 6.914(6), & a_{\varepsilon'} c_{\sigma\sigma\varepsilon'} &= 2.261(2), & a_{\varepsilon''} c_{\sigma\sigma\varepsilon''} &= 0.187(1), \\ a_{\varepsilon'''} c_{\sigma\sigma\varepsilon'''} &= 0.0046(1), & a_\sigma^2 &= 6.757(4)\,, & b_{\sigma\mathrm{D}}^2/C_{\mathrm{D}} &= 0.06282(3); \end{aligned} \tag{2.25}$$

where we denoted with C_{D} the Zamolodchikov norm of the displacement operator.

We can easily generalize this solution to the $O(N)$ spin models using as input the data of table 2.1. For the XY model ($N = 2$) we obtain

$$N = 2\,, \qquad \Delta_{\varepsilon''} = 6.002(33)\,, \qquad \Delta_{\varepsilon'''} = 10.96(4). \tag{2.26}$$

with

$$\begin{aligned} a_\varepsilon c_{\sigma\sigma\varepsilon} &= 5.585(11), & a_{\varepsilon'} c_{\sigma\sigma\varepsilon'} &= 1.466(10), & a_{\varepsilon''} c_{\sigma\sigma\varepsilon''} &= 0.307(9), \\ a_{\varepsilon'''} c_{\sigma\sigma\varepsilon'''} &= 0.0162(3), & a_\sigma^2 &= 5.495(9)\,, & b_{\sigma\mathrm{D}}^2/C_{\mathrm{D}} &= 0.06741(7). \end{aligned} \tag{2.27}$$

[5] Here and in the rest of this section we use as input parameters of the Ising model the values $\Delta_\sigma = 0.518154(15)$, $\Delta_\varepsilon = 1.41267(13)$ and $\Delta_{\varepsilon'} = 3.8303(18)$ taken from [44].

[6] In the entire chapter the estimate of the statistical error due to the uncertainty on the input parameters is obtained by means of a statistical bootstrapping procedure.

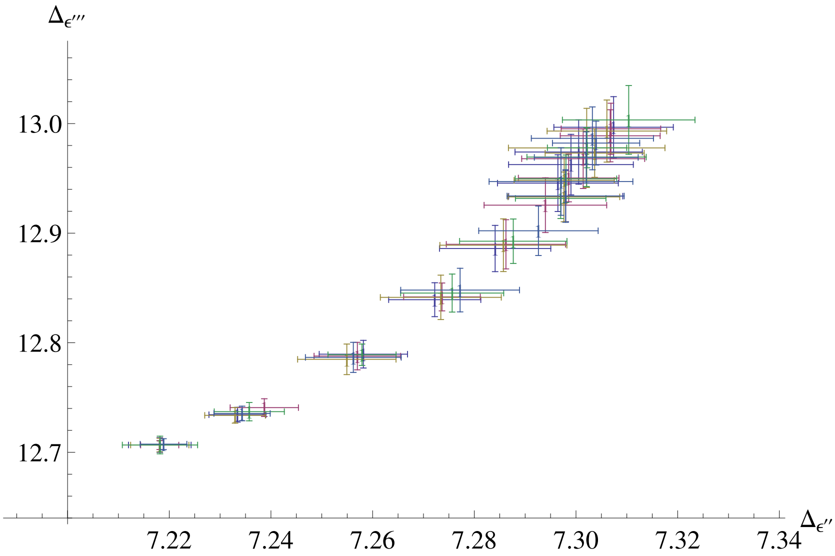

Figure 2.2: Parametric plot of the scaling dimensions of $\Delta_{\varepsilon''}$ and $\Delta_{\varepsilon'''}$ generated by the unknown parameter $\widehat{\Delta}$. Here we see the effect of the statistical errors on the input data, namely Δ_σ, Δ_ε and $\Delta_{\varepsilon'}$ as well as the effect of the spread of the solutions. Some of these data are presented in table 2.1.

Similarly for the Heisenberg model ($N = 3$) we get

$$N = 3\,, \qquad \Delta_{\varepsilon''} = 5.285(17)\,, \qquad \Delta_{\varepsilon'''} = 10.48(1). \tag{2.28}$$

with

$$\begin{aligned} a_\varepsilon c_{\sigma\sigma\varepsilon} &= 5.019(8), \qquad a_{\varepsilon'} c_{\sigma\sigma\varepsilon'} = 0.846(13), \qquad a_{\varepsilon''} c_{\sigma\sigma\varepsilon''} = 0.505(10), \\ a_{\varepsilon'''} c_{\sigma\sigma\varepsilon'''} &= 0.0207(1), \qquad a_\sigma^2 = 4.874(8)\,, \qquad b_{\sigma \mathrm{D}}^2/C_{\mathrm{D}} = 0.06919(10). \end{aligned} \tag{2.29}$$

It is interesting to notice that the value of $\Delta_{\varepsilon''} = 3 + \omega_2$ as a function of N is close to that first calculated in [114] with the functional renormalization group method.

In the case of the Ising model, where we used the more precise input data of [44], we probed the stability of the solution by adding a new conformal block in the boundary channel. It turns out that the truncation (4,2,1) defines a one-dimensional family of solutions, where the free parameter is the dimension of the added surface operator, which can vary in the range $0 < \widehat{\Delta} \leq \infty$. In the limit $\widehat{\Delta} \to \infty$ we recover, as expected in a stable solution, the truncation (4,1,1). The dimensions of the two bulk operators $\Delta_{\varepsilon''}$ and $\Delta_{\varepsilon'''}$ vary as functions of $\widehat{\Delta}$ in a narrow range: the net effect of the unknown parameter is to reduce a bit the scaling dimensions of these bulk operators. Eliminating $\widehat{\Delta}$ we obtain the plot in fig. 2.2. The uncertainty on the actual value of $\widehat{\Delta}$ forces us to enlarge the errors in the bulk dimensions. Fig. 2.2 roughly suggests

$$\Delta_{\varepsilon''} = 7.27[5]\,, \qquad \Delta_{\varepsilon'''} = 12.90[15], \tag{2.30}$$

which supersede eq. (2.24). We used square brackets to indicate that this is not a statistical error, but a sum of the uncertainties.

Unfortunately one can find in literature a wide range of proposed values for $\Delta_{\varepsilon''}$ and $\Delta_{\varepsilon'''}$ which strongly depend on the method employed (see for instance table 3 of [44]). What is especially disturbing is that the method of determinants applied to the four-point

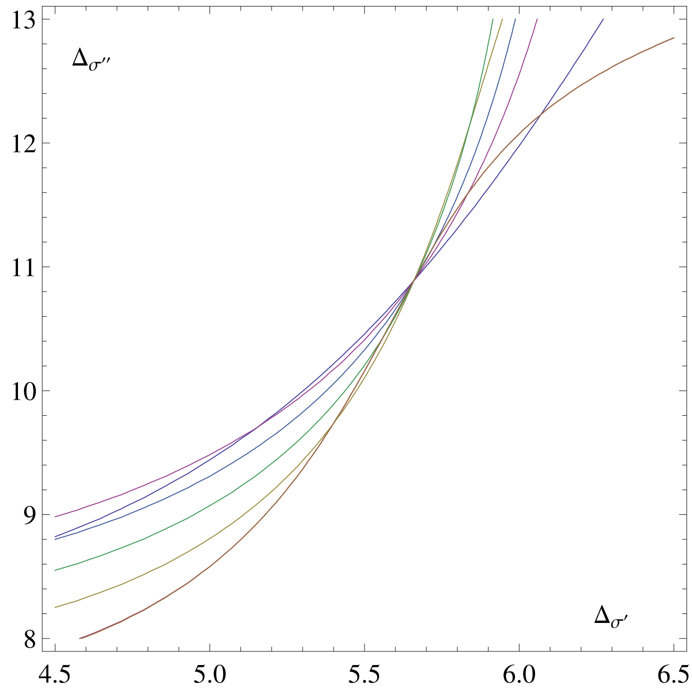

Figure 2.3: Plot of the zeros of some 5×5 determinants associated with the fusion rules (2.31) and (2.32).

function gave very different values for these quantities [105], so we decided to reanalyze the bootstrap equations for the four-point function in the bulk in order to see whether there is also a solution compatible with the spectrum suggested by the boundary bootstrap. Out of this study we can confirm the existence of a scalar of dimension ~ 7.2 with a positive coupling. We were unable to find a proper solution for the scalar at ~ 13, all solutions being characterized by a coupling that is very small, negative and nearly always compatible with zero. The quoted dimensions of these two scalars found with the linear functional method [44] are respectively ~ 7 and ~ 10.5.

Another interesting two-point function to be studied in the extraordinary transition of the Ising model is the spin-energy correlator $\langle \sigma(x)\varepsilon(y)\rangle$ which is different form zero only in this phase, being the only surface transition where the $\mathbb{Z}_2$ symmetry of the model is broken. The fusion rule of the bulk sector contains odd operators only:

$$\sigma \times \varepsilon \sim \sigma + \sigma' + \sigma'' + \dots , \tag{2.31}$$

while in the boundary sector the first primary operator contributing, besides the identity, is the displacement operator:

$$\sigma \sim 1 + \mathrm{D} + \dots , \qquad \varepsilon \sim 1 + \mathrm{D} + \dots \tag{2.32}$$

The first stable solution corresponds to the truncation (3,1,1) defined by the above fusion rules. It is associated with the (apparently) common intersection of the zeros of the 5×5 determinants made with the derivatives of the 5 conformal blocks involved (see fig. 2.3):

$$\Delta_{\sigma'} \simeq 5.66 \; ; \qquad \Delta_{\sigma''} \simeq 10.89 \; ; \tag{2.33}$$

$$a_\sigma c_{\sigma\varepsilon\sigma} \simeq 0.148\,\kappa \; ; \quad a_\varepsilon a_\sigma \simeq 0.927\,\kappa \; ; \quad b_{\sigma\mathrm{D}} b_{\varepsilon\mathrm{D}}/C_{\mathrm{D}} \simeq 0.0196\,\kappa \; . \tag{2.34}$$

The parameter κ arises because now the bootstrap equations are homogeneous, that is, they do not contain the information about the normalization of the external operators. The normalization of the order parameter is contained in the correlator $\langle\sigma\sigma\rangle$, while the normalization of the energy follows from assuming symmetry of the OPE coefficient $c_{\sigma\sigma\varepsilon} = c_{\sigma\varepsilon\sigma}$. Therefore, combining (2.34) with the analogous couplings in (2.25), we can compute the unknowns $a_\varepsilon,\ a_\sigma,\ b_{\sigma\mathrm{D}}/\sqrt{C_\mathrm{D}},\ \mu_{\varepsilon\mathrm{D}}/\sqrt{C_\mathrm{D}},\ \kappa,\ c_{\sigma\varepsilon\sigma}$.

In order to probe the stability of the solution and to evaluate the errors we upgraded the solution to (5,1,1), which corresponds to a one-parameter family of solutions. We used as a free parameter the heaviest bulk scalar σ_4. A solution exists for $18 \leq \Delta_{\sigma_4} \leq 28$. As expected for a stable solution, this parameter has no visible effect on the OPE coefficients and only slightly affects the scale dimensions of the two scalar σ'' and σ'''. The results of this analysis can be found in table 2.2

$\Delta_{\varepsilon''}$	$\Delta_{\varepsilon'''}$	$\Delta_{\sigma'}$	$\Delta_{\sigma''}$	$\Delta_{\sigma'''}$	$\lambda_{\sigma\sigma\varepsilon}$
7.27[5]	12.90[15]	5.49(1)	10.6[3]	16[1]	1.046(1)

a_ε	$b_{\varepsilon\mathrm{D}}/\sqrt{C_\mathrm{D}}$	a_σ	$b_{\sigma\mathrm{D}}/\sqrt{C_\mathrm{D}}$
6.607(7)	1.742(6)	2.599(1)	0.25064(6)

Table 2.2: The main results of the combined analysis of $\langle\sigma\sigma\rangle$ and $\langle\sigma\varepsilon\rangle$ in the extraordinary transition are split in two parts. The top table refers to data of the bulk channel, while the bottom table contains OPE coefficients specific to the boundary channel of the extraordinary transition. Errors in square brackets refer to data whose uncertainties depend on an unknown parameter; the other errors simply reflect the statistical errors of the input data, namely, Δ_σ, Δ_ε and $\Delta_{\varepsilon'}$.

It turns out that $\Delta_{\sigma'}$ is nicely close to the bound $\Delta_{\sigma'} \leq 5.41(1)$ found in [45]. Notice also that the resulting OPE coefficient $c_{\sigma\sigma\varepsilon}$ is in perfect agreement with the estimate of a recent Monte Carlo calculation [115] which gives $c_{\sigma\sigma\varepsilon} = 1.07(3)$ and the value $(c_{\sigma\sigma\varepsilon})^2 = 1.10636(9)$ found in [44] through the study of the four-point function with the linear functional method.

There is another very impressive check of these results. The Ward identity associated with the displacement operator tells us that the quantity $x_O = \Delta_O \frac{a_O}{b_{O\mathrm{D}}}\sqrt{C_\mathrm{D}}$ does not depend on the specific bulk operator O but only on the surface transition, as described in section 2.4. The above results yield

$$x_\sigma = 5.3727(27)\ ; \quad x_\varepsilon = 5.358(15)\ , \tag{2.35}$$

showing, within the errors, a reassuring fulfillment of the Ward identities.

2.3.3 The special transition

According to our discussion at the beginning of this section, solutions ascribed to the special transition are associated with truncations of the form $(m, n, 0)$ in which all the OPE coefficients are non-negative. By consistency with the results of the previous subsection we have to use the same bulk spectrum determined in the extraordinary transition. We found solutions of the form (3,3,0) and (4,3,0) with similar properties. Here we only discuss the latter.

Instead of an isolated solution, in this case we find a one-parameter family in the three-dimensional space of the boundary scale dimensions ($\widehat{\Delta}_1 < \widehat{\Delta}_2 < \widehat{\Delta}_3$). The lowest-

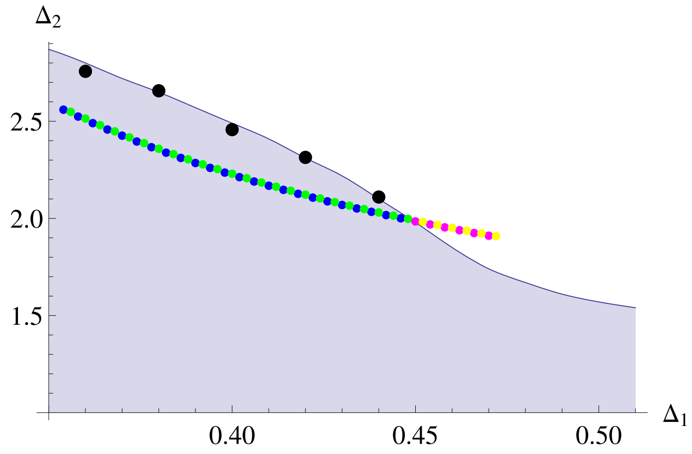

Figure 2.4: Plot of the one-parameter family of the truncation (4,3,0) in the plane $(\widehat{\Delta}_1, \widehat{\Delta}_2)$, superimposed to the upper unitarity bound found in[49]. The blue and green dots correspond respectively to the minimal and the maximal choice of the pair $(\Delta_{\varepsilon''}, \Delta_{\varepsilon'''})$, as determined in fig. 2.2. These dots are replaced by ones respectively magenta and yellow when some OPE coefficient become negative. For the black dots on the unitarity bound see explanation in the text.

dimensional operator has to be identified with $\widehat{\sigma}$ and according with the two-loop calculation of [107] we expect $\widehat{\Delta}_{\widehat{\sigma}} \sim\ 0.42$. In our case a unitary solution exists only for $0.34 \leq \widehat{\Delta}_1 \leq 0.45$. Below 0.34 the solution disappears abruptly; above 0.45 it becomes non-unitary.

Using $\widehat{\Delta}_3$ as a free parameter, we obtain the plot of fig. 2.4, which is superimposed to the unitarity upper bound found in [49]. As expected, the transition to the non-unitary region coincides with the unitarity boundary found by the linear functional method. Consistency requires that the spectrum of our solution at the intersection should agree with the one extracted from the zeros of the linear functional[97] calculated at the same point. In fact, the first zero of the linear functional at the intersection point, in the bulk sector, is (see fig. 2.5) around ~ 6.7, which is consistent with our result for $\Delta_{\varepsilon''}$. Similarly, the zero of the extremal functional for the boundary sector (besides $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$) is perfectly consistent with the value $\widehat{\Delta}_3 \sim 4.44$ at the crossing point.

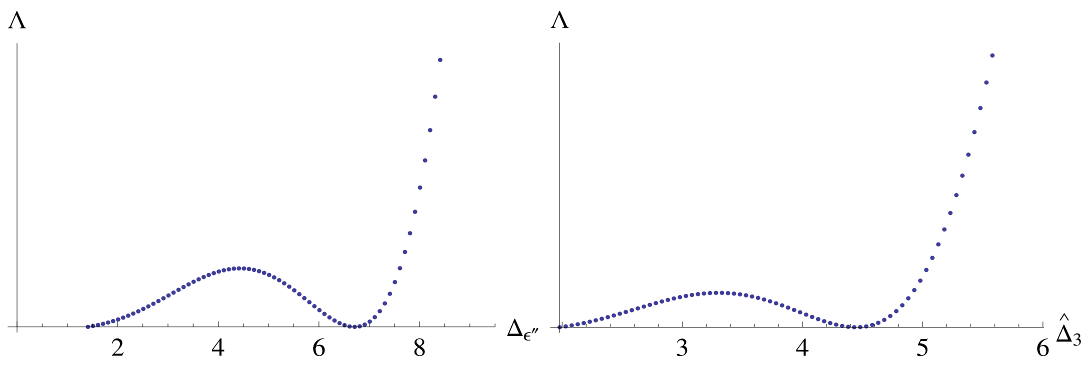

Figure 2.5: Linear functionals for the bulk and boundary channels in the special transition.

Such a boundary required by unitarity could also be seen as the locus were one or more OPE coefficients change sign. Our solution leads us to conjecture that the couplings vanishing at the unitarity bound are $a_{\varepsilon'}$ and $a_{\varepsilon'''}$. In the construction of the upper unitarity bound in [49] it is assumed that the first bulk primary is the Ising energy ε and it follows that the subsequent primary has scale dimension larger than $\Delta_{\varepsilon'}$, as suggested by our conjecture.

The knowledge of the linear functional leading to the bound of fig. 2.4 suggests another interesting cross-check of the two methods: given a value of $\widehat{\Delta}_1$ we insert in the (4,3,0) truncation the first four zeros of the linear functional on the bulk channel and evaluate with the method of determinants the corresponding boundary values $\widehat{\Delta}_2$ and $\widehat{\Delta}_3$. It turns out that in the plane $\widehat{\Delta}_1, \widehat{\Delta}_2$ such a solution lies on the unitarity bound, as consistency requires (see black dots in fig. 2.4).

2.4 Renormalization group domain wall for the $O(N)$ model

Before starting the exploration of a specific conformal interface, let us recall the relevant CFT data that one needs to collect in order to completely describe the generic system. Conformal interfaces are closely related to boundaries. In fact, as we mentioned in section 2.2, an interface between a CFT_1 and a CFT_2 can be mapped to a boundary problem using the folding trick. One turns the original setup into a boundary for the theory $\text{CFT}_1 \times \overline{\text{CFT}}_2$, where the bar means that a reflection $x^d \to -x^d$ has been applied to one of the theories. We see that the natural bulk CFT data is given by the value of the two point functions of operators placed in mirroring points with respect to the interface: they are mapped to expectation values of operators in the folded CFT. This also identifies the needed operators as primaries of the folded theory, which in particular include all bulk primaries of the two CFTs. The latter are not sufficient, though, because they do not play any role as building blocks of correlators across the interface. Another way of understanding this circumstance is provided by the north-south pole quantization, obtained by conformally mapping the theory to a d-dimensional sphere. Local operators at the north or south pole create a state belonging to the Hilbert space of either CFT. The interface is a linear map between the Hilbert spaces, and the correlators of operators placed in mirroring points - that is, at the north and south poles - are the matrix elements of this map. Analogous considerations are valid for the bulk-to-defect couplings. Let us now turn to the specific interface we shall study in this section.

The *Renormalization group domain walls* are interfaces between two CFTs which lie at the top and at the bottom of an RG flow. More precisely, there is an easy operational definition: start with a CFT on the whole space, and modify the action by integrating a relevant operator over half of the space. Far away in this region, the long distance physics will be dominated by the CFT at the bottom of the flow triggered by the perturbation. This definition can be employed literally when the coupling is only mildly relevant, and perturbation theory makes sense. In order to single out a unique gluing condition, it is also necessary to specify which defect deformations are turned on along with the bulk flow. In the case of interest for us, we shall argue that no marginal deformations exist on the defect, and so we just choose to fine tune perturbatively the relevant defect couplings. As usual, near the interface the critical behaviour is modified with respect to both the UV and the IR homogeneous fixed points, with new critical exponents arising. RG domain walls have been mainly studied in two dimensions [116, 117, 118, 119, 120]. In a general non perturbative setting, the determination of the defect spectrum and the computation of correlators is a very difficult task. In some limiting cases, however, some of the answers might be found with little effort. For instance, a relevant operator may force the bulk to flow towards a trivial theory. In this case, the RG interface is reduced to a boundary condition for the ultraviolet CFT. As an example, consider giving a mass to a free boson on half of the space, in any dimension greater than two. Correlators on the perturbed side are exponentially damped, and at large distances the theory is empty. From an RG point of view, the coupling grows in the IR, and the configurations of non-zero field on the perturbed side are suppressed in the partition function. As a consequence, a Dirichlet boundary condition is imposed to the massless free boson on the other side.

A more interesting case is the RG domain wall corresponding to the Wilson-Fisher fixed point of the $O(N)$ model with $(\phi^2)^2$ interaction. This interface is captured by the following bare action:

$$\mathcal{S} = \int d^d x \, \frac{1}{2S_d(d-2)} \, \partial_\mu \phi^i \, \partial_\mu \phi^i + \theta(x^d) \frac{g}{4!} (\phi^i \phi^i)^2, \tag{2.36}$$

where $\theta(x^d)$ is the Heaviside function, $S_d = 2\pi^{d/2} / \, \Gamma(d/2)$ and we chose to normalize the elementary field so that it has a canonical two-point function in free theory. As we pointed out, a question that needs to be answered concerns the stability of this interface. One needs to know how many relevant operators must be fine-tuned, and if marginal deformations exist. The interface possesses a weakly coupled description in $4-\epsilon$ dimensions, and, at the classical level, the only relevant defect primary in the singlet sector is $\widehat{\phi^2}$. Once we tune it to zero, unlike the situation in the special transition, we do not impose Neumann boundary conditions, but only continuity of $\partial_z \widehat{\phi^i}$ on the interface. Hence, the classically marginal operator $\partial_z \widehat{\phi^2}$ does not vanish, and should be taken into account. We shall show that this operator becomes irrelevant at one loop. Therefore, the RG interface appears to be isolated in perturbation theory.

In the following, we characterize the correlations of scalar primaries in the presence of the domain wall at lowest order in ϵ-expansion. Along the way, we point out that correlations across the interface encode at this order the mixing induced by the RG flow among nearly degenerate operators [118]. This is true in the larger class of perturbation interfaces constructed by means of a nearly marginal deformation. We then focus on the RG domain wall between the three dimensional free theory and the Ising model, and study the two-point function of the field σ using the method of determinants. We also provide some non-perturbative information on generic conformal interfaces involving the free theory, by noticing that some of the crossing constraints can be solved analytically.

2.4.1 The ϵ-expansion and the role of the displacement operator

Since the UV side of this RG interface is a free theory, the interface itself is not captured by mean-field theory: the CFT data related to it is $O(\epsilon)$ in perturbation theory. One can easily obtain general results at leading order by exploiting the Ward identity eq. (2.9), which defines the displacement operator. The identity tells us that we can move the interface in the orthogonal direction by integrating the displacement in the action. This is encoded in eq. (1.151), which implies

$$\int d^{d-1}y \, \langle \mathrm{D}(y^a) \, O_1(x_1) \ldots O_n(x_n) \rangle = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i^d} \langle O_1(x_1) \ldots O_n(x_n) \rangle \, . \tag{2.37}$$

Since the violation of translational invariance happens at order g - see eq. (2.42) - the relation (2.37) rephrases some information about an n-point function of order g^L in terms of the integral of a $(n+1)$- point function of order g^{L-1}. In general, knowledge of the variation with respect to the position of the interface is obviously insufficient for reconstructing the full correlator. However, all configurations of two points are conformally equivalent to the one in which the points are aligned on a line perpendicular to the defect. Therefore a two-point function can be traded for the integrated three-point function on the l.h.s. of eq. (2.37). The advantage is that the integral does not generate additional divergences: one only needs to renormalize the theory at order g^{L-1}. On the other hand, it is still necessary to determine a primitive of the l.h.s. of eq. (2.37) as a function of the position of the interface. We shall see that this is possible at lowest order: the tree level 2-point correlator, which is just the homogeneous one, can be used to compute the one loop correction in the presence of the interface.

It is simple to derive from (2.37) a new scaling relation. As pointed out, when two operators are placed in mirroring points, in which case $\xi = -1$, their correlator is equivalent, through the folding trick, to a one-point function:

$$\langle O_\mathrm{L}(x)\, O_\mathrm{R}(\mathcal{R}x)\rangle = \frac{a_{\mathrm{L\,R}}}{|2x^d|^{\Delta_\mathrm{L}+\Delta_\mathrm{R}}}, \qquad \mathcal{R}x = \left(x^a, -x^d\right). \tag{2.38}$$

Here we think of O_L and O_R as scalars belonging respectively to the UV and IR spectrum. Similarly, the three-point function $\langle O_\mathrm{L}\, O_\mathrm{R}\, D\rangle$ is fixed up to a number:

$$\langle O_\mathrm{L}(x)\, O_\mathrm{R}(\mathcal{R}x)\, \mathrm{D}(y^a)\rangle = \frac{b_{\mathrm{L\,R\,D}}}{|2x^d|^{\Delta_L+\Delta_R-d}\;|x-y|^{2d}} \tag{2.39}$$

Using the fact that in this geometry ξ is stationary with respect to orthogonal displacements of the interface, it is easy to derive the following relation between these pieces of CFT data

$$\frac{(\Delta_\mathrm{R}-\Delta_\mathrm{L})a_{\mathrm{L\,R}}}{S_d} = b_{\mathrm{L\,R\,D}}. \tag{2.40}$$

In the particular case where one of the bulk operators is the identity, one recovers a special case of the relation in eq. (1.157):

$$\pm\,\frac{\Delta_k a_k}{S_d} = b_{k\mathrm{D}}, \tag{2.41}$$

where the plus/minus sign is valid for the interacting/free side respectively. We start by using eq. (2.41) to determine the a_k's. The answer at order ϵ is quite simple: only one operator acquires expectation value, on both sides of the interface. To see this, let us identify the displacement. Looking at the action (2.36), we see that the interface is displaced at leading order by integrating the bare operator $g(\phi^2)^2/4!$, that is[7]

$$\mathrm{D} = \frac{g}{4!}(\phi^2)^2 + \mathcal{O}(g^2) = \frac{1}{8(N+8)\pi^2}\epsilon\,(\phi^2)^2 + \mathcal{O}(\epsilon^2), \tag{2.42}$$

where we plugged the fixed point value of the coupling at order ϵ:

$$g^* = \frac{3}{(N+8)\pi^2}\,\epsilon. \tag{2.43}$$

Now, since $(\phi^2)^2$ is a primary of the free theory, and no other primary mixes with it at order one, its correlation function with any other primary is zero at leading order. This means that all coefficients $b_{O\mathrm{D}} = \mathcal{O}(\epsilon^2)$, but for the case $O = (\phi^2)^2$. Using the relation (2.41), we conclude that the only non vanishing expectation value at this order is $\langle(\phi^2)^2\rangle$.

[7]Notice that at higher orders the interacting stress-tensor needs to be improved to be kept finite and traceless [121]. The improvement is proportional to $(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\phi^2$, so that the displacement receives a contribution from the operator $\partial_a\partial^a\phi^2$.

We can then obtain the number a_{ϕ^4} at order ϵ from a tree level computation. Indeed, the relevant bulk-to-defect coupling is given at leading order by

$$b_{\phi^4 D} = |x|^8 \left\langle \frac{(\phi^2)^2(x)}{\sqrt{8N(N+2)}} \mathrm{D}(0) \right\rangle = \frac{\sqrt{2N(N+2)}}{4(N+8)\pi^2}\,\epsilon. \tag{2.44}$$

Therefore

$$a^{\mathrm{IR}}_{\phi^4} = -a^{\mathrm{UV}}_{\phi^4} = \frac{\sqrt{2N(N+2)}}{8(N+8)}\,\epsilon. \tag{2.45}$$

Let us make a comment. It was obvious from the start that only a small class of operators could exhibit a one-point function at first order in the coupling: four powers of the elementary field are needed to contract a single vertex, and of course the operator must be in the singlet of $O(N)$. However, infinitely many scalar primaries can be constructed in free theory which fulfill these requirements, involving an increasing number of derivatives of the fields[8].

The simplest use of eq. (2.45) is the determination of the most general two-point function of operators lying on the same side of the interface at order ϵ. Sticking for simplicity to the case of external scalars, one simply writes

$$\begin{aligned}&\langle O_1(x) O_2(x')\rangle \\ &\quad= \frac{1}{(2x_d)^{\Delta_1}(2x'_d)^{\Delta_2}}\,\xi^{-(\Delta_1+\Delta_2)/2}\Big(\delta_{12} + c_{12\phi^4} a_{\phi^4} f^{d=4}(\Delta_{12}, \Delta=4, \xi)\Big) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{2.46}$$

Notice that $c_{12\phi^4}$ is guaranteed to belong to the 4d free theory only when O_1 and O_2 are on the UV side. Indeed, primaries on the interacting side are in general a mixture of classically degenerate renormalized operator, and when the mixing happens at leading order $c_{12\phi^4}$ becomes a linear combination of UV OPE coefficients. For completeness, we compare this derivation with some direct one loop computations in appendix 2.A.

As pointed out in the introduction to this section, in order to capture correlations across the interface we would need all the one-point functions of the folded theory. This set encompasses the a_{LR} defined in (2.38), and is much bigger. It is in fact more viable to reach for the two-point functions of primaries directly through the integrated Ward identity eq. (2.37), specified to the case of interest:

$$\int d^{d-1}y\, \langle O_{\mathrm{L}}(x)\, O_{\mathrm{R}}(x') \mathrm{D}(y)\rangle = -\left(\frac{\partial}{\partial x^d} + \frac{\partial}{\partial x'^d}\right)\langle O_{\mathrm{L}}(x)\, O_{\mathrm{R}}(x')\rangle\,. \tag{2.47}$$

We pick for the left hand side the three-point function of primaries in the translational invariant theory, and we get the one-loop two-point function by integrating over the position of the displacement. Notice that in doing so we disregard the mixing of primaries with descendants. In the cases in which this happens at order one, on the left hand side of eq. (2.47) additional terms needs to be taken into account, which have the form of a three-point function involving derivatives of a primary operator. Consider first two operators which are degenerate in the free theory. In other words,

$$\Delta_{\mathrm{LR}} \equiv \Delta_{\mathrm{L}} - \Delta_{\mathrm{R}} = \mathcal{O}(\epsilon). \tag{2.48}$$

[8] That these primaries must exist can be seen independently from their expression in terms of elementary fields, for instance from the asymptotics of the two point function of ϕ^2 in a free theory with a boundary. The presence of the identity in the boundary channel can only be balanced by an infinite number of conformal blocks in the bulk channel. Only one primary can be built with two powers of the fields, so the rest are the ones we are interested in. The explicit conformal block decomposition for this case can be found in [49]. It is also amusing to notice that, analogously to the case at hand, this tower of operators does not contribute at order ϵ to the two-point function of ϕ with Dirichlet or Neumann boundary conditions. As noticed in [49], in that case the OPE coefficients $c_{\phi\phi\,\partial^{2k}\phi^4}$ are the vanishing quantities at order ϵ.

In this case eq. (2.47) can only be used to determine the one loop correlator up to a constant. Indeed, since both $b_{\mathrm{L\,R\,D}}$ and $\Delta_{\mathrm{L}} - \Delta_{\mathrm{R}}$ are of order ϵ, one needs the one loop three-point function to determine $a_{\mathrm{L\,R}}$ from eq. (2.40). This is the familiar effect of degeneracies in perturbative computations, and is related to the mixing of operators along the RG flow (see section 2.4.2). Integration of (2.47) is straightforward, and one gets

$$\langle O_{\mathrm{L}}(x)\, O_{\mathrm{R}}(x')\rangle = \frac{a_{\mathrm{L\,R}}}{|2x^d|^{\Delta_{\mathrm{L}}}\,(2x'^d)^{\Delta_{\mathrm{R}}}}(-\xi)^{-\Delta_{\mathrm{L}}}\left(1+\frac{\Delta_{\mathrm{LR}}}{2}\log(-\xi)\right), \quad \Delta_{\mathrm{LR}} = \mathcal{O}(\epsilon). \tag{2.49}$$

Comparing with the form (2.13) we can write at this order

$$f_{\mathrm{L\,R}}(\xi) = (-\xi)^{-\frac{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}{2}} a_{\mathrm{L\,R}} \qquad \Delta_{\mathrm{LR}} = \mathcal{O}(\epsilon). \tag{2.50}$$

A comment is in order. The presence of a logarithmic singularity compatible with exponentiation is somewhat natural, since turning the coupling off one recovers the short distance power low divergence proper of the homogeneous theory. However, there is no reason for this to happen when considering the OPE limits in the Euclidean defect CFT. The exponentiation agrees in the large ξ limit with the defect OPE, as it is easy to verify using the formulae given in subsection 2.4.2. On the other hand, no small ξ limit exists for primaries on opposite sides of the domain wall, and in fact the folded cross-ratio is $\xi_{\text{folded}} = -(1+\xi)$, which vanishes when the operators are placed in mirroring points. We decide to keep the exponentiated form, and notice that it might be fruitful to look for a justification in Lorentzian signature, where the small ξ limit corresponds to light-like separated operators.

In the case of operators with dimension differing in the UV limit, the two-point functions at one loop can be fixed completely. Due to $O(N)$ and rotational symmetry, Δ_{LR} is an even integer in $d=4$, which provides a simplification. The computation is slightly more involved than in a previous case, and we give some details in appendix 2.A. The result in the case $|\Delta_{\mathrm{LR}}| = 2$ is different from all the others:

$$f_{\mathrm{L\,R}}(\xi) = \frac{\pi^2}{2} b_{\mathrm{L\,R\,D}}\,\mathrm{sign}(\Delta_{\mathrm{LR}})\,(-\xi)^{-\frac{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}{2}}\,(\xi-1), \qquad |\Delta_{\mathrm{LR}}| = 2, \tag{2.51}$$

while

$$\begin{aligned} f_{\mathrm{L\,R}}(\xi) = &-\frac{\pi^2\Gamma(2k+3)}{(k-1)k^2\Gamma(k+2)^2} b_{\mathrm{L\,R\,D}}\,\mathrm{sign}(\Delta_{\mathrm{LR}}) \\ &(-\xi)^{-\min(\Delta_{\mathrm{L}},\Delta_{\mathrm{R}})+1}\left\{\left((4k+2)\xi^2+3(k+1)\xi+1\right){}_2F_1\left(-k-1,-k,-2(k+1);\frac{1}{\xi}\right)\right. \\ &\left.-\left((4k+2)\xi^2+(k+2)\xi\right){}_2F_1\left(-k-1,-k-1,-2(k+1);\frac{1}{\xi}\right)\right\}, \\ &|\Delta_{\mathrm{LR}}| \equiv 2k > 2. \end{aligned} \tag{2.52}$$

As one might have expected, the hypergeometric functions in eq. (2.52) are in fact polynomials.

These results complete the analysis of bulk correlations at order ϵ, if knowledge of the c_{123} is assumed: n-point functions of bulk operators are determined by taking successive OPEs on the two sides until one is left with a one-point function or a two-point function across the interface. We shall content ourselves of this leading order solution, but we would like to comment on the possibility of generalizing the procedure. Unfortunately, the number of non vanishing one-point functions is infinite already at next to leading order[9]. Therefore, once the displacement has been correctly normalized, one has to compute the relevant three-point functions at one loop and integrate them to find the two loop two-point functions.

Let us now consider the defect spectrum at order ϵ. The dimensions of the operators can be extracted through the defect OPE decomposition of eq. (2.46). When nearly degenerate operators are present in the UV theory, also the defect operators mix, and the spectrum is given by the eigenvalues of the matrix of anomalous dimensions. We shall deal with this more general case in the next subsection. Here we comment on some features of the spectrum focusing for simplicity on the non-mixing operators. The lightest defect scalar in the OPE of a bulk operator O has dimension

$$\begin{aligned}\widehat{\Delta}_O &= \Delta_O^{\mathrm{UV}} - 2c_{OO\phi^4}\, a_{\phi^4}^{\mathrm{UV}} + \mathcal{O}(\epsilon^2) \\ &= \frac{1}{2}(\Delta_O^{\mathrm{UV}} + \Delta_O^{\mathrm{IR}}) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{2.53}$$

The second equality in eq. (2.53), which agrees with first order conformal perturbation theory, says that the defect primary stands half way between the corresponding infrared and ultraviolet operators in the bulk. Let us make some more specific comments. $\widehat{\Delta}_{\phi^4} = 4 - \epsilon$ is the protected dimension of the displacement operator. This is expected, even if there are degenerate operators in free theory. Two primaries exist with dimension near to four, but both of them are protected, the second one being the displacement of the folded theory. The second interesting scale dimension is obtained by going one step further in the defect OPE of ϕ^2. We encounter the operator $\partial_z\widehat{\phi^2}$, and since no other scalars exist which could mix with it, we can safely read off his dimension from the boundary block decomposition: $\widehat{\Delta}_{\partial\phi^2} = 3 - \frac{N+14}{2(N+8)}\epsilon$. We see that this scalar is irrelevant at the Wilson-Fisher fixed point, so that the stability of the interface is not altered by its presence. A third remark concerns the odd spectrum. Since the anomalous dimension of ϕ^i starts at two loops, or equivalently the bulk OPE does not contain $(\phi^2)^2$ on either side of the interface, the dimensions of $\widehat{\phi^i}$ and $\partial_z\widehat{\phi^i}$ remain classical. Moreover, at this order all fields of the kind $\partial_z^k\widehat{\phi^i}$ can be converted to descendants of $\widehat{\phi^i}$ and $\partial_z\widehat{\phi^i}$ by means of the tree level equations of motion. Hence, the latters are the only primaries appearing with an OPE coefficient of order one. The interesting fact is that $\widehat{\Delta}_\phi$ and $\widehat{\Delta}_{\partial\phi}$ do not receive loop corrections at all, as we review in subsection 2.4.3. A last comment on the one-loop odd spectrum is in order. The two-point function of $\phi^2\phi^i$ should obey eq. (2.46) only on the free side, where the operator is a primary. This two-point function contains a tower of defect operators which we might identify with $\widehat{\phi^2\phi^i}$ and its transverse derivatives. The dimension of $\widehat{\phi^2\phi^i}$ is consistently half-way between $\phi^2\phi^i$ and its image under RG flow, that is, $\Box\phi^i$, and turns out to be marginal at this order. Since we could not devise a mechanism to protect this operator from quantum corrections, we believe this feature will disappear from the spectrum at higher orders. The fact that $\widehat{\phi^2\phi^i}$ is independent from the conformal families of $\widehat{\phi^i}$ and $\partial_z^2\widehat{\phi^i}$ is naturally justified by defining the defect fields as the limit of the *free* bulk fields approaching the interface. Notice that this happens automatically in a hard-core regularization, where all integrals are cut-off at a small distance from the interface.

The considerations leading to eq. (2.46) apply in fact to the leading order in conformal perturbation theory of any interface obtained by a nearly marginal bulk perturbation. Indeed, the key point is that the Zamolodchikov norm of the displacement operator equals the square of the coupling at leading order. We turn now to this more general setting in

[9]This statement again follows immediately from the fact that the operator ϕ^2 acquires an expectation value at order ϵ^2.

order to discuss the leading order mixing of bulk and defect primaries. On the contrary, notice that eqs. (2.51) and (2.52) do not generalize trivially, because we used the fact that UV scale dimensions are (nearly) even-integer separated: formulae get a bit more messy in the general case.

2.4.2 Leading order mixing of primary operators

Consider a conformal field theory in any number of dimensions d, whose spectrum includes one[10] mildly relevant operator φ, that is $\epsilon = d - \Delta_\varphi$ is a small positive number. The interface constructed by integrating $g\varphi$ on one half of the space has an infrared fixed point in which $g = g^* \sim \mathcal{O}(\epsilon)$. The two-point functions of operators on the same side of the interface obey the obvious generalization of eq. (2.46):

$$\begin{aligned}&\langle O_1(x) O_2(x')\rangle \\ &\quad= \frac{1}{(2x_d)^{\Delta_1}(2x'_d)^{\Delta_2}}\,\xi^{-(\Delta_1+\Delta_2)/2}\Big(\delta_{12} + c_{12\varphi}a_\varphi f^d(\Delta_{12}, \Delta = d, \xi)\Big) + \mathcal{O}(\epsilon^2). \quad (2.54)\end{aligned}$$

Here a_φ is of order ϵ and at this order

$$a^{\mathrm{IR}} = -a^{\mathrm{UV}} = g^*\,\frac{S_d}{d}, \tag{2.55}$$

as dictated by eq. (2.41). We would like to study the effect of the mixing of bulk primaries on the defect operators. Let us choose a set of UV scalar primaries O_i^{UV} which are degenerate up to terms of order ϵ. Their defect OPE, restricted to the lowest lying primaries, is encoded in the fusion rule

$$O_i^{\mathrm{UV}} \sim {\mu_i}^j\,\widehat{O}_j + \ldots \tag{2.56}$$

These defect operators are connected by the RG flow to the UV operators themselves, that is there exists a family of renormalized operators $\widehat{O}_i(g)$ such that $O_i^{\mathrm{UV}} = \widehat{O}_i(0)$ and $\widehat{O}_i = \widehat{P}_i{}^j\,\widehat{O}_j(g^*)$. The matrix $\widehat{P}^i{}_j$ depends on the definition of the renormalized operators, that is on the regularization scheme. However, in what follows we shall only need the fact that $\widehat{P}^i{}_j$ is orthogonal at order one. Comparing with eq. (2.56) we see that

$${\mu_i}^j = \widehat{P}^j{}_i + \mathcal{O}(\epsilon). \tag{2.57}$$

The relevant part of the defect OPE decomposition of the correlator $\langle O_i^{\mathrm{UV}} O_j^{\mathrm{UV}}\rangle$ is determined by the following asymptotic behavior for large ξ:

$$f^d(\Delta_{12} = 0, \Delta = d, \xi) \;\sim\; \frac{d}{2}\big(\log\xi + \gamma - \psi(d/2)\big) + \mathcal{O}(\xi^{-1}). \tag{2.58}$$

Comparing this with the large ξ and small ϵ limit of the boundary blocks, we get

$$\sum_k {\mu_i}^k \mu_{jk}\left(\widehat{\Delta}_k - \frac{\Delta_i^{\mathrm{UV}} + \Delta_j^{\mathrm{UV}}}{2}\right) = -\frac{d}{2}\,c_{ij\varphi}^{\mathrm{UV}}\,a_\varphi^{\mathrm{UV}}. \tag{2.59}$$

Since the quantity in parenthesis is of order ϵ, we can make the substitution $\mu \to \widehat{P}$. The latter matrix was defined to be the orthonormal change of basis which diagonalizes the matrix of anomalous dimensions $\widehat{\gamma}_i{}^j$ of the boundary operators $\widehat{O}_j(g)$, so that we get

$$\widehat{\gamma}_{ij} = \Delta_i^{\mathrm{UV}}\delta_{ij} - \frac{d}{2}\,c_{ij\varphi}^{\mathrm{UV}}\,a_\varphi^{\mathrm{UV}} = \Delta_i^{\mathrm{UV}}\delta_{ij} + \frac{S_d}{2}\,c_{ij\varphi}^{\mathrm{UV}}\,g^*. \tag{2.60}$$

[10]We consider for simplicity the case of a one parameter RG flow. The general case proceeds along the same lines.

