Università degli Studi di Pavia
Dipartimento di Fisica

Dottorato di Ricerca in Fisica – XXIX ciclo

Local approach to Casimir effect in axiomatic quantum field theory

Gabriele Nosari

Submitted to the Graduate School of Physics
in partial fulfillment of the requirements for the degree of

DOTTORE DI RICERCA IN FISICA
DOCTOR OF PHILOSOPHY IN PHYSICS

at the University of Pavia

Supervisor: Claudio Dappiaggi

Cover: A pictorial (and frivolous) representation of a static wedge-shaped region W_α, with opening angle α, to which it is assigned the algebra of observables associated to a quantum field theory, $\mathcal{W}^{wed}(W_\alpha)$. (See Chapter 4)

"Local approach to Casimir effect in axiomatic quantum field theory"
Gabriele Nosari
PhD thesis - University of Pavia
Pavia, Italy, December 2016.

Abstract

The Casimir effect offers an interesting test of our understanding of regularization. The diverging zero-point energy of a quantum field confined between Dirichlet boundary conditions can be cured by different paradigms often without a clear criterion to select one of them in particular. Starting from the energy density as the main object of interest (local approach), we revise systematically the regularization of such quantity by framing it in an axiomatic formulation of the underlying quantum field theory. For our purposes, it is natural to work in the framework of quantum field theory on curved spacetimes, based on the locality principle proposed by Haag. This allows for a generalization, regarding the physical setting of the Casimir effect as a quantum field theory on spacetimes with boundaries. A generalization of the usual framework of quantum field theory on globally hyperbolic spacetimes is thus required in order to include geodesically incomplete manifolds. In particular we need a suitable definition of Hadamard states, which are the main tool of the regularization paradigm we deal with. We repeat the analysis for three paramount cases of manifold with boundary: the half Minkowski spacetime, a slab confined between two parallel plates and a wedge-shaped region, confined between two intercepting plates. We exploit the method of images in order relate our theory to a suitable counterpart in Minkowski spacetime. Despite we apply a constructive technique which make use of Minkowski spacetime, our approach provides an intrinsic theory, since all objects and results refers only to elements of the system. We obtain a full characterization of the quantum field theory in each case study, inspecting its axiomatic structure and defining consistently the regularized observables, included the energy density.

Contents

Introduction

One of the most peculiar experimentally verified fingerprints of quantum field theory is undoubtedly the Casimir effect. In its original formulation [Cas48], it predicts the existence of an attracting force generated by quantum fluctuations of the electromagnetic field between uncharged, perfectly conducting metal plates in vacuum. It is argued in [Cas48] that such phenomenon can be ascribed to the response to confinement of a quantized field in the ground state. This conclusion is neither tied to a specific geometry nor to the choice of a precise field theoretical model. There is a vast literature on this topic and offering an overview is beyond the goals of this work. An interested reader could refer for example to [Mil01] and [BKMM09]. Let us just outline the main aspects which are of interest in this thesis.

The theoretical setting of the Casimir effect is based upon a Lagrangian quantum field theory confined in a bounded region of Minkowski space, where the confinement is modeled by suitable boundary conditions on the field equation. At a computational level the attracting force at the heart of the Casimir effect can be evaluated in two different ways. The first one, the original one, starts from estimating the total energy associated to a quantum field. In this *global approach*, the force is derived as the gradient of the energy. The second one is a *local approach*, since it focuses on the energy density. The latter is defined as the temporal component of the stress-energy tensor, which is built as the variation of the Lagrangian action with respect to the background metric. The Casimir force arises as the gradient of the energy density integrated over a suitable (spacelike) volume. The two approaches are equivalent in principle, since the energy contained in a volume is defined as the latter integration over the volume itself. Nonetheless, the quantization introduces some subtleties as we are going to explain.

Both the total energy and the energy density are evaluated as the expectation value of suitable (regularized) quantum observables, with respect to the ground state, that is the state of minimal energy, usually associated to the concept of vacuum. A quantum field on Minkowski space has a vanishing expectation value both for the energy and for the energy density in the ground state. This is not the case for a confined field. The non vanishing expectation value of its components represents ultimately the Casimir effect itself and it has a pure quantum nature since it is not predicted by any classical theory. Indeed, with reference specifically to the total energy, the Casimir effect is commonly considered as one of the paramount manifestations of the so called zero-point energy. With zero-point energy it is meant the energy associated to the ground state of any quantum mechanical system, and, as a feature of quantization, it has a non vanishing expectation value. When dealing with quantization of fields this general feature has even more dramatic effects, since a direct attempt to evaluate such energy yields a divergent quantity. In quantum field theory on the whole Minkowski spacetime this issue is cured by a regularization procedure, the normal ordering, which ultimately amounts to subtracting the divergent part of the ground energy, leaving hence a vanishing expectation value. An infinite zero-point energy is still present when dealing with a spatial boundary. Yet we observe that the diverging contribution to the total energy in the ground state is removed by subtracting that of the vacuum energy on the space without boundaries. The non vanishing energy underlying the Casimir effect is indeed the (finite) contribution surviving to such subtraction. From a physical perspective, this could be understood with the idea that only differences of energy are relevant, so that the subtraction represents a rescaling of a potential energy. From a formal point of view, the subtraction is rather part of a regularization procedure, which involves a wider class of local observables, in particular all those defined in terms of polynomials of the second quantized field observable. On Minkowski spacetime, such objects are the well known Wick polynomials, built by means of normal ordering. The observables associated to the energy density and to all other components of the stress energy tensor are naturally in this class, being proportional to a squared field (and to its second derivatives). When a boundary (condition) is introduced, as for the total energy, a regularization of the energy density can be achieved by subtracting the divergent contribution of the Minkowski counterpart. The subtraction is part of the point-splitting regularization scheme, in which the divergent squared field is redefined as a coincidence limit of a bilinear form. This approach yields finite values, but it is not intrinsic, since it requires the introduction of a reference quantum field, set in a larger spacetime. Many regularization schemes have been proposed – see [BKMM09] – in order to find a procedure which depends only on the system properties, such as the geometry of boundaries. At this stage the choice of the global or local approach does matter. On the one hand, regularizations of the total energy, like the global zeta function regularization, are not defining local observables[1]. On the other hand, the regularized energy density may diverge approaching the boundary. This divergence, surviving to the regularization procedure, should be considered as a direct consequence of the boundary itself, rather than an effect of the quantum zero-point energy – [Ful89]. In this thesis, we are inclined to consider

[1] In this sense they are often considered as *renormalizations* (of the total energy) rather then proper regularization schemes.

the stress-energy tensor as a fundamental quantity, to which it is associated a quantum observable. We will thus focus on the local approach. We regard the total energy as a classical object, defined by an integration of the expectation value of the temporal diagonal component of the stress-energy tensor. A possible divergence of such integral would trigger a reinterpretation of what we consider *physically meaningful* rather than *formally correct*.

The aim of this thesis is to frame the regularization of the quantum stress-energy tensor in an axiomatic formulation of the underlying quantum field theory. For our purposes, it is natural to work in the framework of quantum field theory on curved spacetimes, whose long history can be traced back in some monographs, [DeW75, BD84, Ful89, KW91, Wal94]. It is formalized within the approach of local quantum field theory – for recent reviews see [BDH13, HW14] and the book [BDFY15]. This is an axiomatic framework, first introduced by Haag & Kastler in the sixties – see [HK64], which divides the quantization of a physical system in two separate steps. The first consists of assigning to any region of spacetime the collection of all observables localized therein, which are structured in a unital $*$-algebra, encoding all information on the dynamics of the system. Such assignment is made in such a way to fulfil compatibility between overlapping regions as well as causality. The collection of all possible local observables form the algebra of observables for the quantum field. In the second step, one identifies a quantum state, that is a positive, normalized linear functional on the algebra of observables. Via the renowned GNS theorem, one can recover the standard probabilistic interpretation of all quantum theories. In this framework, different regularization schemes have been proposed, but nowadays it is widely accepted that the most effective relies on the concept of Hadamard states. They are a particular class of algebraic states being characterized by a fixed singular structure of the underlying two-point correlation function – [Rad96a, Rad96b]. Restricting to Hadamard states allows us to approach regularization by means of a point-splitting procedure, subtracting the divergent part due to the state. This procedure is intrinsic, since it turns out that the singularities of Hadamard states are fully characterized by local geometric quantities and by the dynamics of field. To apply such general analysis to the Casimir effect, we need a slight shift in perspective. Instead of considering a quantum field theory on Minkowski spacetime with boundary conditions, we want to regard it as a quantum field theory on a spacetime with non-empty boundaries[2]. Unfortunately this poses the first main obstacle. Models of quantum field theory covered by the algebraic approach are in general required to be defined on globally hyperbolic spacetimes, a geometri-