One may proceed order by order in the large ξ expansion. The resulting defect spectrum includes in general nearly degenerate scalars with dimension close to $\Delta + k$, Δ being the scale dimension of a bulk primary. A primary of level k of course originates from linear combinations of transverse and parallel derivatives of a UV primary. But when nearly integer separated bulk primaries exist, further mixing is expected to take place.

To complete the analysis, we would like to show that by matching the defect spectrum with the IR bulk primaries, we get back the known mixing matrix between UV and IR operators of the homogeneous theory [122]. We restrict ourselves to the case in which the mixing only involves primary operators. We consider the set of IR primaries O_i^{IR} which are related to the O_i^{UV} through a matrix $P^i{}_j$ whose definition is analogous to the one we gave for $\widehat{P}$. The leading part of the defect fusion rule is

$$O_i^{\mathrm{IR}} \sim \nu_i{}^j \widehat{O}_j + \dots \tag{2.61}$$

where we required that the defect spectrum coincides with the one of the UV counterparts. This time we have

$$\nu_i{}^j = P_i{}^k \widehat{P}^j{}_k + \mathcal{O}(\epsilon). \tag{2.62}$$

The same steps as before now lead to a relation identical to eq. (2.59), up to the substitutions $\mu \to \nu$ and UV $\to$ IR. The combination of eqs. (2.55), (2.60), (2.62) with the statement

$$c^{\mathrm{IR}}_{ij\varphi} = P_i{}^m P_j{}^n c^{\mathrm{UV}}_{mn\varphi} + \mathcal{O}(\epsilon), \tag{2.63}$$

leads to

$$\Delta_i^{\mathrm{IR}} \delta_{ij} = P_i{}^m P_j{}^n \left(\Delta_m^{\mathrm{UV}} \delta_{mn} + S_d\, c^{\mathrm{UV}}_{mn\varphi}\, g^* \right). \tag{2.64}$$

Since the matrix P diagonalizes by hypothesis the matrix of bulk anomalous dimensions, we recover the formula

$$\gamma_{ij} = \Delta_i^{\mathrm{UV}} \delta_{ij} + S_d\, c^{\mathrm{UV}}_{ij\varphi}\, g^*. \tag{2.65}$$

Notice that the anomalous part of the defect mixing matrix is one half of the bulk one.

As a last comment, by means of eq. (2.40), we can verify that the pairing of UV and IR primaries matches the matrix P at leading order [118]:

$$a_{ji} = P_{ij} + \mathcal{O}(\epsilon). \tag{2.66}$$

Indeed, eq. (2.66) is immediately obtained starting from the equality

$$(\Delta_i^{\mathrm{IR}} - \Delta_j^{\mathrm{UV}}) a_{ji} = S_d\, P_i{}^k c^{\mathrm{UV}}_{jk\varphi}\, g^*, \tag{2.67}$$

which is valid at leading order, and using the definition (2.65) of the mixing matrix.

2.4.3 The interface bootstrap

In order to single out a solution to the crossing equation which corresponds to our interface, we shall again concentrate on the 3d Ising model, and in particular on the two-point functions involving the lowest lying odd primaries ϕ and σ, on the free and interacting side respectively. The bootstrap constraints involving ϕ can be in fact completely solved in any number of dimensions by requiring the correlation functions to be annihilated by the Laplace operator. Therefore, we start by collecting some general facts about free bosonic theories in the presence of codimension one conformal defects. Let us first of all consider the two-point function $\langle \phi\phi \rangle$. As it is well known, one can prove by applying the equations of motion to the $\phi \times \phi$ OPE that it contains only operators of twist $d-2$, and in particular:

$$\phi \times \phi \sim 1 + \phi^2 + (\text{primaries with zero expectation value}). \tag{2.68}$$

The same method can be applied for establishing that only two primaries appear in the defect OPE of the field (this was first noticed in [87]). Indeed, when the Laplace operator is applied to the r.h.s. of the defect OPE, the parallel derivatives give descendants and we can disregard them. The derivative orthogonal to the defect imposes a constraint on the scale dimension of allowed primaries:

$$0 = \Box\phi(\boldsymbol{x}, x^d) \sim \sum_{\widehat{O}} (\Delta_{\widehat{O}} - \Delta_\phi)\,(\Delta_{\widehat{O}} - \Delta_\phi - 1)\,\frac{\widehat{O}(\boldsymbol{x})}{(x^d)^{\Delta_\phi - \Delta_{\widehat{O}} + 2}} + \text{descendants} \tag{2.69}$$

Hence, there are only two primaries, the limiting value of the field $\widehat{\phi}$ and of its derivative $\widehat{\partial\phi}$. These primaries have protected dimensions $\Delta_{\widehat{\phi}} = \frac{d}{2} - 1$ and $\Delta_{\widehat{\partial\phi}} = \frac{d}{2}$. We see that the most general defect CFT featuring the free theory on half of the space, bounded by any codimension one defect, satisfies the following crossing equation:

$$1 + c_{\phi\phi\phi^2}\, a_{\phi^2} f^d(\Delta_{\phi^2}, \xi) = \xi^{\Delta_\phi}\left(b^2_{\phi\widehat{\phi}}\, \widehat{f}^d(\Delta_{\widehat{\phi}}, \xi) + b^2_{\phi\widehat{\partial\phi}}\, \widehat{f}^d(\Delta_{\widehat{\partial\phi}}, \xi)\right). \tag{2.70}$$

All conformal blocks reduce to elementary function:

$$\widehat{f}^d\left(\frac{d-2}{2}, \xi\right) = \frac{1}{2}\xi^{-\Delta_\phi}\left(1 + \left(\frac{\xi}{\xi+1}\right)^{\Delta_\phi}\right) \tag{2.71}$$

$$\widehat{f}^d\left(\frac{d}{2}, \xi\right) = \frac{2}{d-2}\xi^{-\Delta_\phi}\left(1 - \left(\frac{\xi}{\xi+1}\right)^{\Delta_\phi}\right), \tag{2.72}$$

so the crossing equation is equivalent to the following:

$$\frac{1}{2}\, b^2_{\phi\widehat{\phi}} + \frac{2}{d-2}\, b^2_{\phi\widehat{\partial\phi}} = 1, \tag{2.73a}$$

$$\frac{1}{2}\, b^2_{\phi\widehat{\phi}} - \frac{2}{d-2}\, b^2_{\phi\widehat{\partial\phi}} = c_{\phi\phi\phi^2}\, a_{\phi^2}. \tag{2.73b}$$

The solution is parametrized by an angle:

$$b_{\phi\widehat{\phi}} = \sqrt{2}\,\cos\alpha, \quad b_{\phi\widehat{\partial\phi}} = \sqrt{\frac{d-2}{2}}\,\sin\alpha, \quad c_{\phi\phi\phi^2}\, a_{\phi^2} = \cos 2\alpha. \tag{2.74}$$

The solution of this particular crossing equation is only a necessary condition for the existence of a full fledged defect CFT, therefore the question arises whether for any value of α such a theory exists. Vice versa, a given value of α might be realized in more than one defect CFT, which differ elsewhere. We can restrict α to take values in the interval $[0, \pi/2]$, since sending the defect fields $\widehat{\phi}$ and $\widehat{\partial\phi}$ to minus themselves does not spoil their canonical normalization. At the extrema of this interval one finds Neumann ($\alpha = 0$) and Dirichlet ($\alpha = \pi/2$) boundary conditions, and at the center ($\alpha = \pi/4$) the trivial interface between the free theory and itself. The RG interface with the $O(N)$ model with ϕ^4 interaction lies perturbatively near to the no-interface value, in ϵ-expansion, and fills an interval if N is allowed to take value over the reals.

Since any two-point function involving the field ϕ has to contain only the same two blocks in the defect channel, one can generalize the previous procedure to any correlator of this kind. The general fusion rule with a primary O with dimension Δ is

$$\phi \times O_\Delta \sim O_- + O_+ + (\text{spinning primaries}),$$
$$\Delta^- = \Delta - \Delta_\phi, \quad \Delta^+ = \Delta + \Delta_\phi. \tag{2.75}$$

Notice that degenerate primaries may exist with the right dimensions to enter the r.h.s. of eq. (2.75), as it happens in the $O(N)$ model for $N > 1$. Denoting $c_+ = c_{\phi O_\Delta O_+}$ and $c_- = c_{\phi O_\Delta O_-}$, the solution to the bootstrap equation is

$$b_{\phi\widehat{\phi}}\, b_{O\widehat{\phi}} = c_-\, a_- + c_+\, a_+, \qquad \frac{4}{d-2}\, b_{\phi\widehat{\partial\phi}}\, b_{O\widehat{\partial\phi}} = c_-\, a_- - c_+\, a_+. \tag{2.76}$$

This includes the system (2.73), in particular. The relations (2.76) also apply when the operator O is a primary on the interacting side of the interface. In this case, the OPE happens in the folded picture, and turns out to be a simple way to choose the solution of the Laplace equation with the appropriate asymptotics. Specifically, no singularities should arise when the operators are placed in mirroring points, and this prompts us to eliminate O_- from the r.h.s. of eq. (2.75). In other words,

$$\phi \times O \sim :\!\phi O\!:, \tag{2.77}$$

and the two-point function is simply

$$\langle \phi(x) O(x') \rangle = \frac{a_{\phi O}}{(2|x^d|)^{\Delta_\phi}(2x'^d)^{\Delta}}\ {}_2F_1\left(\Delta_\phi, \Delta, \Delta, -\xi_{\text{folded}}\right) = \frac{a_{\phi O}}{(2x'^d)^{\Delta-\Delta_\phi}(x-x')^{2\Delta_\phi}}, \tag{2.78}$$

where ξ_{folded} is just obtained by replacing x^d with minus itself. The relation (2.76) reduces to

$$a_{\phi O} = b_{\phi\widehat{\phi}}\, b_{O\widehat{\phi}} = -\frac{4}{d-2} b_{\phi\widehat{\partial\phi}}\, b_{O\widehat{\partial\phi}}. \tag{2.79}$$

This relation is potentially useful in bootstrapping the interacting side of the interface. Indeed, the defect OPE of every operator which couples with ϕ contains $\widehat{\phi}$ and $\widehat{\partial\phi}$, and the ratio $b_{O\widehat{\phi}}/b_{O\widehat{\partial\phi}} = -2\, \tan\alpha/\sqrt{d-2}$ does not depend on the operator, and may be used to match solutions for different external primaries. From eq. (2.73), we see that this ratio among coefficients of the interacting theory is determined by the expectation value of ϕ^2 on the free side. In particular, as we pointed out, this one-point function deviates from zero only at order ϵ^2 in the case we are interested in. We compute the leading order value in appendix 2.A for generic N, and find

$$\alpha = \frac{\pi}{4} - \frac{3}{1024\pi^6}\frac{N+2}{(N+8)^2}\epsilon^2. \tag{2.80}$$

In sum, the signature of the RG domain wall in the conformal block decomposition of $\langle\sigma\sigma\rangle$ is the presence of two protected defect operators, with a ratio of OPE coefficients near to the free theory value.

In fact, we found in $3d$ a numerical solution for a (4,4,0) truncation of $\langle\sigma\sigma\rangle$ which has the expected features. The defect channel is formed by the two operators $\widehat{\sigma}$ and $\widehat{\partial_z\sigma}$ of protected dimensions $\frac{1}{2}$ and $\frac{3}{2}$ and two unprotected operators $\widehat{O}_3$ and $\widehat{O}_4$ of dimensions $\widehat{\Delta}_3 \sim 3.11$ and $\widehat{\Delta}_4 \sim 6.17$. The precise value of these quantities as well as the estimates of the relative OPE coefficients depend on the choice of the bulk spectrum. For the sake of consistency we put in the same bulk spectrum obtained in the (4,2,1) solution of the extraordinary transition. The values of $\Delta_{\varepsilon''}$ and $\Delta_{\varepsilon'''}$ depend on the scale dimension $\widehat{\Delta}$ of a surface operator which acts as a free parameter. Therefore, our interface solution also depends on it, though the dependence is very mild, as a stable solution requires (see the discussion on the stability of the solutions on section 2.2). Table 2.1 shows the relevant data of such a solution. Note that the ratio $b_{\sigma\widehat{\sigma}}/b_{\sigma\widehat{\partial_z\sigma}}$ follows the trend suggested by the ϵ expansion.

$\widehat{\Delta}$	$\Delta_{\epsilon''}$	$\Delta_{\epsilon'''}$	$b^2_{\sigma\widehat{\sigma}}$	$b^2_{\sigma\widehat{\partial_z\sigma}}$
3.9	7.235(6)(3)	12.736(7)(4)	1.00612(11)(5)	0.27138(5)(2)
7.9	7.274(10)(2)	12.843(17)(4)	1.00644(15)(4)	0.27123(7)(1)
11.1	7.287(11)(4)	12.892(22)(8)	1.00657(17)(6)	0.27117(7)(2)
15.1	7.297(11)(2)	12.932(23)(4)	1.00668(16)(3)	0.27112(7)(2)
19.9	7.298(11)(1)	12.948(24)(2)	1.00667(16)(2)	0.271127(68)(5)
25.5	7.302(11)(2)	12.968(25)(5)	1.00672(16)(3)	0.27110(7)(2)
31.9	7.303(11)(3)	12.980(25)(7)	1.00674(16)(5)	0.27110(7)(1)
39.1	7.307(12)(4)	12.995(28)(8)	1.00679(17)(5)	0.27108(7)(2)

$\widehat{\Delta}$	$\widehat{\Delta}_3$	$\widehat{\Delta}_4$	$b^2_{\sigma\widehat{O}_3}$	$b^2_{\sigma\widehat{O}_4}$
3.9	3.1190(8)(4)	6.1816(9)(4)	0.002555(5)(2)	0.00002387(4)(2)
7.9	3.1151(12)(3)	6.1757(15)(4)	0.002572(7)(2)	0.00002408(6)(2)
11.1	3.1136(14)(5)	6.1734(17)(6)	0.002579(7)(3)	0.00002417(7)(3)
15.1	3.1123(14)(3)	6.1715(17)(3)	0.002584(7)(2)	0.00002424(7)(2)
19.9	3.1121(14)(1)	6.1710(17)(2)	0.0025850(74)(10)	0.00002426(7)(1)
25.5	3.1115(14)(3)	6.1701(18)(4)	0.002588(7)(1)	0.00002429(7)(1)
31.9	3.1113(14)(4)	6.1697(17)(5)	0.002589(7)(2)	0.00002431(7)(3)
39.1	3.1108(15)(5)	6.1689(19)(6)	0.002591(8)(2)	0.00002433(8)(2)

Table 2.1: Data of the (4,4,0) solution of the 3d Ising interface with the free UV theory. The first column is the free parameter of the solution which is the scale dimension of a surface operator contributing to the extraordinary transition discussed in sec. 2.3. The data are affected by two kinds of errors. The first parenthesis reflects the statistical error of the input data (namely Δ_σ and Δ_ε), while the second parenthesis indicates the spread of the solutions.

Let us make some final remarks. When the bulk OPE coefficients and the scale dimensions are exactly known on one side of an interface, one may extract the one-point functions from the crossing equations involving operators placed on this side. The same data enter various correlators, and the interplay between different solutions to the crossing equations may be used to detect systematics, or to reduce the unknowns. We leave this for future work. For now, we notice that the even spectrum on the free side of our interface is made by an increasing number of degenerate primaries of integer dimension, so it is foreseeable that a reliable truncation would require the inclusion of many bulk primaries. Furthermore, since the parameter N only enters the determinants through the unknown defect spectrum, one expects to find a one-parameter family of solutions. Studying two-point functions of free even primaries is important in particular if one is interested in the Zamolodchikov norm of the displacement operator. Indeed, two defect primaries exist with dimension d, one of which might be identified with the displacement of the folded theory. Given two primaries O_{L} and O_{R} with non-vanishing one-point function, it is not difficult to see that, in order to isolate the displacement, one needs to know $\langle O_{\mathrm{L}} O_{\mathrm{L}} \rangle$, $\langle O_{\mathrm{R}} O_{\mathrm{R}} \rangle$ and $\langle O_{\mathrm{L}} O_{\mathrm{R}} \rangle$. Unfortunately, we have not been able to identify a solution for $\langle \epsilon\, \epsilon \rangle$ which satisfactorily reproduces the domain wall.

2.5 Conclusions and outlook

In this work we explored some consequences of crossing symmetry for defect CFTs. We focused our study on the cases where the defect is a codimension one hyperplane, i.e. a flat interface or a boundary. In the latter case our main results concern the surface transitions of 3d Ising model.

The numerical solutions to the bootstrap equations with the method of determinants turn out to be particularly effective in the ordinary transition, where it suffices to know the scale dimensions of the first few bulk primaries to obtain the dimension of the relevant surface operator of this transition as well as its OPE coefficient. This analysis has been extended to the $O(N)$ models with $N = 0, 1, 2, 3$ where a comparison can be made with the results of a two-loop calculation [107], finding a perfect agreement (see table 2.1).

In the extraordinary transition the contribution of the boundary channel is dominated by the first two low-lying operators, namely the identity and the displacement, thus we used this fact to extract more information on the even and odd spectrum contributing to the bulk channel. We obtained in this way also an accurate determination of the OPE coefficient $c_{\sigma\sigma\varepsilon}$ which compares well with other estimates based on a recent Monte Carlo calculation [115] or on conformal bootstrap [44]. We also obtained some OPE coefficients of one-point and two-point functions (see table 2.2) which allow to verify the impressive fulfillment of the Ward identities associated with the displacement operator.

The solution corresponding to the special transition contains a free parameter, hence we don't get precise numerical results. This case is still very useful for an accurate cross-check of the consistency of the method of determinants with the linear functional method. Together with the just mentioned Ward identities, this check provides evidence for the fact that the systematic error is rather small when a truncation is stable. In this work we investigated the stability of the truncations through the sensitivity to the addition of heavier operators. It would be important to establish more rigorous bounds on the systematic error, maybe along the lines of [93].

The next example of a codimension one defect studied in this chapter is an interface between the $O(N)$ model and the free theory. We tackled the problem both in $4-\epsilon$ and in three dimensions. The weak coupling analysis of the two-point functions was carried out in a way which is trivially adapted to general perturbation interfaces. A preeminent role is played by the displacement operator, whose small Zamolodchikov norm signals the transparency of the interface, in the sense that operators with nearly degenerate dimension are allowed to couple at order one across the interface, while the opposite is true for primaries well separated in the spectrum. This intuition can be made precise in $2d$, where the norm of the displacement coincides up to a normalization with the reflection coefficient defined in [82].[11] It is certainly interesting to look for a similar interpretation of the displacement in higher dimensions, possibly in relation to the correlators of polarized stress-tensors. However, it is worth emphasizing that while in $2d$ the reflection coefficient of a boundary is unity, in dimensions greater than two the norm of the displacement depends on the boundary conditions. The results of the perturbative analysis also confirm that this kind of interfaces encode information about the RG flow that links the theories on the two sides: specifically, the coupling of UV and IR primaries reproduces the leading order mixing of operators, as does the one-dimensional domain wall constructed non-perturbatively in [118]. On the numerical side, we found a solution to the crossing equation consistent with the features of the two-point function of σ in three dimensions. The analysis can be

[11] In particular, it is not difficult to prove unitarity bounds for reflection and transmission in function of the central charges, just by diagonalizing the defect spectrum.

extended in various directions. It would be interesting go to second order in perturbation theory [123], or to study the setting at large N, and see whether the displacement operator still provides important simplifications. We already pointed out that it is viable to bootstrap correlators on the free side, and it would be important in particular to give a prediction for the norm of the displacement in $3d$, to compare it with the estimates for the boundary transitions. We would also like to emphasize that the interface can be realized on the lattice, for instance as a Gaussian model with the addition of a quartic potential on one-half of the lattice.

As we mentioned in the introduction, a complete description of the CFT data cannot be reached, even in principle, only through the study of bulk two-point functions. Four-point functions of defect operators should be studied, and in this case both the method of determinants and the linear functional might be employed. Along the same lines, in both the boundary and the interface setups one may study the crossing constraints coming from correlators of the kind $\langle O_1 O_2 \widehat{O} \rangle$, or two-point functions of tensors. The necessary tools for the latter were developed in [49]. It is of course viable to use the method of determinants for the study of generic defects, and in particular it would be nice to complement the bootstrap analysis carried out in [50] for the twist line in the Ising model.

Appendix

2.A RG domain wall: details on the ϵ-expansion

2.A.1 One loop computations

Two regularization procedures have been preferred in the literature, in dealing with the ϕ^4 model in the presence of a defect of co-dimension one. Dimensional regularization has been especially used for the systematic renormalization of the Lagrangian and for extracting the critical exponents [18, 124, 125]. More recently, fully real space computations were carried out in [9, 10], with a short distance cutoff. Both series of works were concerned with the ϕ^4 theory in the presence of a plain boundary. We follow the latter technique.

We start by checking eq. (2.46) through the two-point function $\langle \phi^2 \phi^2 \rangle$ on the free side of the domain wall. At one loop, the only diagram contributing is shown in fig. 2.A.1. Since the correlator depends only on one cross-ratio, it is sufficient [9] to compute the

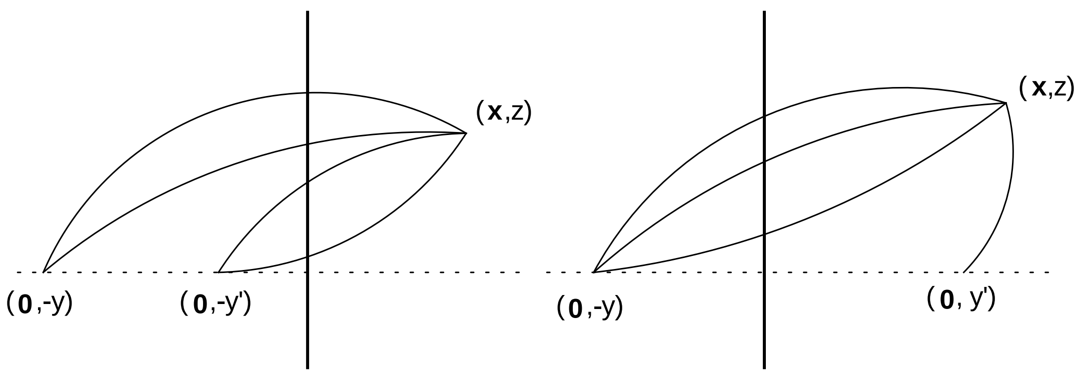

Figure 2.A.1: One loop contributions to $\langle \phi^2(x)\phi^2(x')\rangle$ and to $\langle \phi^2\phi^i(x)\phi^j(x')\rangle$. The free side is the left one, and $y, y', z > 0$.

two-point function in the collinear geometry of fig. 2.A.1, for which

$$\xi \to \frac{(y-y')^2}{4yy'}. \tag{2.81}$$

The corresponding integral is

$$\langle \phi^2(x)\phi^2(x')\rangle_{\text{one-loop}} \\ = -\frac{1}{3}N(N+2)g^* \int_0^\infty dz \int d^{d-1}\boldsymbol{x}\, \frac{1}{\{(\boldsymbol{x}^2+(z+y)^2)\,(\boldsymbol{x}^2+(z+y')^2)\}^{d-2}}. \tag{2.82}$$

Notice that we chose y, $y' > 0$. The integral does not diverge in the UV. This is expected, since the coupling constant renormalizes at $O(\epsilon^2)$, and the lowest lying interface operator that might be needed as a counterterm is ϕ^4, which however - barring mixing which appears at higher orders - equals the displacement operator and is therefore irrelevant. Since the fixed point coupling constant g^* is of order ϵ, we can plug $d=4$ in the integral to obtain the leading order correction, which is easily computed. The result is

$$\langle \phi^2(x)\phi^2(x')\rangle_{\text{one-loop}} = \frac{N(N+2)}{3}\, g^* \frac{\pi^2}{(y-y')^4}\left(\frac{\xi}{\xi+1} - \log(1+\xi)\right) + O(\epsilon^2). \tag{2.83}$$

Plugging into this expression the fixed point value for g^* (2.43) and adding the tree level contribution, one obtains the correlator at first order in ϵ-expansion:

$$\langle \phi^2(x)\phi^2(x')\rangle = \frac{2N}{(x-x')^{2(d-2)}}\left\{1 + \frac{1}{2}\frac{N+2}{N+8}\,\epsilon\left(\frac{\xi}{\xi+1} - \log(1+\xi)\right)\right\}. \tag{2.84}$$

Notice that the one point function of ϕ^2 is $O(\epsilon^2)$, therefore this is the full correlator - not just the connected part - at order ϵ. Comparing the result with the form of the conformal block of ϕ^4, evaluated in $d=4$ at this order:

$$f^{d=4}(\Delta_{\phi^4};\xi) = -2\left(\frac{\xi}{\xi+1} - \log(1+\xi)\right), \tag{2.85}$$

and using that in free theory

$$c_{\phi^2\phi^2\phi^4} = \sqrt{\frac{2(N+2)}{N}}, \tag{2.86}$$

we see agreement with the general result (2.46) and with the one-point function in eq. (2.45).

Let us compare also the general formula (2.51) with an explicit one-loop example. We focus on the correlator between the field ϕ^i on the interacting side and the free primary $\phi^2\phi^i$. The one-loop contribution is encoded in the diagram on the right in fig. 2.A.1, which is UV finite. Including the combinatorics, the result is

$$\langle \frac{\phi^2\phi^i(x)}{\sqrt{2(N+2)}}\, \phi^j(x')\rangle = \frac{\delta^{ij}}{(-2y)^3(2y')}\,\xi^{-2}\,\frac{\sqrt{N+2}}{2\sqrt{2}(N+8)}\,\epsilon\,(\xi-1). \tag{2.87}$$

It is easy to compute the tree level three-point function needed to fix $b_{\mathrm{L\,RD}}$, and see that eq. (2.87) matches eq. (2.51).

Next, we compute the first non-trivial contribution to the two-point function of ϕ^i on the free side, which departs from its free theory value at order ϵ^2. The only diagram contributing is the sunset (fig. 2.A.2). As explained in subsection 2.4.3, we actually only need to know a_{ϕ^2}, which amounts to colliding the two external operators in the diagram. The computation only slightly simplifies at this order, but the statement is valid at any loop (and of course, for any interface involving the free theory). The bulk conformal block of the operator ϕ^2 of dimension $\Delta_{\phi^2} = d-2$ is

$$f(\Delta_{\phi^2};\xi) = \left(\frac{\xi}{1+\xi}\right)^{d-2} . \tag{2.88}$$

Therefore we find

$$\langle \phi^i(x)\phi^j(x')\rangle = \frac{\delta^{ij}}{(x-x')^{d-2}}\left(1 + c_{\phi\phi\phi^2} a_{\phi^2}\left(\frac{\xi}{\xi+1}\right)^{d-2}\right) . \tag{2.89}$$

The integral to be evaluated is the following:

$$I = \int_0^\infty dz \int_0^\infty dz' \int d^3\boldsymbol{x} \int d^3\boldsymbol{x'} \frac{1}{\big(\boldsymbol{x}^2 + (y+z)^2\big)\big(\boldsymbol{x'}^2 + (y+z')^2\big)} \times \frac{1}{\big((\boldsymbol{x}-\boldsymbol{x'})^2 + (z-z')^2\big)^3} . \tag{2.90}$$

Along the computation, which is straightforward, we encounter two divergences. A bulk divergence requires a mass counterterm, and a second divergence arises when the interaction vertices hit the interface. This is compensated by integrating $\widehat{\phi^2}$ along the interface. Relevant operators are required because our cut-off breaks scale invariance. Their renormalized couplings, however, must be fine-tuned in order to reach the critical point. Hence, requiring scale invariance of the one-point function is sufficient to fix the subtraction unambiguously. After renormalization, one finds

$$I = \frac{3\pi^4}{16}\frac{1}{y^2} . \tag{2.91}$$

Taking the combinatorics into account, the expectation value at leading order is

$$\langle \frac{\phi^2(-y)}{\sqrt{2N}} \rangle \equiv \frac{a_{\phi^2}}{(2y)^2} = \frac{3}{512\pi^6}\sqrt{\frac{N}{2}}\frac{N+2}{(N+8)^2}\epsilon^2 \frac{1}{(2y)^2} . \tag{2.92}$$

Substituting back in (2.89), and using

$$c_{\phi\phi\phi^2} = \sqrt{\frac{2}{N}} , \tag{2.93}$$

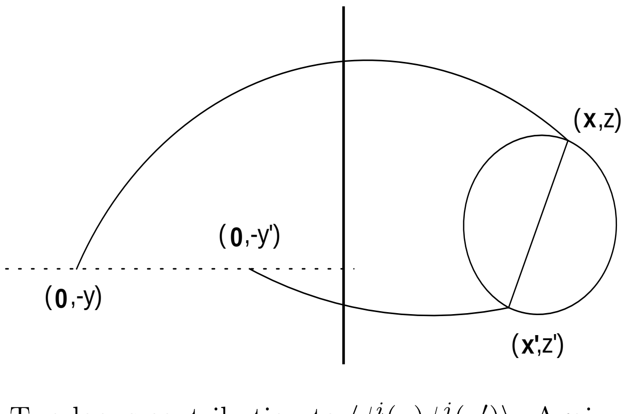

Figure 2.A.2: Two loops contribution to $\langle \phi^i(x)\phi^j(x')\rangle$. Again, $y, y', z, z' > 0$.

we find at this order

$$\langle \phi^i(x)\phi^j(x')\rangle = \frac{\delta^{ij}}{s^{d-2}}\left(1+\frac{3}{512\pi^6}\frac{N+2}{(N+8)^2}\epsilon^2\left(\frac{\xi}{\xi+1}\right)^2\right). \tag{2.94}$$

One can now extract some CFT data. By using the relations (2.73) one finds the defect OPE coefficients

$$b_{\phi\widehat{\phi}} = 1+\frac{3}{1024\pi^6}\frac{N+2}{(N+8)^2}\epsilon^2, \quad b_{\phi\,\widehat{\partial\phi}} = \sqrt{\frac{d-2}{4}}\left(1-\frac{3}{1024\pi^6}\frac{N+2}{(N+8)^2}\epsilon^2\right). \tag{2.95}$$

We also obtain a piece of information about the defect OPE of any primary on the interacting side which couples with ϕ^i, through the equalities (2.79):

$$\frac{b_{O\,\widehat{\phi}}}{b_{O\,\widehat{\partial\phi}}} = -\frac{4}{d-2}\frac{b_{\phi\,\widehat{\partial\phi}}}{b_{\phi\,\widehat{\phi}}} = -\frac{2}{\sqrt{d-2}}\left(1-\frac{3}{512\pi^6}\frac{N+2}{(N+8)^2}\epsilon^2\right). \tag{2.96}$$

We use this result in subsection 2.4.3 as a check of the solutions to the approximate crossing equation for $\langle\sigma\sigma\rangle$.

2.A.2 Two-point functions across the interface

We give some details on the formulae (2.49), (2.51) and (2.52). Let us call $x^d = y_i$ the position of the interface. We choose again the collinear geometry for the two operators and we place one on either side of the interface, at the points $x = (\boldsymbol{x}, y_L < y_i)$ and $x' = (\boldsymbol{x}, y_R > y_i)$. After plugging the free theory three-point function in eq. (2.47), we shall find the two-point function by solving the following equation:

$$\frac{b_{\mathrm{L\,R\,D}}}{(y_R-y_L)^{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}-4}}\int d^3\boldsymbol{z}\left(\boldsymbol{z}^2+(y_L-y_i)^2\right)^{-\frac{4+\Delta_{\mathrm{LR}}}{2}}\left(\boldsymbol{z}^2+(y_R-y_i)^2\right)^{-\frac{4-\Delta_{\mathrm{LR}}}{2}} = \frac{d}{dy_i}\langle O_{\mathrm{L}}(x)O_{\mathrm{R}}(x')\rangle\,. \tag{2.97}$$

First of all, we briefly comment on (2.49), that is, on the case $\Delta_{\mathrm{L\,R}} = \mathcal{O}(\epsilon)$. Since $b_{\mathrm{L\,R\,D}}$ is also at least of order ϵ, we can plug $\Delta_{\mathrm{L}} = \Delta_{\mathrm{R}}$ in (2.97). The integrals are easily evaluated and we get

$$\langle O_{\mathrm{L}}(x)O_{\mathrm{R}}(x')\rangle = -\frac{\pi^2 b_{\mathrm{L\,R\,D}}}{|y_L-y_R|^{2\Delta_{\mathrm{L}}}}\log\left|\frac{y_R-y_i}{y_L-y_i}\right| + c(y_R,y_L). \tag{2.98}$$

The constant of integration $c(y_R, y_L)$ does not depend on the position of the interface. One way to fix it is to require that when the interface stands half-way between the points the correlator takes the form (2.38):

$$c(y_R,y_L) = \frac{a_{\mathrm{L\,R}}}{|y_L-y_R|^{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}}. \tag{2.99}$$

By asking for conformal invariance of this result, one gets back at first order the scaling relation (2.40). Eq. (2.49) is then obtained by reconstructing the correlator for generic choice of the two points through conformal invariance.

Let us now tackle the case of external dimensions differing at order one. The integration in the translational invariant directions is easily recast as the Euler representation of a hypergeometric function:

$$\int d^3\boldsymbol{z}\left(\boldsymbol{z}^2+(y_L-y_i)^2\right)^{-\frac{4+\Delta_{\mathrm{LR}}}{2}}\left(\boldsymbol{z}^2+(y_R-y_i)^2\right)^{-\frac{4-\Delta_{\mathrm{LR}}}{2}}$$
$$=\frac{\pi^2}{8}\,|y_L-y_i|^{-1-\Delta_{\mathrm{LR}}}\,|y_R-y_i|^{-4+\Delta_{\mathrm{LR}}}\;{}_2F_1\!\left(\frac{3}{2},\,2-\frac{\Delta_{\mathrm{LR}}}{2};\,4;\,1-\Big(\frac{y_L-y_i}{y_R-y_i}\Big)^2\right) \tag{2.100}$$

Internal and spacetime symmetries allow to restrict ourselves to the case $\Delta_{\mathrm{L\,R}}=2k$, for integer k, at this order. Furthermore, there is a clear symmetry for the exchange L $\leftrightarrow$ R, so we only consider the case $k>0$. Since for $k>1$ the hypergeometric function is a polynomial, we treat separately the case $k=1$. Eq. (2.51) is obtained integrating the position of the interface and again fixing the integration constant in accordance with conformal invariance. When $k=2,3,\ldots$ one can write eq, (2.97) as

$$\langle O_{\mathrm{L}}(x)O_{\mathrm{R}}(x')\rangle=\frac{b_{\mathrm{L\,R\,D}}}{(y_R-y_L)^{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}}\times$$
$$\int_{(y_L+y_R)/2}^{y_0}dy_i\,\frac{3\pi^{3/2}\Gamma\left(k-\frac{1}{2}\right)}{2\Gamma(k+2)}\frac{(y_R-y_L)^4}{(y_i-y_L)^5}\;{}_2F_1\left(\frac{5}{2},2-k;\frac{3}{2}-k;\frac{(y_R-y_i)^2}{(y_L-y_i)^2}\right)$$
$$+\frac{a_{\mathrm{L\,R}}}{(y_R-y_L)^{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}}. \tag{2.101}$$

One can exploit the fact that the hypergeometric function is a polynomial and integrate addend by addend the second line of (2.101). In particular, we can choose to put the interface in $y_0=0$. Some simplifications occur because of the following observation. As already pointed out, the value of $a_{\mathrm{L\,R}}$ is fixed by the requirement of conformal invariance. On the other hand, any constant piece in the integration has the only effect of shifting $a_{\mathrm{L\,R}}$. Therefore, we disregard such pieces, and fix the constant in the end. All together, introducing the scale invariant variable $r=y_L/y_R$, we find

$$\langle O_{\mathrm{L}}(x)O_{\mathrm{R}}(x')\rangle=\frac{b_{\mathrm{L\,R\,D}}}{(y_R-y_L)^{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}}\,\frac{(-1)^k\pi^{5/2}}{(k-1)k^2\Gamma(k+2)\Gamma(-k-1/2)}\,\frac{(r-1)^2}{4r}\times$$
$$\Bigg\{\left(2k(r^2-r+1)+(r+1)^2\right)\,{}_2F_1\left(\frac{1}{2},-k;-k-\frac{1}{2};\frac{1}{r^2}\right)$$
$$-\left(2k(r^2-r)+(1+r)^2\right)\,{}_2F_1\left(\frac{3}{2},-k;-k-\frac{1}{2};\frac{1}{r^2}\right)\Bigg\}$$
$$+\frac{\tilde a}{(y_R-y_L)^{\Delta_{\mathrm{L}}+\Delta_{\mathrm{R}}}}. \tag{2.102}$$

We only need to enforce invariance under inversions, which amounts to sending $y_R\to 1/y_R$ and $y_L\to 1/y_L$. With the help of standard hypergeometric identities one can check that the first three lines in (2.102) are invariant, therefore

$$\tilde a=0. \tag{2.103}$$

Alternatively, one may simply verify that with this choice the relation (2.40) is fulfilled. The result is not yet explicitly a function of the cross-ratio. The final form eq. (2.52) can be obtained at the price of some more massage.

Chapter 3

A twist line in the $3d$ Ising model

In this chapter we shall be concerned with a first example of a defect of codimension higher than one. Precisely, we shall study on the lattice a line defect embedded in the $3d$ Ising model. It is obtained through a twisting procedure that can be applied to any CFT possessing a flavor symmetry, in any number of dimensions, therefore this should just be considered as the simplest example of a general setup. Nevertheless, it is of immediate interest under a condensed matter perspective, thanks to the large number of systems that fall into the universality class of the $3d$ Ising model. The main results collected in the chapter, which is based on [39], include the determination of the low lying defect spectrum, a lattice realization of the displacement operator, and a lattice computation of a bulk-to-defect two-point function, which probes the conformal invariance of both the critical $3d$ Ising model and of the line defect.

3.1 A monodromy defect on the lattice

We would like to define a class of conformal defects associated to a monodromy. Consider a CFT which is equipped with a flavor symmetry group G. One can define a class of topological domain walls $\mathcal{D}_g$ associated to elements of the flavor group g as follows: when the domain wall is swept over a local operator, the operator is transformed into the image under the action of g. As a consequence, to compute a correlation function one starts from the correlator without the domain wall and acts with g on all the operators on one side of $\mathcal{D}_g$. In particular, displacing the domain wall leaves the partition function unchanged - but not necessarily the action. Also, all correlation functions involving the stress-tensor, which is not charged under the flavor symmetry, are continuous when this operator is dragged over the domain wall: this fact assesses the topological nature of the defect. If there are no topological obstructions, the domain wall can be collapsed and eliminated. The simplest situation in which its presence matters is obtained when there is a non contractible loop in the spacetime manifold. An infinite domain wall which intersects this path is equivalent to a boundary condition for the fields that makes them multivalued around the loop. Besides the customary setup of a manifold with a non trivial topology, one can use this observation to build a codimension 2 conformal defect, by letting $\mathcal{D}_g$ end on it. Local operators which transform non-trivially under g will be multi-valued around the monodromy defect. In particular, this means that the OPE of a bulk operator charged under g will contain defect local operators with fractional spin under the $SO(2)$ group of transverse rotations. For example, if ϕ is the angular coordinate around the defect, r the radial coordinate in the plane perpendicular to the defect, and $G = \mathbb{Z}_2$, a $\mathbb{Z}_2$ odd operator O of conformal dimension

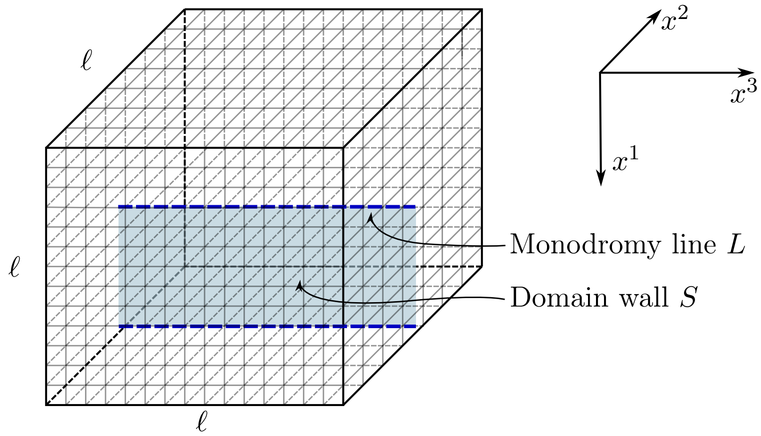

Figure 3.1: In our set-up, the domain wall S is the surface across which the links are frustrated. It ends on two defect lines. We will mostly consider the proximity of one of such lines, which we take to be aligned with the x^3 axis.

Δ will have OPE

$$O(r,\phi) \sim \sum_{n,a} \mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\phi} r^{\Delta_a - \Delta}\, \widehat{O}_{n+\frac{1}{2},a} \tag{3.1}$$

involving defect local operators $\widehat{O}_{n,a}$ of conformal dimension Δ_a and half-integral $SO(2)$ spin $s = n + 1/2$.

It is pretty easy to implement this construction on the $3d$ Ising model. Indeed, the topological domain wall can be engineered in the UV through a frustration. Consider the lattice Hamiltonian

$$H\left(\{\sigma_x\}\right) = -\beta \sum_{\langle xy \rangle} J_{\langle xy \rangle}\, \sigma_x \sigma_y \ , \tag{3.2}$$

where the sum runs over nearest-neighbor sites and the $\mathbb{Z}_2$ variables σ_x are defined on the sites. In what follows, the coupling will always be set to the best known critical value [126]:

$$\beta_c = 0.22165455. \tag{3.3}$$

The twist is introduced by switching the sign of the interaction: we set $J_{\langle xy \rangle} = 1$ but for the bonds that intersect a surface S, for which $J_{\langle xy \rangle} = -1$. This is sown in fig. 3.1, where a realization of the system on a three-torus, which is suitable for Monte Carlo simulations, is depicted. The surface ends on two defect lines which are placed on the dual lattice, that is the one whose links cross every plaquette of the original lattice[1] This construction is well known within statistical mechanics, in the context of Kramers-Wannier duality [127]. This is a lattice duality which exchange high and low temperature, the lattice with its dual, and the symmetry group with the group of its representations. The $2d$ Ising model, for instance, is sent to itself by the duality, and this fact allows to easily find its critical temperature as the self-dual point. $3d$ statistical models with nearest-neighbor interactions, instead, are mapped to theories in which the Hamiltonian contains a sum over plaquettes, and the dynamical variables are the bonds of the dual lattice. These are lattice gauge theories, and in particular the image of the $3d$ Ising model under the duality has $\mathbb{Z}_2$ as a discrete gauge group. The basics observables in this theory are as usual correlation functions of Wilson lines, and it is not difficult to show that their description in terms of the dual statistical

[1] More generally, every k-simplex is substituted by a $(d-k)$-simplex.

variables makes use of the frustration we just described. Precisely, the correlator of two Wilson lines equals the ratio of the partition functions of the Ising model with and without a frustrated surface ending on the two lines. In other words, our monodromy defect is just a Wilson line when the bulk theory is critical.

Since we aim at studying the conformal defect through numerical simulations, we consider a system with finite size ℓ. The frustrated surface cannot extend to infinity, therefore there are two twist lines. This setup breaks conformal invariance, and finite size effects must be taken into account for a precision study of the critical domain. We shall content ourselves of putting the second line defect at the boundary of the lattice, and find an optimal size such that the dependence of observables on ℓ is negligible within the accuracy we can reach. This turns out to be $\ell = 70$ for the observables belonging to the even sector of the spectrum, and $\ell = 120$ for the odd ones.

The piece of CFT data which is the easiest to extract through a Monte Carlo approach is the spectrum of anomalous dimensions of defect operators, so this is going to be our immediate interest.

3.2 The spectrum of defect operators

Since the scale dimensions of operators in the Ising model lie not too far away from the free field value, it is worth to give a mean-field description of the line defect, as a guide to the analysis. We noticed that defect operators carry charge with respect to the group of transverse rotations. In a theory of a free scalar field, the OPE of the scalar field with a monodromy defect would contain spin $s = n+1/2$ primary operators of dimension $|s|+1/2$. Indeed, we can consider a general OPE

$$\phi(x) \sim \sum_a f_a(z, \bar{z})\widehat{O}_a + \text{descendants} \tag{3.4}$$

where $z = x^1 + ix^2 = r\,\mathrm{e}^{\mathrm{i}\phi}$ is a complex coordinate in the plane orthogonal to the defect (we are taking the defect line along the x^3 direction). By applying the equation of motion and ignoring derivatives along the defect, which give descendants, we see that the OPE coefficients must be harmonic functions of $z, \bar{z}$, and the OPE must take the form

$$\phi(x) \sim \sum_n \left(\bar{z}^{n+1/2}\widehat{O}_{n+1/2} + c.c\right) + \cdots \tag{3.5}$$

As for the even sector, the OPEs of odd defect operators contain bilinears $\widehat{O}_{n+1/2}\widehat{O}_{m+1/2}$ of dimension equal to $|s| + 1$. Among these one finds the components of the displacement operator $\mathrm{D} = \mathrm{D}_1 + \mathrm{i}\mathrm{D}_2$ and $\bar{\mathrm{D}} = \mathrm{D}_1 - \mathrm{i}\mathrm{D}_2$, which have spin 1 ($-1$) and dimension 2. Interactions are going to change this picture, but we still expect to find a leading spin 1/2 odd operator of dimension close to 1, and a spin 1 operator of dimension exactly equal to 2.

In order to single out the right correlators, we need to find a lattice realization of these composite fields. Since the defect line lies on the dual lattice, we cannot directly measure defect excitations, and so we have to rely on the defect OPE of the bulk field. In other words, we shall construct defect operators by measuring combinations of the lattice spin σ on the sites of elementary plaquettes linked to the twist line (see fig. 3.1). These combinations should be chosen according to their transformation properties under the symmetries preserved by the lattice. In the absence of the defect, the subgroup of the rotation group preserved by the square lattice would be the D_4 dihedral group, generated

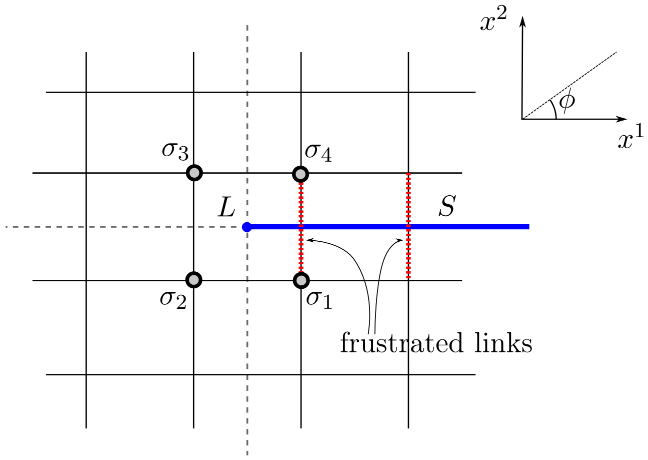

Figure 3.1: The 2d lattice of a plane transverse to the defect line. The projection of the defect plane is the heavier line, crossed by frustrated links, which plays the rôle of a $\mathbb{Z}_2$ monodromy cut.

by the rotation of $\pi/2$ (say, counter-clockwise) and the reflection about any of the symmetry axes (say, the horizontal one in fig. 3.1). However, the monodromy cut must be brought back to the original position after a $\pi/2$ rotation, thus going past a spin variable which switches sign. We denote as a this transformation, which acts on the elementary spins σ_i ($i = 1, 2, 3, 4$) at the corners of a plaquette linked with the defect as follows:

$$a \,:\, (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \mapsto (\sigma_2, \sigma_3, \sigma_4, -\sigma_1) \ . \tag{3.6}$$

The reflection b with respect to the axis through the origin containing the projection of the frustrated plane (the x^1 axis) does not displace the cut and acts as follows:

$$b \,:\, (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \mapsto (\sigma_4, \sigma_3, \sigma_2, \sigma_1) \ . \tag{3.7}$$

These transformations satisfy $a^8 = \mathbb{1}$, $b^2 = \mathbb{1}$ and $(ab)^2 = \mathbb{1}$, and generate thus the dihedral group D_8.

Since the defect is one-dimensional, there are no rotations in the parallel directions to be taken into account. However, there is a discrete symmetry involving the coordinate x^3, namely the reflection

$$\mathcal{S} \,:\, x^3 \mapsto -x^3 \ . \tag{3.8}$$

We call $\mathcal{S}$-parity the eigenvalue of operators with respect to this symmetry.

One can now determine the lattice realization of the defect operators as the combinations of spins transforming according to the irreducible representations of the symmetry group. The D_8 group has order 16 and possesses seven irrepses, four of dimension 1 and three of dimension 2. The 4-dimensional representation acting on the spins σ_i of fig. 3.1 decomposes into two bi-dimensional representations, which we denote as $H_{1/2}$ and $H_{3/2}$. A basis for $H_{1/2}$ is given by (ψ, ψ^*), with

$$\psi = \sigma_1 + \omega\sigma_2 + \omega^2\sigma_3 + \omega^3\sigma_4 \,, \tag{3.9}$$

where $\omega = \exp(\mathrm{i}\pi/4)$. In this representation we have

$$a \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \,, \quad b \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = \begin{pmatrix} 0 & \omega^3 \\ \omega^{-3} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \ . \tag{3.10}$$

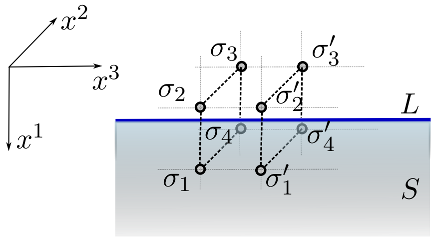

Figure 3.2: The spin variables around the monodromy line in terms of which the defect operators of lowest dimensions of table 3.1 can be realized, as described in the text.

We see that ψ (ψ^*) carries spin $J = 1/2$ ($J = -1/2$). The basis for the representation $H_{3/2}$ is instead given by $(\psi_{3/2}, \psi^*_{3/2})$, with

$$\psi_{3/2} = \sigma_1 + \omega^3\sigma_2 + \omega^6\sigma_3 + \omega\sigma_4\,, \tag{3.11}$$

for which

$$a\begin{pmatrix}\psi_{3/2}\\ \psi^*_{3/2}\end{pmatrix} = \begin{pmatrix}\omega^{-3} & 0\\ 0 & \omega^3\end{pmatrix}\begin{pmatrix}\psi_{3/2}\\ \psi^*_{3/2}\end{pmatrix}\,,\quad b\begin{pmatrix}\psi_{3/2}\\ \psi^*_{3/2}\end{pmatrix} = \begin{pmatrix}0 & \omega\\ \omega^{-1} & 0\end{pmatrix}\begin{pmatrix}\psi_{3/2}\\ \psi^*_{3/2}\end{pmatrix}\,. \tag{3.12}$$

Both the representations $H_{1/2}$ and $H_{3/2}$ are odd under the the flavor group $\mathbb{Z}_2$. All other irreducible representations of D_8 have integral spin and are therefore $\mathbb{Z}_2$ even. They can be obtained by decomposing the direct product of $H_{1/2}$'s. Three of them can be realized in terms of the bilinears $\sigma_i\sigma_j$ corresponding to the four links $\langle ij\rangle$ of fig. 3.1. Indeed the 4-dimensional representation $\mathcal{L}_4$ acting on the four links can be decomposed as the sum of a 2-dimensional representation V and two unidimensional representations: $\mathcal{L}_4 = V \oplus S \oplus T_+$. Here S is the trivial representation, which acts on

$$s = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 - \sigma_4\sigma_1 \tag{3.13}$$

by $a\,s = s$, $b\,s = s$. The leading operator contributing to the spin 1 sector is the displacement, therefore we denote as $(\mathrm{D}_1, \mathrm{D}_2)$ a basis for the vectorial representation V:

$$\mathrm{D}_1 = \sigma_1\sigma_2 + \sigma_2\sigma_3 - \sigma_3\sigma_4 + \sigma_4\sigma_1\,,\quad \mathrm{D}_2 = -\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1\,. \tag{3.14}$$

In this representation the generators act as follows:

$$a\begin{pmatrix}\mathrm{D}_1\\ \mathrm{D}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathrm{D}_1\\ \mathrm{D}_2\end{pmatrix}\,,\quad b\begin{pmatrix}\mathrm{D}_1\\ \mathrm{D}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\mathrm{D}_1\\ \mathrm{D}_2\end{pmatrix}\,. \tag{3.15}$$

Notice that the combination $\mathrm{D} = \mathrm{D}_1 + \mathrm{i}\mathrm{D}_2$ has spin 1, while, of course, $\bar{\mathrm{D}} = \mathrm{D}_1 - \mathrm{i}\mathrm{D}_2$ has spin -1. Finally, T_+ is a representation of spin $J = 2$ acting on

$$t_+ = \sigma_1\sigma_2 - \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1, \tag{3.16}$$

by $a\,t_+ = -t_+$, $b\,t_+ = t_+$

The D_8 representations constructed up to now are defined on a single plane orthogonal to the defect line, thus are all even under $\mathcal{S}$-parity (3.8). We shall denote as P and T_- the remaining unidimensional representations of D_8. P is generated by antisymmetric products of two representations of $H_{1/2}$ type, and cannot thus be realized in terms of the links $\sigma_i\sigma_j$ of the single plaquette in fig. 3.1. The simplest realization of this representation

s	p^o	D	D^o
$\bar{\psi}\psi$	$\Im m(\bar{\psi}\overleftrightarrow{\partial}\psi)$	$\mathrm{i}\psi\psi$	$\bar{\psi}\overleftrightarrow{\partial}\psi_{3/2}$

t_+	t_-	t^o_+	t^o_-
$\Im m(\psi\psi_{3/2})$	$\Re e(\psi\psi_{3/2})$	$\Im m(\psi\overleftrightarrow{\partial}\psi_{3/2})$	$\Re e(\psi\overleftrightarrow{\partial}\psi_{3/2})$

Table 3.1: Schematic description of several lattice operators, built as bilinears in ψ and $\psi_{3/2}$ in analogy to the free field approximation to the primary operators of the continuum theory. ∂f denotes the finite difference $\partial f(x) \equiv f(x+1) - f(x)$ and $g\overleftrightarrow{\partial}f = g\partial f - f\partial g$. Using the transformation properties of ψ and $\psi_{3/2}$ one can verify at once the transformation properties of these bilinears, in accordance with the decomposition of representations described in (3.26) and Fig. 3.4. As described in the text, some of these operators can be built explicitly from the spins at the vertices of a single plaquette, some require us to use spins from the vertices of a cube. These lattice operators provide natural candidates for the corresponding operators in the continuum theory, up to some ambiguity due to the fact that the spin J on the lattice is defined modulo 4.

is obtained by considering the corners of the cube of fig. 3.2). Note that the $\mathcal{S}$-reflection defined in (3.8) now is

$$\mathcal{S} \, : \sigma_i \leftrightarrow \sigma_i' \, . \tag{3.17}$$

The anti-symmetric combinations of the diagonals of the four faces which do not intersect the defect line define a 4-dimensional reducible representation of D_8 which can be decomposed as $P \oplus V \oplus T_-$, where now the representations P, V and T_- act on $\mathcal{S}$-odd operators, that we call respectively p^o,D^o, and t^o_-:

$$p^o =[\sigma_1\sigma_2'] + [\sigma_2\sigma_3'] + [\sigma_3\sigma_4'] - [\sigma_4\sigma_1'] \, , \tag{3.18}$$

$$D^o_1 =- [\sigma_1\sigma_2'] + [\sigma_2\sigma_3'] + [\sigma_3\sigma_4'] + [\sigma_4\sigma_1'] \, , \tag{3.19}$$

$$D^o_2 =[\sigma_1\sigma_2'] + [\sigma_2\sigma_3'] - [\sigma_3\sigma_4'] + [\sigma_4\sigma_1'] \, , \tag{3.20}$$

$$t^o_- =[\sigma_1\sigma_2'] - [\sigma_2\sigma_3'] + [\sigma_3\sigma_4'] + [\sigma_4\sigma_1'] \, , \tag{3.21}$$

with $[\sigma_i\sigma_j'] = \sigma_i\sigma_j' - \sigma_i'\sigma_j$. We have $a\,p^o = p^o$, $b\,p^o = -p^o$ and $a\,t^o_- = -t^o_-$, $b\,t^o_- = -t^o_-$. Similarly the anti-symmetric combinations of the principal diagonals of this cube can be decomposed in the sum $T_+ \oplus P$. The pseudoscalar representation T_+ now acts on

$$t^o_+ = [\sigma_1\sigma_3'] - [\sigma_2\sigma_4'], \tag{3.22}$$

and pseudoscalar representation P on the $\mathcal{S}$-odd operator

$$p^{o'} = [\sigma_1\sigma_3'] + [\sigma_2\sigma_4'] \, , \tag{3.23}$$

which is however less efficient in numerical simulations than p^o introduced in (3.18).

The operators described so far can be expressed as bilinears in ψ and $\psi_{3/2}$, as shown in tab. 3.1. On the other hand, for any irreducible representation of D_8 one can construct primary operators of both $\mathcal{S}$-parities. The primaries of opposite $\mathcal{S}$-parity with respect to those already constructed can be expressed as multi-linear products of σ_i or as operators involving more nodes of the lattice; they are thus difficult to deal with in numerical simulations and are expected to have larger anomalous dimensions.

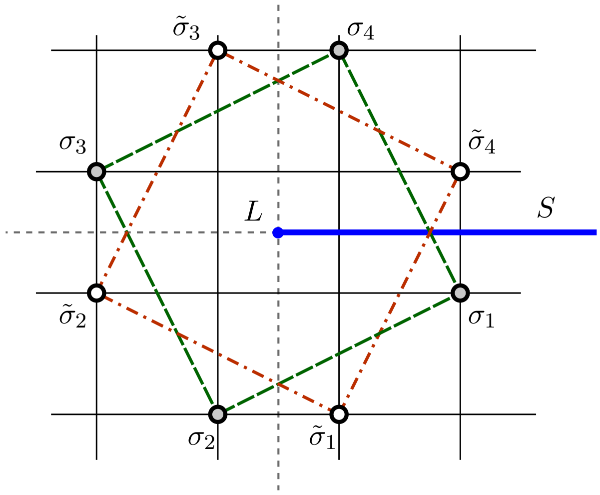

Figure 3.3: An alternative set of spin variables in terms of which we construct some of the defect operators, as described in the text. These spins are lying around the monodromy line in plane orthogonal to it, just as in Fig. 3.1. In particular, the $\mathcal{S}$-even pseudo-scalar representation P can be realized in terms of the bilinears $\sigma_i\sigma_{i+1}$ and $\tilde{\sigma}_i\tilde{\sigma}_{i+1}$, indicated in the drawing by the two sets of diagonal segments.

As an example, an $\mathcal{S}$-even pseudoscalar $\tilde{p}$ can be realized by considering eight spin variables lying around the line defect as in fig. 3.3. The generators act on these variables as follows:

$$\begin{aligned} a &: (\sigma_1,\sigma_2,\sigma_3,\sigma_4) \mapsto (\sigma_2,\sigma_3,\sigma_4,-\sigma_1) \ , \quad (\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_4) \mapsto (\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_4,-\tilde{\sigma}_1) \ ; \\ b &: \sigma_1 \leftrightarrow \tilde{\sigma}_4 \ , \quad \sigma_4 \leftrightarrow \tilde{\sigma}_1 \ , \quad \sigma_2 \leftrightarrow \tilde{\sigma}_3 \ , \quad \sigma_3 \leftrightarrow \tilde{\sigma}_2 \ . \end{aligned} \tag{3.24}$$

It is easy to check that the $\mathbb{Z}_2$-even operator

$$\tilde{p} = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 - \sigma_1\sigma_4 - (\tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_2\tilde{\sigma}_3 + \tilde{\sigma}_3\tilde{\sigma}_4 - \tilde{\sigma}_1\tilde{\sigma}_4) \tag{3.25}$$

transforms in the pseudoscalar representation P, i.e., we have $a\tilde{p} = \tilde{p}$ and $b\tilde{p} = -\tilde{p}$.

A useful tool to summarize the D_8 irrepses is the graph associated with the decomposition of the tensor product of any irreducible representation R_i with the two-dimensional representation $H_{1/2}$, see fig. 3.4. The incidence matrix of this graph, which turns out to correspond to the extended Dynkin diagram of the $\mathcal{D}_6$ Lie Algebra, is given by the Clebsh-Gordan coefficients c_{ij} in the decomposition

$$H_{1/2} \otimes R_i = \sum_j c_{ij} R_j \ . \tag{3.26}$$

3.3 Simulations and results

A simple way to compute scale dimensions of operators on the lattice is the direct fit of the large distance fall-off of the two-point functions. All results are reported in tab. 3.1. As for the even spectrum, one can measure the correlators of pairs of link variables $\sigma_i\sigma_j$ lying on the nearest plaquettes to the monodromy defect independently for every mutual distance along the line and for every mutual orientation. One can then reconstruct the

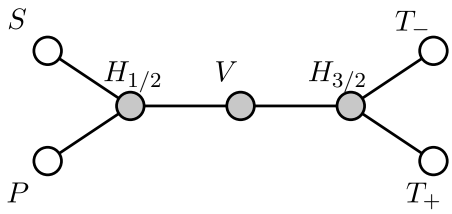

Figure 3.4: The irrepses of the D_8 dihedral group are encoded in the extended Dynkin diagram of the $\mathcal{D}_6$ algebra, as described in the text. Open circles denote one-dimensional representations while grey circles are associated to two-dimensional representations.

$\widehat{\mathcal{O}}$	D_8 irrep	$\mathbb{Z}_2$ parity	$O(2)$ spin	$\mathcal{S}$-parity	Δ
s	S	+	0^+	+	2.27(1)
p^o	P	+	0^-	−	2.9(2)
$\tilde{p}$	P	+	0^-	+	3.7(2)[3]
ψ	$H_{1/2}$	−	$\frac{1}{2}$	+	0.9187(6)
D	V	+	1	+	2
D^o	V	+	1	−	3.3(2)[3]
$\psi_{3/2}$	$H_{3/2}$	−	$\frac{3}{2}$	+	1.99(5)
t_+	T_+	+	2	+	3.1(5)[3]
t_+^o, t_-^o	T_+, T_-	+	2	−	≥4.2(1)

Table 3.1: The lowest anomalous dimensions of the local operators at the defect line. The round brackets indicate the statistical error, while the square brackets for $\Delta \geq 3$ denote an estimate of the systematic error (see more details in the text). For the operator t_+^o, realized on the diagonal of elementary cubes, we did not use a direct Monte Carlo evaluation which is too noisy, but a lower bound obtained by considering only the contributions of the spin-spin two-point functions involved (see [128] for a proof of the bound).

two-point function of the operators described in the previous subsection. The best result in this sector is attained through the correlator $\langle \bar{\mathrm{D}}(0)\mathrm{D}(x)\rangle$, which is subject to a one parameter fit, that also confirms the protected dimension $\Delta_{\mathrm{D}} = 2$. The interpolation is shown in fig. 3.1. The other operators in the even sector are heavier, as expected, and this makes the lattice determination of the scale dimensions difficult. The two-point function of the scalar s has to be fitted with the law

$$\langle s(0)s(x)\rangle = c_s\, x^{-2\Delta_s} + \text{constant}. \tag{3.27}$$

The constant, that is the square of the one-point function, is allowed by the $D_8 \times \mathbb{Z}_2$ symmetry preserved by the lattice, while it would be forbidden by scale invariance. It is present, indeed, because the lattice breaks scale invariance both in the UV and in the IR. The infrared contribution to the one-point function, in particular, can be used for a more convenient determination of the scale dimension, due to the smaller exponent:

$$\langle s\rangle = \tilde{a}_s\, \ell^{-\Delta_s} + \text{constant}\,. \tag{3.28}$$

Furthermore, one can extract one more piece of CFT data from the combined measurement:

$$a_s = \frac{\tilde{a}_s}{\sqrt{c_s}} = 0.33(1). \tag{3.29}$$

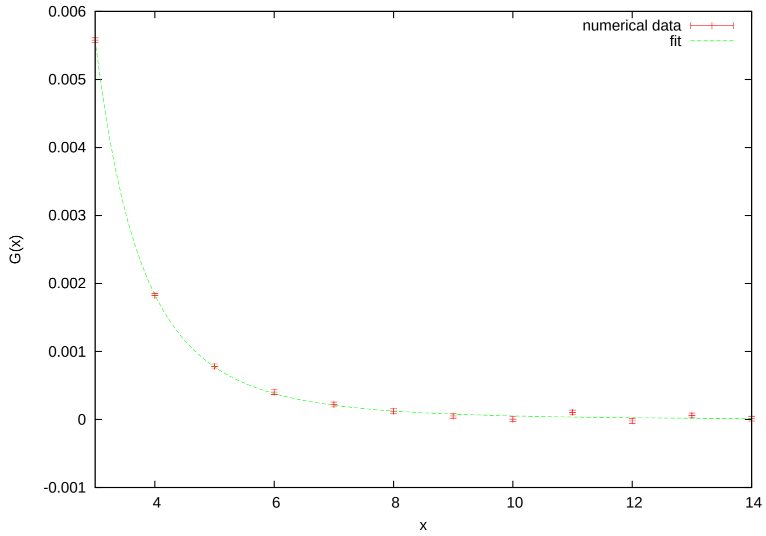

Figure 3.1: The correlation function of the displacement operator. The solid curve is the one parameter fit to $\tilde{C}_{\mathrm{D}}/x^4$. Notice that the operator is not normalized in such a way to respect the Ward identity (1.149).

One more observation about the even spectrum is in order. At small values of the spin J, the identification between lattice operators in a given D_8 representation and continuum operators in appropriate $O(2)$ representations is clear. The operators with high spin in the continuum theory are expected to have large conformal dimension, and thus give subleading contributions to the lattice operators. The one exception is the operator $\tilde{p}$. Although it has no spin from the point of view of D_8, it is built from alternating spins in a way which would closely resemble a $J = 4$ operator of the continuum theory. In the free theory, a $J = 0$ operator with the quantum numbers of $\tilde{p}$ would be a descendant of p^o. The $J = 4$ primary would have dimension close to 5. But while the scale dimensions might point towards the identification of the spin 0 descendant as the leading contribution to $\tilde{p}$, the correlation function of p^o and $\tilde{p}$ is very small. We need to conclude that the $J = 4$ contribution to $\tilde{p}$ must be very large, and the numerical estimates correspondingly poor.

Let us also stress that the determination of the scale dimension of the heavier operators is sometimes affected by systematic error, in the sense that since the signal is rapidly drowned into noise, different fitting procedures yield different results. In order to give a rough estimate of at least one source of systematic error, one can fit the data including the contribution of the descendant $\partial_3\partial_3\widehat{O}$, that is with the law $a/x^{2\Delta} + b/x^{2(\Delta+1)}$. The difference between the values of Δ with and without the second contribution is the source of the error reported in square brackets in 3.1.

The situation is better for the odd spectrum, where the lightest operators lie. Since Δ_ψ turns out to be slightly less than one, the two-point function of the spin 1/2 operator is different from zero for many lattice spacings, within statistical errors which are particularly small (see tab. 3.2).

Up to now, we have tested only the scale invariance of the theory living on the line defect. The simplest observable which is fixed only with the help of the full set of unbroken generators of the conformal group is the two-point function of a scalar primary in the bulk

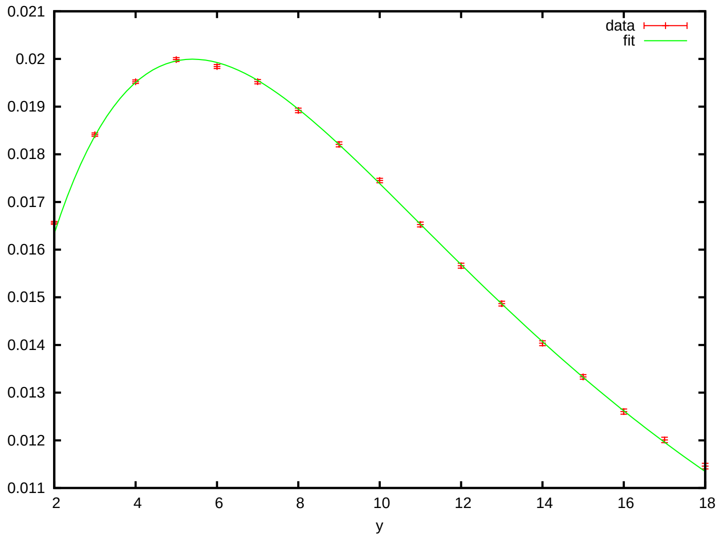

Figure 3.2: The correlation function between the lattice spin in the bulk and a $J = \frac{1}{2}$ defect local operator in a cubic lattice of size ℓ^3 with $\ell = 120$ with periodic boundary conditions. The solid line is a one-parameter fit to (3.30) with $d = 10$. In this fit we also considered the contribution of the nearest copy of the defect line, see the discussion in the main text.

x	$G_{1/2}(x)$	x	$G_{1/2}(x)$
2	0.86752(3)	11	0.04529(5)
3	0.46136(4)	12	0.03873(5)
4	0.28125(4)	13	0.03341(5)
5	0.18892(4)	14	0.02925(5)
6	0.13592(4)	15	0.02592(4)
7	0.10287(4)	16	0.02301(5)
8	0.08066(4)	17	0.02071(5)
9	0.06518(4)	18	0.01873(5)
10	0.05383(4)	19	0.01692(5)

Table 3.2: The values of the spin-1/2 correlation function $G_{1/2}(x) = \Re \langle \psi(x)\psi^*(0)\rangle$ on the defect line.

and a primary on the defect. Its functional form was fixed in eq. (1.22):

$$\langle \sigma(d,r,\phi)\widehat{\mathcal{O}}_J(0)\rangle = b_{\sigma\widehat{\mathcal{O}}_J}\, \mathrm{e}^{-\mathrm{i}\phi J} \frac{r^{\Delta_{\widehat{\mathcal{O}}_J}-\Delta_\sigma}}{(r^2+d^2)^{\Delta_{\widehat{\mathcal{O}}_J}}}\,, \tag{3.30}$$

where the geometric setup is the one of fig. 3.3. We measure the correlation function $\langle \sigma(d,r,\pi)\psi(0)\rangle$ fixing $d = 10$, and we fit the data with 3.30. We fix $\Delta_\psi = 0.9187(6)$ through the previous measurements, and $\Delta_\sigma = 0.5182(2)$, which is a conservative average between the best existing estimates $\Delta_\sigma = 0.51813(5)$ from Monte Carlo simulations [129] and $\Delta_\sigma = 0.51819(7)$ from Strong Coupling Expansions [130]. This is therefore a one parameter fit. In this case we need to take into account the fact that replicas of the defect are present because of the periodic boundary conditions. In particular, the nearest replica

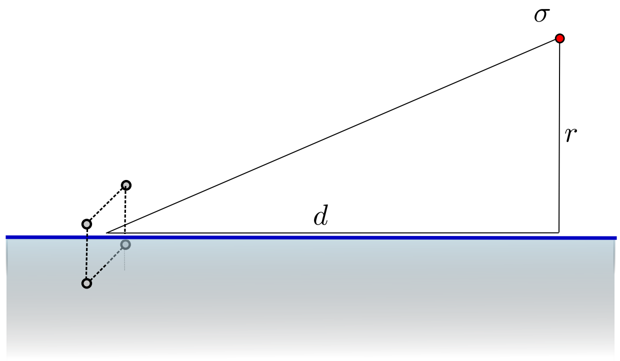

Figure 3.3: Set-up for the mixed correlation function of the scalar spin on the bulk and the $\mathbb{Z}_2$- odd local operator on the defect. This local operator is built with the spins associated to the corners of the plaquette wrapped around the defect line, following the prescriptions of Eq.s (3.9) and (3.11).

of the ψ defect operator lies at distance ℓ from the original monodromy defect, that is at distance $r' = \ell - r$ and at an angle $\phi' = 0$ with respect to the bulk spin. This is the only replica included in the fit. The nice agreement of the data with the interpolating curve is a check of both conformal invariance of the bulk and of the line defect. After measuring separately the correlators $\langle\sigma\sigma\rangle$ without the defect and $\langle\psi\psi\rangle$ and $\langle\psi_{3/2}\psi_{3/2}\rangle$ on the defect, one also obtains the Defect OPE coefficients

$$|b_{\sigma\psi}| = 0.968(2), \qquad \left|b_{\sigma\psi_{3/2}}\right| = 0.61(9). \tag{3.31}$$

One more OPE coefficient can be measured with a good precision in the even sector. This is the one-point function of the energy operator. We choose the link variable $\sigma_i\sigma_j$, which receives contributions also from all the higher weight even operators ϵ', ϵ'', ..., and we look for the coefficient $\tilde{A}_\epsilon$ in

$$\langle\sigma_i\sigma_j(x)\rangle = \text{constant } + \frac{\tilde{a}_\epsilon}{|x^i|^{\Delta_\epsilon}} + \text{higher order terms}. \tag{3.32}$$

We use $\Delta_\epsilon = 1.4130(5)$, again combining the Monte Carlo value $\Delta_\epsilon =\ 1.41275(25)$ [129] with the Strong Coupling estimate $\Delta_\epsilon = 1.4130(4)$ [130]. Dividing by the square root of the two-point function in homogeneous space one gets

$$a_\epsilon = -0.167(4). \tag{3.33}$$

As a concluding remark, the Monte Carlo results presented here testify that the lattice is suitable for direct investigation of fixed points, even if sources of violation of conformal invariance - in particular, finite-size effects - would probably become a more serious issue in high precision measurements. It would be interesting to see how much information one can get from the lattice about poorly explored parts of the conformal data, such as OPE coefficients of the homogeneous system. As for the twist line defects, as we emphasized the construction can be applied generally, and a next step will be to study the generalization of this setup to the XY model[2]. The twist defect in the Ising model has been recently studied also in ϵ-expansion and through the positive-functional bootstrap, applied for the four-point function of ψ operators [50]. It would be interesting to repeat the analysis bootstrapping the two-point function of the spin σ in the bulk, compare the two methods and possibly get more precise combined predictions.

[2]This is work in progress with C. Bonati.

Chapter 4

Rényi entropies as conformal defects

The last two chapters of this thesis are dedicated to a kind of defect that appears in the study of entanglement in quantum field theory. In the words of Schrödinger, entanglement is «not [...] *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought» [131]. It is remarkable that the framework of defect conformal field theory allows to efficiently address questions in this context. More precisely, an extended operator arises in studying one of the possible measures of entanglement in quantum systems, the family of Rényi entropies, which we define below in eq. (4.12). Although the history of entanglement is practically as old as the history of quantum mechanics, recently there has been a growing interest in entanglement and Rényi entropies as probes of complex interacting quantum systems in a variety of areas ranging from condensed matter physics, *e.g.,* [132, 133, 134] to quantum gravity, *e.g.,* [135, 136, 137, 138]. Furthermore, the past year has seen remarkable experimental advances, and the Rényi entropy, as well as quantum purity and mutual information, of a system of delocalized interacting particles have been measured in the laboratory [139]. This experimental breakthrough strengthens the motivation to develop further theoretical insight into these entanglement measures, particularly in the framework of quantum field theory (QFT). As mentioned, in this work we focus our attention on Rényi entropies [140, 141] in the context of conformal field theories (CFTs).

For the sake of completeness, we start in section 4.1 with an introduction to entanglement in quantum mechanics and quantum field theory and to its most acclaimed measure, that is entanglement entropy. In section 4.2 we introduce our main player, a (conformal) defect known as the twist operator, and we rephrase in the language of defect CFT the following question: how do the Rényi entropies of a region of space depend on the shape of its boundary? This problem has attracted some interest in the literature, as we review below, and a number of conjectures have been recently made concerning small variations around a flat or spherical boundary. In the rest of the chapter, we use conformal symmetry to compute the shape dependence in a variety of situation, up to a single coefficient, which is the norm of the displacement operator C_{D}. Along the way, we show that the various conjectures are summarized by a relation between C_{D} and the coefficient of the one-point function of the stress tensor in the presence of a flat twist operator. This coefficient, which we would call a_T according to the notation established in chapter 1 - see for instance eq. (1.70) - is dubbed the *conformal weight* of the twist operator and denoted h_n in the context of Rényi entropies. In accordance to the literature, we adopt the latter convention in what follows. Let us anticipate that the conjecture as been proven to fail in the four dimensional holographic dual of Einstein gravity [52]. This result can be in fact extended to other dimensions, and this is the subject of chapter 5. As a final remark, we would like to stress that most of what is contained in the present chapter applies to any conformal defect: reference to the twist operator is only contained in the specific value of the CFT data C_{D} and h_n. Their value will become important only in the next chapter, so what follows can also be read as one more piece of the general defect CFT toolbox that we have been building in the first part of the thesis.

4.1 Entanglement in quantum systems

Entanglement is an extremely simple consequence of the fact that quantum states can be linearly combined. Yet, it plays a crucial role in quantum information and quantum computation [142]. Quantum entanglement is also object of a lot of interest in the context of condensed matter physics [143]. Indeed, the vacuum wave function contains all the information concerning the various phases of a given many body system at zero temperature, and entanglement can be used as a diagnostic tool next to the traditional local order parameters.

In order to define (bipartite) entanglement, consider a composite system AB whose Hilbert space factorizes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{4.1}$$

The two subsystems are said to be entangled if the total state cannot be expressed as a tensor product of a state belonging to $\mathcal{H}_A$ times another one belonging to $\mathcal{H}_B$. The simplest example of such a situation is the so called EPR (Einstein-Podolski-Rosen) pair [144]. This is a system of two qubits, placed in the following state:

$$|\mathrm{EPR}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \tag{4.2}$$

It is easy to show that this state cannot be written as a product state:

$$\nexists\ |A\rangle \in \mathcal{H}_A,\ |B\rangle \in \mathcal{H}_B, \quad \text{such that} \quad |\mathrm{EPR}\rangle = |A\rangle\, |B\rangle\,. \tag{4.3}$$

The first striking consequence of this fact is that a local measurement applied to system A influences system B: in particular, the value of the z component of the spin of the two systems is completely correlated. For our purposes, it is useful to stress another feature of this state. We possess a complete description of the EPR state, including the relative phase between $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. However, this is not so if we are allowed only local measurement in one of the subsystems: we know that a spin measurement on A will yield $+1$ half of the times, but there is no information about the relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$. Of course, the best technical description of this fact requires the use of the density matrix formalism. Recall that a density matrix is a Hermitian operator with unit trace and non-negative eigenvalues. The *reduced* density matrix which describes subsystem A is obtained by tracing out the Hilbert space of B:

$$\rho_A \equiv \mathrm{Tr}_B \rho_{\mathrm{EPR}} = \mathrm{Tr}_B\, |\mathrm{EPR}\rangle\, \langle \mathrm{EPR}| = \frac{1}{2}\left(|\uparrow\rangle\,\langle\uparrow| + |\downarrow\rangle\,\langle\downarrow|\right). \tag{4.4}$$

We see that the state is mixed, that is, ρ_A cannot be written as the density matrix of a pure state $\rho_{\mathrm{pure}} = |s\rangle\,\langle s|$ (this simply follows from invariance of the rank under change of basis). Let us notice, *en passant*, that a criterium to distinguish a mixed state, *i.e.,* a way to detect entanglement, is the *quantum purity*, defined as

$$\mathrm{Tr}\rho^2 < 1, \quad \text{iff } \rho \text{ is mixed.} \tag{4.5}$$

We have seen that the notion of entanglement is closely related to the one of lack of information, uncertainty. It is not surprising, then, that an entropy is the best measure of quantum entanglement. Indeed, entropy measures how much information can we gain by executing an experiment on the system, or equivalently, the uncertainty in its outcome - see for instance [145]. Let us make a comment on what kind of uncertainty we refer to in the quantum case. Even when a complete description of a state is available, the good old indetermination principle enforces some amount of uncertainty in the outcome of an experiment. What we are interested in, is the *additional* uncertainty when the state is a mixture. In particular, if we know that the system lies in a given pure state, there are experiments which is not worth executing: for instance, measuring the z-component of the spin of the state $|\uparrow\rangle$. On the contrary, if the state is a mixture, there is no projector whose trace over ρ gives one, that is, no experiment whose outcome is absolutely certain. This is because there is a classical uncertainty about which state in the mixture is going to be pinned by the experiment.[1] It is the latter uncertainty that we are after. The entropy that measures it is the Von Neumann entropy of the density matrix ρ:

$$S_{\text{Von Neumann}}(\rho) = -\text{Tr}\rho \log \rho. \tag{4.6}$$

In particular, $S_{\text{Von Neumann}} = 0$ for a pure state. The Von Neumann entropy of a reduced density matrix, when the total state is pure, is exclusively due to entanglement, is called *entanglement entropy*, and is symmetric in the components of a bipartite system:

$$S_{\text{EE}}(\rho_A) = -\text{Tr}\rho_A \log \rho_A = -\text{Tr}\rho_B \log \rho_B = S_{\text{EE}}(\rho_B). \tag{4.7}$$

Von Neumann entropy, as well as its classical analogue, the *Shannon entropy*, enjoys many properties, and many other quantities related to it are of interest in quantum information theory. We will not need most of these things, therefore we refer to [142] for a more complete treatment of the subject. Here, we limit ourselves to two possible operative interpretations of entanglement entropy, both related to the amount of information stored in an entangled state. The first is a simple observation. From (4.4), we deduce that the entropy associated to N maximally entangled qubits is

$$S_{\text{EE}}\left({\rho_A}^{\otimes N}\right) = \log 2^N, \quad \rho_A = \text{Tr}_B \rho_{\text{EPR}}. \tag{4.8}$$

An information theoretic phrasing of this fact goes as follows: the exponential of S_{EE} counts the minimal number of auxiliary states we need to entangle to a system A in order to obtain ρ_A as the reduced density matrix of a pure state of the total system. The second property of entanglement entropy that we would like to highlight in the quantum information context is less trivial. It can be proven that $\exp S_{\text{EE}}$ is the minimal dimension of a Hilbert space $\mathcal{H}_{\text{min}}$, such that the density matrix ρ_A can be reliably compressed into an operator acting on $\mathcal{H}_{\text{min}}$. This is the content of the Schumacher's noiseless coding theorem - see [142] for a proof.

Entanglement entropy is uniquely fixed by a set of properties that a measure of entanglement should intuitively satisfy. By relaxing one of them, we arrive to the definition of a more general family of quantities, the *Rényi entropies*, which will be the main object of interest in the next two chapters. Let us first list the fundamental properties. This list can be found, in the classical case of Shannon entropy, in Rényi's original paper [146]. We report the quantum version spelled out in [147]. We consider the space $\mathcal{S}$ of positive operators ρ with $\text{Tr}\rho \in [0, 1]$. Then entanglement entropy is singled out by the following features:

[1] For simplicity, we have in mind a diagonal density matrix, which is a classical mixture of pure quantum states. This is actually general, given that density matrices are Hermitian.

i. **Continuity:** $S(\rho)$ is continuous in $\mathcal{S}$.

ii. **Unitary invariance:** $S(U\rho U^\dagger) = S(\rho)$.

ii. **Additivity:** $S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$.

iv. **Normalization:** $S(\mathbb{1}/2) = \log 2$.

v. **Arithmetic mean:** $S(\rho \oplus \sigma) = \frac{\mathrm{Tr}\rho}{\mathrm{Tr}(\rho\oplus\sigma)} S(\rho) + \frac{\mathrm{Tr}\sigma}{\mathrm{Tr}(\rho\oplus\sigma)} S(\sigma)$, if $\mathrm{Tr}(\rho \oplus \sigma) \leq 1$.

The solution is $S(\rho) = -\mathrm{Tr}(\rho \log \rho)/\mathrm{Tr}\rho$, and reduces to S_{EE} for unit normalized density matrices. Let us consider the last condition, and to fix ideas consider the case of diagonal ρ and σ, whose traces sum to one. Condition (v) says that two orthogonal subspaces of the Hilbert space contribute to the total uncertainty each via its own uncertainty, weighted by the probability of a measurement to collapse the system in one or the other subspace. This is the condition to be relaxed in order to obtain the Rényi entropies. Indeed, one might consider more general means than the arithmetic one. In the classsical case, given a continuous monotonic function g, a mean over a set of variables x_i weighted by probabilities p_i is defined in general as

$$\bar{x}_g = g^{-1}\left(\sum_i p_i g(x_i)\right). \tag{4.9}$$

We may therefore substitute:

v'. **Generalized mean** If $\mathrm{Tr}(\rho + \sigma) \leq 1$,

$$S(\rho \oplus \sigma) = g^{-1}\left(\frac{\mathrm{Tr}\rho}{\mathrm{Tr}(\rho \oplus \sigma)} g\left(S(\rho)\right) + \frac{\mathrm{Tr}\sigma}{\mathrm{Tr}(\rho \oplus \sigma)} g\left(S(\sigma)\right)\right). \tag{4.10}$$

It turns out that the most general mean compatible with postulates (i)-(iv) involves a one-parameter family of functions

$$g(x) = e^{(1-n)x}, \tag{4.11}$$

which in turns leads to the definition of the Rényi entropies:

$$S_n(\rho) = \frac{1}{1-n} \log \frac{\mathrm{Tr}\rho^n}{\mathrm{Tr}\rho}. \tag{4.12}$$

For diagonal density matrices, it is easy to make sense of this definition for all values of n. In general, for the sake of this work we will assume that it is possible to compute S_n for integer n and analytically continue the result. In this case we also obtain entanglement entropy as a limit:

$$\lim_{n\to 1} S_n = S_{\mathrm{EE}}. \tag{4.13}$$

Many of the properties listed above for entanglement entropy are valid for Rényi entropies as well. In particular, $S_n(A) = S_n(A^c)$ if the total state is pure, and also eq. (4.8) remains valid substituting S_n to S_{EE}. As we already commented in the introduction to the chapter - see references there - the Rényi entropies provide a useful way of detecting entanglement, and are often much easier to compute than entanglement entropy, which is then obtained via eq. (4.13) (the AdS/CFT context is an exception to this rule, and the last chapter of the present work is in fact devoted to some progress in this direction). Here, we would like to stress two interesting aspects of this family of quantities. On the theoretical side, Rényi entropies contain more information than entanglement entropy: knowledge of S_n for every

n is equivalent to knowledge of the eigenvalues of the reduced density matrix, the so-called *entanglement spectrum*. The latter is important, for instance, in detecting phases of matter from the groundstate wave function [134]. Furthermore, from the quantum information point of view, the entanglement spectrum offers a way of comparing states according to their entanglement: given a bipartite system globally described by a pure state ρ, one may ask if it it is possible to convert ρ to a different state σ using only local operations on the subsystems and classical communication between them (LOCC). The latter condition is required to forbid the creation of entanglement in the communication. It turns out that it is possible to do this only if the entanglement spectrum of the reduced density matrix σ_A majorizes[2] the one of ρ_A [148]. On the experimental side, the second Rényi is the same thing as the quantum purity defined in eq. (4.5), and the quantum purity has been recently measured in the laboratory in a many-body system [139]. These are sufficient reasons to be interested in the study of Rényi entropies.

As we anticipated, the Rényi entropies in a quantum field theory can be computed via the insertion of an extended operator in the path-integral, which in a CFT becomes a conformal defect. Before making this connection precise, let us dedicate the second part of this section to some remarks on entanglement in continuous systems, for which we loosely follow the outline given in Hartman's lecture notes [149].

4.1.1 Entanglement in quantum field theory and holography

In a many-body system, and in its continuous version, that is a QFT, it is often experimentally relevant and theoretically interesting to ask what is the entanglement between two regions of space. The subsystems A and B are then composed by the degrees of freedom residing in the two regions, which are separated by a co-dimension two surface in space-time, which is named the *entangling surface*, and we shall always denote Σ. In this case, we shall sometimes rename $B = A^c$. Here, we would like to make some comments about the dependence of entanglement entropy on the size and shape of the entangling surface. We shall also make similar remarks about Rényi entropies, although more will follow in the rest of the chapter.