[2]A similar point of view is not unprecedented. It has been considered since the early attempts of quantization on curved spacetime as a model of non trivial background [DeW75]. Curiously enough, Casimir effect has found its main application in the study of regularization of the stress-energy tensor, as a prototypical example ([BD84, Ful89, Kay79]). In a sense, we are introducing quantum field theory on curved spacetime in the analysis of Casimir effect backwardly. Unfortunately in none of these analyses it has been faced a rigorous construction of the quantum theory in presence of a boundary.

cal requirement which ensures the well-posedness of the Cauchy problem, out of which these models are defined. A spacetime with boundaries in general does not fulfil this property, leaving us with the need of including boundary conditions in our framework. An early mathematically rigorous analysis of quantum field theories on spacetimes with boundaries can be found in [DC79]. Preliminary investigations in the algebraic framework can be found in [Kay79, Appendix] and [Küh05, Nie09, Som06]. It is especially noteworthy the analysis in [Som06] which associates to a quantum field theory on a region with boundaries the algebra generated by the subalgebras of properly embedded globally hyperbolic subregions (with a construction known as universal algebra, [BDF87] and [BFM09, Appendix B]). Boundary conditions are taken into account via appropriate algebraic ideals. Although this is a viable alternative, we shall not focus on it, leaving a comparison to our methods to future investigations. Another notable analysis of quantum field theory in spacetime with boundaries is represented by [Her04, Her05, Her10], although our approach should be seen as parallel and complementary rather than a continuation of these works. Before going into details of the main body of this thesis let us mention the alternative regularization of the Casimir energy density by means of local zeta function techniques. This method, which has been proved to be equivalent to point-splitting in [HM12], was first introduced in [Haw77], in the context of an Euclidean approach (Wick rotation) to field theories on curved space-times. The interested reader may refer to [FP14b] (and to the related series of papers). Even though we will not focus on this regularization, we will often refer to its results, being the real alternative to point-splitting in the context of local approach to Casimir effect.

Let us outline more in detail this thesis. We are interested in three idealized scenarios of a real scalar field on different bounded regions of Minkowski spacetime: a half space, bounded by a single infinitely extended hyperplane, a slab, bounded by two parallel infinitely extended hyperplanes, and a wedge, defined by two incident infinitely extended hyperplanes. Dynamics is ruled by the Klein-Gordon equation with Dirichlet boundary conditions. The first two scenarios represent standard examples, usually dubbed respectively as *Casimir-Polder system*, in analogy with [CP48], and as *Casimir system* with reference to [Cas48]. The third one is less known consisting of two intercepting plates dividing the space in wedges. We will focus on one of the wedge-shaped regions, dubbing the scenario built on it *wedge-shaped Casimir system*.

We address several issues concerning specific structural aspects of these systems. The first concerns which is the correct algebra of observables to associate to a free quantum field theory in a confined region such as those considered. This is not an obvious question since the standard procedure in the algebraic approach relies heavily on the underlying manifold being globally hyperbolic and on finding the smooth solutions to the equation(s) of motion, seen as an initial value problem. Both features are not present in our model(s). In order to tackle this problem we adapt to the case at hand the so-called functional formalism which has been used successfully in the algebraic framework in the past few years – see for an introduction [BDF09, BF09, FR15]. The net advantage of this procedure is the following: Observables are seen as functionals on a space of kinematical/dynamical configurations and the algebraic structure is obtained by deforming the standard pointwise product so to include the information of the canonical commutation relations. As soon as one either wants to deal with interactions at a perturbative level or is interested in the expectation value of quantities such as the stress-energy tensor, Wick polynomials are needed. Although their rigorous construction is known since more than a decade [HW01], the functional formalism allows for an easier identification not only of the polynomials themselves but also of the underlying algebraic structure via an additional deformation of the pointwise product. In order to select a specific class of functionals we adapt to the case at hand a procedure which was already successfully applied recently to the analysis of Abelian gauge theories [BDS12, BDS13, SDH12] and of linearized gravity [BDM14]: We start by constructing the space of all possible configurations allowed by the underlying dynamics, by means of a well-known procedure in PDE theory, the method of images. At this stage the analysis of the third case calls for an extension of the method itself. Most notably, we exploit the existence of an embedding of the wedge-shaped region into a second manifold whose geometry allows for the application of the method of images as in the two parallel plate case. Subsequently we identify a set of linear functionals on the collection of dynamical configurations which plays the role of the generators for the underlying algebra of observables. In order to justify our choice we will argue that they answer to a set of minimal requirements which need to be met, following the analysis of [Ben16]. At this stage the analysis of a Casimir-Polder, of a Casimir and of a wedge Casimir system will start to diverge considerably. While in the first case we will show that the generators are, up to an isomorphism, a subset of those for a Klein-Gordon field in Minkowski spacetime, in the second, this feature is lost. The third situation, despite its peculiarities in the application of the method of images, shares common features with the second, once the underlying quantum scalar field theory is realized on the embedding spacetime. Additionally we verify that the algebra of observables enjoys also the so-called *F-locality property* introduced by Kay in [Kay92]. It states that the restriction of such algebra to any globally hyperbolic subregion of the underlying manifold should be $*$-isomorphic to the one built directly on the region itself seen as a genuine globally hyperbolic manifold of its own. An important novel point, which our investigation shall uncover, is that the algebra of observables for the three cases enjoys the same structural properties of the standard Minkowski counterpart, especially the time-slice axiom, a feature which was not considered before.