Let us first consider the case of a gapped many-body system, a system defined on a lattice, for instance. We take the lattice size to be infinite, and consider a bipartition in which subsystem A is large but finite. If we had to randomly guess a wavefunction for the whole system, it would be very probable to entangle degrees of freedom in A with the (infinitely many) ones outside. This intuition can be made precise, and goes under the name of Page's theorem [150] - see also [151]. A typical state in the Hilbert space features maximal entanglement between the subsystems A and A^c. This means that $S_{\text{EE}}(\rho_A) \sim \log \mathcal{D}_A$, $\mathcal{D}_A$ being the dimension of $\mathcal{H}_A$ (cfr. with eq. (4.8)). If the Hilbert space of a single site of the lattice has dimension q, $\mathcal{D}_A = q^N$, N being the lattice sites in A. We obtain this way that the entanglement entropy scales like the volume of the subsystem, in a generic state.

$$S_{\text{EE}} \sim \text{Volume}, \quad \text{in a generic state.} \tag{4.14}$$

A randomly generated state, however, will be almost certainly a combination of excited states. Indeed, in the ground state of a gapped-system correlations die off exponentially,

[2] Recall that the spectrum of a matrix M majorizes the one of N if, once the eigenvalues λ_i^M and λ_i^N are labeled in decreasing order,

$$\sum_{i=1}^{k} \lambda_i^M \geq \sum_{i=1}^{k} \lambda_i^N, \quad \forall k.$$

and it is not expected that far away lattice sites are entangled at all. Only degrees of freedom close to each other, on opposite sides of the entangling surface, are going to be entangled. We see that ground states are special, in the sense that the entanglement entropy of a subsystem scales like the area in this case:[3]

$$S_{\text{EE}} \sim \text{Area}, \quad \text{in the ground state.} \tag{4.15}$$

Notice that both eq. (4.14) and eq. (4.15) are in fact valid for the Rényi entropies as well.

The situation may be different at a critical point, where long distance correlations are not exponentially suppressed. We would like to use the intuition gained so far to address the latter question directly in a quantum field theory. In practice, this amounts to take the lattice spacing to zero. As we do that, we meet some obstructions. The presence of degrees of freedom at shorter and shorter scales makes it impossible to literally split the Hilbert space of the two subsystem in a tensor product. The situation gets worse for gauge theories, see [153], but we'll ignore this issue here. We can still keep the cut-off, and ask how does entanglement behave when we make it smaller and smaller. Correlations of short distance degrees of freedom diverge, and so we reach the conclusion that, in any state, entanglement is dominated by UV physics close to the entangling surface. The same divergence will show up in the measures of entanglement: both entanglement entropy and the Rényi entropies are UV divergent, and independently of the perhaps complicated infrared structure of the finite part, the divergent contribution is local. We are going to phrase the following statements in terms of Rényi entropies. A pleasant way to organize the divergences requires choosing a diffeomorphism invariant cutoff ϵ_{UV} as opposed to a simple lattice spacing. For instance, in practical situations one could use a Pauli-Villar regulator, or dimensional regularization. As promised, we also restrict ourselves to scale invariant theories, which are anyway the focus of this work. Diffeomorphism invariance and locality imply that the divergences are local functionals of the geometric invariants associated with the entangling surface. In particular, from appendix 1.C we know that we can construct them from contractions of the extrinsic curvature K_{ab}. Notice that we are considering for now only a constant time slice: the entangling surface is a boundary, and its extrinsic curvature has no orthogonal index. Still, we know that in a pure state $S_n(A) = S_n(A^c)$, while the extrinsic curvature is odd under switching A and A^c. Therefore only even powers of K can appear. Of course, in a Lorentz invariant theory we can Wick rotate to Euclidean time, and then we should use the full spacetime curvature K^I_{ab}: contraction of indices leads to the same result. By scale invariance, a typical divergent term has the following schematic structure:[154]

$$S_n \supset c_m \frac{1}{\epsilon_{\text{UV}}^{d-2-2m}} \int_\Sigma K^{2m}. \tag{4.16}$$

We see that the most divergent term is proportional to the area of the surface. This general fact is what is called the area law. Notice that this is valid also in an excited state: the volume term (4.14) is due to IR, *i.e.* finite, correlations, and is not enhanced in the continuum limit. If R is some typical dimension of the entangling surface - which for simplicity we take to be connected - then we can rewrite the structure of divergences as

[3] This is a theorem in $2d$, where S_{EE} goes like a constant in a gapped system, and is the basis of the DMRG (density matrix renormalization group) technique, widely used to compute the ground state of $(1+1)$d systems [152].

follows:[154]

$$S_n = \begin{cases} \left(\frac{R}{\epsilon_{\rm UV}}\right)^{d-2} + \cdots + \frac{R}{\epsilon_{\rm UV}} + (-1)^{\frac{d-1}{2}} s_d^{(\Sigma)} + \frac{\epsilon_{\rm UV}}{R} + \cdots & \text{odd d} \\ \left(\frac{R}{\epsilon_{\rm UV}}\right)^{d-2} + \cdots + \left(\frac{R}{\epsilon_{\rm UV}}\right)^{2} + (-1)^{\frac{d-2}{2}} s_d^{(\Sigma)} \log\frac{R}{\epsilon_{\rm UV}} + \text{const} + \left(\frac{\epsilon_{\rm UV}}{R}\right)^2 + \cdots & \text{even d} \end{cases} \tag{4.17}$$

where for notational simplicity we have suppressed most of the coefficients c_m of eq. (4.16). We kept the finite piece in odd dimensions and the logarithm in even dimensions, for reasons that will be clear in a moment, and we used for them the notation of [154]. The presence of a logarithmic term in even dimension is easy to understand. The term with $m = (d-2)/2$ in eq. (4.16) has an $(\epsilon_{\rm UV})^0$. But the integral over "K^{d-2} " cannot be accompanied by a finite coefficient: the assumption of locality only holds for UV divergences. On the other hand, scale invariance allows a logarithmic divergence to appear only if a dimensionless geometric coefficient is available. The integral over "K^{d-2} " precisely provides such a coefficient. We shall have more to say about this divergence is section 4.5 below, where we will relate it to a conformal anomaly.

This description of the divergences is in fact not completely satisfactory. Most of the coefficients c_m in eq. (4.16) are scheme dependent. The area law finds a justification in the context of gravity: the entropy of a black hole does in fact obey an area law, where the divergence is automatically regulated by $\epsilon_{\rm UV}^{-1} \sim M_{\rm Plank}$. In the usual QFT framework, however, one is interested in quantities that do not depend on the way the theory is regulated, that is, on what the theory looks like at very short scales. It is easy to see that the only terms independent of a rescaling of the cut-off are the $s_d^{(\Sigma)}$ in eq. (4.17). More general shifts of the cutoff are not allowed in a scale invariant theory, but may arise, for instance, when comparing the values of $s_d^{(\Sigma)}$ at the top and at the bottom of an RG flow [154]. This endangers the universality of this coefficient in d odd, where it is especially important because, when the entangling surface is a sphere, it is conjecture to be monotonically decreasing [155, 156, 154]. The conjecture has been proven in $d = 3$ [157]. What makes $s_d^{(\Sigma)}$ well defined also in this case is its non local character. Any shift of the cut-off which pollutes the finite part is still going to depend on R in the way prescribed by eq. (4.17), so it is easy to subtract all the divergent pieces unambiguously due to their dependence on the size of the entangling surface, rather than on the form of the cutoff (see [154] for an extensive discussion). In what follows we shall only be interested in a fixed CFT, so this issue will never arise, and we shall consider finite parts as universal in computations that do not involve logarithms, as customary. It is worth mentioning a different issue, which could also modify the answer for $s_d^{(\Sigma)}$ in odd dimensions. A short distance cutoff might prevent from measuring the size of the entangling surface with precision greater then $\epsilon_{\rm UV}$. It is easy to see that a shift $R \to R + \alpha\epsilon_{\rm UV}$ would affect $s_d^{(\Sigma)}$ [157]. This happens for instance in measuring entanglement entropy across a sphere using a square lattice [158]. A good geometric cutoff should therefore allow to resolve the shape of the entangling surface with arbitrary good precision, a fact that we already assumed in writing eq. (4.16). Let us finally mention that a way to define the universal part of S_n in odd dimensions, which does not assume a class of preferred regulators, exists, and is based on the computation of the mutual (Renyi) information [159]. Mutual information between two regions A and B is defined as

$$I_n(A,B) = S_n(A) + S_n(B) - S_n(A \cup B). \tag{4.18}$$

It is easy to see that local divergences cancel in this expression, which is therefore UV finite. We refer to [159] for the relation between $I_n(A,B)$ and s_d^{Σ} in odd dimensions.

This bird's eye view of entanglement in quantum theories wouldn't be complete without mentioning how entanglement and Rényi entropies can be computed holographically. An introduction to gauge-gravity duality is beyond the scope of this review: we shall limit ourselves to sketching what it is about, and how it can be practically used in the present context. There are various formulations of the holographic principle, some stronger and vaguer than others. One form is as follows. Take any theory of quantum gravity which reduces to ordinary gravity and matter in the IR, and put it on a spacetime which is asymptotically Anti de Sitter in $d+1$ dimensions. This theory is dual to some conformal field theory on a conformally flat manifold. Here, *dual* means that for every observable in the gravity theory, there exists an observable in the CFT which has the same value. The dictionary between these quantities is only partially known. Much of it supports the following view of the relation between gravity and CFT: the field theory lives on the boundary of AdS. In fact, quantum gravity and quantum field theory are two descriptions of the same physical theory, and do not coexist, strictly speaking. Nevertheless, the intuition about a boundary and a bulk theory is usually of great help to intuition. In general, the duality is a weak/strong one, *i.e.,* it relates semiclassical gravity to strongly coupled conformal field theories, and vice versa. This fact makes the conjecture difficult to verify, but also extremely useful in investigating quantum gravity and strongly coupled quantum field theories. Today there are many concrete examples of dual pairs of theories, and an enormous amount of evidence which supports the conjecture. The prototypical example, the first concrete embodiment of the holographic principle, is the duality between Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ supersymmetric Yang-Mills in $d=4$ [3].

In the following, we shall use holography to gain access to the strongly coupled regime of some class of CFTs with many degrees of freedom. In other words, we shall only be concerned with (semy)classical gravity. We shall also only need two basic ingredients from the holographic dictionary. The first one is the prescription for computing correlation functions from gravity. Let us place the boundary of AdS - or whatever spacetime M which is asymptotically AdS - at $z=0$, for some coordinate z. The coordinates on the boundary are denoted as x^μ. For every field $\phi(x,z)$ in the gravitational theory, there is an operator $O(x)$ in the CFT which acts as a source for ϕ. In particular, the bulk metric is sourced by the stress-tensor. CFT correlation functions are computed by correlators of the gravitational fields in the limit in which insertions approach the boundary. More precisely, the generating functional of correlators of the low lying operators in the CFT can be written in terms of the boundary value of the light fields of the gravity theory:

$$Z_{\text{grav}}[\phi_0^i;\partial M] = \left\langle \exp\left(-\sum_i \int d^d x \phi_0^i O_i\right)\right\rangle_{\text{CFT on }\partial M}. \tag{4.19}$$

This is known as the GKPW dictionary [160, 161]. Let us explain the notation. The gravitational partition function on the left hand side depends on the prescribed boundary conditions for the fields, which is determined by ϕ_0^i up to a z dependent prefactor. Even including the metric among the fields, Z_{grav} still depends on the topology of the boundary, so we denoted explicitly the dependence on the boundary manifold ∂M. In chapter 5 we will need the one-point function of the stress-tensor. This is related to the expectation value of specific components of the metric. Since one-point functions of elementary fields obey the equations of motions of the theory, we will in practice only need to solve Einstein's equations. The exact procedure for extracting the relevant boundary value of the metric is due to [162] and will be recalled in section 5.2.2.

Of course, the second piece of dictionary needed in this work is related to entanglement. Even if Rényi entropies are going to be our main focus, here we would like to recall the famous formula for the computation of holographic entanglement entropy across a region A. If the bulk is described by classical Einstein gravity, [163]

$$S_A = \frac{\text{area}(\gamma_A)}{4G_N}, \tag{4.20}$$

where G_N is the Newton constant and γ_A is a codimension 2 spacelike extremal surface extending into the bulk, which is homologous to the entangling surface $\partial A \equiv \Sigma$:

$$\partial\gamma_A = \Sigma. \tag{4.21}$$

This prescription, known as the *Ryu-Takayanagi formula*, has been generalized in various ways [164, 165, 166, 167, 168]. A proof of eq. (4.20) was given in [169]. If we apply the Ryu-Takayanagi formula to a static black-hole in space time, and we choose the region A to be all of spacetime, we easily see that it reduces to the Bekenstein-Hawking area law for black-hole entropy:

$$S_{\text{B.H.}} = \frac{\text{area(horizon)}}{4G_N}. \tag{4.22}$$

Since this formula was the cornerstone of the early ideas about the holographic principle [170, 171], it is clear how important is the role of (thermal and entanglement) entropy in the context of holography. In fact, the interplay between entanglement and emergence of spacetime is deep and a very active field of research - see for instance [135, 136, 137, 138].

While in weakly coupled theories it is normally easier to compute the Rényi entropies and then extract entanglement entropy, the opposite is true in holography. Although the recent derivation [169] of the Ryu-Takayanagi formula presents a generalization to holographic Rényi entropies in principle, explicit holographic calculations of the Rényi entropy have been largely restricted to a spherical entangling surface [156, 172]. Recently, a formula with the same general character as the Ryu-Takayanagi formula (4.20) has been proposed [173]:

$$n^2\partial_n\left(\frac{n-1}{n}S_n\right) = \frac{\text{area(Cosmic Brane}_n)}{4G_N}. \tag{4.23}$$

In this case, this particular derivative of the Rényi entropies is computed by a bulk codimension two cosmic brane homologous to the entangling region, and whose tension is fixed so that the backreaction on the ambient geometry creates a conical deficit angle $\Delta\phi = 2\pi(n-1)/n$. The intuitive reason for this is related to the so called *replica trick*, which we will describe in section 4.2. We will not have more to say about this formula. The holographic part of this thesis, contained in chapter 5, will essentially make use of (a slight variation of) the specific prescription valid for spherical entangling surfaces [172]. We delay its detailed description until section 5.2.

This concludes the introductory material on entanglement. The rest of the chapter focuses on Rényi entropies in a generic CFT, and is based on [40].

4.2 Shape dependence of Rényi entropies in a CFT

In what follows, we build on conformal defect techniques to develop a field theoretic framework which allows for quantitative studies of the Rényi entropy. In particular, we shall use the basic properties of the displacement operator for perturbative calculations of the Rényi entropy when small modifications are made in the geometry of the entangling surface. Our focus on the displacement operator in the following arises because recently there has been a great deal of interest in the shape dependence of Rényi and entanglement entropies, *e.g.*, [174, 175, 176, 177, 178, 179, 180, 181]. First, we show below for a planar or spherical entangling surface that the second order variation of the Rényi entropy is fixed by the two-point function of the displacement operator. Given this framework, one of our key results is then to identify a simple relation between the coefficient defining the two-point function of the displacement operator and the conformal weight of the twist operator, which unifies a number of distinct conjectures with regards to the shape dependence of the Rényi entropy. In particular, in section 4.3, we apply this approach to evaluate the second order variation of the Rényi entropy for a spherical region. In the limit that the Rényi index goes to one, using the new relation, we precisely recover Mezei's conjecture for variations of the entanglement entropy [174] — see also [175]. Further the displacement operator can also be used to examine the variation of the Rényi entropy for small but 'singular' deformations of the entangling surface. In section 4.4, we consider the case of a planar entangling surface which undergoes a singular deformation to create a small conical singularity. With the previous relation, our result for the change in the Rényi entropy matches previous conjectures with regards to cusp and cone geometries in the limit that the entangling surface is almost smooth [176, 177, 178, 179]. In section 4.5, we focus on the Rényi entropy across an arbitrary entangling surface in four spacetime dimensions. We are able to relate two coefficients in the universal contribution to the Rényi entropy to the conformal weight of the twist operator and to the coefficient in the two-point function of the displacement operator, respectively. Our relation between the latter two quantities then yields the equality of these coefficients, as was conjectured for all four-dimensional CFTs in [180]. Hence, interestingly, the relation between the coefficient of the two-point function of the displacement and the conformal weight of the twist operator underlies a number of existing conjectures in the literature about the Rényi entropy.

However, at this point, we must stress that recent holographic calculations [52] imply that the proposed relation does *not* hold for general values of the Rényi index in four-dimensional holographic CFTs dual to Einstein gravity.[4] Hence it becomes an interesting question to ask for precisely which CFTs does such a constraint hold. For example, our calculations in Appendix 4.A confirm that it does in fact hold for free massless scalars in four dimensions.

In section 4.6, we pose the question whether it may be possible to define the twist operator through the operator product expansion generically available in the presence of defects, and we point out an intriguing universal feature of the fusion of the stress-tensor with this specific extended operator. We conclude with a discussion of our results and possible future directions in section 4.7. We devote Appendix 4.A to the example of a four-dimensional free scalar field, where the displacement operator can be given a precise identity in terms of the elementary field.

4.2.1 The replica trick and the twist operator

A central object for our discussion will be the twist operator which naturally arises in evaluating Rényi entropies in quantum field theory [182, 183, 184].[5] Therefore, let us start by recalling the definition of our main player. We begin with a generic QFT in flat d-dimensional spacetime. On a given time slice, the QFT is in a global state described by the density matrix ρ — in fact, shortly we will restrict our attention to the vacuum state. We consider the density matrix ρ_A obtained when the state is restricted to a particular region A, of which we recall the definition, together with the one of the Rényi entropies

[4] Of course, [52] appeared after [40], on which this chapter is based. As we shall see in the next chapter, the defect CFT framework allows to generalize the results of [52] to dimensions different than four.

[5] In two-dimensional CFTs, twist fields associated to branch points were first introduced in [185, 186].

associated to it:

$$\rho_A = \mathrm{Tr}_{A^c}(\rho)\,. \tag{4.24}$$

$$S_n = \frac{1}{1-n}\,\log \mathrm{Tr}(\rho_A^n). \tag{4.25}$$

Specifically for integer n (with $n > 1$), a path integral construction, which is widely known as the replica trick, allows us to evaluate the Rényi entropies for a QFT. An analytic continuation is then required to make contact with the entanglement entropy but we will have nothing to add about the conditions under which the continuation is reliable.

The replica trick begins by evaluating the reduced density matrix ρ_A in terms of a (Euclidean) path integral on $\mathbb{R}^d$ but with independent boundary conditions fixed over the region A as it is approached from above and below in Euclidean time, *e.g.,* with $t_{\mathrm{E}} \to 0^{\pm}$. The expression $\mathrm{Tr}(\rho_A^n)$ is then evaluated by extending the above to a path integral on a n-sheeted geometry [183, 184], where the consecutive sheets are sewn together on cuts running over A. Denoting the corresponding partition function as Z_n, we can write the Rényi entropy (4.25) as

$$S_n = \frac{1}{1-n}\,\log\frac{Z_n}{Z_1^n}\,, \tag{4.26}$$

where the denominator Z_1^n is introduced here to ensure the correct normalization, *i.e.,* $\mathrm{Tr}[\rho_A] = 1$. The partition function Z_n has an important symmetry. That is, even if in the above construction we chose to glue the copies together along the codimension-one submanifold A on the $t_{\mathrm{E}} = 0$ slice, the precise location of the cut between different sheets is meaningless — see for instance section 3.1 of [187]. Hence the only source of breaking of translational invariance on each sheet is at the location of the entangling surface, *i.e.,* the boundary of A. Since the modification is local in this sense, it can be reinterpreted as the insertion of a *twist operator* τ_n. In defining τ_n, the above construction is replaced by a path integral over n copies of the underlying QFT on a single copy of the flat space geometry. The twist operator is then defined as a codimension-two surface operator in this n-fold replicated QFT, which extends over the entangling surface and whose expectation value yields

$$\langle\tau_n\rangle \equiv \frac{Z_n}{Z_1^n} = e^{(1-n)S_n}. \tag{4.27}$$

Hence eq. (4.27) implies that τ_n opens a branch cut over the region A which then connects consecutive copies of the QFT in the n-fold replicated theory. Note that here and in the following, we omit the A dependence of τ_n to alleviate the notation.

In proceeding, we restrict our attention to the case where the QFT of interest is a conformal field theory and the state is simply the flat space vacuum state. Now let us take a closer look at the residual symmetry group in the presence of the twist operator. In doing so, we restrict ourselves to a very symmetric situation where we choose A to be half of the space. That is, we choose τ_n to lie on a flat $(d{-}2)$-dimensional plane, which we denote as Σ. For concreteness, we parametrize $\mathbb{R}^d$ with coordinates $(x^1,\dots,x^d)$, and we locate the twist operator at $x^1 = 0 = x^2$. In accordance with the notation used so far, we will denote directions parallel to Σ with Latin indices from the beginning of the alphabet $(a,\,b,\dots)$ and orthogonal directions with Latin indices from the middle of the alphabet $(i,\ j,\dots)$, while $\mu = (a,i)$. As usual, since a spherical entangling surface can be obtained from the planar one by means of a conformal transformation, the following applies equally well to the spherical case.

Let us choose the cut to lie along a half-plane in $\mathbb{R}^d$, *e.g.,* $x^1 < 0$ (and $x^2 = 0$), then a moment's thought is sufficient to realize that the gluing condition is preserved only if the same conformal transformation is applied to all the copies at the same time.[6] The rotations in the transverse plane, on the other hand, move the cut, which can be brought back to the original position through the symmetry of the partition function which we referred to above. This leads to a remark on the structure of the symmetry group. A rotation of an angle 2π has the net effect of shifting by one the labeling of the replicas: in a correlation function, an operator inserted in the i-th copy ends up in the $(i+1)$-th one. Therefore, the $u(1)$ algebra of orthogonal rotations exponentiates in the n-fold cover of the group $O(2)$. Up to this subtlety, we see that the symmetry group preserved by the twist operator is the same as the one preserved by a flat conformally invariant extended operator, *i.e.*, a conformal defect.

The twist operator is a conformal defect placed in the tensor product $(\text{CFT})^n$ rather than in the original conformal field theory. Therefore, it is especially interesting to consider the consequences of interactions among replicas, which distinguish this setup from a mere local modification of a CFT on $\mathbb{R}^d$: these are probed by correlation functions of operators belonging to different copies of the theory. Such correlators do not escape the defect CFT framework and in particular can be handled with the classical tool available in any conformal field theory: the existence of an OPE. Consider the defect OPE of a bulk scalar of scaling dimension Δ:

$$O(x^a, x^i) \sim b_{\widehat{O}_0}\, r^{\widehat{\Delta}_0-\Delta}\widehat{O}_0(x^a) + \dots, \qquad r \equiv |x^i|. \tag{4.28}$$

The meaning of the label 0 given to the defect operator will become clear in a moment. Let us stress that operators in our formulae will always be thought of as inserted on a single copy of the CFT and when present, sums over replicas will be written explicitly. Now consider a correlator of bulk primaries which belong to different factors of the n-fold replicated CFT. We can substitute to each of them the respective defect OPE, and since the latter converges inside correlation functions, the resulting sum over two-point functions of defect operators must reproduce the original correlator. In particular, we see that the expression on the right hand side of eq. (4.28) must retain the information about the copy in which the primary on the left hand side was inserted. This is possible thanks to the global structure of the symmetry group — that is, the fact that rotations around the defect are combined non-trivially with the $\mathbb{Z}_n$ replica symmetry. The rotational symmetry around an extended operator is a global symmetry from the point of view of the theory on the defect. As a consequence, defect operators carry a $u(1)$ quantum number s. In our case, this transverse spin is rational: $s = k/n$, k being an integer. We see that the defect OPE of a bulk scalar contains in general terms of the form

$$O(x^a, x^i) \sim b_{\widehat{O}_s}\, |x^i|^{\widehat{\Delta}_s-\Delta} e^{is\phi}\widehat{O}_s(x^a) + \dots \tag{4.29}$$

where $\phi \in [0, 2\pi n)$ is the angle in a plane orthogonal to the defect and provides the information about the replica on which the bulk primary has been inserted. In appendix 4.A, we shall see explicit examples of OPEs of the form (4.29) in the free scalar theory, and how they allow us to decompose correlation functions of bulk primaries placed in arbitrary positions.

As usual, the breaking of translational invariance in the directions transverse to the entangling surface gives rise to an operator of transverse spin $s = 1$, the displacement operator D^i which we normalize to correspond to an insertion of the total stress-tensor of

[6]In the density matrix language, this is rephrased in the statement that the transformation $U\rho_A U^{-1}$ is a symmetry of Z_n only if applied to the n factors of ρ_A appearing in the trace (4.25).

the replicated theory:

$$\sum_{\mathrm{m}=1}^{n} \partial_\mu T^{\mu a}_{(\mathrm{m})}(x^\nu) = \delta_\Sigma(x)\ \mathrm{D}^i(x^a), \tag{4.30}$$

where the index m runs over the replicas[7] and δ_Σ denotes the delta function in the transverse space with support on the Σ. The sum over replicas appears because, as mentioned, symmetry transformations should be applied to all the sheets in the same way, resulting in a sum over insertions of the stress-tensor. Recall that the displacement operator of a codimension two defect has scaling dimension $\Delta = d-1$:

$$\langle \mathrm{D}^i(x^a)\, \mathrm{D}^j(0)\rangle_n = C_{\mathrm{D}}(n)\ \frac{\delta^{ij}}{|x^a|^{2(d-1)}}\,. \tag{4.31}$$

Here and in the following the subscript n applied to expectation values implies the presence of the twist operator:

$$\langle O\rangle_n \equiv \frac{\langle \tau_n\, O\rangle}{\langle \tau_n\rangle}. \tag{4.32}$$

As described in section 1.5, a consequence of the Ward identity (4.30) is that a small deformation $\delta x^i(x^a)$ of the defect is obtained by integrating the displacement operator in the action. The first order variation under such a deformation can be written

$$\delta\,\langle X\rangle_n = -\int d^{d-2}x\, \delta x^i(x^a)\, \langle \mathrm{D}^i(x^a) X\rangle_n\,. \tag{4.33}$$

where X is an arbitrary product of local operators. Since defect primaries have no expectation value, the first order variation of the partition function (4.27) vanishes for a flat (or spherical) entangling surface. The second order variation is then related directly to C_{D}. Indeed, denoting the variation as $\epsilon\,\delta x^i$, we find

$$\frac{1}{\langle\tau_n\rangle}\frac{d^2}{d\epsilon^2}\,\langle\tau_n\rangle\bigg|_{\epsilon=0} = \int d^{d-2}x \int d^{d-2}x'\, \langle \mathrm{D}^i(x)\mathrm{D}^j(x')\rangle_n\, \delta x_i \delta x'_j. \tag{4.34}$$

The double integration will contain divergences which must be regulated. However, power-law divergences can be unambiguously tuned away, and finite or logarithmically divergent parts are universal well defined quantities, proportional to C_{D}. Eq. (4.34) shows very explicitly that the displacement operator governs the shape dependence of the Rényi entropy, which has been extensively studied in the recent literature *e.g.,* [174]–[181]. A key result of this chapter is unify a variety of conjectures related to this shape dependence in terms of a constraint on C_{D}, the coefficient defining the two-point function (4.31) of the displacement operator. In particular, these conjectures imply that the value of C_{D} is entirely determined by the one-point function of the stress tensor in presence of the defect, also called the conformal dimension of the twist operator. The latter, dubbed h_n, is defined by the leading singularity of the one-point function $\langle T_{\mu\nu}\rangle_n \equiv \langle T_{\mu\nu}\,\tau_n\rangle\,/\,\langle\tau_n\rangle$. For a planar conformal defect in Euclidean flat geometry, this leading singularity is easily identified as it is completely fixed by symmetry - recall eq. (1.79):

$$\langle T_{ab}\rangle_n = -\frac{h_n}{2\pi n}\frac{\delta_{ab}}{r^d}\,, \qquad \langle T_{ia}\rangle_n = 0\,, \qquad \langle T_{ij}\rangle_n = \frac{h_n}{2\pi n}\frac{(d-1)\,\delta_{ij} - d\, n_i n_j}{r^d}\,. \tag{4.35}$$

[7]We stress again that in general, our calculations will refer to bulk operators in a single copy of the replicated CFT. Hence $T^{\mu\nu}_{(\mathrm{m})}$ here denotes the stress tensor in the m'th copy of the CFT and the total stress tensor for the full theory would be given by $T^{\mu\nu}_{\mathrm{tot}} = \sum_{\mathrm{m}=1}^n T^{\mu\nu}_{(\mathrm{m})}$. However, in order to reduce clutter in expressions below, we will drop the subscript (m) but $T^{\mu\nu}$ still denotes the single-copy stress tensor. The total stress tensor will always be denoted as $T^{\mu\nu}_{\mathrm{tot}}$.

Here n_i is a unit normalized vector normal to the entangling surface and $r = |x^i|$ the transverse distance. The factor n in the denominator appears so that h_n is the coefficient in the one-point function for the total stress tensor (summed over all of the replicas), *e.g.,* as defined in [182]. In the following, we demonstrate that if, in a d-dimensional CFT, the values of $C_{\rm D}(n)$ and h_n are constrained to obey the following equality

$$C_{\rm D}(n) = d\,\Gamma\big(\tfrac{d+1}{2}\big)\,\left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1} h_n, \tag{4.36}$$

then the Rényi entropy satisfies a number of interesting properties, outlined below, with regards to shape dependence.

One immediate consequence of this relation is $C_{\rm D}(1) = 0$, which must hold since the defect disappears for $n = 1$. Further, if we analytically continue (4.36) to real n, we can consider the first order variation around $n = 1$:

$$\partial_n C_{\rm D}|_{n=1} = d\,\Gamma\big(\tfrac{d+1}{2}\big)\,\left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1} \partial_n h_n|_{n=1} = \frac{2\pi^2}{d+1} C_T, \tag{4.37}$$

where we used the relation

$$\partial_n h|_{n=1} = 2\pi^{\frac{d}{2}+1}\frac{\Gamma(\frac{d}{2})}{\Gamma(d+2)} C_T, \tag{4.38}$$

first observed in [172] for holographic theories and then proven in [182] for general CFTs. Implicity, the recent results of [188] imply that eq. (4.37) holds for generic CFTs. Hence in the vicinity of $n = 1$, the proposed relation (4.36) is a constraint that holds for general CFTs.

Moving away from $n = 1$, the constraint in eq. (4.36) produces an number of interesting properties for the shape dependence of the Rényi entropy, which have appeared previously in the literature as conjectures:

- In section 4.3, we calculate the second order correction to the Rényi entropy induced by small perturbations of a perfect sphere. In the limit $n \to 1$, the formula (4.37) reproduces the variation of the entanglement entropy across a deformed sphere conjectured in [174] for arbitrary dimensions, which was recently proven in [188].

- Eq. (4.37) also allows one to compute the universal contribution to the Rényi entropy for an entangling surface with a (hyper)conical singularity of opening angle Ω. The leading coefficient in an expansion around the smooth entangling surface has been conjectured to be related the conformal weight h_n [179] — see also [176, 177, 178]. In section 4.4, we prove the equivalence of that conjecture and formula (4.37).

- With $d = 4$, eq. (4.36) implies the equivalence of the coefficients $f_b(n)$ and $f_c(n)$ in the universal part of the four-dimensional Rényi entropy for general n, as discussed in [189, 180]. This is demonstrated in section 4.5 by relating f_b to $C_{\rm D}$, and f_c to h_n. However, we re-iterate that [52] recently showed that the proposed equivalence $f_b(n) = f_c(n)$ does not hold for four-dimensional holographic CFTs dual to Einstein gravity.

The latter result demonstrates that eq. (4.36) is *not* a universal relation that holds in all CFTs (for general values of n). However, it is then interesting to ask for precisely which CFTs does such a constraint hold. It seems that free field theories are a good candidate for such a theory. Certainly, the results of [178, 190, 191] imply that eq. (4.36) holds for free scalars and fermions in three dimensions. Further, our calculations in Appendix 4.A confirm that it also holds for free massles scalars in four dimensions. We hope to return to this question in future work [192].

4.3 Rényi and entanglement entropy across a deformed sphere

In this section, we calculate the second order correction to the Rényi entropy induced by small perturbations of a perfect sphere. In the limit $n \to 1$, our findings agree with the holographic results previously found in [174], but are of course valid for a generic CFT.

Starting from (4.34), we note that upon slightly deforming a spherical entangling surface with $\epsilon\, \delta x^i$, the leading correction to S_n appears at second order and is given by

$$\delta S_n = \frac{\epsilon^2}{2(1-n)} \int_\Sigma \int_{\Sigma'} \langle \mathrm{D}^i(x) \mathrm{D}^j(x') \rangle_n \, \delta x_i \, \delta x'_j + \mathcal{O}(\delta x^3) \; . \tag{4.39}$$

Here, the two integrals run over the original spherical entangling surface of radius R. We will restrict the deformation to the $t_{\mathrm{E}} = 0$ time slice and denote $\delta \vec{x} = f(x)\, \hat{r}$ where $\hat{r}$ is a unit vector in the radial direction. The relevant correlator (4.31) then beomes

$$\langle \mathrm{D}^r(x)\, \mathrm{D}^r(x') \rangle_n = \frac{C_{\mathrm{D}}}{(x-x')^{2(d-1)}} = \frac{C_{\mathrm{D}}}{(2R^2)^{d-1}} \frac{1}{(1-\cos\gamma)^{d-1}} \; , \tag{4.40}$$

with γ being the angle between $x, x' \in S^{d-2}$.

Let us now represent the two-point correlator (4.40) in the basis of spherical harmonics on S^N $(N \equiv d-2)$

$$Y_{\ell_N \ldots \ell_1}(\theta_N \ldots \theta_1) = \frac{1}{\sqrt{2\pi}} e^{i \ell_1 \theta_1} \prod_{n=2}^{N} {}_n c_{\ell_n}^{\ell_{n-1}} \, (\sin\theta_n)^{\frac{2-n}{2}} \, P_{\ell_n + \frac{n-2}{2}}^{-\left(\ell_{n-1} + \frac{n-2}{2}\right)}(\cos\theta_n) \tag{4.41}$$

where $\ell_N \geq \ell_{N-1} \geq \ldots \geq |\ell_1|$ are integers and

$$ds_N^2 = d\theta_N^2 + \sin^2\theta_N ds_{N-1}^2 \; , \quad ds_1^2 = d\theta_1^2 \; , \tag{4.42}$$

$$\sqrt{g} = \sin^{N-1}\theta_N \sin^{N-2}\theta_{N-1} \cdots \sin\theta_2 \; , \tag{4.43}$$

$$P_\nu^{-\mu}(x) = \frac{1}{\Gamma(1+\mu)} \left(\frac{1-x}{1+x} \right)^{\mu/2} {}_2F_1\left(-\nu \; , \; \nu+1 \; ; \; 1+\mu \; ; \; \frac{1-x}{2} \right) \quad , \tag{4.44}$$

$${}_n c_L^l = \left[\frac{2L+n-1}{2} \frac{(L+l+n-2)!}{(L-l)!} \right]^{1/2} \quad . \tag{4.45}$$

For simplicity, we assume that one of the points is sitting at the north pole, in which case only spherical harmonics with $\ell_{N-1} = \ell_{N-2} = \ldots = \ell_1 = 0$ contribute

$$Y_{\ell_N 0 \ldots 0}(\theta_N) = \sqrt{\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}} \, {}_N c_{\ell_N}^0 \, (\sin\theta_N)^{\frac{2-N}{2}} \, P_{\ell_N + \frac{N-2}{2}}^{-\frac{N-2}{2}}(\cos\theta_N) \; . \tag{4.46}$$

Hence, by assumption $\gamma = \theta_N$ in (4.40), and the following identity holds

$$\begin{aligned} \langle \mathrm{D}(x) \mathrm{D}(x') \rangle_n &= \frac{C_{\mathrm{D}}}{(2R^2)^{d-1}} \sum_{\ell_N} A_{\ell_N} Y_{\ell_N 0 \ldots 0}(\gamma) \; , \\ A_{\ell_N} &= \sqrt{\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}} \, {}_N c_{\ell_N}^0 \int_{-1}^{1} dz \, \frac{(1-z^2)^{\frac{N-2}{4}}}{(1-z)^{N+1}} \, P_{\ell_N + \frac{N-2}{2}}^{-\frac{N-2}{2}}(z) \end{aligned} \tag{4.47}$$

where we introduced a new variable $z = \cos\gamma$.

The above integral diverges at $z = 1$. This is not surprising given that the coefficients A_{ℓ_N} correspond to a spherical harmonic representation of a singular function (4.40). To regulate these coefficients let us modify the power of $(1 - \cos\gamma)$ in (4.40) by introducing a new parameter α such that A_{ℓ_N} takes the form

$$A_{\ell_N} = \frac{\pi^{\frac{N}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N+1}{2}}\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)} \lim_{\alpha\to 0}\int_0^1 dy\, y^{\alpha-\frac{N+4}{2}}\ {}_2F_1\Big(-\ell_N - \frac{N}{2}+1\ ,\ \ell_N + \frac{N}{2}\ ;\ \frac{N}{2}\ ;\ y\Big)\ . \quad (4.48)$$

where $y = (1 - z)/2$ and we used (4.45) to express the associated Legendre polynomial in terms of the hypergeometric function. Now the integral can be readily evaluated assuming that α is large enough[8]

$$A_{\ell_N} = \frac{\pi^{\frac{N}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N+1}{2}}\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)} \lim_{\alpha\to 0} \frac{{}_3F_2\Big(\alpha-\frac{N}{2}-1\ ,\ -\ell_N-\frac{N}{2}+1\ ,\ \ell_N+\frac{N}{2}\ ;\ \alpha-\frac{N}{2}\ ,\ \frac{N}{2}\ ;\ 1\Big)}{\alpha-\frac{N}{2}-1} \quad (4.49)$$

For odd N (odd d) the limit $\alpha \to 0$ is finite. However, it diverges for even N (even d). Therefore we analyze these cases separately.

Odd d

For odd d, we have

$$\begin{aligned} A_{\ell_N} &= -\frac{\pi^{\frac{N}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N-1}{2}}\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)} \frac{{}_3F_2\Big(-\frac{N}{2}-1\ ,\ -\ell_N-\frac{N}{2}+1\ ,\ \ell_N+\frac{N}{2}\ ;\ -\frac{N}{2}\ ,\ \frac{N}{2}\ ;\ 1\Big)}{N+2} \\ &= (-1)^{\frac{N-1}{2}} \frac{\pi^{\frac{N+4}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N-3}{2}}\,N(N+2)\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)\Gamma(N+1)} \prod_{k=1,\dots,N+2}(\ell_N+k-2) \end{aligned} \quad (4.50)$$

Using now the addition theorem for spherical harmonics

$$Y_{\ell_N 0\dots 0}(\gamma) = \frac{1}{{}_N c^0_{\ell_N}}\sqrt{\frac{(4\pi)^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)}{2}} \sum_{\ell_{N-1},\dots,\ell_1} Y^*_{\ell_N\dots\ell_1}(x)Y_{\ell_N\dots\ell_1}(x')\ , \quad (4.51)$$

we obtain from (4.47)

$$\begin{aligned} \langle \mathrm{D}(x)\mathrm{D}(x')\rangle &= C_{\mathrm{D}}\,\frac{(-1)^{\frac{N-1}{2}}\,\pi^{\frac{N+2}{2}}}{2(2R^2)^{N+1}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)} \\ &\times \sum_{\ell_N,\dots,\ell_1} Y^*_{\ell_N\dots\ell_1}(x)Y_{\ell_N\dots\ell_1}(x') \prod_{k=1,\dots,N+2}(\ell_N+k-2)\ . \end{aligned} \quad (4.52)$$

Substituting this result into (4.39), yields

$$\delta S_n = \epsilon^2\,\frac{C_{\mathrm{D}}}{(n-1)}\,\frac{(-1)^{\frac{d-1}{2}}\pi^{\frac{d}{2}}}{2^{d+1}\,\Gamma(d-1)\Gamma\left(\frac{d}{2}+1\right)} \sum_{\ell_N,\dots,\ell_1}|a_{\ell_N\dots\ell_1}|^2 \prod_{k=1,\dots,d}(\ell_N+k-2)+\mathcal{O}(\epsilon^3)\ , \quad (4.53)$$

where $a_{\ell_N\dots\ell_1}$ are the coefficients of $f(x)$ in a spherical harmonics representation. This result agrees with [174] for any odd d provided that (4.37) holds.

[8] As usual, small values of α are treated by analytic continuation.

Even d

For even d, the limit $\alpha \to 0$ in (4.49) is singular due to logarithmic divergence. To extract the numerical coefficient of this divergence, we expand the integrand in (4.47) around $z = 1$ and keep only the logarithmically divergent term:

$$\frac{(1-z^2)^{\frac{N-2}{4}}}{(1-z)^{N+1}}\, P^{-\frac{N-2}{2}}_{\ell_N+\frac{N-2}{2}}(z) = \frac{(-1)^{\frac{N}{2}} \prod_{k=1,\ldots,N+2}(\ell_N+k-2)}{2^{\frac{N+2}{2}}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)}\,\frac{1}{z-1} + \ldots\,, \tag{4.54}$$

The ellipsis denotes terms which do not generate logarithms upon integration. Hence,

$$A_{\ell_N} = (-1)^{\frac{N+2}{2}} \sqrt{\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}}\; {}_N c^0_{\ell_N}\, \frac{\prod_{k=1,\ldots,N+2}(\ell_N+k-2)}{2^{\frac{N}{2}}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)} \log(R/\delta) + \ldots\,, \tag{4.55}$$

with $\delta = R\cdot\delta\gamma$ being the short-distance cut-off. Using now (4.51), we obtain

$$\begin{aligned}\langle \mathrm{D}(x)\mathrm{D}(x')\rangle_n &= \log(R/\delta)\, C_{\mathrm{D}}\, \frac{(-1)^{\frac{N+2}{2}}\,\pi^{\frac{N}{2}}}{(2R^2)^{N+1}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)} \\ &\times \sum_{\ell_N,\ldots,\ell_1} Y^*_{\ell_N\ldots\ell_1}(x) Y_{\ell_N\ldots\ell_1}(x') \prod_{k=1,\ldots,N+2} (\ell_N+k-2) + \ldots\,. \end{aligned} \tag{4.56}$$

Substituting this result into (4.39), yields

$$\begin{aligned}\delta S_n &= \epsilon^2\, \frac{C_{\mathrm{D}}}{(n-1)}\, \frac{(-\pi)^{\frac{d-2}{2}}}{2^d\,\Gamma(d-1)\Gamma\left(\frac{d}{2}+1\right)} \log(R/\delta) \\ &\times \sum_{\ell_N,\ldots,\ell_1} |a_{\ell_N\ldots\ell_1}|^2 \prod_{k=1,\ldots,d} (\ell_N+k-2) + \ldots\,, \end{aligned} \tag{4.57}$$

where $a_{\ell_N\ldots\ell_1}$ are coefficients of $f(x)$ in a spherical harmonics representation. Combined with (4.37), this result is again in full agreement with [174]. Let us notice that the cutoff (in)dependence of eqs. (4.53) and (4.57) precisely coincides with the one proposed in subsection 4.1.1 - see in particular eq. (4.17). We see that the general considerations invoked there are backed by explicit computations, in the case of small deformations away from a spherical geometry. We also stress that the result only uses conformal invariance, and is in fact independent from the information theoretical interpretation of the divergences. In fact, the logarithmic divergence in even dimensions has a simple interpretation from the point of view of the defect CFT: it is the effect of a conformal anomaly, *i.e.,* of contributions to the trace of the stress tensor localized on the position of the defect. We shall come back to this in section 4.5. Finally, one might ask if we imposed any restriction to our short distance cutoff δ, to make it coincide with the geometric one which we called ϵ_{UV} in subsection 4.1.1. In fact, implicitly we did: our δ is a cut-off *on the entangling surface* whose geometry - the radius R and the coefficients $a_{\ell_N\ldots\ell_1}$ - are supposed to be known exactly.