The second question to which we wish to give an answer concerns the choice of an algebraic quantum state of Hadamard form. The microlocal characterization of the Hadamard condition was formulated by [Rad96a, Rad96b] for scalar field theories on globally hyperbolic spacetimes. Here we extend the definition so that it can be applied also to theories in bounded regions. In particular we shall call a state Hadamard if such property is satisfied by its restriction to any globally hyperbolic submanifold of the underlying spacetime, extending at a level of states the above mentioned F-locality property. Subsequently we investigate explicit examples. Also at this stage, the three examples, that we consider, differ greatly. In the Casimir-Polder one, it turns out that algebraic states can be constructed via pull-back from those in the whole Minkowski spacetime inheriting, moreover, the Hadamard property. In the Casimir system, the situation is far more complicated. Here our main goal is to make contact with the procedures often followed in the standard physics literature, where states are constructed either with the method of Green operators or, thanks to the special geometry of the system, via the method of images – for a preliminary investigation see [Nie09]. The aim, especially of the latter, is to show that one can construct states for a Casimir system starting directly from those for a Klein-Gordon field on the whole Minkowski spacetime. We stress one additional advantage, which is almost never mentioned: The method of images does not rely on modes and hence on a Fourier transform, being thus a natural candidate to be used for a generalization of our results to (a suitable class of) curved backgrounds. We investigate how to translate rigorously this procedure in the algebraic framework and we show that, in the case of a massless real scalar field, if we start from the Poincaré vacuum, we obtain a full-fledged Hadamard state for a Casimir system. At the same time we show that we can consider a larger class of states on the whole Minkowski spacetime as starting point. More precisely we give sufficient conditions to identify them and we show that KMS states at finite temperature meet them. As a byproduct, it turns out that the corresponding state for the Casimir system preserves the KMS condition. In a wedge-shaped region the construction of states by the method of images presents a further complication, since, in order to implement the Hadamard property we need to prove whether it is possible to define Hadamard states on the larger spacetime. An answer is not obvious, since there are obstructions in extending states fulfilling locally the Hadamard property to states defined on the whole algebra of observables. This question is left open, but an insightful example of a Hadamard state with a global explicit extension is given.

Eventually, in view of the microlocal characterization of the Hadamard states for the two systems under investigation, we are able to construct the extended algebra of Wick polynomials. Noteworthy is the fact that, in order to embed the local Wick polynomials, *i.e.*, those constructed in a globally hyperbolic subregion, into a global extended algebra, a non-local deformation of the ordinary star product is necessary. In this respect, we recall that the local extended algebra depends only on the choice of the Hadamard function, used to deform the star-product. Different choices of Hadamard functions yield isomorphic algebras and the intertwining isomorphism is a regular deformation

[BDF09]. Yet, in the systems under investigation, contrary to the Minkowski case, it is impossible to construct a global Hadamard function which depends only on local geometric properties. Hence, the Wick polynomials constructed out of local property of the spacetime can be represented in a global algebra only after applying a local deformation. The necessity for such deformation becomes manifest since, without it the correlations between local observables constructed on certain different globally hyperbolic subregions is ill-defined. The extended algebra of observables allows us to introduce the stress-energy tensor and make contact with previous results in literature. In particular, we obtain at the boundary the same divergences for the energy density. These prevent the total energy from being defined by integrating the energy density over the whole region. Often this difficulty is by-passed by *ad hoc* regularizations, which seem unsatisfactory in the present framework, where Wick polynomials are defined locally. Yet, we tend not to worry about such issue, ascribing it to the idealization of strong confinement.