4.4 The cone conjecture

In this section, we consider the relation of the proposed constraint (4.36) to various conjectures about the universal contribution to the Rényi entropy coming from singular deformations of entangling surfaces. In particular, [176, 177] proposed a conjecture for the universal corner contribution to the entanglement entropy in three-dimensional CFTs, and this conjecture was then extended to Rényi entropy in [178]. Finally, the discussion was extended to higher dimensions in [179]. In order to introduce the claim of these conjectures, we consider a deformation of a flat entangling surface which consists in creating a conical singularity. The three- and four-dimensional cases are shown in figure 1 of ref. [179]. The universal contribution to the Rényi (and consequently the entanglement) entropy is affected by this modification. In particular, if the twist operator is smooth, the universal contribution would be logarithmically divergent in even dimensions and constant (*i.e.*, regulator independent) in odd dimensions. When a conical singularity is present an additional logarithm emerges and the universal contribution to the Rényi entropy takes the form

$$S_n^{\text{univ}}(A) = \begin{cases} (-1)^{\frac{d-1}{2}}\, a_n^{(d)}(\Omega)\, \log(\ell/\delta) & d \text{ odd} \\ (-1)^{\frac{d-2}{2}}\, a_n^{(d)}(\Omega)\, \log^2(\ell/\delta) & d \text{ even} \end{cases} \tag{4.58}$$

Here Ω is the opening angle of the cone, varying in the interval $[0, \frac{\pi}{2}]$ and approaching $\frac{\pi}{2}$ in the limit of smooth surface.[9] The function $a_n^{(d)}$ is the universal contribution to the Rényi entropy and depends on the angle Ω only. Further ℓ and δ are the IR and UV regulators, respectively. As usual, the former can be thought of as a (macroscopic) length scale characterizing the geometry of the entangling region A (*i.e.*, the region enclosed by the twist operator), whereas the latter can be taken to be a short-distance cut-off originating from the infinite number of short-distance correlations in proximity of the twist-operator. The cusp conjecture, in the most general formulation of [179], states that, for an arbitrary conformal field theory, the leading contribution to $a_n^{(d)}$ for $\Omega \to \frac{\pi}{2}$ is controlled by the constant h_n introduced in (4.35). Explicitly,

$$a_n^{(d)}(\Omega) \overset{\Omega\to\pi/2}{\sim} 4\,\sigma_n^{(d)}(\Omega-\tfrac{\pi}{2})^2\,, \quad \sigma_n^{(d)} = \frac{h_n}{n(n-1)}\,\frac{(d-1)(d-2)\,\pi^{\frac{d-4}{2}}\,\Gamma\left[\frac{d-1}{2}\right]^2}{16\,\Gamma[d/2]^3} \times \begin{cases} \pi & d \text{ odd}\,, \\ 1 & d \text{ even}\,. \end{cases} \tag{4.59}$$

Restricting to the case $n = 1$ and using (4.38), one finds the following relation between the small angle contribution to the entanglement entropy and the central charge C_T of a CFT

$$\sigma_1^{(d)} \equiv \sigma^{(d)} = C_T\,\frac{\pi^{d-1}(d-1)(d-2)\Gamma[\frac{d-1}{2}]^2}{8\,\Gamma[d/2]^2\,\Gamma[d+2]} \times \begin{cases} \pi & d \text{ odd}\,, \\ 1 & d \text{ even}\,. \end{cases} \tag{4.60}$$

In the following, we will apply the theoretical framework introduced in section 4.2 to this particular deformation and find a connection between σ_n and C_{D}. This allows us to prove the equivalence of the cusp conjecture and eq. (4.36).

4.4.1 Conical deformation from the displacement operator

One of the appealing features of the displacement operator is that equation (4.34) is valid for any kind of deformation of the defect, regardless of whether or not it is smooth. It is then clear that the response (4.58) of the Rényi entropy to a conical singularity in the limit $\Omega \to \frac{\pi}{2}$ can be related to the two-point function of the displacement operator (4.34) integrated over a planar defect with the appropriate profile. In particular combining (4.27) and (4.59), we obtain

$$\frac{1}{2}\Sigma^{(d)} \equiv \frac{1}{2}\,\frac{1}{\langle\tau_n\rangle}\frac{d^2}{d\epsilon^2}\,\langle\tau_n\rangle\Big|_{\epsilon=0} = 4(n-1)\,\sigma_n^{(d)} \times \begin{cases} (-1)^{\frac{d+1}{2}}\,\log(\ell/\delta) & d \text{ odd} \\ (-1)^{\frac{d}{2}}\,\log^2(\ell/\delta) & d \text{ even} \end{cases} \tag{4.61}$$

[9] The angle Ω actually varies over the full range $[0, \pi]$, but, since the Rényi entropy evaluated for a pure state is equal for the region A or for its complement $\bar{A}$, the function $a_n^{(d)}$ is symmetric for reflections with respect to $\Omega = \frac{\pi}{2}$, i.e. $a_n^{(d)}(\Omega) = a_n^{(d)}(\pi - \Omega)$ and we can consistently focus on the interval $[0, \frac{\pi}{2}]$

where the first equality is just the definition of $\Sigma^{(d)}$. In the following, we will compute $\Sigma^{(d)}$ in terms of $C_{\rm D}$ using (4.34). Then, exploiting the conjectured relation (4.36), we will reproduce the cusp conjecture (4.59).

Consider a planar defect, parametrized by parallel coordinates x^a with $a = 3, \dots, d$, and its deformation into a configuration with a conical singularity at the origin. The two coordinates for the orthogonal directions are x^i with $i = 1, 2$. To deform the plane into a cone, we introduce spherical coordinates $\{r, \theta^1, \dots, \theta^{d-3}\}$ in the directions parallel to the entangling surface and we consider a variation $\epsilon\, \delta x^i$ in the direction 2 proportional to the radius r, *i.e.*,

$$\delta x^i = \delta^i_2\, r\,. \tag{4.62}$$

Plugging this expression into (4.34) combined with (4.31) and using the symmetries of the problem to perform the angular integrations, we are left with

$$\Sigma^{(d)} = C_{\rm D}\, \Omega_{d-3}\Omega_{d-4} \int dr_1\, dr_2 \int_0^{2\pi} d\theta_{12}\, \frac{r_1^{d-2} r_2^{d-2} \sin^{d-4}\theta_{12}}{(r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta_{12})^{d-1}}\,. \tag{4.63}$$

where θ_{12} is the angle described by the position of the two displacement operators in the plane defined by them and the origin. Further $\Omega_m = 2\pi^{\frac{m+1}{2}}/\Gamma(\frac{m+1}{2})$ is the volume of a unit m-sphere. The integration over θ_{12} yields

$$\begin{aligned}\Sigma^{(d)} =\ & C_{\rm D} \frac{2^{d-3}\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-1)} \\ & \times \int dr_1\, dr_2 \left[\left|r_1^2 - r_2^2\right|^{-d-1} r_1^{d-2} r_2^{d-2} \Big((d-2) r_1^4 + 2d\, r_1^2 r_2^2 + (d-2) r_2^4 \Big) \right]\end{aligned} \tag{4.64}$$

One has to be particularly careful in the integration over r_1 and r_2 since we expect a singularity along the line $r_1 = r_2$. Therefore it is useful to note the symmetry of the integral under the exchange $r_1 \leftrightarrow r_2$ and restrict the integration contour to the region $r_1 > r_2$. We then regulate the divergences for $r_1 \to r_2$ and for $r_1, r_2 \to 0$ with a UV cut-off δ, and the divergence for $r_1, r_2 \to \infty$ with an IR cut-off ℓ. Introducing the variables $x = r_1 + r_2$ and $y = r_1 - r_2$, the integral takes the form

$$\begin{aligned}\Sigma^{(d)} =\ & C_{\rm D} \frac{2^{3-2d}\pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \int_\delta^\ell dx \\ & \times \int_\delta^x dy \left[\left(x^2 - y^2\right)^{d-2} (xy)^{-d-1} \left((d-1)x^4 + 2(d-3)x^2y^2 + (d-1)y^4 \right) \right]\end{aligned} \tag{4.65}$$

An additional change of variables $w = (y/x)^2$ yields

$$\begin{aligned}\Sigma^{(d)} =\ & C_{\rm D} \frac{2^{2-2d}\pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \\ & \times \int_\delta^\ell \frac{dx}{x} \int_{\left(\frac{\delta}{x}\right)^2}^1 dw \left[(1-w)^{d-2}\, w^{-1-\frac{d}{2}} \left((d-1)w^2 + 2(d-3)w + d - 1 \right) \right]\end{aligned} \tag{4.66}$$

Since the treatment of this integral differs substantially in even and odd dimensions, it is convenient to analyze the two cases separately.

Even dimension

It is useful to note that, for integer d, the binomial $(1-w)^{d-2}$ can be converted in a finite sum over powers of w. Furthermore, if d is even also the exponent of $w^{1-d/2}$ is an integer, which implies that the integral over w contains a first logarithmic divergence for small w. We focus on that contribution and we perform the first integration, which yields

$$\Sigma^{(d)}_{\text{even}} = C_{\text{D}} \frac{(-1)^{\frac{d}{2}} 2^{5-2d} \pi^{d-2} \Gamma\left(d\right)}{d(d-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)^3} \int_\delta^\ell \frac{dx}{x} \log\frac{x}{\delta} + \cdots \tag{4.67}$$

where the missing terms contain power-law divergences. The last integration can be trivially carried out and the final result is

$$\Sigma^{(d)}_{\text{even}} = C_{\text{D}} \frac{(-1)^{\frac{d}{2}} 2^{-d} \pi^{d-\frac{5}{2}} d\, \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}+1\right)^2} \log^2(\ell/\delta) + \cdots . \tag{4.68}$$

Comparing this result with eqs. (4.60) and (4.61), we find perfect agreement when using (4.36) for C_{D}.

One aspect of the computation deserves a comment: Each of the integrations in eq. (4.66) contribute one of the logarithmic factors to the final expression (4.68). We can see then that one of the logarithmic singularities arises from $x \sim 0$, which corresponds to the region near the tip of the cone (since $x = r_1 + r_2$). Further, the second comes from $w \sim 0$, which corresponds to the collision of the two displacement operators (since $w \sim r_1 - r_2$). Implicitly then, the latter appears everywhere along the entangling surface and is sensitive to the geometry far from the tip of the cone. Of course, this fits in nicely with the lore that in even dimensions, the Rényi entropy contains a (universal) logarithmic factor that is geometric in nature. In a certain sense then, the presence of the cone is completely encoded in the logarithm coming from the integration over x in eq. (4.66), while the second logarithm is sensitive to the smooth geometry far from the tip of the cone and is largely unaware of this singular feature. We also note that S_n may also contain contributions with a single logarithmic factor but these are no longer universal in the presence of the conical singularity [193], *i.e.,* they will be modified when the cut-off changed because of the logarithm-squared term. As we shall see below, similar comments apply for odd dimensions as well. However, the 'universal' factor coming from the w integration is simply a constant (independent of δ) and also receives contributions from configurations in which the two displacement operators are separated by a finite distance.

Odd dimension

In odd dimensions, it is still true that the binomial $(1-w)^{d-2}$ can be expanded as a finite sum but $1-\frac{d}{2}$ is not an integer anymore. Hence (4.66) becomes an integral of the form

$$\begin{aligned} \Sigma^{(d)}_{\text{odd}} = C_{\text{D}} & \frac{2^{2-2d}\pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \int_\delta^\ell \frac{dx}{x} \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^k \\ & \times \int_{\left(\frac{\delta}{x}\right)^{\frac{1}{2}}}^1 dw \left((d-1)w^{k-1-\frac{d}{2}} + 2(d-3)w^{k-\frac{d}{2}} + (d-1)w^{k+1-\frac{d}{2}}\right) \end{aligned} \tag{4.69}$$

For odd d, all the exponents in the last bracket are half-integers, and the integration over w only leads to power-like divergences. The only logarithmic term comes from the integration over x, combined with the finite part of the integration over w, *i.e.,*

$$\Sigma^{(d)}_{\rm odd} = C_{\rm D}\frac{2^{2-2d}\pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)}\log(\ell/\delta) \tag{4.70}$$

$$\times\sum_{k=0}^{d-2}\binom{d-2}{k}(-1)^k\left(\frac{d-1}{k-\frac{d}{2}}+2\frac{d-3}{k+1-\frac{d}{2}}+\frac{d-1}{k+2-\frac{d}{2}}\right)+\cdots\,. \tag{4.71}$$

Performing the finite sums, we find

$$\Sigma^{(d)}_{\rm odd} = C_{\rm D}\frac{(-1)^{\frac{d+1}{2}}2^{-d}\pi^{d-\frac{3}{2}}d\,\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}+1\right)^2}\log(\ell/\delta)+\cdots\,. \tag{4.72}$$

Again, substituting for $C_{\rm D}$ using (4.36) produces precise agreement with eqs. (4.60) and (4.61).

4.4.2 Wilson lines in supersymmetric theories and entanglement in $d = 3$

The relation between the expectation value of the stress tensor and the two-point function of the displacement operator has been explored, in fact, at least in one other example of a defect CFT, *i.e.,* for Wilson lines [194]. In that context, $C_{\rm D}$ is better known as the Bremsstrahlung function. Indeed, a sudden acceleration of a charged source creates a cusp in the Wilson line that describes its trajectory, and it can be shown that the coefficient of the two-point function of the displacement operator measures the energy emitted in the process [195]. The precise relation between the two quantities is

$$C_D^{WL} = 12\,B\,, \tag{4.73}$$

where B is the Bremsstrahlung function. The authors of [194] observed that the ratio between B and h (the conformal dimension of the Wilson line) is theory dependent. However, a restricted form of universality is valid within a certain class of conformal gauge theories, whose Bremsstrahlung function is related to the one-point function of the stress tensor through a coefficient that only depends on the dimension of spacetime. This class includes theories with $\mathcal{N} = 4$ [194] and four-dimensional $\mathcal{N} = 2$ [196, 197] supersymmetry. In particular, in three dimensions, the general formula conjectured in [194] yields

$$C_D^{WL} = 24\,h^{WL}, \tag{4.74}$$

where h^{WL} is the constant entering the one-point function of the stress-tensor in the presence of a Wilson line.

Now the three-dimensional case is especially interesting for us, because twist operators become one-dimensional line operators as well. Furthermore, if we consider holographic CFTs, the calculation of the Wilson line [198, 199] and the Ryu-Takayanagi prescription [163, 200] for holographic entanglement entropy both reduce to evaluating the area of extremal surfaces anchored on the AdS boundary. The only difference in the two calculations is the overall factor multiplying the extremal area in evaluating the final physical quantity, but this constant factor will cancel out in the ratio between $C_{\rm D}$ and h. Hence for theories which possess a holographic dual and belong to the class for which (4.74) is valid, *e.g.,* ABJM theory [201], the relation between $\partial_n C_{\rm D}|_{n=1}$ and $\partial_n h_n|_{n=1}$ has to coincide with (4.74) — at strong coupling. Hence it is a nontrivial check that, indeed, formula (4.36) reduces to (4.74) for $d = 3$.

4.5 Rényi entropy and anomalies in 4d Defect CFTs

In any even number of dimensions, the universal contribution to the Rényi entropy (4.25) depends only on the shape of the spatial region A through local geometric quantities. In four dimensions, in particular, when the theory is conformal, Weyl invariance fixes the universal contribution up to three functions of n. If we denote by ℓ a characteristic length scale of the entangling surface Σ, then the Renyi entropy takes the form[10]

$$S_n = \left(-\frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma - \frac{f_b(n)}{2\pi} \int_\Sigma \tilde{K}^i_{ab}\tilde{K}^i_{ab} + \frac{f_c(n)}{2\pi} \int_\Sigma \gamma^{ab}\gamma^{cd} C_{acbd} \right) \log(\mu\ell) + \lambda_n \ , \quad (4.75)$$

where γ^{ab} is the inverse of the induced metric on the entangling surface, μ is an arbitrary mass scale typically chosen to be of order of the inverse cut-off, and λ_n is a non-universal constant. Further, $\tilde{K}^i_{ab}$ is a traceless part of the extrinsic curvature of Σ

$$\tilde{K}^i_{ab} = K^i_{ab} - \frac{K^i}{2}\gamma_{ij} \ , \quad (4.76)$$

with $K^i = \gamma^{ab}K^i_{ab}$. Now, two of the coefficients appearing in (4.75) were conjectured to be equal to each other [180]:

$$f_b(n) = f_c(n) \, . \quad (4.77)$$

From our defect CFT point of view, the expression (4.75) has the form of a conformal anomaly, which simply arises because the presence of a defect in the vacuum provides additional ways to violate Weyl invariance. Since the a and c coefficients of the trace anomalies in a generic even dimensional CFT appear in correlation functions of the stress tensor, one might wonder if the same happens in a defect CFT. In this section we show that this is indeed the case, in the sense that f_b and f_c are directly related to C_{D} and h, respectively, *i.e.,*

$$f_c(n) = \frac{3\pi}{2}\frac{h_n}{n-1} \, , \qquad f_b(n) = \frac{\pi^2}{16}\frac{C_{\mathrm{D}}(n)}{n-1} \ . \quad (4.78)$$

The relation between f_c and h_n was recently found in the context of entanglement entropy [181], but both equalities turn out to be true in a generic defect CFT.[11] In the case of the replica defect, they also establish the equivalence of the conjecture (4.77) with the four-dimensional version of eq. (4.36). The relation (4.77) has been proven for $n = 1$, but has been proven to fail in general [52]. The failure of this specific statement, in turns, proves that the general conjecture (4.36) is not valid for generic CFTs, and so the cone conjecture is disproven as well - see section 4.4. As a first step towards eq. (4.78), we notice that by dimensional analysis (or direct calculation), we have

$$\mu\frac{\partial}{\partial\mu}S_n - \ell\frac{\partial}{\partial\ell}S_n = 0 \quad \Leftrightarrow \quad \mu\frac{\partial}{\partial\mu}S_n = S_n^{\mathrm{univ}} \ , \quad (4.79)$$

where S_n^{univ} denotes the universal Renyi entropy

$$S_n^{\mathrm{univ}} = -\frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma - \frac{f_b(n)}{2\pi} \int_\Sigma \tilde{K}^i_{ab}\tilde{K}^i_{ab} + \frac{f_c(n)}{2\pi} \int_\Sigma \gamma^{ab}\gamma^{cd} C_{acbd} \ . \quad (4.80)$$

[10] In what follows we suppress the area law $\sim (\mu\ell)^2$. Its coefficient is scheme dependent and thus non-universal. In particular, it vanishes within dimensional regularization scheme which we employ throughout this chapter.

[11] See for instance [202] for a discussion of anomalies in the context of surface operators in $\mathcal{N} = 4$ SYM. The relations reported in eq. (4.78) clearly apply for those defects as well.

Varying both sides of (4.79) with respect to the metric and using (4.26), yields

$$\frac{1}{1-n}\,\mu\frac{\partial}{\partial\mu}\sum_{\mathrm{m}}\Big(\langle T^{\mu\nu}_{(\mathrm{m})}(x)\rangle_n-\langle T^{\mu\nu}(x)\rangle_1\Big) = \frac{-2}{\sqrt{g(x)}}\frac{\delta S^{\mathrm{univ}}_n}{\delta g_{\mu\nu}(x)}\,, \tag{4.81}$$

$$\frac{1}{1-n}\,\mu\frac{\partial}{\partial\mu}\Big(\sum_{\mathrm{l,m}}\langle T^{\mu\nu}_{(\mathrm{l})}(x)T^{\alpha\beta}_{(\mathrm{m})}(y)\rangle_n-n\langle T^{\mu\nu}(x)T^{\alpha\beta}(y)\rangle_1\Big) = \frac{4}{\sqrt{g(y)}}\frac{\delta}{\delta g_{\alpha\beta}(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta S^{\mathrm{univ}}_n}{\delta g_{\mu\nu}(x)}, \tag{4.82}$$

where indices m and n run over the replicas. In the next subsection, we build on eq. (4.81) to prove that f_c appears in the one-point function of the stress-tensor, while eq. (4.82) will be needed in subsection 4.5.2 to match f_b with the two-point function of the displacement operator.

4.5.1 f_c and the expectation value of the stress tensor

Substituting $d = 4$ into eq. (4.35), the nontrivial terms in the one-point function of the stress tensor become[12]

$$\begin{aligned}\langle T^{ab}_{\mathrm{tot}}\rangle_n &= -\frac{h_n}{2\pi}\frac{\delta^{ab}}{r^4}+\dots\,,\\ \langle T^{ij}_{\mathrm{tot}}\rangle_n &= \frac{h_n}{2\pi}\frac{3\,\delta^{ij}\,r^2-4\,x^ix^j}{r^6}+\dots\,,\end{aligned} \tag{4.83}$$

where as usual the indices a, b and i, j denote the two parallel directions and the two transverse directions to the entangling surface, respectively. Further, r denotes the transverse distance from the defect with $r^2 = x^i x_i$. Note that h_n in the above expression is a constant, *i.e.,* we are in the regime when the surface and the background are flat and thus all curvatures can be ignored. While eq. (4.35) was written for a planar twist operator, this expression also coincides with the leading singularity for general entangling surfaces if x is sufficiently close to Σ. In particular, the same constant appears for the conformal weight h_n independently of the geometry of the entangling surface.

Of course, (4.83) is independent of μ, and thus one might think that we reached a contradiction with (4.81). However, this conclusion is too fast. The right hand side of (4.81) vanishes unless $r = 0$, but $r = 0$ corresponds to a singular point of (4.83). This singularity should be carefully defined as distribution. As we will see, this results in a dependence on a mass scale μ.

In what follows we use dimensional regularization and expand all the results around $d = 4$. In particular, we start from the analog of (4.83) with dimension of the entangling surface being fixed (*i.e.,* two in our case), while the transverse space to the entangling surface is assumed to have dimension $d - 2$ (rather than two, as in four dimensions). Hence, the analog of (4.83) reads

$$\begin{aligned}\langle T^{ab}_{\mathrm{tot}}\rangle_n &= -\frac{h_n}{2\pi}\frac{\delta^{ab}}{r^d}\,,\\ \langle T^{ij}_{\mathrm{tot}}\rangle_n &= \frac{h_n}{2\pi}\frac{1}{d-3}\frac{3\delta^{ij}\,r^2-d\,x^ix^j}{r^{d+2}}=\frac{h_n}{2\pi(d-2)(d-3)}\left(\delta^{ij}\partial^2_\perp-\partial^i\partial^j\right)\frac{1}{r^{d-2}}\,,\end{aligned} \tag{4.84}$$

where $\partial^2_\perp = \delta^{ij}\partial_i\partial_j$ is Laplace operator in the transverse space.

[12] For convenience in this section, we work with the total energy-momentum tensor of the replicated CFT: $T^{\mu\nu}_{\mathrm{tot}} = \sum_{\mathrm{m}=1}^{n} T^{\mu\nu}_{(\mathrm{m})}$.

Now using the standard Fourier integral

$$\int \frac{d^{d-2}k}{(2\pi)^{d-2}} \frac{e^{ik\cdot x}}{(k^2)^\alpha} = \frac{\Gamma(d/2-\alpha-1)}{(4\pi)^{(d-2)/2}\Gamma(\alpha)} \left(\frac{4}{x^2}\right)^{d/2-\alpha-1} , \tag{4.85}$$

we deduce

$$\frac{1}{(r^2)^{d/2-\alpha-1}} = \frac{4^\alpha \pi^{(d-2)/2}\Gamma(\alpha)}{\Gamma(d/2-\alpha-1)} (-\partial_\perp^2)^{-\alpha} \delta_\Sigma , \tag{4.86}$$

where equality holds between the distributions and we recall that δ_Σ denotes the delta function in the transverse space with support on Σ. Now examining the cases $\alpha = -1+\epsilon$ and $\alpha = 0+\epsilon$ with $\epsilon \ll 1$, and replacing $(-\partial_\perp^2)^\epsilon \to \mu^{2\epsilon}$ yields

$$\begin{aligned} \frac{1}{r^{d-2}} &= -\Omega_{d-3} \left(\frac{1}{2\epsilon} + \log(\mu\, r) + \dots\right) \delta_\Sigma , \\ \frac{1}{r^d} &= -\frac{\Omega_{d-1}}{4\pi} \left(\frac{1}{2\epsilon} + \log(\mu\, r) + \dots\right) \partial_\perp^2 \delta_\Sigma . \end{aligned} \tag{4.87}$$

where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ and ellipses correspond to a finite μ-independent constant as $\epsilon \to 0$. Consequently, r^{-d} and $r^{-(d-2)}$ although defined by analytic continuation in d are singular when $d=4$. Hence, to define (4.84) as a sensible distribution, one has to subtract the singular part,

$$\begin{aligned} \mathcal{R}\frac{1}{r^{d-2}} &= -\Omega_{d-3} \left(\log(\mu\, r) + a\right) \delta_\Sigma , \\ \mathcal{R}\frac{1}{r^d} &= -\frac{\Omega_{d-1}}{4\pi} \left(\log(\mu\, r) + a\right) \partial_\perp^2 \delta_\Sigma , \end{aligned} \tag{4.88}$$

with a an arbitrary constant (which may be absorbed into μ). Note that such a subtraction modifies (4.84) in the limit of coincident points only. Furthermore, the details of this subtraction are not important as long as the result is used in (4.81)

$$\begin{aligned} -\frac{h_n}{4(n-1)} \delta^{ab} \partial_\perp^2 \delta_\Sigma &= -2 \frac{\delta S_n^{\rm univ}}{\delta g_{ab}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} , \\ \frac{h_n}{2(n-1)} \left(\delta^{ij}\partial_\perp^2 - \partial^i\partial^j\right) \delta_\Sigma &= -2 \frac{\delta S_n^{\rm univ}}{\delta g_{ij}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} . \end{aligned} \tag{4.89}$$

Next we use (4.80) to evaluate the variation on the right hand side. We start from noting that the term proportional to $f_a(n)$ is topological, and therefore its variation vanishes. Hence, in general we need only vary $f_b(n)$ and $f_c(n)$ terms. Now in four dimensions, the following relations hold

$$\begin{aligned} C^\lambda{}_{\mu\sigma\nu} &= R^\lambda{}_{\mu\sigma\nu} - \left(g^\lambda_{[\sigma} R_{\nu]\mu} - g_{\mu[\sigma} R^\lambda_{\nu]}\right) + \frac{1}{3} R\, g^\lambda_{[\sigma} g_{\nu]\mu} , \\ \gamma^{ab}\gamma^{cd} C_{acbd} &= \gamma^{ab}\gamma^{cd} R_{acbd} - \gamma^{ab} R_{ab} + \frac{1}{3} R \\ &= \frac{1}{3}\left(\gamma^{ab}\gamma^{cd} R_{acbd} - \gamma_{ab}\, g^\perp_{\mu\nu} R^{a\mu b\nu} + g^\perp_{\mu\nu} g^\perp_{\alpha\beta} R^{\mu\alpha\nu\beta}\right) , \end{aligned} \tag{4.90}$$

where $g^\perp_{\mu\nu} = n^i_\mu n^j_\nu \delta_{ij}$ is the metric in the transverse space to Σ, *i.e.*, $g_{\mu\nu} = \gamma_{\mu\nu} + g^\perp_{\mu\nu}$. One can use the Gauss-Codazzi relation

$$\gamma^{ab}\gamma^{cd} R_{acbd} = R_\Sigma + K^i_{ab} K^{ab}_i - K^i K_i , \tag{4.91}$$

where R_Σ is the intrinsic curvature of the entangling surface, to write

$$\gamma^{ab}\gamma^{cd}C_{acbd} = \frac{1}{3}\left(R_\Sigma + K^i_{ab}K^{ab}_i - K^iK_i - \gamma_{ab}\, g^\perp_{\mu\nu}R^{a\mu b\nu} + g^\perp_{\mu\nu}g^\perp_{\alpha\beta}R^{\mu\alpha\nu\beta}\right) \tag{4.92}$$

Now recall that (4.84) is valid in the limit when all curvatures (extrinsic, intrinsic and background) are negligibly small. Hence, we expand the relevant curvature components around the flat space, $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ - see appendix B of [40],

$$\begin{aligned} \delta R^{ai}{}_{ai} &= \frac{1}{2}\left(2\,\partial^a\partial^i h_{ai} - \partial^2_\perp h^a{}_a - \partial^a\partial_a h^i{}_i\right) + \mathcal{O}(h^2)\ , \\ \delta R^{ij}{}_{ij} &= \partial^i\partial^j h_{ij} - \partial^2_\perp h^i{}_i + \mathcal{O}(h^2)\ . \end{aligned} \tag{4.93}$$

Next we again use the fact that the integral of the intrinsic curvature over a two-dimensional manifold is a topological invariant, and therefore its variation vanishes. As a result, we obtain[13]

$$\begin{aligned} -2\frac{\delta S^{\rm univ}_n}{\delta g_{ab}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= -\frac{f_c(n)}{6\pi}\delta^{ab}\partial^2_\perp\delta_\Sigma\ , \\ -2\frac{\delta S^{\rm univ}_n}{\delta g_{ij}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= -\frac{f_c(n)}{3\pi}\left(\partial^i\partial^j - \delta^{ij}\partial^2_\perp\right)\delta_\Sigma\ , \end{aligned} \tag{4.94}$$

where we have used the following identities

$$\frac{\delta g_{\alpha\beta}(y)}{\delta g_{\mu\nu}(x)} = \delta^\mu_{(\alpha}\delta^\nu_{\beta)}\delta(x-y)\ , \qquad \int_\Sigma \delta(x-y) = \delta_\Sigma(x_a) \text{ for } y\in\Sigma\ . \tag{4.95}$$

Comparing (4.89) and (4.94), yields

$$f_c(n) = \frac{3\pi}{2}\frac{h_n}{n-1}\ . \tag{4.96}$$

In full agreement with the existing results for free fields [180]. As we mentioned above, this result was also found with a complementary argument in [181].

4.5.2 f_b and the two-point function of the displacement operator

We now turn to the second equality in eq. (4.78). Since we like to find the appearance of $C_{\rm D}$, we begin by considering the following Ward identity:

$$\langle {\rm D}^i(x){\rm D}^j(y)\rangle_n\delta_\Sigma(x)\,\delta_\Sigma(y) = \langle\nabla_\mu T^{\mu i}_{\rm tot}(x)\nabla_\nu T^{\nu j}_{\rm tot}(y)\rangle_n \quad \text{for} \quad x\neq y\in\Sigma\ . \tag{4.97}$$

Of course, when either x or y are away from Σ, the correlator on the right hand side vanishes identically. However, as we will see, it does not vanish when x and y hit the entangling surface Σ. This is why the δ_Σ's are explicitly included on the left hand side of the above identity. In particular, we are interested in the leading order singularity of $\langle {\rm D}^i(x){\rm D}^j(y)\rangle_n$ when x approaches y. In this limit curvature corrections are subleading, *i.e.,* both the entangling surface and the background can be regarded as flat. From (4.82), we have

$$\begin{aligned} \frac{1}{1-n}\,\mu\frac{\partial}{\partial\mu}\langle\partial_\mu T^{\mu i}_{\rm tot}(x)\ \partial_\nu T^{\nu j}_{\rm tot}(y)\rangle_n\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \frac{\partial^2}{\partial y^\nu\,\partial x^\mu}\left(4\frac{\delta^2 S^{\rm univ}_n}{\delta g_{\nu j}(y)\delta g_{\mu i}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}\right. \\ &\quad \left. -\ 2\,\delta^{\nu j}\,\delta(x-y)\,\frac{\delta S^{\rm univ}_n}{\delta g_{\mu i}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}\right)\ . \end{aligned} \tag{4.98}$$

[13] Note that the third term in (4.93) is a total derivative.

One can also compute [40]

$$
\begin{aligned}
\delta^2 R^{ai}{}_{ai} =& \frac{1}{4}\left(2\partial_i h_{a\mu}\partial^a h^{i\mu} + \partial_i h_{a\mu}\partial^i h^{a\mu} - 2\partial_i h_{a\mu}\partial^\mu h^{ia} + \partial_a h_{i\mu}\partial^a h^{i\mu}\right. \\
& \left. -2\partial_a h_{i\mu}\partial^\mu h^{ia} + \partial_\mu h_{ia}\partial^\mu h^{ia}\right) \\
& -\frac{1}{4}\left(4\partial^i h_{i\mu}\partial_a h^{a\mu} - 2\partial_i h^{i\mu}\partial_\mu h^a_a - 2\partial_\mu h^i_i \partial_a h^{a\mu} + \partial_\mu h^i_i \partial^\mu h^a_a\right) \\
& +\frac{1}{2}h^{ij}\left(\partial_i\partial_j h^a_a - 2\partial_a\partial_j h^a_i + \partial_a\partial^a h_{ij}\right) \\
& +\frac{1}{2}h^{ab}\left(\partial^i\partial_i h_{ab} - 2\partial_i\partial_b h^i_a + \partial_a\partial_b h^i_i\right) \\
& + h^{ia}\left(\partial_i\partial_a h^b_b - \partial_i\partial_b h^b_a - \partial_a\partial_b h^b_i + \partial^b\partial_b h_{ia}\right).
\end{aligned}
\tag{4.99}
$$

Similarly,

$$
\begin{aligned}
\delta^2 R^{ij}{}_{ij} =& \frac{1}{2}\left(\partial_i h_{j\mu}\partial^j h^{i\mu} + \partial_i h_{j\mu}\partial^i h^{j\mu} - \partial_i h_{j\mu}\partial^\mu h^{ij} - \partial_j h_{i\mu}\partial^\mu h^{ij} + \frac{1}{2}\partial_\mu h_{ij}\partial^\mu h^{ij}\right) \\
& - \partial^i h_{i\mu}\partial_j h^{j\mu} + \partial_i h^{i\mu}\partial_\mu h^j_j - \frac{1}{4}\partial_\mu h^i_i \partial^\mu h^j_j \\
& + h^{ij}\left(\partial_i\partial_j h^k_k - 2\partial_j\partial_k h^k_i + \partial_k\partial^k h_{ij}\right) \\
& - 2h^{ai}\left(\partial_i\partial_j h^j_a - \partial_j\partial^j h_{ai} - \partial_a\partial_i h^j_j + \partial_a\partial_j h^j_i\right).
\end{aligned}
\tag{4.100}
$$

and

$$
\begin{aligned}
\delta^2\left(K^i_{ab}K^{ab}_i\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \frac{1}{2}\partial_a h_{ib}\partial^a h^{ib} + \frac{1}{2}\partial_b h_{ai}\partial^a h^{ib} - \partial_i h_{ab}\partial^a h^{ib} + \frac{1}{4}\partial_i h_{ab}\partial^i h^{ab}\,, \\
\delta^2\left(K^i K_i\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \partial_a h^{ai}\partial_b h^b_i - \partial_a h^{ai}\partial^i h^b_b + \frac{1}{4}\partial_i h^a_a\,\partial^i h^b_b\,.
\end{aligned}
\tag{4.101}
$$

These expansions together with (4.93) are sufficient to evaluate the variation on the right hand side of (4.98). There is, however, a significant simplification if we notice that the general term of this variation contains: two delta functions, δ_Σ, which restrict the final answer to the entangling surface, one delta function intrinsic to the entangling surface and 4 derivatives, ∂_a and ∂_i, which act on these delta functions. Among all such terms only those with four derivatives parallel to the entangling surface will contribute to the leading singularity of $\langle \mathrm{D}^i(x)\mathrm{D}^j(y)\rangle_n$ as x approaches y. Hence, the relevant part of the variations are

$$
\begin{aligned}
\delta^2 R^{ai}{}_{ai} &= \frac{1}{2}\left(\partial_a h_{ib}\partial^a h^{ib} - \partial_a h_{ib}\partial^b h^{ai}\right) + h^{ai}\left(-\partial_a\partial_b h^b_i + \partial^b\partial_b h_{ai}\right) + \ldots \\
\delta^2\left(K^i_{ab}K^{ab}_i\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \frac{1}{2}\left(\partial_a h_{ib}\partial^a h^{ib} + \partial_b h_{ai}\partial^a h^{ib}\right) + \ldots\,, \\
\delta^2\left(K^i K_i\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \partial_a h^{ai}\partial_b h^b_i + \ldots
\end{aligned}
\tag{4.102}
$$

where the ellipses encode terms which do not contribute to the leading singularity of $\langle \mathrm{D}^i(x)\mathrm{D}^j(y)\rangle_n$ as x approaches y.

Now it follows from (4.92) that the term proportional to $f_c(n)$ in (4.80) does not contribute to the leading singularity of $\langle \mathrm{D}^i(x)\mathrm{D}^j(y)\rangle_n$ while $f_b(n)$ gives

$$
4\,\frac{\partial^2}{\partial y^\nu\,\partial x^\mu}\,\frac{\delta^2 S_n^{\text{univ}}}{\delta g_{\nu j}(y)\delta g_{\mu i}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} = -\frac{f_b(n)}{2\pi}\,\delta^{ij}\,(\partial_a\partial^a)^2\delta_{\|}(x-y)\,\delta_\Sigma(x)\,\delta_\Sigma(y)\,, \tag{4.103}
$$

where $\delta_{\|}(x-y)$ is the delta function intrinsic to Σ. Substituting into (4.98) and using (4.97), yields[14]

$$\frac{1}{1-n}\mu\frac{\partial}{\partial\mu}\langle \mathrm{D}^i(x)\mathrm{D}^j(y)\rangle_n = \frac{f_b(n)}{2\pi}\,\delta^{ij}\,(\partial_a\partial^a)^2\delta_{\|}(x-y) \quad \text{for} \quad x,y\in\Sigma\ . \tag{4.104}$$

Now we should use the analog of (4.86) to interpret the two-point function of the displacement in the limit $x\to y$:[15]

$$\langle \mathrm{D}^i(x)\mathrm{D}^j(y)\rangle_n = C_{\mathrm{D}}\frac{\delta^{ij}}{|x-y|^6} = C_{\mathrm{D}}\,\frac{\delta^{ij}}{(x-y)^{2(d-1)}} \quad \text{for} \quad x,y\in\Sigma\ . \tag{4.105}$$

The final answer takes the form

$$\frac{1}{r^6} = -\frac{\pi}{32}\left(\frac{1}{2\epsilon}+\log(\mu\, r)+\dots\right)(\partial_a\partial^a)^2\,\delta_{\|}(r)\ . \tag{4.106}$$

Combining altogether, yields

$$\frac{C_{\mathrm{D}}}{n-1} = \frac{16}{\pi^2}\,f_b(n)\ . \tag{4.107}$$

Further, let us note that this result is in full agreement with (4.37) since $f_b(1)=c=\pi^4\,C_T/40$.

4.6 Twist operators and the defect CFT data

In the most general sense, a conformal field theory is defined by a set of data, whose knowledge is sufficient to compute all the observables in the theory. A minimal definition of the CFT data includes the spectrum of scaling dimensions of local operators and the OPE coefficients which regulate their fusion. Knowledge of such a set of numbers is sufficient to compute correlation functions with any number of points. However, one might argue that a more complete definition of the CFT data should include those associated to non-local probes, *i.e.,* defects: certainly, they are part of the observables of a theory. A possible objection is that the set of defects that can be inserted in a higher dimensional conformal field theory may be very large, even nearly as large as the set of lower dimensional conformal field theories. We may point out that in two dimensions the study of boundaries and interfaces has uncovered a beautiful and simple picture — see *e.g.,* [12, 13, 14, 203, 82]. However, even in $d=2$, a complete classification of the defect lines which can be placed in a given CFT is a difficult problem, without a solution for the generic case. The situation is better in the special case of topological defects [16], which have been classified for the Virasoro minimal models [204] and for the free boson [205]. In higher dimensions, it is perhaps better to think of a theory with a defect as a separate problem, more similar in spirit to the question of which new fixed points can be obtained by coupling two CFTs together.

It is then natural to ask what is the set of CFT data which characterizes the twist operator. This question is not only a simple curiosity. The definition of the replica defect is through a boundary condition in the path-integral. This is often sufficient, but a new definition in terms of CFT data would apply to any conformal field theory, irrespectively of the availability of a path-integral description.[16] Again, some care is needed in setting

[14]It follows from (4.94) that the first variation of S_n^{univ} does not have the same singularity structure as $\langle \mathrm{D}^i\mathrm{D}^j\rangle$, and therefore it does not contribute.

[15]The analog is obtained by replacing δ_Σ and $\partial_\perp^2$ with $\delta_{\|}$ and $\partial^a\partial_a$ respectively.

[16]We thank Davide Gaiotto for a discussion on this point.

up this question. The large majority of the OPE coefficients appearing in formulae such as (4.29) will depend on the theory in which the twist operator is inserted. However, if an unambiguous characterization exists, it should be possible to single out some universal pattern, unique to this defect and independent of the CFT. The present work can also be seen as a first attempt in exploring this question. First, we highlighted that the CFT data associated to a flat twist operator always includes a spectrum of defect primaries with rational spin under rotations around the defect. Second, we showed that a set of conjectures motivated by free theory results can also be rephrased in terms of CFT data: the coefficient of the two-point function of the displacement and the one of the expectation value of the stress-tensor might be constrained to obey eq. (4.36). Both these facts are theory independent, but both of them are not unique to the twist operator. Codimension-two defects supporting operators with non-integer transverse spin can be easily constructed - see for instance the defect in the 3d Ising model that we constructed in chapter 3 - while the relation (4.36) is shared by Wilson lines in a class of three-dimensional supersymmetric gauge theories, as we discussed in section 4.4. Even worse, we now understand that the latter constraint will only be obeyed within a special class of CFTs. Eq. (4.36) is nevertheless remarkable, and one might wonder whether it is possible to understand it from the abstract perspective that we are adopting here. This is interesting for one more reason, which will become more clear in the next chapter. While twist operators in strongly coupled theories do not generically obey eq. (4.36), the amount by which the conjecture fails is at most around 2% for all values of $n > 1$, in the examples that we will consider - that is, Einstein and Gauss-Bonnet gravity in various dimensions. Since the conjecture is satisfied in various free theories, one might wonder if such a moderate dependence on the coupling is accidental or not. We will not be able to give an answer to this question, but here we will point out that, in fact, something special does happen in the defect OPE of the stress tensor, when this relation (4.36) is fulfilled: a certain number of singular contributions to this OPE disappear.