Let us outline a synopsis of this work. In the first chapter we will present a condensed introduction to the main aspects of algebraic quantum field theory and functional formalism.In the second section, instead we focus on a Casimir-Polder system. To start with, we classify all dynamically allowed configurations, constructing out of them the $*$-algebra of fields and relating it to a subalgebra of the one for a Klein-Gordon field on Minkowski spacetime. Subsequently we give a notion of Hadamard states for a Casimir-Polder system and we show how they are related to those on the whole Minkowski spacetime. Eventually we discuss the notion of Wick polynomials and of Hadamard regularization pointing out the differences with the standard approach. We show how one can recover, starting from the Poincaré vacuum, the usual results for the two-point function and for the regularized energy density. In the third section instead we focus on a Casimir system. Mimicking the same procedure of the second section, first we construct all dynamical configurations and then the unital $*$-algebra of fields. After giving the notion of Hadamard states, we investigate how to construct them starting from those on the whole Minkowski spacetime. In particular we discuss the method of images and we show that it gives well-defined results if we start either from the Poincaré vacuum or from a KMS state at finite temperature, if we consider a massless Klein-Gordon field. In this respect we extend to the algebraic framework earlier analyses, see in particular [BM69, FR87, KCD79] for the thermal case and [Ful89] for the vacuum case. Eventually we compute also in this case the expectation value of the two-point function and of the regularized energy density. The last chapter is devoted to the case of a wedge-shaped Casimir system. Here the analysis deviates to an ancillary problem: The quantization of Klein-Gordon field on Dowker space. Dowker space is obtained, heuristically, from Minkowski space, considered with cylindrical coordinates, where the angular coordinate is taken ranging all over $\mathbb{R}$. This space represents the base for the application of the method of images in wedge geometry. On it the construction of algebraic quantization of Casimir system may be repeated almost identically for wedge-shaped Casimir system. To define a quantum field theory on Dowker space however we loose completely the advantage of the method of images. We are then forced to adapt some result from the theory of normally hyperbolic operators on globally hyperbolic spacetimes. The construction of an algebraic quantum field theory on a wedge-shaped region is eventually built from the one on Dowker, by means of the same construction applied to the case of a Casimir system.

Part of this work has been published in a joint work with Dappiaggi and Pinamonti, [DNP16], as part of the research activity of the author during his Ph. D. studentship. Apart from some modifications necessary to introduce it in this thesis, the mentioned paper became the second and the third chapters. As well, the introduction has been readjusted and integrated to the present thesis. The same matter has appeared in the proceeding of Marcel Grossmann workshop on Algebraic Quantum Field Theory - XIV, [Nos15].

Chapter 1

Algebraic quantum field theory and functional approach

The algebraic approach provides an axiomatic description of quantum field theory which allows for a very effective extension to arbitrary globally hyperbolic spacetimes of the canonical quantization procedure for Minkowski spacetime. The latter approach, based on assigning of a Hilbert space to the physical system, fails on curved background in face of several obstacles. The absence of a large isometry group such as the Poincaré group implies the lacking of a fundamental tool in the construction of a classical free field theory, Fourier transform, on which is based the construction of a mode expansion for field operators. Whenever the metric coefficients are explicitly time dependant, even concepts such as positive and negative frequency parts are lost. Furthermore one loses a preferred criterion for the identification of a ground state for the quantum theory leading to problems in selecting the right Hilbert space on which observables are represented.

The perspective offered by an algebraic approach shifts the attention from the identification of Hilbert space, to the definition of an abstract algebra of observables. It is organized in two steps:

1. The recollection of all possible observables in a single body with the structure of an algebra encoding the main properties of locality and causality, the canonical commutation relations and the dynamics of the system;

2. The identification of a quantum state, that is a positive, normalized linear functional on the algebra of observables.

The choice of a state yields the standard Hilbert space picture via the GNS theorem. In this sense Algebraic Quantum Field Theory does not represent an alternative approach to field quantization, but rather a complementary viewpoint.

In this chapter we introduce the algebraic quantization of a free scalar field theory on a globally hyperbolic spacetime. The goal of this presentation is providing the reader with the main concepts and tools of algebraic quantum field theory and with a guideline to be followed in the forthcoming case studies of a scalar field theory on manifolds with boundaries.