Let us recollect here some formulae from chapter 1, both to ease the reading, and because of two minor changes in notation, with respect to chapter 1. In particular, the displacement has opposite sign - compare eq. (4.30) with eq. (1.149) - and the conformal weight of the twist operator is denoted there as $a_T = -d\,h/2\pi$. The appearance of the displacement operator in the defect OPE of the stress tensor is constrained by Lorentz and scale invariance to take the following form:

$$T^{ab}(x) \sim \cdots + \alpha \frac{x_j \mathrm{D}^j \, \delta^{ab}}{r^2} + \beta \, x_j \, \partial_a \partial_b \mathrm{D}^j + \gamma \, x_j \, \delta^{ab} \, \partial_c \partial^c \mathrm{D}^j + \dots \tag{4.108a}$$

$$T^{aj}(x) \sim \cdots + \delta \, \frac{x^j x_k \, \partial^a \mathrm{D}^k}{r^2} + \epsilon \, \partial^a \mathrm{D}^j + \dots \tag{4.108b}$$

$$T^{jk}(x) \sim \cdots + \zeta \, \frac{x^j x^k x_i \mathrm{D}^i}{r^4} + \eta \, \frac{\delta^{jk} x_i \mathrm{D}^i}{r^2} + \lambda \frac{\mathrm{D}^j x^k + \mathrm{D}^k x^j}{r^2} + \dots \tag{4.108c}$$

where r denotes the transverse distance from the defect, as usual. The first ellipsis in each line alludes to the identity and to operators which might be lighter than the displacement, and the second ellipsis indicates less-singular contributions, including higher descendants of the displacement itself. As we saw in chapter 1, conformal invariance and conservation of the energy-momentum tensor place constraints on the coefficients in eq. (4.108), and allow to express them as linear combinations of the conformal weight h and the coefficient C_{D}. Recall that the two-point function of the displacement operator with the stress-tensor is fixed by conformal symmetry up to three coefficients:[17]

$$
\begin{aligned}
\langle \mathrm{D}^i(x_1) T^{ab}(x_2)\rangle &= \frac{x_2^i}{(x_{12}^2)^{d-1} r^2}\frac{1}{d}\left\{ b^1_{\mathrm{D}T}\left(\frac{4d\, r^2\, x_{12}^a x_{12}^b}{x_{12}^4} - \delta^{ab}\right) + b^2_{\mathrm{D}T}\,\delta^{ab}\right\}, \\
\langle \mathrm{D}^i(x_1) T^{aj}(x_2)\rangle &= \frac{x_{12}^a}{(x_{12}^2)^{d}}\left\{ 2b^1_{\mathrm{D}T}\,\frac{x_2^i x_2^j}{r^2}\left(1-\frac{2r^2}{x_{12}^2}\right) + b^3_{\mathrm{D}T}\left(\delta^{ij} - \frac{x_2^i x_2^j}{r^2}\right)\right\}, \\
\langle \mathrm{D}^i(x_1) T^{jk}(x_2)\rangle &= \frac{1}{(x_{12}^2)^{d-1} r}\left\{ b^1_{\mathrm{D}T}\,\frac{x_2^i}{r}\left[\frac{x_2^j x_2^k}{r^2}\frac{\left(x_{12}^2 - 2r^2\right)^2}{x_{12}^4} - \frac{1}{d}\delta^{jk}\right]\right. \\
&\quad + b^2_{\mathrm{D}T}\,\frac{x_2^i}{r}\left(\frac{x_2^j x_2^k}{r^2} - \frac{d-1}{d}\delta^{jk}\right) \\
&\quad \left. + b^3_{\mathrm{D}T}\left(1-\frac{2r^2}{x_{12}^2}\right)\left(\frac{\delta^{ij}x_2^k + \delta^{ik}x_2^j}{2r} - \frac{x_2^i x_2^j x_2^k}{r^3}\right)\right\}.
\end{aligned}
\tag{4.109}
$$

Here x_1 is, of course, confined to the defect while r denotes the transverse distance of $T^{\mu\nu}$ from the defect, and $x_{12}^\mu = x_1^\mu - x_2^\mu$. These coefficients are related to C_{D} and h as follows:

$$ b^2_{\mathrm{D}T} = \frac{1}{d-1}\left(\frac{d}{2} b^3_{\mathrm{D}T} - b^1_{\mathrm{D}T}\right), \tag{4.110a} $$

$$ b^3_{\mathrm{D}T} = 2^d d\, \pi^{-\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)\frac{h}{2\pi}. \tag{4.110b} $$

$$ b^2_{\mathrm{D}T} = \frac{1}{d-2}\left(\frac{d}{2}\, b^3_{\mathrm{D}T} - 2\frac{C_{\mathrm{D}}}{2\pi}\right). \tag{4.110c} $$

Let us now consider the most singular contributions in every component in (4.108). In Euclidean signature, all terms in the the OPE of T^{aj} and T^{jk} have the same degree of singularity. We can still define the most singular terms in Lorentzian signature, by considering a spacelike defect — this is especially natural when talking about Rényi entropy. Now as the insertion approaches the null cone, the individual x^i may remain finite while r approaches zero. In this circumstance, the most singular terms are those multiplied by α, δ and ζ. Comparing eqs. (4.108) and (4.109), we easily find

$$ \alpha = \frac{1}{d\, C_{\mathrm{D}}}(b^2_{\mathrm{D}T} - b^1_{\mathrm{D}T}), \quad \delta = \frac{1}{2(d-1)C_{\mathrm{D}}}(2b^1_{\mathrm{D}T} - b^3_{\mathrm{D}T}), \quad \zeta = \frac{1}{C_{\mathrm{D}}}(b^1_{\mathrm{D}T} + b^2_{\mathrm{D}T} - b^3_{\mathrm{D}T}). \tag{4.111} $$

Remarkably, the three constants vanish when eq. (4.36) holds, *i.e.,*

$$ \alpha = \delta = \zeta = 0 \iff C_{\mathrm{D}}(n) = d\,\Gamma\left(\tfrac{d+1}{2}\right)\left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1} h_n \tag{4.112} $$

This observation is appealing, even if its meaning remains somewhat obscure. One may speculate that the twist operator is a “mild” defect, in some sense. It is obtained through a modification of the geometry, rather than the addition of local degrees of freedom, and now we see that the OPE of the stress-tensor is less singular than for a generic defect. However, this idea should not be taken too literally. The identity appears in the same defect OPE, with a more severe singularity. Moreover, lighter defect operators with respect to the displacement might exist — in fact, they do in a free scalar theory, as discussed

[17]The switch in the sign of the displacement has the only effect of switching the sign of the the constants $b^i_{\mathrm{D}T}$. Compare eqs. (4.110) with eqs. (1.161) and (1.165).

in Appendix 4.A. Some of them may also appear in the defect OPE of the stress tensor. Finally, again, the relation (4.36) is not exactly satisfied in general. Whatever the right interpretation may be, it is worth emphasizing that it would have been probably difficult to recognize the special character of the relation (4.36), without adopting the defect CFT perspective.

4.7 Discussion

Twist operators were originally defined in examining Rényi entropies in two-dimensional CFTs [183, 184] and they are easily understood in this context since they are local primary operators. As discussed in section 4.2, twist operators are formally defined for general QFTs through the replica trick, as in eq. (4.27). In higher dimensions then, they become nonlocal surface operators and their properties are less well understood. In the present work, we have begun to explore twist operators for CFTs in higher dimensions from the perspective of conformal defects. This approach naturally introduces a number of tools that are unfamiliar in typical discussions of Rényi entropies. In particular, our discussion has focused on the displacement operator D^i, which appears with the new contact term in the Ward identity (4.30).

A key role of the displacement operator is to implement small local deformations of the entangling surface, as in eq. (4.33). As shown in eq. (4.34), the expectation value of the twist operator itself only varies at second order for such deformations of a planar (or spherical) entangling surfaces and is determined by the two-point function (4.31) of the displacement operator. This behaviour was previously seen in holographic studies of the so-called entanglement density [206] and more recently in [188]. These results correspond to the special case of the $n \to 1$ limit in eq. (4.34). We might also like to note that the connection with Wilson lines in holographic conformal gauge theories discussed in section 4.4.2 would also relate these entanglement variations to the wavy-line behaviour of Wilson lines [207].

Our main result was to unify a variety of distinct conjectures, summarized at the end of section 4.2, about the shape dependence of Rényi entropy to a constraint (4.36) relating the coefficient defining the two-point function of the displacement operator and the conformal weight of the twist operator. While the connections between these conjectures, were already considered in [179] — see also discussion in [188] — eq. (4.36) appears to provide the root source with a relation between two pieces of CFT data characterizing the twist operators.

One of these conjectures was the equivalence of the coefficients $f_b(n)$ and $f_c(n)$ appearing in the universal part of the four-dimensional Rényi entropy for general n [189, 180]. However, it was very recently shown that this equivalence does not hold for four-dimensional holographic CFTs dual to Einstein gravity [52]. As a consequence, it follows that eq. (4.36) does not hold for general n in these holographic CFTs either. On the other hand, this relation does hold in the vicinity of $n = 1$ for general CFTs. That is, the recent results of [188] demonstrate that the first order expansion of eq. (4.36) about $n = 1$ is a constraint which holds for generic CFTs. Despite the fact that this relation does *not* hold for all values of n for all CFTs, it is still interesting to ask for precisely which CFTs does this constraint hold. It seems that free field theories are a good candidate for such a theory. The results of [178, 190, 191] for the universal corner contribution to the Rényi entropy in three dimensions imply that eq. (4.36) holds for free scalars and fermions in this dimension. Further, our calculations in Appendix 4.A confirm that it holds for free massles scalars in four dimensions. We hope to return to this question in future work [192].

While eq. (4.36), and hence the related conjectures, are not completely universal, it is nevertheless a remarkable relation. It may still be interesting to explore other implications which this relation has for Rényi entropies in other geometries and other dimensions. For example, it could provide a relation (for arbitrary n) between different coefficients appearing in the universal contribution to the Rényi entropy in $d = 6$ or higher even dimensions, along the lines of our four-dimensional discussion in section 4.5.

Recalling that the twist operator is a local primary in two-dimensional CFTs, we might ask how the displacement operator appears in this context. Here, the natural object is the first descendant, *i.e.,* derivative, of the twist operator which would be analogous to the combination of the displacement and twist operators together. This matches the appropriate contact term in the two-dimensional version of the Ward identity (4.30). Here we refer to an analogy (rather than a precise match) keeping in mind that as a local operator, the two-dimensional twist operator can be moved but not deformed. Still one might make sense of the two-point correlator (4.31) by considering a "spherical" entangling surface. In two dimensions, the (zero-dimensional) sphere would correspond to two points whose separation defines the diameter of the sphere. Hence eq. (4.31) would be given by taking derivatives of the correlator of two twist operators and hence one finds that the corresponding $C_{\rm D}$ is indeed proportional to the conformal weight h_n.

Our discussion has highlighted h_n and $C_{\rm D}$ as two pieces of CFT data which characterize twist operators. With this perspective of regarding the twist operator as a conformal defect, we began in section 4.6 to consider the question of what are the defining characteristics of the twist operator? Certainly the relation (4.36) would be an important feature since, as we noted there, it has an interesting impact on the defect OPE with the stress tensor. However, this relation is not completely universal and, as described in section 4.2, this property is also shared by Wilson line operators in certain superconformal gauge theories. Another important property discussed in section 4.2 is that the spectrum of defect operators can contain operators with fractional spins k/n. Our analysis of the free scalar theory in appendix 4.A explicitly reveals the presence of such operators. But again twist operators are not unique in this regard. Another interesting point that arises in our discussion is that the twist operators are naturally defined for integer n but in discussing h_n and $C_{\rm D}$, as well as the Rényi entropy, one continues the results to real n almost immediately. Here derivatives of correlators with respect to the Rényi entropy index are naturally defined in terms of the modular Hamiltonian [182, 208]. This seems to point to a unique characteristic of twist operators in higher dimensions. In any event, better understanding the definition of the twist operator as a conformal defect remains an open question. Undoubtedly it is a question whose answer will produce a better understanding of the entanglement properties of CFTs, and perhaps QFTs more generally.

Appendix

4.A Displacement operator for the free scalar

In this appendix, we consider the theory of a free scalar in four dimensions, and we explore the defect OPE of the low lying bulk primaries. In doing so, we give a concrete expression for the displacement operator in terms of Fourier modes of the fundamental field and we verify the conjecture (4.36) for this particular case. Given the Lagrangian of a four-dimensional free massless boson

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 \,, \tag{4.113}$$

the propagator in presence of a conical singularity with an angular excess $2\pi(n-1)$ placed in $r=0$ can be derived [209]:

$$\langle\phi(x)\phi(x')\rangle_n = \frac{\sinh(\frac{\eta}{n})}{8\pi^2 n r r' \sinh\eta \left(\cosh(\frac{\eta}{n}) - \cos(\frac{\theta}{n})\right)}, \tag{4.114}$$

where

$$\cosh\eta = \frac{r^2 + r'^2 + y^2}{2rr'}. \tag{4.115}$$

We use alternatively polar coordinates around the defect with $x = (r, \theta, y^1, y^2)$, $x' = (r', 0, 0, 0)$ or complex coordinates $x = (z, \bar{z}, y^i)$, $x' = (z', \bar{z}', 0)$ with $z = re^{i\theta}$ and $z' = r'$. Assuming integer values of n and expanding (4.114) in the defect OPE limit, *i.e.,* for $r \to 0$ and $r' \to 0$, one finds

$$\langle\phi(r,\theta,y^i)\phi(r',0,0)\rangle_n = \frac{1}{4n\pi^2}\left(\frac{1}{y^2} + 2\sum_{k=1}^{n-1}\frac{r^{\frac{k}{n}} r'^{\frac{k}{n}}}{(y^2)^{1+\frac{k}{n}}}\cos\frac{k\theta}{n} - \frac{r^2 + r'^2 - 2rr'\cos\theta}{y^4} + \cdots\right) \tag{4.116}$$

where $y^2 = (y^1)^2 + (y^2)^2$ and the ellipsis indicates terms with higher powers of r/y and r'/y. This result can be precisely reproduced by the following OPE expansion for the field ϕ [18]

$$\phi(z,\bar{z}) = \phi(0) + \frac{1}{2\pi\sqrt{n}}\sum_{k\in\mathbb{N}}\left(z^{\frac{k}{n}} O_{\frac{k}{n}} + \bar{z}^{\frac{k}{n}} \bar{O}_{\frac{k}{n}}\right) + \cdots \tag{4.117}$$

where the operators $O_{\frac{k}{n}}$ are defect primaries with transverse spin $s = \frac{k}{n}$ and scaling dimension $\Delta = s + 1$ and the ellipsis indicates contributions from the descendants. This spectrum of twist-one[19] defect primaries can be easily understood through the requirement that every conformal family appearing on the r.h.s. of (4.117) is annihilated by the Laplace operator. Indeed, the latter reduces to the two-dimensional $\partial_z\partial_{\bar{z}}$ differential operator once we disregard descendants, and the holomorphicity property of the contribution of defect primaries to the OPE quickly follows. On the other hand the possible values of the spin are fixed by the symmetry preserved by the defect, *i.e.,* a n-fold cover of $SO(2)$. The normalization of the operators is fixed by

$$\langle O_{\frac{k}{n}} \bar{O}_{\frac{k}{n}}\rangle_n = \frac{1}{(y^2)^{1+\frac{k}{n}}} \tag{4.118}$$

Let us make one more comment on the nature of the defect spectrum. The twist operator is responsible for the presence of a tower of primaries with non-integer transverse spin. While these Fourier modes do not possess a local expression in terms of the elementary field, this is not so for the defect operators with integer spin. Their contribution to the defect OPE is modified by the defect, but we can still identify them with derivatives of ϕ in directions orthogonal to the defect.[20] In particular, it will be important in a moment that a defect operator $O_1 = \partial_z\phi$ exists.

We expect to find evidence of the presence of the displacement operator in the defect OPE expansion of the scalar operator ϕ^2. Therefore we consider the connected correlator

$$\langle\phi(x)^2\phi(x')^2\rangle_n - \langle\phi(x)^2\rangle_n \langle\phi(x')^2\rangle_n = 2\,{\langle\phi(x)\phi(x')\rangle_n}^2 \tag{4.119}$$

[18]The two contributions proportional to r^2 and r'^2 in (4.116) originate from the descendant $\partial_a\phi$.

[19]We are calling twist the difference between the scaling dimension and the charge under a transverse rotation. However, let us stress that the latter is a global symmetry from the point of view of the defect theory

[20]This is somewhat loose: a defect primary will in general be a combination of derivatives orthogonal and parallel to the defect. The one exception is $\partial_i\phi$, for which no mixing happens.

in the defect OPE limit and we extract the contribution given by operators of dimension 3 (spin 1), which reads

$$\langle\phi(x)^2\phi(x')^2\rangle_n\big|_{\text{spin }1} \sim \frac{rr'\cos\theta}{4n^2\pi^4y^6}(n+1) \tag{4.120}$$

This formula can be interpreted in terms of the OPE expansion of ϕ^2, which can be obtained by studying the fusion of two ϕ OPEs. In particular at dimension 3 one has several possible contributions coming from the combination of all the possible spins summing to 1 and the result is

$$\phi^2(z,\bar z) \sim \cdots + \frac{1}{4\pi^2 n}\sum_{k=0}^{n}\left(z\,O_{(\frac{k}{n},\frac{n-k}{n})} + \bar z\,\bar O_{(\frac{k}{n},\frac{n-k}{n})}\right) + \cdots \tag{4.121}$$

where $O_{\left(\frac{k}{n},\frac{k'}{n}\right)} = O_{\frac{k}{n}}O_{\frac{k'}{n}}$ and the ellipses indicate that we are focusing only on the spin-one contribution. Notice that the sum in (4.121) is redundant since $O_{\left(\frac{k}{n},\frac{k'}{n}\right)} = O_{\left(\frac{k'}{n},\frac{k}{n}\right)}$, nevertheless we keep this notation so as not to clutter the following expressions. Inserting the OPE in the two-point function and performing the Wick contractions, one obtains

$$\langle\phi(x)^2\phi(x')^2\rangle_n\big|_{\text{spin }1} = \frac{2(z\bar z' + z'\bar z)}{16\pi^4 n^2}\sum_{k=0}^{n}\langle O_{\frac{k}{n}}\bar O_{\frac{k}{n}}\rangle\,\langle O_{\frac{n-k}{n}}\bar O_{\frac{n-k}{n}}\rangle = \frac{z\bar z' + z'\bar z}{8n^2\pi^4y^6}(n+1)\,, \tag{4.122}$$

in agreement with (4.120). The degeneracy we just observed complicates the task of singling out the displacement operator. In the following we will start from a general ansatz and derive a set of constraints which allows to fix the precise form of the displacement operator for $n \le 5$ and to extrapolate a general pattern for higher n. In the process we will also prove that for this specific theory the relation (4.36) holds for any n.

We start from the general linear combination [21]

$$D = \frac{1}{2\pi n^2}\sum_{k=0}^{n} c_k O_{(\frac{k}{n},\frac{n-k}{n})} \tag{4.123}$$

where the normalization factor has been introduced for future convenience. The redundancy of the sum gives the first constraint on the coefficients

$$c_k = c_{n-k} \tag{4.124}$$

In order to find further constraints we compute the coupling of the displacement with ϕ^2 and with the stress tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\partial\phi\cdot\partial\phi - \frac{1}{6}\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\right)\phi^2\,. \tag{4.125}$$

The former is fixed by the Ward identity

$$\int d^2y\,\langle\phi^2(z,\bar z,0)D(y^i)\rangle_n = \partial_z\,\langle\phi^2(z,\bar z,0)\rangle_n \tag{4.126}$$

whereas the latter is determined in terms of C_{D} and h_n by equations (4.109), (4.110a), (4.110b) and (4.110c).

[21] Here we discuss only the holomorphic part of the displacement operator, but analogous considerations are valid for the anti-holomorphic component $\bar D$.

We start with the coupling to ϕ^2. The bulk-defect correlator $\langle \phi^2 O_{(\frac{k}{n},\frac{n-k}{n})}\rangle_n$ is fixed by symmetry up to a normalization which can be extracted from the OPE (4.121). The result is

$$\langle \phi^2(z,\bar z,0)\, O_{(\frac{k}{n},\frac{n-k}{n})}(y)\rangle_n = \frac{\bar z}{2\pi^2 n (y^2+z\bar z)^3} \tag{4.127}$$

On the other hand the one-point function $\langle \phi^2(z,\bar z,0)\rangle_n$ on the r.h.s. of (4.126) is simply

$$\langle \phi^2(z,\bar z,0)\rangle_n = \frac{1-n^2}{48 n^2\pi^2 z\bar z} \tag{4.128}$$

It is then clear that the Ward identity (4.126) gives a constraint on the sum of the coefficients c_k. Explicitly

$$\sum_{k=0}^{n} c_k = \frac{(n-1)n(n+1)}{6} \tag{4.129}$$

Notice that the r.h.s. of this expression is always an integer.

We now move to the computation of the two-point function of the displacement with the stress tensor. By standard Wick contraction one can compute the coupling of $O_{(\frac{k}{n},\frac{n-k}{n})}$ with the parallel components of the stress tensor T^{ab}. This gives

$$\langle T^{ab}(z,\bar z,0)\, O_{(\frac{k}{n},\frac{n-k}{n})}(y)\rangle = \frac{2k(n-k)}{n^3\pi^2}\frac{\bar z\, y^a y^b}{(y^2+z\bar z)^5} \tag{4.130}$$

Comparing this expression with equation (4.109) we notice the absence of a term proportional to δ^{ab} which implies that, regardless of the explicit form of the displacement operator, the most singular part of the defect OPE of T^{ab} has to vanish. The immediate consequence of that is

$$b^1_{DT} - b^2_{DT} = 0 \qquad \Rightarrow \qquad C_{\mathrm{D}}(n) = \frac{24\, h_n}{\pi} \tag{4.131}$$

for any value of n. Hence we have verified that (4.36) holds for the four-dimensional free scalar!

The result (4.130) provides also an additional constraint. Indeed comparing with (4.109) we can extract

$$b^1_{DT} = \sum_{k=0}^{n} \frac{c_k k(n-k)}{2n^4\pi^3} \tag{4.132}$$

and using (4.110a), (4.110c), (4.131) and the value of h_n for a free scalar in four dimensions [182]

$$h_n = \frac{n^4-1}{720\,\pi\, n^3}, \tag{4.133}$$

we obtain

$$b^1_{DT} = \frac{n^4-1}{60 n^4\pi^3} \tag{4.134}$$

Equating the two expressions for b^1_{DT} we get

$$\sum_{k=0}^{n} c_k k(n-k) = \frac{n(n^4-1)}{30}\,. \tag{4.135}$$

Once more, rather non-trivially, the r.h.s. is an integer. Since we have determined the exact value of C_{D} we can also use the two-point function of the displacement to put a quadratic constraint on the coefficients

$$\sum_{k=0}^{n} c_k^2 = \frac{n(n^4-1)}{30} \tag{4.136}$$

The constraints collected so far allow to compute the exact values of c_k for $n \leq 5$. The result is

$$n=2 \qquad c_0 = c_2 = 0 \qquad c_1 = 1 \tag{4.137}$$

$$n=3 \qquad c_0 = c_3 = 0 \qquad c_1 = c_2 = 2 \tag{4.138}$$

$$n=4 \qquad c_0 = c_4 = 0 \qquad c_1 = c_3 = 3 \qquad c_2 = 4 \tag{4.139}$$

$$n=5 \qquad c_0 = c_5 = 0 \qquad c_1 = c_4 = 4 \qquad c_2 = c_3 = 6 \tag{4.140}$$

Based on these results and on the structure of the constraints it is very natural to assume that the coefficient c_k are integers. In this case they admit the unique solution

$$c_k = k(n-k) \qquad \Rightarrow \qquad \mathrm{D} = \frac{1}{2\pi n^2}\sum_{k=0}^{n} k(n-k) O_{\left(\frac{k}{n},\frac{n-k}{n}\right)} \tag{4.141}$$

It may be possible to explicitly verify this expression for higher values of $n\,(>5)$ by examining the two-point function of the displacement with the higher spin currents [192].

Chapter 5

Shape dependence of holographic Rényi entropies

The final chapter of this thesis is again dedicated to the study of Rényi entropies. We refer back to section 4.1 for some introductory comments on entanglement, its measures, and its role in quantum field theory and holography. The content of the present chapter is work in progress with Lorenzo Bianchi, Shira Chapman, Xi Dong, Damian Galante and Rob Myers, and it was unpublished at the time in which this thesis was completed.

In the previous chapter, we described how, in a CFT, the leading order variation of the Rényi entropies due to the deformation of a flat or spherical entangling surface is encoded in one coefficient. We identified this coefficient as the Zamolodchikov norm $C_{\rm D}$ of the displacement operator associated to a twist defect, and we analyzed various examples of smooth or singular deformations. We also expressed in a unified and general way a set of conjectures which were made at various points in the literature [174, 176, 178, 179, 40]. These conjectures can be summarized as a linear relation between $C_{\rm D}$ and the coefficient of the one-point function of the stress tensor h_n, also known as the conformal weight of the twist operator - we refer to the previous chapter for the exact definition:

$$C_{\rm D}(n) = d\,\Gamma\!\left(\tfrac{d+1}{2}\right)\,\left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1}\,h_n. \tag{5.1}$$

However, as we anticipated, after [40] appeared, it was shown that eq. (5.1) is not satisfied in the holographic dual of Einstein gravity in four dimensions. As we shall explain in a moment, the method employed in [52] to find this result meets some obstruction when applied to dimensions different than four. On the other hand, the defect CFT formalism developed so far allows to extend the recipe to arbitrary number of dimensions: this is the main result of this chapter. In practice, we will compute $C_{\rm D}(n)$ explicitly for $3 \leq d \leq 6$, in the holographic dual of Einstein gravity, and confirm that the result does not obey eq. (5.1) in any of these dimensions, although in all of them the discrepancy turns out to be very mild.

In the holographic context, our work is part of the effort to extend the computational tools to study the Rényi entropies, for which the results are still rare away from the free theory limit. Numerical techniques have been developed to evaluate the Rényi entropies in lattice models describing critical theories, *e.g.,* [210, 211, 212] but these are demanding and must be adapted for the specifics of a given model. Beyond these numerical studies, the existing literature considers primarily the Rényi entropy across a spherical entangling surface for a CFT living in flat space, *e.g.,* [156, 213, 214, 182, 215, 189]. In particular, as we review in section 5.2, the authors of [172] extended the prescription for entanglement entropy, found in [156], and showed that Rényi entropies across spheres, at strong coupling, equal the free energy of a hyperbolic black hole in AdS. A variety of higher derivatives theories of gravity admit these solutions. Given that the relevance of Rényi entropies transcends the one of a mere tool to compute entanglement entropy, it is important to understand how to compute them holographically for more generic shapes of the entangling surface. For a generic deformation of a flat or spherical entangling surface, the dual geometry is not known, but in the limit of a small deformation progress was recently made [52]. We shall review the construction of [52] in section 5.2; for now, we would like to highlight the fact that the geometry is known only at leading order in the size of the deformation. This raises the question whether this knowledge is sufficient to compute the variation of the Rényi entropies at the first non-trivial order with respect to a flat or spherical entangling surface. As we explained in section 4.2, the first non-trivial order is the quadratic one, and yet the answer was found to be positive in [52] in four dimensions. In this paper, we provide an understanding of this fact in CFT terms, in a way that allows a generalization of the results of [52] to any number of dimensions.

Before introducing the general story, let us briefly describe the situation in four dimensions. Recall that, in even dimensions, the universal part of the Rényi entropies for any smooth shape of the entangling surface is given by the coefficient of a logarithmic UV divergence. In four dimensions - see section 4.5 - the coefficient depends on three functions of n [216]:

$$S_n = \left(-\frac{f_a(n)}{2\pi}\int_{\partial\Sigma} R_\Sigma - \frac{f_b(n)}{2\pi}\int_{\partial\Sigma} \tilde{K}^i_{ab}\tilde{K}^i_{ab} + \frac{f_c(n)}{2\pi}\int_{\partial\Sigma} \gamma^{ab}\gamma^{cd}C_{acbd}\right)\log(\ell/\delta) + c\ . \quad (5.2)$$

Recall that we denote via a, b, ... indices in directions tangent to the entangling surface Σ, while normal directions are denoted by i, j, ... R_Σ is the Ricci scalar of the entangling surface, $\tilde{K}^i_{ab} = K^i_{ab} - \frac{1}{d-2}\delta_{ab}K^{b\,i}_b$ is the traceless part of the extrinsic curvature of the same surface, and $C_{\mu\nu\rho\sigma}$ is the spacetime Weyl tensor. Finally, δ and ℓ are UV and IR cutoffs respectively. Formula (5.1) is equivalent to $f_b(n) = f_c(n)$.

In flat space, the shape dependence of S_n is determined by $f_b(n)$. In particular, a deformation of order ϵ of a flat entangling surface has an extrinsic curvature of order ϵ, so we recover that the Rényi entropies are affected at order ϵ^2. For brevity, we shall sometimes refer to an entangling surface deformed away from a flat or spherical configuration simply as a "deformed entangling" surface. In [52], $f_b(n)$ was extracted holographically from knowledge of the dual geometry at leading order in the following way. Recall that the replica trick maps S_n to the path-integral over an n-fold branched cover of the original spacetime:

$$S_n = \frac{1}{1-n}\log\frac{Z_n}{Z^n}, \quad (5.3)$$

where Z is the path-integral on the original manifold. Given an entangling surface of a general shape, a deformation thereof can be triggered by a deformation in the metric, and the latter results at leading order in the insertion of the stress tensor in the partition function Z_n. Now, the expectation value of the stress tensor can be computed holographically with an entangling surface deformed at leading order, and then change in S_n at second order is computed by the integration of $\langle T_{\mu\nu}\rangle\,\delta g^{\mu\nu}$ on the branched manifold. Logarithmically divergent terms in the integration can then be matched with formula (5.2). In doing so, reference [52] found in particular that $f_b(n) \neq f_c(n)$ in the holographic dual of four dimensional Einstein gravity.

In higher even dimension, the universal part of S_n is still logarithmically divergent, but its general form is not known (in $d = 6$ there is a proposal [217], corroborated by various holographic examples). On the other hand, in odd dimension the universal part is finite, and therefore not associated to the local geometry of the entangling surface. This makes it non obvious how to compute $C_{\rm D}$ holographically in dimensions other than four, and this is the issue that we address in section 5.1. In particular, we show that $C_{\rm D}$ appears in the expectation value of the stress tensor in the presence of a deformed defect. In section 5.2, we briefly review the construction of the holographic dual of a deformed planar entangling surface, and we compute $C_{\rm D}$ numerically in $3 \leq d \leq 6$ by simply extracting the expectation value of the stress tensor in this background. Some technical details are relegated to the appendices. Before proceeding, let us finally emphasize that our procedure applies equally well to any other conformal defect: the only place in which the information about Rényi entropy enters the computation is in the specific form of the dual metric.

5.1 The CFT Story

Although planar and spherical entangling surfaces are conformally equivalent, to fix ideas and notations we focus on the computation of the Rényi entropies across a plane, which we call Σ. Let us recall a couple of formulae from the previous chapter. If we denote via $X^\mu(y)$ the position of the twist operator in space - in the present (planar) case, $X^\mu = (0, y^a)$ - the response of a defect to a displacement:

$$\delta X^\mu = \delta^\mu_i f^i, \tag{5.4}$$

is given by repeated insertions of the displacement operator:

$$(1-n)\,\delta S_n = \frac{1}{2}\int_\Sigma dw \int_\Sigma dw'\, f^i(w) f^j(w')\, \langle D_i(w) D_j(w')\rangle_n + O(f^4). \tag{5.5}$$

The two point function of the displacement operator is fixed up to a single coefficient:

$$\langle D_i(w) D_j(w')\rangle_n = \delta_{ij}\frac{C_{\rm D}}{(w-w')^{2(d-1)}}. \tag{5.6}$$

The number $C_{\rm D}$ is the one we are after. Extracting it from a direct computation of δS_n in eq. (5.5) would involve second order perturbation theory around a flat entangling surface. Luckily, $C_{\rm D}$ appears in other observables, some of which are linear in the displacement operator, and so will require only a leading order perturbation. In particular, it is convenient to focus on the correlation function between the displacement operator and the stress tensor, for which we refer to eqs. (4.109) and ss. As proven in section 1.5 and reviewed in section (4.6), $C_{\rm D}$ appears in said correlator, along with the *conformal weight* h_n, defined by its appearance in the expectation value of the stress tensor with a planar entangling surface:

$$\langle T_{ab}(x^i, y^a)\rangle_n = -\frac{h_n}{2\pi}\frac{\delta_{ab}}{|x|^d}, \qquad \langle T_{ij}(x^i, y^a)\rangle_n = \frac{h_n}{2\pi}\frac{1}{|x|^d}\left((d-1)\,\delta_{ij} - d\,\frac{x_i x_j}{x^2}\right). \tag{5.7}$$

Here we split the coordinates of the insertion into orthogonal (x^i) and parallel (y^a) ones. We shall also sometimes regroup them as $z^\mu = (x, y)$. Now recall also that, for an arbitrary product of insertions X,

$$\delta \langle X\rangle_n = -\int_\Sigma dw\, f^i(w)\, \langle {\rm D}^i(w) X\rangle_n + O(f^2). \tag{5.8}$$

We use (5.8) to compute the expectation value of $T_{\mu\nu}$ in the presence of the deformed entangling surface ($f\Sigma$):

$$\langle T_{\mu\nu}(z)\rangle_{n,f\Sigma} = \langle T_{\mu\nu}(z)\rangle_n - \int_\Sigma dw\, \langle \mathrm{D}_i(w) T_{\mu\nu}(z)\rangle_n f^i(w) + O\big(f^2\big)\,. \tag{5.9}$$

Clearly, the integral cannot be performed explicitly if we do not specify the deformation f^i. However, it turns out that the singular terms in the short distance expansion $|x| \to 0$ can be written down explicitly. This is due to the following property of the correlation function (4.109). When the limit $|x| \to 0$ is taken in the weak sense, the first few coefficients in the expansion are distributions with support at $w = 0$. More precisely, the following formula is proven in appendix 5.A:

$$\begin{aligned} \langle D_i(w) T_{ab}(z)\rangle_n = {} & \frac{x_i}{|x|^d}\left(A_n\left(\frac{\delta^{d-2}(w)}{|x|^2} + \frac{\partial^2\delta^{d-2}(w)}{2(d-2)}\right)\delta_{ab}\right. \\ & \left. + D_n\left(\partial_a\partial_b - \frac{\delta_{ab}}{d-2}\partial^2\right)\delta^{d-2}(w)\right) + \dots, \end{aligned} \tag{5.10a}$$

$$\begin{aligned} \langle D_i(w) T_{aj}(z)\rangle_n = {} & -\frac{\partial_a\delta^{d-2}(w)}{|x|^d} A_n\left(\delta_{ij} - \frac{x_i x_j}{x^2}\right) \\ & - \frac{1}{d-2}\frac{\partial_a\partial^2\delta^{d-2}(w)}{|x|^{d-2}}\left(\frac{A_n}{2}\delta_{ij} + C_n\frac{x_i x_j}{x^2}\right) + \dots, \end{aligned} \tag{5.10b}$$

$$\begin{aligned} \langle D_i(w) T_{jk}(z)\rangle_n = {} & -\frac{\delta^{d-2}(w)}{|x|^{d+2}} A_n\left[2\delta_{i(j}x_{k)} + x_i\left((d-1)\delta_{jk} - \frac{x_b x_c}{|x|^2}(d+2)\right)\right] \\ & + \frac{\partial^2\delta^{d-2}(w)}{|x|^d}\frac{A_n}{2(d-2)}\left[2\delta_{i(j}x_{k)} - x_a\left((d-1)\delta_{jk} - \frac{x_j x_k}{|x|^2}(d-2)\right)\right] + \dots, \end{aligned} \tag{5.10c}$$

with:

$$\begin{aligned} A_n &= \frac{d h_n}{2\pi}, \qquad B_n = \frac{(d-1)\Gamma\left(\frac{d}{2}-1\right)\pi^{\frac{d}{2}-2}}{2\Gamma(d+1)} C_{\mathrm{D}} \\ C_n &= \left(B_n - \frac{d\,A_n}{2(d-2)}\right), \qquad D_n = B_n - \frac{A_n}{d-2}. \end{aligned} \tag{5.11}$$

The dots in equations (5.10a)-(5.10c) stand for terms which are less singular in the distance from the entangling surface, and which we will not need in this work. These terms can however be fully expressed using the formulas of appendix 5.A.

Let us make a comments. The one-point function in eq. (5.7) refers to a flat entangling surface. Of course, the one-point function in the presence of a defect obtained from this one via a conformal transformation is still proportional to h_n. Correspondingly, in equations (5.10a)-(5.10c), C_{D} only appears as a coefficient of the traceless part of $\partial_a\partial_b\delta(w)$ and of the third derivative $\partial_a\partial^2\delta(w)$. Indeed, recall that at leading order the extrinsic curvature $K^i_{ab} = -\partial_a\partial_b f^i$, and that conformal transformations map planes into spheres, whose extrinsic curvature is diagonal and constant.

In view of the holographic computation in the next section, it is useful to write the one-point function (5.9) in a coordinate system adapted to the shape of the deformed entangling surface. Such coordinates can be defined perturbatively in the distance $\rho \equiv |x^a|$ from the entangling surface. The new set of coordinates is related to the Cartesian one as follows:

$$\left\{\begin{array}{l} x'^i = x^i - f^i(y) - \frac{1}{d-2}\left(x^i K^j x_j - \frac{1}{2}K^i x^2\right) + O(\rho^4)\,, \\ y'^a = y^a + \partial^a f^i(y) x_i - \frac{1}{2(d-2)} x^2 \partial^a K^i x_i + O(\rho^5)\,, \end{array}\right. \tag{5.12}$$

and the metric becomes (to reduce the clutter, we neglect the primes in the following but this metric is understood to be in the new adapted coordinate system)

$$ds^2 = \left(1 + \tfrac{2K^c x_c}{d-2}\right)\left(\rho^2 d\tau^2 + d\rho^2 + [\delta_{ij} + 2\tilde{K}^i_{ab}x_i]dy^a dy^b + \tfrac{4}{d-2}\partial_a K^j\, x_j \rho d\rho dy^a\right) + \mathcal{C}\,, \tag{5.13}$$

where $x^i = (\rho\cos\tau, \rho\sin\tau)$, and $\mathcal{C}$ represents the higher order terms with

$$\mathcal{C} = O(\rho^3)d\rho^2 + O(\rho^5)d\tau^2 + O(\rho^4)d\rho d\tau + O(\rho^4)\, d\rho dy^a + O(\rho^5)\, d\tau dy^a + O(\rho^3)\, dy^a dy^b\,. \tag{5.14}$$

We consistently kept track of the corrections to the metric coming from any further change of coordinates allowed by symmetries, at leading order in the deformation. We do not need to make any assumptions on the coordinate system at higher order in ρ, because the order at which we work already allows to determine $C_{\rm D}$.

Notice that the change of coordinates (5.12) simplifies for traceless extrinic curvature, $K^i = 0$. When $d > 3$, in order to determine $C_{\rm D}$, it is sufficient to consider a deformation with vanishing trace of the extrinsic curvature (at least locally). However the choice of the frame defined by (5.12) has two advantages. It is convenient in $d = 3$, where the extrinsic curvature is a scalar in parallel space. And in general number of dimensions, the possibility of handling deformations for which K^i is non vanishing will allow for a consistency check on our computation of $C_{\rm D}$.

As a last step, we apply two consecutive Weyl transformations. The first with scale factor $\Omega_1 = (1 - K^i x'_i/(d-2))$ to remove the prefactor in the metric (5.13) and the second with $\Omega_2 = \frac{1}{\rho}$ in view of the holographic computation. After the first rescaling the metric exhibits the reasons for the change of coordinates (5.12). Indeed if $f^i(y)$ implements a conformal transformation, eq. (5.12) is the inverse transformation. Hence, after the Weyl rescaling f^i correctly appears in the metric only via $\tilde{K}^i_{ab}$ and derivatives of K^i, such that, when simply mapping the planar defect onto a sphere, the metric trivializes to the original Cartesian one. Furthermore the position of $\tilde{K}^i_{ab}$ is fixed by contraction of the indices, whereas the last term in the second line of eq. (5.12) forces the trace of the extrinsic curvature to appear in as few places as possible.

The second Weyl rescaling does not provide an equivalent simplification, but it will turn out to be useful for the holographic computation in section 5.2. After the transformation $G_{\mu\nu} \to \frac{1}{\rho^2}G_{\mu\nu}$ we find the conformally equivalent metric:

$$ds^2 = \frac{1}{\rho^2}\left(d\rho^2 + g_{\tau\tau}\, d\tau^2 + g_{ab}\, dy^a dy^b + g_{\rho a}\, d\rho dy^a\right) + \dots\,, \tag{5.15}$$

where the dots stand for the various orders in ρ specified in (5.13). From now on we remove the primes on our coordinates to avoid cluttering the notation. The metric above describes a slightly deformed version of the manifold $S^1 \times H^{d-1}$ which we denote $\tilde{H}_n$.

In these coordinates the stress tensor one-point function looks particularly simple. However, one has to keep in mind that in even spacetime dimensions the stress tensor does not transform as a primary under Weyl transformations. In particular under the rescaling $G_{\mu\nu} \to \widetilde{G}_{\mu\nu} = \Omega^2 G_{\mu\nu}$ (here $\Omega = \Omega_1\Omega_2$) the stress tensor one-point function changes as follows:

$$\langle \widetilde{T}_{\mu\nu}\rangle_n = \Omega^{2-d}\,\langle T_{\mu\nu}\rangle_n + \mathcal{A}_{\mu\nu}, \tag{5.16}$$

where $\langle \widetilde{T}_{\mu\nu}\rangle_n$ is the stress tensor expectation value after the rescaling. The anomalous contributiona $\mathcal{A}_{\mu\nu}$ is the higher dimensional analog of the Schwarzian derivative and is independent of n because locally the n–fold branched cover is identical to the original spacetime manifold. Using the fact that in flat space the vacuum expectation value of the stress tensor vanishes – $\langle T_{\mu\nu}\rangle_{n=1} = 0$ – we see that

$$\mathcal{A}_{\mu\nu} = \langle \widetilde{T}_{\mu\nu}\rangle_1 \,. \tag{5.17}$$

We can finally write

$$\begin{aligned}\langle \widetilde{T}_{ij}(x)\rangle_n &= \frac{g_n}{\rho^2}\left((d-1)\delta_{ij} - d\frac{x_i x_j}{\rho^2}\right) + \dots ,\\ \langle \widetilde{T}_{ai}(x)\rangle_n &= \frac{x_i x_j}{\rho^2}\partial_a K^j \frac{k_n}{d-2} + \dots ,\\ \langle \widetilde{T}_{ab}(x)\rangle_n &= \frac{1}{\rho^2}\left(-g_n\delta_{ab} + k_n \tilde{K}^i_{ab} x_i\right) + \dots .\end{aligned} \tag{5.18}$$

where

$$k_n - k_1^{\text{even}} = \frac{(d-1)\Gamma\left(\frac{d}{2}-1\right)\pi^{\frac{d}{2}-2}}{2\Gamma(d+1)} C_{\text{D}} - \frac{3d-4}{d-2}\frac{h_n}{2\pi} \qquad g_n - g_1^{\text{even}} = \frac{h_n}{2\pi} \tag{5.19}$$

and k_1 and g_1 vanish in odd dimensions. It is important that C_{D} appears in the difference $k_n - k_1$: it means that all n independent terms in the computation can be disregarded, because they cancel in the subtraction. We shall make use of this fact in the next section. Equation (5.18), together with the metric (5.13), are the only ingredients entering the holographic computation. Let us also point out that the extrinsic curvature appears in the same components of the metric and of the stress tensor one-point function. This is reminiscent of the way in which the stress tensor arises in holography from the dual metric, and hints to the fact that the ansatz for the bulk metric should be especially simple, as we will confirm in the next section.

5.2 The AdS story

The next step is to use holography to compute the one-point function of the stress tensor and match to (5.18)-(5.19) in order to extract C_{D}. For the purposes of holography it is maybe more natural to think of the replica trick as defining the path-integral over the n-fold branched cover of the original d-dimensional spacetime. In this case, starting from a metric of the form (5.15) the identification $\tau \sim \tau + 2\pi n$ generates a conical singularity at the position of the entangling surface, $\rho = 0$. Rényi entropies are related to the free energy of the QFT on this singular geometry.

Let us briefly review some properties of the metric (5.15) in the undeformed ($K^i_{ab} = 0$) case. In this case the geometry is precisely $H^{d-1} \times S^1$, i.e. a product of a $(d-1)$–dimensional hyperbolic space of unit radius and a τ circle of size $2\pi n$. This space is thermal with temperature $T = 1/(2\pi n)$. Although not offering a significant simplification from the field theoretical point of view, this configuration is extremely useful in the context of the gauge/gravity correspondence. Indeed, the standard expectation is that the dual of a thermal state is a black hole, whose temperature equals the one of the boundary theory. The problem is therefore reduced to finding a class of black hole geometries which, at fixed time, asymptote a hyperbolic slicing of AdS at large radius. It turns out that both Einstein gravity and various higher derivative theories admit such solutions, in the form of topological black holes with hyperbolic horizons [156, 172]. Since the Rényi entropies are proportional to the free energy of the theory on H^{d-1}, they can then be computed from the (on-shell) Euclidean gravitational action evaluated on these dual geometries.

It turns out that the situation is not much modified in the deformed case. We can solve for the dual geometry, order by order in the distance from the entangling surface – ρ. The leading order solution coincides with the geometry for an undeformed entangling surface. One can then move to the next order in ρ to compute the bulk metric at first order in the deformation. Once the bulk metric has been determined one can try and extract $C_{\rm D}$. As emphasized in the previous section, $C_{\rm D}$ appears in the on-shell action only at second order in the deformation (see (5.5)–(5.6)). Nevertheless, the approach adopted here only requires to compute the one point function of the stress tensor at first order in the deformation (see e.g. (5.18)). We compute the stress tensor one-point function using holographic renormalization along the lines of [162].