1.1 Algebraic preliminaries

We start this brief summary of algebraic quantum field theory presenting the main definitions needed and introducing the GNS theorem, which provides a bridge with the canonical quantization scheme on Hilbert spaces.

Definition 1.1.1. An **algebra** $\mathcal{A}$ over the field $\mathbb{C}$ is a $\mathbb{C}$-vector space with a bilinear operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ called product such that:

(*i*) it is associative, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in \mathcal{A}$;

(*ii*) it is distributive with respect to product in $\mathbb{C}$, $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$, for all $a, b, c \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

An algebra $\mathcal{A}$ is called unital if there exists an element $\mathbb{I} \in \mathcal{A}$ such that $\mathbb{I}a = a\mathbb{I} = a$, for all $a \in \mathcal{A}$.
An **algebra morphism** is a map $\phi : \mathcal{A} \to \mathcal{B}$ which is compatible with the vector space structure and the product, that is for all $a, b \in \mathcal{A}$ and for all $\alpha \in \mathbb{C}$,

$$\phi(\alpha a + b) = \alpha\phi(a) + \phi(b), \qquad \phi(a \cdot b) = \phi(a) \cdot \phi(b). \tag{1.1}$$

We need a notion of continuity of functionals on $\mathcal{A}$, which is provided by introducing a notion of topology:

Definition 1.1.2. A topological vector space X is a vector space over $\mathbb{C}$, endowed with a topology such that vector addition and scalar multiplication are continuous functions. An algebra $\mathcal{A}$ is said *topological* if it is topological as a vector space.

The following definition gives the real minimal structure needed for quantum theory, providing an abstract model of any algebra of linear operators. In particular we shall identify an abstract counterpart for the operation of "taking the adjoint" of an operator.

Definition 1.1.3. A **$*$-algebra** $\mathcal{A}$ is an algebra over $\mathbb{C}$ which is endowed with an involution, that is a map $* : \mathcal{A} \to \mathcal{A}$ which fulfils the following properties:

(*i*) $(a + b)^* = a^* + b^*$, $(a \cdot b)^* = b^* \cdot a^*$, for all $a, b \in \mathcal{A}$;

(*ii*) $(\alpha a)^* = \overline{\alpha} a^*$, for every $\alpha \in \mathbb{C}$ and every $a \in \mathcal{A}$;

(*iii*) $(a^*)^* = a$, for all $a \in \mathcal{A}$.

The physical data relative to the system are encoded in the algebra by adding further suitable structures. We postpone this discussion to a later stage, in order to keep this algebraic prologue as general as possible.

Let us proceed to the definition of algebraic states. In the algebraic approach, algebraic states represents states of a physical system, providing expectation values of observables.

Definition 1.1.4. An **algebraic state** on a unital $*$-algebra $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ such that

- it is positive, $\omega(a^* a) \geq 0$ for all $a \in \mathcal{A}$,
- it is normalized, $\omega(\mathbb{I}) = 1$ being $\mathbb{I} \in \mathcal{A}$ the unit element.

Algebraic states are fundamental ingredients to recover the Hilbert space formulation of a quantum theory. Before bridging this gap, we need the notion of representation of a $*$-algebra.

Definition 1.1.5. A $*$-morphism is a map $\phi : \mathcal{A} \to \mathcal{B}$ between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, which is an algebra morphism compatible with the involution, that is $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$.

Let us consider now $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, a separable Hilbert space together with its Hermitian inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and let $\mathscr{L}(\mathcal{K})$ be the space of linear operators on a dense invariant subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$. We recall that, given a linear operator $A \in \mathscr{L}(\mathcal{H})$, a subspace $\mathcal{K} \subset \mathcal{H}$ is invariant under A if for all $\psi \in \mathcal{K}$, $A\psi \in \mathcal{K}$. Let us suppose, in addiction, that $\mathscr{L}(\mathcal{K})$ forms a $*$-algebra, with a $*$-involution given by the adjoint operation. With these assumptions, we define,

Definition 1.1.6. A **$*$-representation** of a $*$-algebra $\mathcal{A}$ is a unit preserving $*$-homomorphism $\pi : \mathcal{A} \to \mathscr{L}(\mathcal{K})$. Given two different $*$-representations of $\mathcal{A}$, π on $\mathcal{H}$ and π' on $\mathcal{H}'$, we say that they are *unitarily equivalent* if there is a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $U\pi(\mathcal{A}) = \pi'(\mathcal{A})U$.