5.2.1 Holographic Setup

Let us start by introducing an ansatz for the bulk metric. We first observe that the parallel components of the metric (5.15) only depend on the traceless part of the extrinsic curvature, while the $g_{\rho a}$ components contain contributions from the parallel derivatives of the trace of the extrinsic curvature. This was achieved by our choice of coordinates (5.12) (plus Weyl rescaling) and it is convenient in minimizing the number of unknown functions required for the gravitational ansatz. The bulk metric then reads:

$$\begin{aligned} ds^2_{bulk} =& \frac{dr^2}{g(r)} + g(r)d\tau^2 \\ &+ \frac{r^2}{\rho^2}\left(d\rho^2 + [\delta_{ab} + 2\,k(r)\tilde{K}^i_{ab}x_i]dy^a dy^b + \frac{4}{d-2}v(r)\partial_a\,K^j x_j \rho d\rho dy^a\right) + \cdots \,, \end{aligned} \tag{5.20}$$

where the dots stand for higher orders in ρ and dimensions are in units of the AdS radius. The function $g(r)$ will be determined by the equations at leading order in ρ, which are blind to the deformation. Its zero determines the position of the horizon. We will refer to the functions $k(r)$ and $v(r)$ as the traceless and traceful parts of the gravity solution, respectively. Their value will be determined by solving gravitational equations of motion at the first subleading order in ρ. This will produce two second-order differential equations for $k(r)$ and $v(r)$ which we solve analitically in the vicinity of $n = 1$ as well as numerically for a wide range of n-s. We impose as boundary conditions $k(r) \to 1$ and $v(r) \to 1$ as we approach the boundary ($r \to \infty$) to reproduce the boundary metric, as well as regularity at the horizon.

5.2.2 Einstein Gravity

In this subsection, we extract $C_{\rm D}$ for the boundary theories whose holographic dual is described by Einstein gravity. The blackening factor $g(r)$ is determined by the Einstein equations at zeroth order in ρ, and is given by [172, 52]:

$$g(r) = r^2 - 1 - \frac{r_h^d - r_h^{d-2}}{r^{d-2}}, \tag{5.21}$$

where r_h is the position of the horizon (in Lorentzian signature). It is related to n by

$$n = \frac{2r_h}{d\left(r_h^2 - 1\right) + 2}\,. \tag{5.22}$$

The next order in ρ gives a second order differential equation for $k(r)$

$$k''(r) + \left(\frac{d-1}{r} + \frac{g'(r)}{g(r)}\right) k'(r) - \left(\frac{1}{g^2(r)} + \frac{d-3}{g(r)r^2}\right) k(r) = 0\,, \tag{5.23}$$

that correctly reproduces the one in [52] at $d = 4$, as well as a linear equation:

$$k(r) = v(r). \tag{5.24}$$

Other components of the Einstein equations give additional first and second order equations for $v(r)$ which are automatically satisfied when using (5.23)-(5.24). To derive (5.24) we used the Gauss Codazzi relations $\partial_c K^i_{ab} = \partial_b K^i_{ac}$ (at leading order in f^i). Note that the equality (5.24) is in agreement with the expectation that the traceless and traceful deformations both depend on the same linear combination of C_{D} and h_n (see the dependence of (5.18) on k_n). This constitutes a consistency check on our results. The $d = 3$ case is slightly different since the traceless part of the extrinsic curvature vanishes. We therefore find that Einstein equations contain only the second order differential equation for $v(r)$:

$$v''(r) + \left(\frac{2}{r} + \frac{g'(r)}{g(r)}\right) v'(r) - \left(\frac{1}{g^2(r)}\right) v(r) = 0\,, \tag{5.25}$$

which is the same as (5.23) when substituting $v(r)$ for $k(r)$.

Holographic renormalization

Given the bulk metric (5.20), we are interested in the boundary expectation value of the stress tensor. Its computation can be performed using the technique described in [162]. The idea is to write the metric in the Fefferman-Graham (FG) form [218]:

$$ds^2_{bulk} = \frac{1}{z^2}\left(dz^2 + h_{\mu\nu}dx^\mu dx^\nu\right)\,, \tag{5.26}$$

where

$$h_{\mu\nu} = h_{(0)\mu\nu} + z^2 h_{(2)\mu\nu} + \ldots + z^d h_{(d)\mu\nu} + \ldots\,. \tag{5.27}$$

The expectation value for the stress tensor is then determined by the $h_{(i)}$'s, with the following general expression

$$\langle T_{\mu\nu}\rangle_{\tilde{H}_n} = \frac{d}{2}\left(\frac{1}{\ell_{\mathrm{P}}}\right)^{d-1} h_{(d)\mu\nu} + \mathcal{X}_{\mu\nu}\left[h_{(m)\mu\nu}\right]_{m<d}. \tag{5.28}$$

The subscript $\tilde{H}_n$ indicates that the expectation value is taken in the deformed boundary geometry described by eq. (5.15). Here $\mathcal{X}_{\mu\nu}$ is a functional of the lower order $h_{(i)}$ terms, which are completely fixed by the boundary geometry. This contribution is related to the Weyl anomaly and accordingly, it vanishes with an odd number of boundary dimensions. In even d, its explicit expression depends on the dimension. For the cases $d = 4$ and 6, the interested reader is referred to eqs. (3.15) and (3.16) in [162]. We will see that it is not necessary to compute those contributions in order to obtain C_{D}. However, for completeness, we show how to obtain the exact expressions for the expectation value of the stress tensor in appendix 5.B. By comparing eqs. (5.28) and (5.20) with eq. (5.18), we see that the expansions of $k(r)$ and $v(r)$ near the boundary carry the information about the displacement operator. In this limit, the form of the solution to the equations of motion (5.23)-(5.24) reads

$$\begin{aligned}
d=3 \quad &, \quad k(r) = 1 - \frac{1}{2r^2} + \frac{\beta_n}{r^3}, \\
d=4 \quad &, \quad k(r) = 1 - \frac{1}{2r^2} + \frac{\beta_n}{r^4}, \\
d=5 \quad &, \quad k(r) = 1 - \frac{1}{2r^2} - \frac{1}{8r^4} + \frac{\beta_n}{r^5}, \\
d=6 \quad &, \quad k(r) = 1 - \frac{1}{2r^2} - \frac{1}{8r^4} + \frac{\beta_n}{r^6},
\end{aligned} \tag{5.29}$$

Here, β_n is the first coefficient which is not fixed by the boundary conditions at infinity. As one might expect, this coefficient determines C_{D}, and we obtain it numerically in the next subsection. Matching these expansions with eq. (5.18), we find the following relations:

$$\begin{aligned}
k_n &= \left(\frac{1}{\ell_{\mathrm{P}}}\right)^{d-1} \left(r_h^d - r_h^{d-2} + d\beta_n + k_0^{(d)}\right), \\
g_n &= -\left(\frac{1}{\ell_{\mathrm{P}}}\right)^{d-1} \left(\frac{r_h^d - r_h^{d-2} + g_0^{(d)}}{2}\right),
\end{aligned} \tag{5.30}$$

where k_0 and g_0 contain the anomalous contributions. As mentioned before, these vanish for odd dimensions and are independent of n in even dimensions[1] – see appendix 5.B. Note that in order to obtain C_{D} and h_n from eq. (5.19), we only need to consider the differences $k_n - k_1$ and $g_n - g_1$. Then, all the anomalous contributions will cancel.[2]

Comparing eqs. (5.30) and (5.19), we find holographic expressions for C_{D} and h_n,

$$\frac{h_n}{\pi n} = \left(\frac{1}{\ell_{\mathrm{P}}}\right)^{d-1} \left(r_h^{d-2} - r_h^d\right), \tag{5.31}$$

$$\frac{C_{\mathrm{D}}}{n} = \frac{d\,\Gamma(d+1)}{(d-1)\pi^{d/2-2}\Gamma(d/2)} \left((d-2)\left(\frac{1}{\ell_{\mathrm{P}}}\right)^{d-1} (\beta_n - \beta_1) + \frac{h_n}{2\pi n}\right). \tag{5.32}$$

The Planck length ℓ_{P} can be replaced for CFT data as follows, *e.g.*, see [182]:

$$C_T = \left(\frac{1}{\ell_{\mathrm{P}}}\right)^{d-1} \left(2^{d-2}\pi^{-\frac{d+1}{2}} d(d+1)\Gamma\left(\frac{d-1}{2}\right)\right), \tag{5.33}$$

where recall that C_T appears in the two-point function of the vacuum stress tensor [53, 219],

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{C_T}{x^{2d}} \mathcal{I}_{\mu\nu,\rho\sigma}(x)\,. \tag{5.34}$$

In order to obtain C_{D}, we now only need to solve numerically the equations of motion (5.23) and extract β_n. We will compare C_{D} with the value in eq. (5.1) related to previous conjectures [40]

$$C_{\mathrm{D}}^{\mathrm{conj}}(n) = d\,\Gamma\left(\frac{d+1}{2}\right)\left(\frac{2}{\sqrt{\pi}}\right)^{d-1} h_n\,. \tag{5.1}$$

[1] In particular, one can find that for $d=4$, $k_0^{d=4} = 3/4$ and $g_0^{d=4} = 1/4$, and for $d=6$, $k_0^{d=6} = 5/8$ and $g_0^{d=6} = -3/8$.

[2] Notice that in our conventions the stress tensor has lowered indexes, contrary to the one in [52]. The dictionary between the two conventions is as follows: $P_n = -g_n$ and $\alpha_n = k_n + 4g_n$, with $\ell_{\mathrm{P}}^{d-1} = 8\pi G_N$. This gives precise agreement between both expressions in $d=4$.

We will find that the conjecture is violated for holographic theories in any spacetime dimension. This conclusion will be supported numerically for $3 \leq d \leq 6$ with arbitrary n in section 5.2.2, and also with analytic results near $n = 0, 1$ in general dimensions in section 5.2.2. In particular, the expected agreement with eq. (5.1) is reproduced only at linear order in $(n-1)$, but we see C_{D} will depart from eq. (5.1) at order $(n-1)^2$.

Numerical Solutions

To solve the second order differential equation (5.23), we use a shooting method. The two integration constants will be free coefficients in the asymptotic expansions near both limits of integration. Near the asymptotic boundary we have β_n, while regularity of the solution near the horizon fixes a new integration constant. In particular, near the horizon we need $k(r) \propto (r - r_h)^{n/2}$, where the proportionality constant will provide the second integration constant. It is useful to consider coordinates in which the extreme values are kept fixed. Hence for our numerical integrations, we defined $\tilde{r} \equiv r_h/r$, so that the AdS boundary is at $\tilde{r}_{bdy} = 0$ and the horizon, at $\tilde{r}_{hor} = 1$. For each value of n, we solve the equation numerically both from the boundary and the horizon, fixing the integration constants so that the two curves meet smoothly.

The results for C_{D} are plotted in fig. 5.1. In the figure, we chose to normalize C_{D} by a factor n, in order to exhibit that this combination reaches a fixed value at large values of the Rényi index. Notice that, due to the prefactor in the definition of the Rényi entropies (4.12), this normalization quantifies more precisely the shape dependence of S_n at large n. As one can see from fig. 5.2, C_{D} deviates from $C_{\mathrm{D}}^{\mathrm{conj}}$ away from the linear regime around $n = 1$. Yet, notice that curiously, the relative difference $\frac{C_{\mathrm{D}} - C_{\mathrm{D}}^{\mathrm{conj}}}{C_{\mathrm{D}}}$ is fairly small for all $n > 1$. Although we are sure that this difference is bigger than our numerical accuracy, the analytic solution of the differential equation (5.23) close to $n = 1$ confirms that eq. (5.1) fails (for general dimensions), as does the analytic result for the limit $n \to 0$.

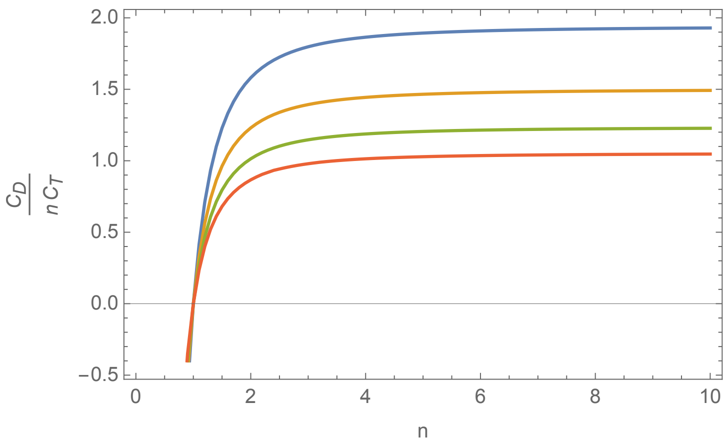

Figure 5.1: $C_{\mathrm{D}}/(nC_T)$ as a function of n. Different curves correspond to $d = 3$ (blue), $d = 4$ (yellow), $d = 5$ (green) and $d = 6$ (red).

Analytic Solutions

It is also possible to produce an analytic treatment of eq. (5.23) near $n = 1$. We can solve the equation analytically order by order in powers of $(n-1)$ and then fix the integration constants by providing the boundary expansion for $k(r)$ and regularity near the horizon.

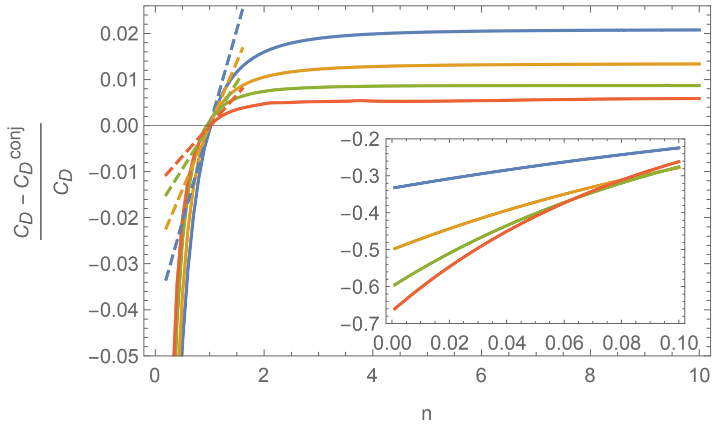

Figure 5.2: Relative mismatch between C_{D} and the conjectural value (5.1) as a function of n for $d=3$ (blue), $d=4$ (yellow), $d=5$ (green) and $d=6$ (red). Dashed lines show the leading order analytic solution around $n=1$, supporting the numerical data. In the inset, we show the numerical results near $n=0$, which smoothly approach the value $(2-d)/d$ at $n=0$, as predicted analytically in eq. (5.43).

We find that

$$k(\tilde{r}) = k_0(\tilde{r}) + k_1(\tilde{r})(n-1) + k_2(\tilde{r})(n-1)^2 + O(n-1)^3\,, \tag{5.35}$$

with

$$\begin{aligned} k_0(\tilde{r}) &= \sqrt{1-\tilde{r}^2}\,, \\ k_1(\tilde{r}) &= \frac{(d-1)\left(\tilde{r}^2-1\right)\tilde{r}^d\ {}_2F_1\left(1,\frac{d}{2};\frac{d+2}{2};\tilde{r}^2\right) + d\left(\tilde{r}^d-\tilde{r}^2\right)}{(d-1)d\sqrt{1-\tilde{r}^2}}\,. \end{aligned} \tag{5.36}$$

For $k_2(x)$ we solve separately for each dimension. These results determine β_n perturbatively around $n=1$:

$$\beta_n^d = \beta_1^d + \frac{1}{d(d-1)}(n-1) - \frac{4d^3-8d^2+d+2}{2\,d^2(d-1)^3}(n-1)^2 + O(n-1)^3\,, \tag{5.37}$$

with β_1^d being zero for odd d and $\beta_1^d = -\frac{\Gamma\left(\frac{d-1}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d}{2}+1\right)}$ for even d.

Given this expansion for β_n and the corresponding expansion for r_h from eq. (5.22), it is straightforward to compute C_{D} as a power series in $(n-1)$:

$$C_{\mathrm{D}}/C_T = \frac{2\pi^2}{d+1}(n-1) - \frac{2\pi^2\left(d^2-d-1\right)}{d^3-d}(n-1)^2 + O(n-1)^3\,, \tag{5.38}$$

which, as expected [188], agrees with the conjecture (5.1) at linear order,

$$C_{\mathrm{D}}^{\mathrm{conj}}/C_T = \frac{2\pi^2}{d+1}(n-1) - \frac{\pi^2(2d^2-4d+1)}{(d-1)^2(d+1)}(n-1)^2 + O(n-1)^3\,, \tag{5.39}$$

but not at second order. In fact, the relative mismatch between the two expressions can be easily computed and is given by

$$\frac{C_{\mathrm{D}}-C_{\mathrm{D}}^{\mathrm{conj}}}{C_{\mathrm{D}}} = \frac{(d-2)}{2\,d(d-1)^2}(n-1) + O\left(n-1\right)^2. \tag{5.40}$$

Interestingly, one can also extract the analytic expression for C_D at leading order as $n \to 0$. This result follows from the observation that the β_n contribution in eq. (5.32) is subleading with respect to r_h^d at small n. More precisely, one can verify that $\beta_n/r_h^d \sim n$ in this limit. Then, we do not actually need to solve eq. (5.23) but just expand r_h for small n to find

$$C_\mathrm{D}/C_T = -\left(\frac{1}{dn}\right)^{d-1}\left(\frac{2^{d-1}\pi^2}{d+1}+O(n)\right), \tag{5.41}$$

$$C_\mathrm{D}^{\mathrm{conj}}/C_T = -\left(\frac{1}{dn}\right)^{d-1}\left(\frac{2^{d}\pi^2(d-1)}{d(d+1)}+O(n)\right), \tag{5.42}$$

which yields

$$\frac{C_\mathrm{D}-C_\mathrm{D}^{\mathrm{conj}}}{C_\mathrm{D}} = -\frac{d-2}{d}+O(n)\,. \tag{5.43}$$

Note that the relative error is order one as n goes to zero, contrary to the small differences which were obtained for $n > 1$.

5.3 Discussion

In this chapter, we began exploring the shape dependence of the Rényi entropies holographically. We showed how to compute the norm of the displacement operator at strong coupling in arbitrary dimensions, and we found explicit results in the holographic dual of Einstein gravity, up to six dimensions. The results show that the twist operator is not as "simple" as it had been conjectured - cfr. section 4.6. Indeed, at strong coupling the ratio C_D/h_n depends on the replica parameter n. Nevertheless, some degree of universality still seems to be present. On the one hand, the relative mismatch between the conjectural universal ratio $C_\mathrm{D}^{\mathrm{conj}}/h_n$ in eq. (5.1) and the actual holographic result is smaller than 2% for all values of $n > 1$ - see fig. 5.2. On the other hand, also in the limit $n \to 0$ simplifications occur. The interesting fact in this case is that the complication due to the deformed geometry, encapsulated in the coefficient $(\beta_n - \beta_1)$ in eq. (5.32), disappears, and the ratio C_D/h_n is determined by r_h alone. Let us look in more detail at these two facts, and in the meanwhile discuss some future directions for this ongoing work.

As for the small mismatch when $n > 1$, one might ask if any naturally small parameter is hidden behind this fact. The CFT characterization of the conjecture given in section 4.6 comes short of offering such a crucial ingredient. For instance, small parameters arise in the presence of broken global symmetries, but both the stress tensor and the displacement operator are singlet under global symmetries, therefore this does not seem to be the case. It would certainly be nice to explore more the consequences of the conjecture (5.1), in the hope to gain more intuition about its meaning. It may be also worth it to explore the possible limits of the existing parameters. We know that $1/n$ cannot be the relevant small parameter, since the conjecture is not fulfilled even as n goes to infinity, as it is clear from the plots 5.2. A surviving possibility is the inverse of the number of dimensions: the mismatch does get smaller as the dimension increases. It will be interesting to see if any analytic treatment is possible in this limit. On the other hand, the closeness of the ratio $C_\mathrm{D}/C_\mathrm{D}^{\mathrm{conj}}$ to one might be completely accidental. A question we should ask is whether there are any accidental cancellations that would keep this ratio close to one, independently of the value of the crucial parameter $\beta \equiv (\beta_n - \beta_1)$ - see eq. (5.32) - that we extract numerically. The answer to this question is negative. The relative error on β is related to the one on C_{D} by an order one coefficient:

$$\left|\frac{\beta-\beta^{\mathrm{conj}}}{\beta^{\mathrm{conj}}}\right| = 2\,\frac{d-1}{d-2}\left|\frac{C_{\mathrm{D}}-C_{\mathrm{D}}^{\mathrm{conj}}}{C_{\mathrm{D}}^{\mathrm{conj}}}\right|. \tag{5.44}$$

Therefore the small discrepancy is a direct consequence of the features of eqs. (5.23) and (5.25), *i.e.,* of the dual geometry.

Let us now turn to the small n limit. Let us notice that the region $n < 1$ can hardly be interpreted with the tools of a defect CFT, which applies strictly speaking only for n integer and greater than one. The fact that in this opposite limit the conjecture fails at order one could raise some hope that the "hidden small parameter" problem that we just described may indeed find a solution within the framework of defect CFT. Coming back to the $n \to 0$ result expressed in eq. (5.43), it seems to be quite robust: the only ingredient entering the computation is the observation - which is numerical, for now - that $\beta/h_n \to 0$ in this limit. It is an interesting question that we leave for the future whether it is possible to understand this feature through flat space thermodynamics. Indeed, the $n \to 0$ limit is the high temperature limit on the hyperboloid - even if some subtleties might be hidden in the fact that the deformed geometry is not a simple Cartesian product of a space manifold times a circle anymore. In such a limit, the effect of the curvatures should be negligible, and indeed it is easy to obtain this way the correct scaling of h_n.

We would like to make one more comment about the ongoing work. In order to test more generally the universality of these results, we decided to explore (at least) one example of higher derivative gravity, and we chose to modify the Einstein action with the Gauss-Bonnet coupling. Our results in this case are preliminary, but indicate that the qualitative picture is not modified. The size of the Gauss-Bonnet coupling λ_{GB} is restricted by unitarity bounds [220], and throughout the allowed range the relative mismatch of C_{D} from the conjectured value is at most 3%. Furthermore, in the $n \to 0$ limit one recovers eq. (5.43). One would be tempted to hypothesize that such relation is theory independent, but this proposal quickly runs into troubles. Indeed, we know from appendix 4.A that a free scalar field in $4d$ obeys the conjecture, at least for every integer $n > 1$. The standard lore is that one can analytically continue this result for real n. It would then be surprising to find that such an analytic continuation fails for $n < 1$.

Finally, we would like to stress again one more general aspect of the results of this chapter. The procedure that was set up in section 5.1 is completely agnostic when it comes to the specific nature of the defect. It gives a completely general way to extract C_{D} from knowledge of the dual geometry at leading order in the deformation. The procedure becomes especially useful in all those cases in which the holographic computation becomes increasingly hard when moving away from a flat or spherical defect. It would be interesting to apply our formalism to other situations of this kind.

Appendix

5.A Expanding Two Point Functions as Distributions

In this appendix we provide the details needed to derive equations (5.10a)-(5.10c). As mentioned in the main text the singular terms in the short distance expansion $|x| \to 0$ of the two point function $\langle \mathrm{D}_i(w) T_{\mu\nu}(z)\rangle$ can be written in the weak limit in terms of delta functions in the $d-2$ parallel directions with support at $w = 0$, where we recall that

$z^\mu = (x, y)$, and we further fixed $y = 0$. Keep in mind that the expressions that we write in this appendix only hold inside integrals when multiplied by a test function that decays fast enough at infinity and is regular at zero.

The general expansion *up to terms which are regular as* $|x| \to 0$ reads:

$$\frac{w^{2\alpha}}{(x^2+w^2)^{d-1+\beta}} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})\Gamma(d+\beta-1)} \times \sum_{\substack{n=0 \\ n \text{ even}}}^{d+2(\beta-\alpha)-1} (\partial^2)^{\frac{n}{2}} \delta^{d-2}(w) \left[\frac{(n-1)!!(d-4)!!}{n!(d-4+n)!!}\right] \frac{\Gamma(\frac{d-n}{2}-\alpha+\beta)\Gamma(\frac{d+n}{2}+\alpha-1)}{|x|^{d+2\beta-2\alpha-n}} \tag{5.45}$$

Similar formulas for tensorial structures of w can be derived by differentiating identities of the form (5.45). In the rest of this appendix we present a derivation of the formula (5.45) as well as a list of useful identities that can be deduced from it.

5.A.1 Derivation of the Kernel Formula

Consider the kernel:

$$K(w,x) = \frac{w^{2\alpha}}{(x^2+w^2)^{d-1+\beta}} \tag{5.46}$$

and a test function $f(w)$ which is smooth at $w = 0$ and decays strong enough when $w \to \infty$ and define the Integral:

$$I(x) = \int d^{d-2}w f(w) K(w,x). \tag{5.47}$$

We can split the integration domain between $|w| \leq 1$ and $|w| > 1$. The exterior region is convergent when $|x| \to 0$ and therefore does not contribute to the divergent terms in (5.45). The function $f(w)$ can be Taylor expanded in the inside domain as follows:

$$I(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{a_1} \ldots \partial_{a_n} f(0) \int_{|w|<1} d^{d-2}w\, w^{a_1} \ldots w^{a_n} K(w,x) + \text{regular}, \tag{5.48}$$

where "regular" stands for terms which are regular at $x \to 0$. Using a change of variables $w^a = y^a|x|$ and symmetry considerations on the tensor structure inside the integral we obtain:

$$\begin{aligned} I(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{a_1} \ldots \partial_{a_n} f(0) \frac{1}{|x|^{d-n+2\beta-2\alpha}} \int_{|y|<1/|x|} d^{d-2}y \frac{y^{2\alpha} y^{a_1} \ldots y^{a_n}}{(1+y^2)^{d-1+\beta}} + \text{regular} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{a_1} \ldots \partial_{a_n} f(0) \frac{1}{|x|^{d-n+2\beta-2\alpha}} \frac{\delta^{a_1 a_2} \ldots \delta^{a_{n-1} a_n} + \text{permutations}}{\text{normalization}} \\ &\qquad \times \Omega_{d-3} \int_{|y|<1/|x|} dy \frac{y^{n+d-3+2\alpha}}{(1+y^2)^{d-1+\beta}} + \text{regular} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{a_1} \ldots \partial_{a_n} f(0) \frac{\delta^{a_1 a_2} \ldots \delta^{a_{n-1} a_n} + \text{permutations}}{\text{normalization}} \\ &\qquad \times \frac{1}{|x|^{d-n+2\beta-2\alpha}} \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})} \frac{\Gamma(\frac{d-n}{2}-\alpha+\beta)\Gamma(\frac{d+n}{2}+\alpha-1)}{2\Gamma(d+\beta-1)} + \text{regular}. \end{aligned} \tag{5.49}$$

In the weak limit we replace $f(0)$ by $\delta^{d-2}(w)$ to express the kernel $K(w,x)$ outside the integral as:

$$\frac{w^{2\alpha}}{(x^2+w^2)^{d-1+\beta}} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})\Gamma(d+\beta-1)} \sum_{n=0}^{\infty} \frac{P_n\,\Gamma(\frac{d-n}{2}-\alpha+\beta)\Gamma(\frac{d+n}{2}+\alpha-1)}{|x|^{d+2\beta-2\alpha-n}} \tag{5.50}$$

where

$$P_n = \frac{1}{n!}\partial_{a_1}\dots\partial_{a_n}\delta^{d-2}(w)\left[\frac{\delta^{a_1a_2}\dots\delta^{a_{n-1}a_n} + \text{permutations}}{\text{normalization}}\right], \tag{5.51}$$

and the normalization is chosen such that the term in the square bracket traces to one. Combinatorial arguments then lead to the simplified form (5.45).

5.A.2 List of Formulas for Kernels

We use (5.45) to derive the following:

$$\frac{1}{(x^2+w^2)^{d-1}} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{\Gamma(d-1)}\left(\frac{\delta^{d-2}(w)}{|x|^d} + \frac{\partial^2\delta^{d-2}(w)}{2(d-2)|x|^{d-2}}\right) + \dots. \tag{5.52}$$

$$\frac{1}{(x^2+w^2)^{d}} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{2\Gamma(d)}\left(\frac{d\,\delta^{d-2}(w)}{|x|^{d+2}} + \frac{\partial^2\delta^{d-2}(w)}{2|x|^{d}}\right) + \dots, \tag{5.53}$$

$$\frac{1}{(x^2+w^2)^{d+1}} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{4\Gamma(d)}\left[\frac{\delta^{d-2}(w)}{|x|^{d+4}}(d+2) + \frac{\partial^2\delta^{d-2}(w)}{2|x|^{d+2}}\right] + \dots, \tag{5.54}$$

$$\frac{w^2}{(x^2+w^2)^{d}} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{2\Gamma(d)}\left[\frac{\delta^{d-2}(w)}{|x|^{d}}(d-2) + \frac{d}{2(d-2)}\frac{\partial^2\delta^{d-2}(w)}{|x|^{d-2}}\right] + \dots, \tag{5.55}$$

$$\frac{w^2}{(x^2+w^2)^{d+1}} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{4\Gamma(d)}\left[\frac{\delta^{d-2}(w)}{|x|^{d+2}}(d-2) + \frac{\partial^2\delta^{d-2}(w)}{2|x|^{d}}\right] + \dots, \tag{5.56}$$

$$\frac{w^4}{(x^2+w^2)^{d+1}} = \frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{4\Gamma(d)}\left[\frac{\delta^{d-2}(w)}{|x|^{d}}(d-2) + \frac{d+2}{2(d-2)}\frac{\partial^2\delta^{d-2}(w)}{|x|^{d-2}}\right] + \dots, \tag{5.57}$$

Differentiating (5.45) with respect to w_a we get:

$$\frac{w^{2\alpha}w_a}{(x^2+w^2)^{d+\beta}} = \frac{1}{2(1-d-\beta)}\left[\partial_a\left(\frac{w^{2\alpha}}{(x^2+w^2)^{d-1+\beta}}\right) - \frac{2\alpha\,w^{2(\alpha-1)}w_a}{(x^2+w^2)^{d-1+\beta}}\right]. \tag{5.58}$$

We can then use (5.52) and (5.53) to show

$$\frac{w^a}{(x^2+w^2)^{d}} = -\frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{2\Gamma(d)}\left(\frac{\partial^a\delta^{d-2}(w)}{|x|^d} + \frac{\partial^a\partial^2\delta^{d-2}(w)}{2(d-2)|x|^{d-2}}\right) + \dots, \tag{5.59}$$

and

$$\frac{w^a}{(x^2+w^2)^{d+1}} = -\frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{4\Gamma(d+1)}\left(\frac{d\,\partial^a\delta^{d-2}(w)}{|x|^{d+2}} + \frac{\partial^a\partial^2\delta^{d-2}(w)}{2|x|^{d}}\right) + \dots, \tag{5.60}$$

and (5.55) and (5.59) to get:

$$\frac{w^2w^a}{(x^2+w^2)^{d+1}} = -\frac{\pi^{\frac{d-2}{2}}\Gamma(\frac{d}{2})}{4\Gamma(d+1)}\left[\frac{d\,\partial^a\delta^{d-2}(w)}{|x|^{d}} + \frac{(d+2)}{2(d-2)}\frac{\partial^a\partial^2\delta^{d-2}(w)}{|x|^{d-2}}\right] + \dots. \tag{5.61}$$

Differentiating (5.52) twice and using (5.53) we can also show:

$$\frac{w^aw^b}{(x^2+w^2)^{d+1}} = \frac{\Gamma(\frac{d}{2})\pi^{\frac{d-2}{2}}}{4\Gamma(d+1)}\left[\delta^{ab}\left(\frac{d\,\delta^{d-2}(w)}{|x|^{d+2}} + \frac{\partial^2\delta^{d-2}(w)}{2|x|^{d}}\right) + \frac{\partial^a\partial^b\delta^{d-2}(w)}{|x|^{d}}\right]. \tag{5.62}$$

This concludes the ingredients needed for the derivation of equations (5.10a)-(5.10c).

5.B Details of Holographic Renormalization

In this appendix we give more details on how to obtain the expectation value of the stress tensor in Einstein gravity. As described in section 5.2.2, the expectation value of the stress tensor in the deformed geometry can be computed using eqs. (5.26)–(5.28). As noted in the main text, it is not necessary for our purposes to compute explicitly $\mathcal{X}$ since all $h_{(m)}$ with $m < d$ are independent of r_h (equivalently of the Rényi index n) and then, they will cancel after the subtraction in (5.19). We keep track of the anomalous contributions for completeness.

In order to write the metric (5.20) in the FG coordinates, as in equation (5.26), we work order by order in an expansion in z (that we define perturbatively below). We reach the desired coordinate system by two successive changes of coordinates. First we define $r \equiv 1/\tilde{z}$ and then we redefine $\tilde{z} = z(1 + c_1 z^2 + \ldots + c_d z^d + \ldots)$. Using the blackening factor (5.21) we can fix the constants c_i such that

$$\frac{dr^2}{g(r)} = \frac{dz^2(\partial\tilde{z}/\partial z)^2}{\tilde{z}^4 g(r(z))} = \frac{1^2}{z^2} dz^2 (1 + O(z^{d+1}))\,. \tag{5.63}$$

We obtain

$$\begin{aligned}
d = 3 \quad &, \quad \tilde{z} = z\left(1 - \frac{1}{4}z^2 - \frac{r_h\left(r_h^2 - 1\right)}{6}z^3 + \ldots\right), \\
d = 4 \quad &, \quad \tilde{z} = z\left(1 - \frac{1}{4}z^2 - \frac{r_h^2(r_h^2 - 1) - 1/2}{8}z^4 + \ldots\right), \\
d = 5 \quad &, \quad \tilde{z} = z\left(1 - \frac{1}{4}z^2 + \frac{1}{16}z^4 - \frac{r_h^3\left(r_h^2 - 1\right)}{10}z^5 + \ldots\right), \\
d = 6 \quad &, \quad \tilde{z} = z\left(1 - \frac{1}{4}z^2 + \frac{1}{16}z^4 - \frac{r_h^4(r_h^2 - 1) + 3/16}{12}z^6 + \ldots\right).
\end{aligned} \tag{5.64}$$

Note that the r_h (equivalently n) dependence starts appearing only at order z^d. This is due to the form of the blackening factor (5.21) in which the dependence on r_h starts at the d-th subleading order in the boundary expansion. It is then obvious that only $h_{(d)\mu\nu}$ in (5.27) will depend on n while the lower order $h_{(m)}$'s entering in the anomalous functional $\mathcal{X}$ will not. In order to get our FG metric we now need to expand the components of the original metric (5.20) in powers of z. Equivalently, in terms of the $h_{\mu\nu}$ of equation (5.26) we need to expand the factors in square brackets in the following expression:

$$\begin{aligned}
h_{\mu\nu}dx^\mu dx^\nu = &\left[z^2 g(r(z))\right] d\tau^2 + \left[\left(\frac{z}{\tilde{z}}\right)^2\right]\frac{d\rho^2}{\rho^2} \\
&+ \left[\frac{4}{d-2}\left(\frac{z}{\tilde{z}}\right)^2 v(r(z))\right]\left(\partial_a K^j x_j\right)\frac{d\rho}{\rho}dy^a \\
&+ \frac{1}{\rho^2}\left(\left[\left(\frac{z}{\tilde{z}}\right)^2\right]\delta_{ab} + \left[2\left(\frac{z}{\tilde{z}}\right)^2 k(r(z))\right]\tilde{K}^i_{ab}x_i\right)dy^a dy^b
\end{aligned} \tag{5.65}$$

in powers of z. To complete the expansion we also need to use the asymptotic expansion of $k(r)$ and $v(r)$ which is given in equation (5.29).

We obtain the following expansions for the different dimensions. For odd dimensions we only specify $h_{(d)\mu\nu}$ since it is the only one needed for the stress tensor in equation (5.28) (remember that $\mathcal{X}$ vanishes in odd dimensions):

$$\begin{aligned} h_{(d)\mu\nu}dx^\mu dx^\nu =& \left(\frac{d-1}{d}(r_h^{d-2}-r_h^d)\right)d\tau^2+\left(\frac{r_h^d-r_h^{d-2}}{d}\right)\frac{d\rho^2}{\rho^2}\\ &+\frac{4}{d}\left(\frac{r_h^d-r_h^{d-2}+d\beta_n}{(d-2)}\right)\partial_a K^j x_j\frac{d\rho}{\rho}dy^a\\ &+\frac{2}{d}\left[\left(\frac{r_h^d-r_h^{d-2}}{2}\right)\delta_{ab}+\left(r_h^d-r_h^{d-2}+d\beta_n\right)\tilde{K}^i_{ab}x_i\right]\frac{dy^a dy^b}{\rho^2}+\dots . \end{aligned} \tag{5.66}$$

In $d=3$ the traceless part of the extrinsic curvature should be set to zero. In $d=4$ we obtain

$$h_{(0)\mu\nu}dx^\mu dx^\nu = d\tau^2+\frac{d\rho^2}{\rho^2}+\frac{2}{\rho}\partial_a K^j x_j\,d\rho\,dy^a+\frac{1}{\rho^2}\left[\delta_{ab}+2\tilde{K}^i_{ab}x_i\right]dy^a dy^b+\dots, \tag{5.67}$$

$$h_{(2)\mu\nu}dx^\mu dx^\nu = -\frac{1}{2}d\tau^2+\frac{1}{2}\frac{d\rho^2}{\rho^2}+\frac{1}{2\rho^2}dy^a dy_a+\dots, \tag{5.68}$$

$$\begin{aligned} h_{(4)\mu\nu}dx^\mu dx^\nu =& \left(\frac{-12r_h^4+12r_h^2+1}{16}\right)d\tau^2+\left(\frac{1-2r_h^2}{4}\right)^2\frac{d\rho^2}{\rho^2}\\ &+\left(\frac{16\beta_n+\left(1-2r_h^2\right)^2}{8}\right)\partial_a K^j x_j\frac{d\rho}{\rho}dy^a\\ &+\frac{1}{\rho^2}\left[\left(\frac{1-2r_h^2}{4}\right)^2\delta_{ab}+\left(\frac{16\beta_n+\left(1-2r_h^2\right)^2}{8}\right)\tilde{K}^i_{ab}x_i\right]dy^a dy^b+\dots . \end{aligned} \tag{5.69}$$

In $d=6$ instead

$$h_{(0)\mu\nu}dx^\mu dx^\nu = d\tau^2+\frac{d\rho^2}{\rho^2}+\frac{1}{\rho}\partial_a K^j x_j\,d\rho\,dy^a+\frac{1}{\rho^2}[\delta_{ij}+2K_{aij}x^a]dy^a dy^b+\cdots, \tag{5.70}$$

$$h_{(2)\mu\nu}dx^\mu dx^\nu = -\frac{1}{2}d\tau^2+\frac{1}{2}\frac{d\rho^2}{\rho^2}+\frac{1}{2\rho^2}dy^a dy_a+\cdots, \tag{5.71}$$

$$\begin{aligned} h_{(4)\mu\nu}dx^\mu dx^\nu =& \frac{1}{16}d\tau^2+\frac{1}{16}\frac{d\rho^2}{\rho^2}-\frac{1}{16\rho}\partial_a K^j x_j\,d\rho\,dy^a\\ &+\frac{1}{16\rho^2}[\delta_{ij}-2K_{aij}x^a]dy^a dy^b+\cdots \end{aligned}, \tag{5.72}$$

$$\begin{aligned} h_{(6)\mu\nu}dx^\mu dx^\nu =& -\frac{5(r_h^6-r_h^4)}{6}d\tau^2+\left(\frac{r_h^6-r_h^4}{6}\right)\frac{d\rho^2}{\rho^2}\\ &+\frac{1}{\rho}\left(\beta_n+\frac{8r_h^6-8r_h^4+3}{48}\right)\partial_a K^j x_j\,d\rho\,dy^a\\ &+\frac{1}{\rho^2}\left[\left(\frac{r_h^6-r_h^4}{6}\right)\delta_{ab}+\left(2\beta_n+\frac{8r_h^6-8r_h^4+3}{24}\right)K_{aij}x^a\right]dy^a dy^b+\cdots . \end{aligned} \tag{5.73}$$

Using equations (5.28) and (3.15)-(3.16) of [162] we obtain the expectation value of the stress tensor of the form (5.18) where g_n and k_n are given by eq. (5.30). Note that in order to obtain the values of the anomalous contributions appearing in footnote 1 we need to use the lower-order metrics appearing in this Appendix.

Final remarks and outlook

In these conclusive remarks, we would like to put this work in perspective, and propose possible further directions of research. This thesis is the result of an exploration in the realm of conformal defects in dimensions higher than two. In the introduction, we have argued that there are plenty of phenomenological and theoretical motivations to explore this topic, and yet the abstract literature on the subject is not extensive. In particular, an efficient way of dealing with symmetry constraints on correlation functions had not been worked out before. The first chapter of the thesis is dedicated to filling part of this gap.

There are many ways to expand upon the results presented in chapter 1, where we did not consider different representations of the Lorentz group than the traceless symmetric ones. We believe that formal results should always be pursued and welcomed, even when they are not stimulated by immediate applications. In the CFT context, two of the papers by Dolan and Osborn on the conformal block decomposition of four-point functions [73, 76] provide perhaps the most beautiful example in support of this belief. They were written before the revival of the bootstrap program [41], and became extremely important for the community afterwards, a fact that clearly emerges also from the history of the citations to these two papers.

That said, it is fair to tackle first those technical problems which are better motivated from a practical point of view. Here, let us take a bootstrap perspective. The most obvious goal is to perform the first numerical bootstrap analysis of a scalar two-point function with a defect of codimension higher than one. In the absence of parity odd structures, only traceless symmetric representations can acquire an expectation value: indeed, there is only one position vector in the game, which cannot be antisymmetrized. This means that the results presented in section 1.4 are sufficient to set up the bootstrap analysis for defects of codimension two in dimension four and six. Indeed, in this cases explicit expressions for the conformal blocks of the four-point functions (with no defect) in terms of hypergeometric functions are known [76], and via our eqs. (1.120) and (1.123), this is equivalent to the knowledge of the bulk blocks for the two-point function. As for defects of codimension two in odd dimensions, one may still use the available results for the four-point function. However, in this case no closed form is available, and it is important to check that eq. (1.120) maps physical values of the cross-ratios ξ and ϕ to values of u, v for which the chosen expansion for the four-point function blocks converges.

When the defect has codimension higher than two, we only know the bulk blocks via a recurrence relation in the light-cone limit, see subsection 1.4.2. In implementing the numerical defect bootstrap, so far the choice has been to evaluate derivatives of the blocks at $\xi = 4$.[3] Since the light-cone expansion expresses the blocks as a power series in ξ, it is probably not suitable in this case. One may certainly explore choices of points other than $\xi = 4$. It is also possible to set up the numerical bootstrap by evaluating the blocks at multiple points, instead of using derivatives [92], but in order to get useful constraints,

[3] We are using here the convention of chapter 1. See footnote 14

one should not expect to be able to pick all configurations close to the light-cone limit. In conclusion, this discussion identifies as a priority the search for better representations of the bulk blocks. Let us also stress that the variables ξ and ϕ are probably non optimal, and a closely related task is the one of finding new cross-ratios, with the property that a power series representation of the conformal blocks in the new variables has a fast rate of convergence and can be truncated [72]. This part of the program is, in fact, work in progress [84].

In chapter 2, we studied a set of codimension one defects associated with the $O(N)$ models using the conformal bootstrap, and to a lesser extent ϵ-expansion and conformal perturbation theory. A first set of questions left open is related to the performances of the *determinant method*, which is the only one presently suitable for tackling this kind of problem. At the current status, this approach is not optimized for extensive searches in the space of CFT data, although it might be possible to adapt it to such a task. The results obtained so far, included the ones presented in this work, have been achieved starting from a rudimentary knowledge of the spectrum, for instance through mean field theory expectations. This rendered the search for solutions of the truncated bootstrap equations manageable. It would be interesting to understand whether the determinant method can be freed from this limitation. This approach is affected by another issue: it is not clear how to estimate the systematic error coming from the truncation of the spectrum. In chapter 2, some progress was made in this direction – see in particular the discussion towards the end of section 2.2. This is still unsatisfactory, because our capability of making an estimate relies on the presence of free parameters in the solution of the truncated bootstrap equation, a feature which is non generic. The two problems just mentioned, the difficulties in exploring the space of CFT data and in estimating the error, are in a sense similar in nature. The method of determinants has never been implemented in a systematic way. This is certainly the essence of the first problem: sampling the space of CFT data requires an algorithmic approach, which can be easily fed to a computer. On the contrary, so far we have relied on direct inspection when selecting the physically relevant solution to the truncated bootstrap equations. At the same time, a better control over the space of solutions would also allow to include more and more exchanged operators, thus reducing the systematic error. However, this last issue has a more theoretical side: the estimates of the truncation error available in the literature on the four-point function are not immediately adapted to the two-point function with a defect. Indeed, positivity is required to prove those estimates [93, 79], while, as noticed many times, the bulk channel lacks it. This constitutes one more question left for the future.

The next question concerns candidates of defect CFTs which one may study via numerical bootstrap. Two obvious examples are given by the line defect in the $3d$ Ising model, which was defined in chapter 3, and the Rényi entropy twist operator which we considered in chapters 4 and 5. In particular, in the light of the previous discussion on the determinant method, we expect this approach to be especially helpful in studying Rényi entropy in CFTs which possess a weakly coupled limit, so that some preliminary information on the spectrum is available. Of course, one may try and apply the determinant method to a variety of other defects, starting from the obvious example of a Wilson line.

Before abandoning the bootstrap, let us finally mention that the numerical approach is not the only one which can be used in studying defects. The light-cone bootstrap, which was first introduced in [46, 47], has a rather straightforward application to defects of codimension equal or higher than two. It can be used to constrain a sector of the defect spectrum of a generic theory, namely the one with large charge under rotations around the defect [85].

Chapter 3 was dedicated to a lattice study of the Kramers-Wannier dual of a Wilson line in the 3d Ising model. The same defect can be defined in an $O(N)$ model, and it would be interesting to study its properties at large N (while ϵ-expansion has been carried out at leading order in [50]). This kind of codimension two defects is in fact more general, and can be constructed every time the CFT possesses a global symmetry, at least at weak coupling or on the lattice. If the symmetry group is continuous, the twist lines depend on continuous parameters.

In the two final chapters, we applied defect CFT techniques to the study of Rényi entropies in conformal field theories. We focused on the shape dependence of the Rényi entropies, and we extracted the Zamolodchikov norm of the displacement operator in the holographic dual of Einstein gravity. Our computations greatly benefited from knowledge of the symmetry constraints on correlation functions worked out in chapter 1. The results collected in chapter 5 disproved a conjecture (5.1) relating C_{D} to the expectation value of the stress-tensor (h_n), but also showed that the violation of the conjecture is mild. It would be useful to understand whether this is accidental or not. One way of tackling this question would be to study the stability of this qualitative picture under higher derivative corrections. More generally, our understanding about the conjectured relation $C_{\mathrm{D}} = C_{\mathrm{D}}^{\mathrm{conj}}$ seems incomplete. Although the relation is obeyed by some examples of free theories in $d = 3$ [190, 191] and $d = 4$ [40], it has not been established whether its violation is only a consequence of the presence of interactions. Further investigations in the context of free theories will be required in order to answer this question. More generally, we know that when the conjecture (5.1) is satisfied, the singularities of the defect OPE with the stress tensor are simplified [40], but we do not yet have an understanding of the consequences of this fact on the structure of entanglement. Finally, we note that this problem can be reformulated more broadly: what are the properties required for a defect to obey the conjecture (5.1)? In fact, this issue has been a question of interest in the context of gauge theories as well. A similar relation between the Bremsstrahlung function – which is the analogue of C_{D} in that context – and the conformal weight is obeyed by a class of Wilson lines. However, the theories in which this happens have not been classified yet [194, 196]. In fact, in $d = 3$ the relation conjectured in [194] reduces to our relation $C_{\mathrm{D}} = C_{\mathrm{D}}^{\mathrm{conj}}$.

We would like to stress that questions related to shape deformations are not the only ones, in the context of Rényi entropies, that one can address using defect CFT techniques. Let us mention two examples. The Rényi (and entanglement) mutual information between two spherical regions is a very interesting UV finite quantity which is not known exactly even for free theories. It can be computed perturbatively in the limit of large separations of the two regions. In practice, the strategy is to expand one of the spheres in local operators of the conformal field theory [221]. The conformal blocks appearing in this computation are precisely the ones considered in our subsection 1.4.2. Results available in the literature do not make use of the full defect conformal group to resum the contribution of descendants [222], so our results should allow to considerably improve the perturbative expansion. The second example is related to the ‘phase transitions’ in the Rényi entropy, which have been observed to take place at a critical value of the Rényi index in certain theories [223]. This behaviour has also been observed in a holographic setting for spherical regions [224]. This happens when the CFT has a sufficiently low-dimensional scalar operator, in which case the dual hyperbolic black hole solution would be unstable towards the development of scalar hair at low temperatures (large n). Such phase transitions may correspond to defect RG flows triggered by a relevant defect primary operator. It would be interesting to test this hypothesis in specific examples.

As it is clear from these concluding remarks, the possible directions of research – which are many more than those enumerated here – are multiple and diverse. The field of defect conformal field theory in higher dimensions is rich and mostly unexplored. Correspondingly, both the formal results and the examples contained in this thesis do not constitute a systematic treatment, but rather a choice of topics dictated by the interests of the author and his collaborators. On one hand, the study of specific examples allowed us to develop or adapt techniques that were not previously available in this field. On the other hand, we tried to focus our attention on physically relevant defect CFTs. In conclusion, we hope that this work proved that both the motivations and the tools exist, to conduct a broad research program in defect conformal field theory.

Bibliography

[1] N. Seiberg, *Electric - magnetic duality in supersymmetric nonAbelian gauge theories*, Nucl.Phys. **B435** (1995) 129–146, `arXiv:hep-th/9411149 [hep-th]`.

[2] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl.Phys. **B426** (1994) 19–52, `arXiv:hep-th/9407087 [hep-th]`.

[3] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int.J.Theor.Phys. **38** (1999) 1113–1133, `arXiv:hep-th/9711200 [hep-th]`.

[4] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum Spectral Curve for Planar $\mathcal{N} =$ Super-Yang-Mills Theory*, Phys. Rev. Lett. **112** (2014) no. 1, 011602, `arXiv:1305.1939 [hep-th]`.

[5] B. Basso, S. Komatsu, and P. Vieira, *Structure Constants and Integrable Bootstrap in Planar N=4 SYM Theory*, `arXiv:1505.06745 [hep-th]`.

[6] J. Polchinski, *Scale and Conformal Invariance in Quantum Field Theory*, Nucl.Phys. **B303** (1988) 226.

[7] A. Dymarsky, Z. Komargodski, A. Schwimmer, and S. Theisen, *On Scale and Conformal Invariance in Four Dimensions*, `arXiv:1309.2921 [hep-th]`.

[8] Y. Nakayama, *Is boundary conformal in CFT?*, Phys. Rev. **D87** (2013) no. 4, 046005, `arXiv:1210.6439 [hep-th]`.

[9] D. McAvity and H. Osborn, *Energy momentum tensor in conformal field theories near a boundary*, Nucl.Phys. **B406** (1993) 655–680, `arXiv:hep-th/9302068 [hep-th]`.

[10] D. McAvity and H. Osborn, *Conformal field theories near a boundary in general dimensions*, Nucl.Phys. **B455** (1995) 522–576, `arXiv:cond-mat/9505127 [cond-mat]`.

[11] J. L. Cardy, *Conformal Invariance and Surface Critical Behavior*, Nucl.Phys. **B240** (1984) 514–532.

[12] J. L. Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, Nucl.Phys. **B324** (1989) 581.

[13] J. L. Cardy and D. C. Lewellen, *Bulk and boundary operators in conformal field theory*, Phys.Lett. **B259** (1991) 274–278.

[14] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, Nucl. Phys. **B570** (2000) 525–589, `arXiv:hep-th/9908036 [hep-th]`. [Nucl. Phys.B579,707(2000)].

[15] C. Bachas, J. de Boer, R. Dijkgraaf, and H. Ooguri, *Permeable conformal walls and holography*, JHEP **0206** (2002) 027, `arXiv:hep-th/0111210 [hep-th]`.

[16] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. **B763** (2007) 354–430, `arXiv:hep-th/0607247 [hep-th]`.

[17] M. Hasenbusch, *Thermodynamic Casimir force: A Monte Carlo study of the crossover between the ordinary and the normal surface universality class*,Phys. Rev. B**83** (Apr., 2011) 134425, `arXiv:1012.4986 [cond-mat.stat-mech]`.

[18] H. Diehl and S. Dietrich, *Field-theoretical approach to multicritical behavior near free surfaces*, Phys.Rev. **B24** (1981) 2878–2880.

[19] C. Cosme, J. M. V. P. Lopes, and J. Penedones, *Conformal symmetry of the critical 3D Ising model inside a sphere*, JHEP **08** (2015) 022, `arXiv:1503.02011 [hep-th]`.

[20] A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, JHEP **06** (2001) 063, `arXiv:hep-th/0105132 [hep-th]`.

[21] O. DeWolfe, D. Z. Freedman, and H. Ooguri, *Holography and defect conformal field theories*, Phys. Rev. **D66** (2002) 025009, `arXiv:hep-th/0111135 [hep-th]`.

[22] J. Erdmenger, Z. Guralnik, and I. Kirsch, *Four-dimensional superconformal theories with interacting boundaries or defects*, Phys. Rev. **D66** (2002) 025020, `arXiv:hep-th/0203020 [hep-th]`.

[23] O. Aharony, O. DeWolfe, D. Z. Freedman, and A. Karch, *Defect conformal field theory and locally localized gravity*, JHEP **07** (2003) 030, `arXiv:hep-th/0303249 [hep-th]`.

[24] D. Gaiotto and E. Witten, *Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory*, J. Statist. Phys. **135** (2009) 789–855, `arXiv:0804.2902 [hep-th]`.

[25] M. de Leeuw, C. Kristjansen, and K. Zarembo, *One-point Functions in Defect CFT and Integrability*, JHEP **08** (2015) 098, `arXiv:1506.06958 [hep-th]`.

[26] M. Rapcak, *Nonintegrability of NS5-like Interface in $\mathcal{N} = 4$ Supersymmetric Yang-Mills*, `arXiv:1511.02243 [hep-th]`.

[27] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, Phys.Rev. **D74** (2006) 025005, `arXiv:hep-th/0501015 [hep-th]`.

[28] E. I. Buchbinder, J. Gomis, and F. Passerini, *Holographic gauge theories in background fields and surface operators*, JHEP **12** (2007) 101, `arXiv:0710.5170 [hep-th]`.

[29] S. Gukov and E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, `arXiv:hep-th/0612073 [hep-th]`.

[30] N. R. Constable, J. Erdmenger, Z. Guralnik, and I. Kirsch, *Intersecting D-3 branes and holography,* Phys. Rev. **D68** (2003) 106007, arXiv:hep-th/0211222 [hep-th].

[31] s. J. Dias, G. T. Horowitz, N. Iqbal, and J. E. Santos, *Vortices in holographic superfluids and superconductors as conformal defects*, JHEP **1404** (2014) 096, arXiv:1311.3673 [hep-th].

[32] M. F. Paulos, S. Rychkov, B. C. van Rees, and B. Zan, *Conformal Invariance in the Long-Range Ising Model*, arXiv:1509.00008 [hep-th].

[33] S. Sachdev, C. Buragohain, and M. Vojta, *Quantum Impurity in a Nearly Critical Two Dimensional Antiferromagnet*,eprint arXiv:cond-mat/0004156 (Apr., 2000) , cond-mat/0004156.

[34] A. Kolezhuk, S. Sachdev, R. R. Biswas, and P. Chen, *Theory of quantum impurities in spin liquids*,Phys. Rev. B**74** (Oct., 2006) 165114, cond-mat/0606385.

[35] S. Powell and S. Sachdev, *Excited-state spectra at the superfluid-insulator transition out of paired condensates*,Phys. Rev. A**75** (Mar., 2007) 031601, cond-mat/0608611.

[36] S. Powell and S. Sachdev, *Spin dynamics across the superfluid-insulator transition of spinful bosons*,Phys. Rev. A**76** (Sept., 2007) 033612, cond-mat/0703011.

[37] M. A. Metlitski and S. Sachdev, *Valence bond solid order near impurities in two-dimensional quantum antiferromagnets*,Phys. Rev. B**77** (Feb., 2008) 054411, arXiv:0710.0626 [cond-mat.str-el].

[38] A. Allais and S. Sachdev, *Spectral function of a localized fermion coupled to the Wilson-Fisher conformal field theory*, Phys.Rev. **B90** (2014) 035131, arXiv:1406.3022 [cond-mat.str-el].

[39] M. Billó, M. Caselle, D. Gaiotto, F. Gliozzi, M. Meineri, *et al.*, *Line defects in the 3d Ising model*, JHEP **1307** (2013) 055, arXiv:1304.4110 [hep-th].

[40] L. Bianchi, M. Meineri, R. C. Myers, and M. Smolkin, *Rényi entropy and conformal defects*, arXiv:1511.06713 [hep-th].

[41] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, JHEP **0812** (2008) 031, arXiv:0807.0004 [hep-th].

[42] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, *et al.*, *Solving the 3D Ising Model with the Conformal Bootstrap*, Phys.Rev. **D86** (2012) 025022, arXiv:1203.6064 [hep-th].

[43] F. Gliozzi, *More constraining conformal bootstrap*, Phys.Rev.Lett. **111** (2013) 161602, arXiv:1307.3111.

[44] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, *et al.*, *Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents*, J.Stat.Phys. **xx** (2014) xx, arXiv:1403.4545 [hep-th].

[45] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping Mixed Correlators in the 3D Ising Model*, arXiv:1406.4858 [hep-th].

[46] Z. Komargodski and A. Zhiboedov, *Convexity and Liberation at Large Spin*, JHEP **1311** (2013) 140, arXiv:1212.4103 [hep-th].

[47] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, *The Analytic Bootstrap and AdS Superhorizon Locality*, JHEP **12** (2013) 004, `arXiv:1212.3616 [hep-th]`.

[48] M. Billò, V. Gonçalves, E. Lauria, and M. Meineri, *Defects in conformal field theory*, JHEP **04** (2016) 091, `arXiv:1601.02883 [hep-th]`.

[49] P. Liendo, L. Rastelli, and B. C. van Rees, *The Bootstrap Program for Boundary* CFT_d, JHEP **1307** (2013) 113, `arXiv:1210.4258 [hep-th]`.

[50] D. Gaiotto, D. Mazac, and M. F. Paulos, *Bootstrapping the 3d Ising twist defect*, JHEP **1403** (2014) 100, `arXiv:1310.5078 [hep-th]`.

[51] F. Gliozzi, P. Liendo, M. Meineri, and A. Rago, *Boundary and Interface CFTs from the Conformal Bootstrap*, JHEP **05** (2015) 036, `arXiv:1502.07217 [hep-th]`.

[52] X. Dong, *Shape Dependence of Holographic Rényi Entropy in Conformal Field Theories*, Phys. Rev. Lett. **116** (2016) no. 25, 251602, `arXiv:1602.08493 [hep-th]`.

[53] H. Osborn and A. Petkou, *Implications of conformal invariance in field theories for general dimensions*, Annals Phys. **231** (1994) 311–362, `arXiv:hep-th/9307010 [hep-th]`.

[54] R. Jackiw and S.-Y. Pi, *Tutorial on scale and conformal symmetries in diverse dimensions*,Journal of Physics A Mathematical General **44** (June, 2011) 223001, `arXiv:1101.4886 [math-ph]`.

[55] J. L. Cardy, *Scaling and renormalization in statistical physics*,.

[56] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*,.

[57] K. G. Wilson, *Nonlagrangian models of current algebra*, Phys.Rev. **179** (1969) 1499–1512.

[58] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*,.

[59] J. Holland and S. Hollands, *Associativity of the operator product expansion*, Journal of Mathematical Physics **56** (2015) no. 12, . `http://scitation.aip.org/content/aip/journal/jmp/56/12/10.1063/1.4937811`.

[60] J. Holland and S. Hollands, *Recursive Construction of Operator Product Expansion Coefficients*, Communications in Mathematical Physics **336** (2015) no. 3, 1555–1606. `http://dx.doi.org/10.1007/s00220-014-2274-8`.

[61] M. B. Fröb and J. Holland, *All-order existence of and recursion relations for the operator product expansion in Yang-Mills theory*,ArXiv e-prints (Mar., 2016) , `arXiv:1603.08012 [math-ph]`.

[62] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*,.

[63] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, Nucl.Phys. **B241** (1984) 333–380.

[64] K. Jensen and A. O'Bannon, *A Constraint on Defect and Boundary Renormalization Group Flows*, `arXiv:1509.02160 [hep-th]`.

[65] S. N. Solodukhin, *Boundary terms of conformal anomaly*, Phys. Lett. **B752** (2015) 131–134, `arXiv:1510.04566 [hep-th]`.

[66] P. A. M. Dirac, *Wave equations in conformal space*, Annals Math. **37** (1936) 429–442.

[67] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Correlators*, JHEP **1111** (2011) 071, `arXiv:1107.3554 [hep-th]`.

[68] S. Weinberg, *Six-dimensional Methods for Four-dimensional Conformal Field Theories*, Phys. Rev. **D82** (2010) 045031, `arXiv:1006.3480 [hep-th]`.

[69] M. S. Costa and T. Hansen, *Conformal correlators of mixed-symmetry tensors*, JHEP **02** (2015) 151, `arXiv:1411.7351 [hep-th]`.

[70] M. S. Costa, V. Gonçalves, and J. Penedones, *Spinning AdS Propagators*, JHEP **09** (2014) 064, `arXiv:1404.5625 [hep-th]`.

[71] V. K. Dobrev, V. B. Petkova, S. G. Petrova, and I. T. Todorov, *Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory*, Phys. Rev. **D13** (1976) 887.

[72] M. Hogervorst and S. Rychkov, *Radial Coordinates for Conformal Blocks*, Phys. Rev. **D87** (2013) 106004, `arXiv:1303.1111 [hep-th]`.

[73] F. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, Nucl.Phys. **B599** (2001) 459–496, `arXiv:hep-th/0011040 [hep-th]`.

[74] F. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, `arXiv:1108.6194 [hep-th]`.

[75] M. S. Costa, V. Gonçalves, and J. Penedones, *Conformal Regge theory*, JHEP **12** (2012) 091, `arXiv:1209.4355 [hep-th]`.

[76] F. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, Nucl.Phys. **B678** (2004) 491–507, `arXiv:hep-th/0309180 [hep-th]`.

[77] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Blocks*, JHEP **11** (2011) 154, `arXiv:1109.6321 [hep-th]`.

[78] J. Penedones, E. Trevisani, and M. Yamazaki, *Recursion Relations for Conformal Blocks*, `arXiv:1509.00428 [hep-th]`.

[79] S. Rychkov and P. Yvernay, *Remarks on the Convergence Properties of the Conformal Block Expansion*, `arXiv:1510.08486 [hep-th]`.

[80] D. Simmons-Duffin, *Projectors, Shadows, and Conformal Blocks*, JHEP **1404** (2014) 146, `arXiv:1204.3894 [hep-th]`.

[81] E. Eisenriegler, *Polymers Near Surfaces.* World Scientific, Singapore, 1993.

[82] T. Quella, I. Runkel, and G. M. Watts, *Reflection and transmission for conformal defects*, JHEP **0704** (2007) 095, `arXiv:hep-th/0611296 [hep-th]`.

[83] J. Fuchs, C. Schweigert, and C. Stigner, *The Classifying algebra for defects*, Nucl. Phys. **B843** (2011) 673–723, `arXiv:1007.0401 [hep-th]`.

[84] V. Gonçalves, E. Lauria, M. Meineri, and E. Trevisani Work in progress .

[85] M. Meineri *et al.* Work in progress .

[86] S. Ferrara, A. Grillo, and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, Annals Phys. **76** (1973) 161–188.

[87] T. Dimofte and D. Gaiotto, *An E7 Surprise*, JHEP **1210** (2012) 129, `arXiv:1209.1404 [hep-th]`.

[88] L. P. Eisenhart, *Riemannian geometry.* Princeton University Press, 1997.

[89] Y. Aminov, *The geometry of submanifolds.* Gordon and Breach Science Publishers, 2001.

[90] D. Poland and D. Simmons-Duffin, *The conformal bootstrap*, Nature Phys. (2016) 535–539.

[91] D. Simmons-Duffin, *TASI Lectures on the Conformal Bootstrap*, `arXiv:1602.07982 [hep-th]`.

[92] A. C. Echeverri, B. von Harling, and M. Serone, *The Effective Bootstrap*, `arXiv:1606.02771 [hep-th]`.

[93] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, *OPE Convergence in Conformal Field Theory*, Phys.Rev. **D86** (2012) 105043, `arXiv:1208.6449 [hep-th]`.

[94] V. S. Rychkov and A. Vichi, *Universal Constraints on Conformal Operator Dimensions*, Phys.Rev. **D80** (2009) 045006, `arXiv:0905.2211 [hep-th]`.

[95] R. Rattazzi, S. Rychkov, and A. Vichi, *Central Charge Bounds in 4D Conformal Field Theory*, Phys.Rev. **D83** (2011) 046011, `arXiv:1009.2725 [hep-th]`.

[96] D. Poland and D. Simmons-Duffin, *Bounds on 4D Conformal and Superconformal Field Theories*, JHEP **1105** (2011) 017, `arXiv:1009.2087 [hep-th]`.

[97] S. El-Showk and M. F. Paulos, *Bootstrapping Conformal Field Theories with the Extremal Functional Method*, Phys.Rev.Lett. **111** (2013) no. 24, 241601, `arXiv:1211.2810 [hep-th]`.

[98] C. Beem, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 4$ Superconformal Bootstrap*, Phys.Rev.Lett. **111** (2013) 071601, `arXiv:1304.1803 [hep-th]`.

[99] Y. Nakayama and T. Ohtsuki, *Five dimensional $O(N)$-symmetric CFTs from conformal bootstrap*, Phys.Lett. **B734** (2014) 193–197, `arXiv:1404.5201 [hep-th]`.

[100] S. M. Chester, J. Lee, S. S. Pufu, and R. Yacoby, *The $\mathcal{N} = 8$ superconformal bootstrap in three dimensions*, JHEP **1409** (2014) 143, `arXiv:1406.4814 [hep-th]`.

[101] S. M. Chester, S. S. Pufu, and R. Yacoby, *Bootstrapping O(N) Vector Models in $4 < d < 6$*, `arXiv:1412.7746 [hep-th]`.

[102] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, `arXiv:1412.7541 [hep-th]`.

[103] D. Simmons-Duffin, *A Semidefinite Program Solver for the Conformal Bootstrap*, `arXiv:1502.02033 [hep-th]`.

[104] N. Bobev, S. El-Showk, D. Mazac, and M. F. Paulos, *Bootstrapping the Three-Dimensional Supersymmetric Ising Model*, `arXiv:1502.04124 [hep-th]`.

[105] F. Gliozzi and A. Rago, *Critical exponents of the 3d Ising and related models from Conformal Bootstrap*, `arXiv:1403.6003 [hep-th]`.

[106] Y. Nakayama, *Bootstrapping critical Ising model on three-dimensional real projective space*, Phys. Rev. Lett. **116** (2016) no. 14, 141602, `arXiv:1601.06851 [hep-th]`.

[107] H. Diehl and M. Shpot, *Massive field theory approach to surface critical behavior in three-dimensional systems*, Nucl.Phys. **B528** (1998) 595–647, `arXiv:cond-mat/9804083 [cond-mat]`.

[108] Y. Deng, H. W. J. Blöte, and M. P. Nightingale, *Surface and bulk transitions in three-dimensional O (n) models*,Phys. Rev. E**72** (July, 2005) 016128, `cond-mat/0504173`.

[109] T. Prellberg, *Scaling of Self-Avoiding Walks and Self-Avoiding Trails in Tree Dimensions*, J. Phys. A: Math. Gen. **34** (2001) L599, `arXiv:cond-mat/0108538 [hep-th]`.

[110] M. Hasenbusch, *Finite size scaling study of lattice models in the three-dimensional Ising universality class*, Phys.Rev. **B82** (2010) 174433, `arXiv:1004.4486`.

[111] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, *Critical behavior of the three-dimensional xy universality class*, Phys.Rev. **B63** (2001) 214503, `arXiv:cond-mat/0010360 [cond-mat]`.

[112] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, *Critical exponents and equation of state of the three-dimensional Heisenberg universality class*, Phys.Rev. **B65** (2002) 144520, `arXiv:cond-mat/0110336 [cond-mat]`.

[113] R. Guida and J. Zinn-Justin, *Critical exponents of the N vector model*, J.Phys. **A31** (1998) 8103–8121, `arXiv:cond-mat/9803240 [cond-mat]`.

[114] D. F. Litim, *Critical exponents from optimized renormalization group flows*, Nucl. Phys. **B631** (2002) 128–158, `arXiv:hep-th/0203006 [hep-th]`.

[115] M. Caselle, G. Costagliola, and N. Magnoli, *Numerical determination of OPE coefficients in the 3D Ising model from off-critical correlators*, `arXiv:1501.04065 [hep-th]`.

[116] S. Fredenhagen and T. Quella, *Generalised permutation branes*, JHEP **0511** (2005) 004, `arXiv:hep-th/0509153 [hep-th]`.

[117] I. Brunner and D. Roggenkamp, *Defects and bulk perturbations of boundary Landau-Ginzburg orbifolds*, JHEP **0804** (2008) 001, `arXiv:0712.0188 [hep-th]`.

[118] D. Gaiotto, *Domain Walls for Two-Dimensional Renormalization Group Flows*, JHEP **1212** (2012) 103, `arXiv:1201.0767 [hep-th]`.

[119] A. Konechny and C. Schmidt-Colinet, *Entropy of conformal perturbation defects*, J.Phys. **A47** (2014) no. 48, 485401, `arXiv:1407.6444 [hep-th]`.

[120] G. Poghosyan and H. Poghosyan, *RG domain wall for the N=1 minimal superconformal models*, `arXiv:1412.6710 [hep-th]`.

[121] L. S. Brown, *Dimensional Regularization of Composite Operators in Scalar Field Theory*, Annals Phys. **126** (1980) 135.

[122] A. Zamolodchikov, *Renormalization Group and Perturbation Theory Near Fixed Points in Two-Dimensional Field Theory*, Sov.J.Nucl.Phys. **46** (1987) 1090.

[123] M. R. Gaberdiel, A. Konechny, and C. Schmidt-Colinet, *Conformal perturbation theory beyond the leading order*, J.Phys. **A42** (2009) 105402, `arXiv:0811.3149 [hep-th]`.

[124] H. Diehl and S. Dietrich, *Multicritical behaviour at surfaces*, Zeit. f. Phys B **50** (1983) 117.

[125] H. Diehl, S. Dietrich, and E. Eisenriegler, *Universality, irrelevant surface operators, and corrections to scaling in systems with free surfaces and defect planes*, Phys.Rev. **B27** (1983) 2937–2954.

[126] Y. Deng and H. W. Blote, *Simultaneous analysis of several models in the three-dimensional Ising universality class*, Phys.Rev. **E68** (2003) 036125.

[127] R. Savit, *Duality in Field Theory and Statistical Systems*, Rev.Mod.Phys. **52** (1980) 453.

[128] G. Parisi, *STATISTICAL FIELD THEORY*,.

[129] M. Hasenbusch, *Finite size scaling study of lattice models in the three-dimensional Ising universality class*,Phys. Rev. B**82** (Nov., 2010) 174433, `arXiv:1004.4486 [cond-mat.stat-mech]`.

[130] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, *25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple-cubic lattice*,Phys. Rev. E**65** (June, 2002) 066127, `cond-mat/0201180`.

[131] E. Schrödinger, *Discussion of Probability Relations between Separated Systems*,Mathematical Proceedings of the Cambridge Philosophical Society **31** (10, 1935) 555–563. `http://journals.cambridge.org/article_S0305004100013554`.

[132] M. Levin and X.-G. Wen, *Detecting Topological Order in a Ground State Wave Function*, Phys. Rev. Lett. **96** (2006) 110405.

[133] A. Kitaev and J. Preskill, *Topological entanglement entropy*, Phys. Rev. Lett. **96** (2006) 110404, `arXiv:hep-th/0510092 [hep-th]`.

[134] H. Li and F. Haldane, *Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States*, Phys. Rev. Lett. **101** (2008) no. 1, 010504.

[135] M. Van Raamsdonk, *Building up spacetime with quantum entanglement*, Gen. Rel. Grav. **42** (2010) 2323–2329, `arXiv:1005.3035 [hep-th]`. [Int. J. Mod. Phys.D19,2429(2010)].

[136] E. Bianchi and R. C. Myers, *On the Architecture of Spacetime Geometry*, Class. Quant. Grav. **31** (2014) 214002, `arXiv:1212.5183 [hep-th]`.

[137] J. Maldacena and L. Susskind, *Cool horizons for entangled black holes*, Fortsch. Phys. **61** (2013) 781–811, `arXiv:1306.0533 [hep-th]`.

[138] T. Jacobson, *Entanglement equilibrium and the Einstein equation*, `arXiv:1505.04753 [gr-qc]`.

[139] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. Rispoli, and M. Greiner, *Measuring entanglement entropy through the interference of quantum many-body twins*, `arXiv:1509.01160 [cond-mat]`.

[140] A. Rényi, *On measures of entropy and information*, in *Fourth Berkeley symposium on mathematical statistics and probability*, vol. 1, pp. 547–561. 1961.

[141] A. Rényi, *On the foundations of information theory*, Revue de l'Institut International de Statistique (1965) 1–14.

[142] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Dec., 2010.

[143] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Entanglement in many-body systems*,Reviews of Modern Physics **80** (Apr., 2008) 517–576, `quant-ph/0703044`.

[144] A. Einstein, B. Podolsky, and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*,Physical Review **47** (May, 1935) 777–780.

[145] P. Walters, *An introduction to ergodic theory*. Graduate texts in mathematics. Springer, New York, Berlin, Heidelberg, 1982. `http://opac.inria.fr/record=b1090845`. Autres tirages : 2000, 2001.

[146] A. Rényi, *On Measures of Entropy and Information*, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pp. 547–561. University of California Press, Berkeley, Calif., 1961. `http://projecteuclid.org/euclid.bsmsp/1200512181`.

[147] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, *On quantum Rényi entropies: A new generalization and some properties*,Journal of Mathematical Physics **54** (Dec., 2013) 122203–122203, `arXiv:1306.3142 [quant-ph]`.

[148] M. A. Nielsen, *Conditions for a Class of Entanglement Transformations*,Physical Review Letters **83** (July, 1999) 436–439, `quant-ph/9811053`.

[149] T. Hartman, *Quantum gravity and Black Holes*, Lecture notes (2015) . `http://www.hartmanhep.net/topics2015/`.

[150] D. N. Page, *Average entropy of a subsystem*,Physical Review Letters **71** (Aug., 1993) 1291–1294, `gr-qc/9305007`.

[151] D. Harlow, *Jerusalem Lectures on Black Holes and Quantum Information*, Rev. Mod. Phys. **88** (2016) 15002, `arXiv:1409.1231 [hep-th]`. [Rev. Mod. Phys.88,15002(2016)].

[152] U. Schollwöck, *The density-matrix renormalization group*,Reviews of Modern Physics **77** (Jan., 2005) 259–315, `cond-mat/0409292`.

[153] H. Casini, M. Huerta, and J. A. Rosabal, *Remarks on entanglement entropy for gauge fields*, Phys. Rev. **D89** (2014) no. 8, 085012, `arXiv:1312.1183 [hep-th]`.

[154] H. Liu and M. Mezei, *A Refinement of entanglement entropy and the number of degrees of freedom*, JHEP **04** (2013) 162, `arXiv:1202.2070 [hep-th]`.

[155] R. C. Myers and A. Sinha, *Seeing a c-theorem with holography*, Phys. Rev. **D82** (2010) 046006, `arXiv:1006.1263 [hep-th]`.

[156] H. Casini, M. Huerta, and R. C. Myers, *Towards a derivation of holographic entanglement entropy*, JHEP **1105** (2011) 036, `arXiv:1102.0440 [hep-th]`.

[157] H. Casini and M. Huerta, *On the RG running of the entanglement entropy of a circle*, Phys. Rev. **D85** (2012) 125016, `arXiv:1202.5650 [hep-th]`.

[158] H. Casini and M. Huerta, *Entanglement entropy in free quantum field theory*, J. Phys. **A42** (2009) 504007, `arXiv:0905.2562 [hep-th]`.

[159] H. Casini, M. Huerta, R. C. Myers, and A. Yale, *Mutual information and the F-theorem*, JHEP **10** (2015) 003, `arXiv:1506.06195 [hep-th]`.

[160] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett. **B428** (1998) 105–114, `arXiv:hep-th/9802109 [hep-th]`.

[161] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253–291, `arXiv:hep-th/9802150 [hep-th]`.

[162] S. de Haro, S. N. Solodukhin, and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, Commun. Math. Phys. **217** (2001) 595–622, `arXiv:hep-th/0002230 [hep-th]`.

[163] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, Phys.Rev.Lett. **96** (2006) 181602, `arXiv:hep-th/0603001 [hep-th]`.

[164] V. E. Hubeny, M. Rangamani, and T. Takayanagi, *A Covariant holographic entanglement entropy proposal*, JHEP **0707** (2007) 062, `arXiv:0705.0016 [hep-th]`.

[165] L.-Y. Hung, R. C. Myers, and M. Smolkin, *On Holographic Entanglement Entropy and Higher Curvature Gravity*, JHEP **1104** (2011) 025, `arXiv:1101.5813 [hep-th]`.

[166] J. de Boer, M. Kulaxizi, and A. Parnachev, *Holographic Entanglement Entropy in Lovelock Gravities*, JHEP **07** (2011) 109, `arXiv:1101.5781 [hep-th]`.

[167] X. Dong, *Holographic Entanglement Entropy for General Higher Derivative Gravity*, JHEP **1401** (2014) 044, `arXiv:1310.5713 [hep-th]`.

[168] J. Camps, *Generalized entropy and higher derivative Gravity*, JHEP **1403** (2014) 070, `arXiv:1310.6659 [hep-th]`.

[169] A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, JHEP **1308** (2013) 090, `arXiv:1304.4926 [hep-th]`.

[170] G. 't Hooft, *Dimensional reduction in quantum gravity*, in *Salamfest 1993:0284-296*, pp. 0284–296. 1993. `arXiv:gr-qc/9310026 [gr-qc]`.

[171] L. Susskind, *The World as a hologram*, J. Math. Phys. **36** (1995) 6377–6396, `arXiv:hep-th/9409089 [hep-th]`.

[172] L.-Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, *Holographic Calculations of Renyi Entropy*, JHEP **12** (2011) 047, `arXiv:1110.1084 [hep-th]`.

[173] X. Dong, *An Area-Law Prescription for Holographic Renyi Entropies*, `arXiv:1601.06788 [hep-th]`.

[174] M. Mezei, *Entanglement entropy across a deformed sphere*, Phys. Rev. **D91** (2015) no. 4, 045038, `arXiv:1411.7011 [hep-th]`.

[175] A. Allais and M. Mezei, *Some results on the shape dependence of entanglement and Rényi entropies*, Phys. Rev. **D91** (2015) no. 4, 046002, `arXiv:1407.7249 [hep-th]`.

[176] P. Bueno, R. C. Myers, and W. Witczak-Krempa, *Universality of corner entanglement in conformal field theories*, Phys. Rev. Lett. **115** (2015) no. 2, 021602, `arXiv:1505.04804 [hep-th]`.

[177] P. Bueno and R. C. Myers, *Corner contributions to holographic entanglement entropy*, JHEP **08** (2015) 068, `arXiv:1505.07842 [hep-th]`.

[178] P. Bueno, R. C. Myers, and W. Witczak-Krempa, *Universal corner entanglement from twist operators*, JHEP **09** (2015) 091, `arXiv:1507.06997 [hep-th]`.

[179] P. Bueno and R. C. Myers, *Universal entanglement for higher dimensional cones*, JHEP **12** (2015) 168, `arXiv:1508.00587 [hep-th]`.

[180] J. Lee, L. McGough, and B. R. Safdi, *Rényi entropy and geometry*, Phys. Rev. **D89** (2014) no. 12, 125016, `arXiv:1403.1580 [hep-th]`.

[181] A. Lewkowycz and E. Perlmutter, *Universality in the geometric dependence of Renyi entropy*, JHEP **01** (2015) 080, `arXiv:1407.8171 [hep-th]`.

[182] L.-Y. Hung, R. C. Myers, and M. Smolkin, *Twist operators in higher dimensions*, JHEP **10** (2014) 178, `arXiv:1407.6429 [hep-th]`.

[183] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, J. Stat. Mech. **0406** (2004) P06002, `arXiv:hep-th/0405152 [hep-th]`.

[184] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, J. Statist. Phys. **130** (2008) 129–168, `arXiv:0706.3384 [hep-th]`.

[185] V. G. Knizhnik, *Analytic Fields on Riemann Surfaces. 2*, Commun. Math. Phys. **112** (1987) 567–590.

[186] L. J. Dixon, D. Friedan, E. J. Martinec, and S. H. Shenker, *The Conformal Field Theory of Orbifolds*, Nucl. Phys. **B282** (1987) 13–73.

[187] M. Caraglio and F. Gliozzi, *Entanglement Entropy and Twist Fields*, JHEP **11** (2008) 076, `arXiv:0808.4094 [hep-th]`.

[188] T. Faulkner, R. G. Leigh, and O. Parrikar, *Shape Dependence of Entanglement Entropy in Conformal Field Theories*, JHEP **04** (2016) 088, `arXiv:1511.05179 [hep-th]`.

[189] A. Lewkowycz and E. Perlmutter, *Universality in the geometric dependence of Rényi entropy*, JHEP **01** (2015) 080, `arXiv:1407.8171 [hep-th]`.

[190] J. S. Dowker, *Conformal weights of charged Rényi entropy twist operators for free scalar fields in arbitrary dimensions*, J. Phys. **A49** (2016) no. 14, 145401, `arXiv:1508.02949 [hep-th]`.

[191] J. S. Dowker, *Conformal weights of charged Renyi entropy twist operators for free Dirac fields in arbitrary dimensions*, `arXiv:1510.08378 [hep-th]`.

[192] L. Bianchi and etal, *On the Shape Dependence of Rényi Entropy in Free CFTs*, in preparation .

[193] R. C. Myers and A. Singh, *Entanglement Entropy for Singular Surfaces*, JHEP **09** (2012) 013, `arXiv:1206.5225 [hep-th]`.

[194] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, JHEP **1405** (2014) 025, `arXiv:1312.5682 [hep-th]`.

[195] D. Correa, J. Henn, J. Maldacena, and A. Sever, *An exact formula for the radiation of a moving quark in N=4 super Yang Mills*, JHEP **1206** (2012) 048, `arXiv:1202.4455 [hep-th]`.

[196] B. Fiol, E. Gerchkovitz, and Z. Komargodski, *The Exact Bremsstrahlung Function in N=2 Superconformal Field Theories*, `arXiv:1510.01332 [hep-th]`.

[197] V. Mitev and E. Pomoni, *Exact Bremsstrahlung and Effective Couplings*, `arXiv:1511.02217 [hep-th]`.

[198] J. M. Maldacena, *Wilson loops in large N field theories*, Phys. Rev. Lett. **80** (1998) 4859–4862, `arXiv:hep-th/9803002 [hep-th]`.

[199] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, Eur. Phys. J. **C22** (2001) 379–394, `arXiv:hep-th/9803001 [hep-th]`.

[200] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, JHEP **0608** (2006) 045, `arXiv:hep-th/0605073 [hep-th]`.

[201] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **10** (2008) 091, `arXiv:0806.1218 [hep-th]`.

[202] N. Drukker, J. Gomis, and S. Matsuura, *Probing N=4 SYM With Surface Operators*, JHEP **10** (2008) 048, `arXiv:0805.4199 [hep-th]`.

[203] T. Quella and V. Schomerus, *Symmetry breaking boundary states and defect lines*, JHEP **06** (2002) 028, `arXiv:hep-th/0203161 [hep-th]`.

[204] V. B. Petkova and J. B. Zuber, *Generalized twisted partition functions*, Phys. Lett. **B504** (2001) 157–164, `arXiv:hep-th/0011021 [hep-th]`.

[205] J. Fuchs, M. R. Gaberdiel, I. Runkel, and C. Schweigert, *Topological defects for the free boson CFT*, J. Phys. **A40** (2007) 11403, `arXiv:0705.3129 [hep-th]`.

[206] M. Nozaki, T. Numasawa, A. Prudenziati, and T. Takayanagi, *Dynamics of Entanglement Entropy from Einstein Equation*, Phys. Rev. **D88** (2013) no. 2, 026012, `arXiv:1304.7100 [hep-th]`.

[207] G. W. Semenoff and D. Young, *Wavy Wilson line and AdS / CFT*, Int. J. Mod. Phys. **A20** (2005) 2833–2846, `arXiv:hep-th/0405288 [hep-th]`.

[208] M. Smolkin and S. N. Solodukhin, *Correlation functions on conical defects*, Phys. Rev. **D91** (2015) no. 4, 044008, `arXiv:1406.2512 [hep-th]`.

[209] M. E. X. Guimaraes and B. Linet, *Scalar Green's functions in an Euclidean space with a conical-type line singularity*, Commun. Math. Phys. **165** (1994) 297–310.

[210] M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko, *Measuring Renyi entanglement entropy in quantum Monte Carlo simulations*, Physical review letters **104** (2010) no. 15, 157201.

[211] A. B. Kallin, M. B. Hastings, R. G. Melko, and R. R. Singh, *Anomalies in the entanglement properties of the square-lattice Heisenberg model*, Physical Review B **84** (2011) no. 16, 165134.

[212] A. B. Kallin, E. Stoudenmire, P. Fendley, R. R. Singh, and R. G. Melko, *Corner contribution to the entanglement entropy of an O (3) quantum critical point in 2+ 1 dimensions*, Journal of Statistical Mechanics: Theory and Experiment **2014** (2014) no. 6, P06009.

[213] H. Casini and M. Huerta, *Entanglement entropy for the n-sphere*, Phys.Lett. **B694** (2010) 167–171, `arXiv:1007.1813 [hep-th]`.

[214] E. Perlmutter, *A universal feature of CFT Rényi entropy*, JHEP **03** (2014) 117, `arXiv:1308.1083 [hep-th]`.

[215] J. Lee, A. Lewkowycz, E. Perlmutter, and B. R. Safdi, *Rényi entropy, stationarity, and entanglement of the conformal scalar*, JHEP **03** (2015) 075, `arXiv:1407.7816 [hep-th]`.

[216] S. N. Solodukhin, *Entanglement entropy, conformal invariance and extrinsic geometry*, Phys. Lett. **B665** (2008) 305–309, `arXiv:0802.3117 [hep-th]`.

[217] R.-X. Miao, *Universal Terms of Entanglement Entropy for 6d CFTs*, JHEP **10** (2015) 049, `arXiv:1503.05538 [hep-th]`.

[218] C. Fefferman and C. R. Graham, *The ambient metric*, `arXiv:0710.0919 [math.DG]`.

[219] J. Erdmenger and H. Osborn, *Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions*, Nucl. Phys. **B483** (1997) 431–474, `arXiv:hep-th/9605009 [hep-th]`.

[220] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha, and M. Smolkin, *Holographic GB gravity in arbitrary dimensions*, JHEP **03** (2010) 111, `arXiv:0911.4257 [hep-th]`.

[221] J. Cardy, *Some results on the mutual information of disjoint regions in higher dimensions*,Journal of Physics A Mathematical General **46** (July, 2013) 285402, arXiv:1304.7985 [hep-th].

[222] C. A. Agón, I. Cohen-Abbo, and H. J. Schnitzer, *Large distance expansion of Mutual Information for disjoint disks in a free scalar theory*,ArXiv e-prints (May, 2015) , arXiv:1505.03757 [hep-th].

[223] M. A. Metlitski, C. A. Fuertes, and S. Sachdev, *Entanglement Entropy in the O(N) model*, Phys. Rev. **B80** (2009) no. 11, 115122, arXiv:0904.4477 [cond-mat.stat-mech].

[224] A. Belin, A. Maloney, and S. Matsuura, *Holographic Phases of Renyi Entropies*, JHEP **12** (2013) 050, arXiv:1306.2640 [hep-th].