The celebrated Gelfand-Naimark-Segal (GNS) theorem yields finally the sought junction with Hilbert spaces, showing that every algebraic state can be represented by a vector in a Hilbert space.

Theorem 1.1.1 (GNS). *Let ω be a state on a unital $*$-algebra $\mathcal{A}$. Then there exists a triple $(\mathcal{H}, \pi, \Omega)$ such that:*

(i) *$\mathcal{H}$ is a Hilbert space, π is a $*$-representation of $\mathcal{A}$ in $\mathcal{H}$ and $\Omega \in \mathcal{H}$ is a unit vector,*

(ii) *$\omega(a) = \langle \omega, \pi(a)\omega \rangle$, for all $a \in \mathcal{A}$,*

(iii) *$\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$.*

Furthermore, the triple $(\mathcal{H}, \pi, \Omega)$ is unique up to unitary equivalence.

Remark 1.1.1. For completeness, let us mention the definition of C*-algebras,

Definition 1.1.7. A unital $*$-algebra $\mathcal{A}$ is called a C*-algebra if it is endowed with a norm $||\cdot|| : \mathcal{A} \to \mathbb{R}$ such that $\mathcal{A}$ is complete with respect to $||\cdot||$ and, for all $a, b \in \mathcal{A}$, $||a \cdot b|| \leq ||a|| \, ||b||$. Furthermore, $||a^*|| \, ||a|| = ||a||^2$, for all $a \in \mathcal{A}$.

An important C*-algebra is $\mathscr{BL}(\mathcal{H})$, the algebra of bounded linear operators on Hilbert space $\mathcal{H}$, which is often used in the description of physical systems for its relation with Von Neumann algebras. In our analysis we will refer to $*$-algebras, rather than C*-algebras, since we are eventually interested to introduce Wick polynomials, which are not representable as bounded operators. An interested reader could refer to a vast literature, in particular [BR87, Haa92, BDFY15].

1.2 Cauchy problem for the Klein Gordon equation

We now start to develop the quantization of a real scalar field theory. Before constructing the algebra of observables and identifying suitable states, we need to characterize the classical dynamics. Consider the Klein-Gordon equation for a real scalar field $\phi : M \to \mathbb{R}$ on a Lorentzian manifold (M, g),

$$P\phi = (\Box - m^2 - \xi R)\phi = 0, \tag{1.2}$$

where $\Box = \nabla_a \nabla^a$ is the D'Alambert operator built out of g, $m \geq 0$ is a mass term, $\xi \in \mathbb{R}$ is a coupling with R, the scalar curvature. Notice that, although R vanishes identically on certain manifolds, such as for example Minkowski of Schwarzschild spacetime, the coupling term ξR entails the existence of an additional term in the underlying Lagrangian $\mathcal{L}$,

$$\mathcal{L}_{KG}[\phi] = -\frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \xi R \phi^2\right) \tag{1.3}$$

This has far reaching consequences on the form of notable quantities, first and foremost the stress-energy tensor, which is proportional to the variation of the Lagrangian with respect to the metric. It is customary to characterize those $\phi \in C^\infty(M)$ satisfying (1.2) in terms of an initial value problem. On curved backgrounds, however, it is not guaranteed the well-posedness of Cauchy problems and additional requirements are needed to guarantee the existence of a unique solution.

1.2.1 Causal structure and global hyperbolicity

We consider from now on M, a connected, second countable[1], smooth 4-dimensional manifold, together with g a Lorentzian metric with signature $(-,+,+,+)$. We call Lorentzian manifold the pair (M,g) and, for notational simplicity, we denote it as $M \equiv (M,g)$. We recall that a (smooth) vector field $X \in \mathfrak{X}(M)$ is

- timelike if $g_x(X,X) < 0$ for all $x \in M$,
- null or lightlike if $g_x(X,X) = 0$ for all $x \in M$,
- spacelike if $g_x(X,X) > 0$ for all $x \in M$.

A vector field is called causal if it is either null or timelike. A continuous piecewise C^1-curve $\gamma : I \subseteq \mathbb{R} \to M$ is called timelike, lightlike, spacelike or causal if its tangent vector $\dot{\gamma}$ is at each point of the curve timelike, lightlike, spacelike or causal, respectively. We will assume also that M is orientable and time-oriented, meaning that,

Definition 1.2.1. A Lorentzian manifold (M,g) is **orientable** if it is equipped with a differential form of maximal degree which does not vanish anywhere. It is time-orientable if it admits a vector field $Y \in \mathfrak{X}(M)$ with $g_x(Y,Y) < 0$ for any $x \in M$. A Lorentzian manifold together with such a vector field is called **time-oriented**.

Having fixed a time-orientation Y, we call a timelike or a null vector field X *future-pointing*, if $g_x(X,Y) < 0$ for all $x \in M$, or *past-pointing*, if $g_x(X,Y) > 0$ for all $x \in M$. In what follows an orientable and time-oriented Lorentzian manifold will be referred to as **spacetime**. Time orientation allows for a partial ordering, for any $x, x' \in M$

- $x << x'$ iff there is a future-directed timelike curve in M from x to x',
- $x < x'$ iff there is a future-directed causal curve in M from x to x',
- $x \prec x'$ iff $x < x'$ or $x = x'$.

Causal relations yield a causal structure on M defined as follows,

Definition 1.2.2. The **chronological future** $I_+^M(x)$ and the **causal future** $J_+^M(x)$ of $x \in M$ are the set of points

$$I_+^M(x) = \{x' \in M \mid x < x'\}, \qquad J_+^M(x) = \{x' \in M \mid x \prec x'\}.$$

[1] A manifold M is second countable if there exists a countable collection $\mathscr{O} = \{\mathcal{O}_i\}_{i=1}^{\infty}$ of open subsets of M such that any open subset of M can be written as a union of elements of some subfamily of $\mathscr{O}$. Second countability is a topological property which provides us with a partition of unity.

The chronological future of a subset $A \subset M$ and its causal future are defined to be

$$I_+^M(A) := \bigcup_{x\in A} I_+^M(x)\,, \qquad J_+^M(A) := \bigcup_{x\in A} J_+^M(x).$$

Similarly, we define the **chronological past** $I_-^M(x)$ and the **causal past** $J_-^M(x)$ of $x \in M$ as the set of points

$$I_-^M(x) = \{x' \in M \mid x' < x\}\,, \qquad J_-^M(x) = \{x' \in M \mid x' \prec x\}.$$

The chronological past of a subset $A \subset M$ and its causal past are, accordingly,

$$I_-^M(A) := \bigcup_{x\in A} I_-^M(x)\,, \qquad J_-^M(A) := \bigcup_{x\in A} J_-^M(x).$$

.

We can define accordingly

Definition 1.2.3. Two subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ in M are called **causally separated** if they cannot be connected by a causal curve, *i.e.*, if for all $x \in \mathcal{O}_1$, $J_\pm(x)$ has empty intersection with $\mathcal{O}_2$.

Having defined a causal structure unveils that we have possible concerns with causality. The main pathologies to single out are closed causal curves as well as causal curves that can fall off the edge of a spacetime in a finite coordinate time. The first should make undetermined *consecutio* cause-effect, while the second involves a world line of a causal physical observer which ends for finite value of any of its affine parameters: Both scenarios look unphysical. Furthermore, aiming at defining well-posed Cauchy problems, we should be able to make sense of a foliation of hypersurfaces where to assign initial data. Remarkably all these issues turns out to have the same medicine, global hyperbolicity of spacetime M. Let us introduce additional structures,

Definition 1.2.4. Let (M, g) a spacetime. We say that,

- M is **causal** if it has no closed causal curves.
- Given a subset A of M, the **domain of dependence** of A is the set of all points $x \in M$ such that every inextensible causal curve through x intersects A.
- A subset A of M is **achronal** if no causal curve intersects A more than once.
- The **edge** of a subset A of M is the set of $x \in A$ such that, for all open $\mathcal{O} \subset M$ containing x, there exist $x' \in \mathcal{O} \cap I^+(x)$ and $x'' \in \mathcal{O} \cap I^-(x)$ as well as a timelike curve which joins x' and x'' without intersecting A.
- A **Cauchy surface** Σ for M is a closed achronal set with empty edge and whose domain of dependence coincides with M.

We can define a globally hyperbolic spacetime, after [BGP07, Th. 1.3.10]:

Definition 1.2.5. Given a spacetime M the three following conditions are equivalent:

1. The spacetime has a Cauchy surface;
2. The spacetime is causal, and for every pair of points $x, x' \in M$, the subset $J^M_-(x) \cap J^M_+(x')$ is compact or empty;
3. M is isometric to $\mathbb{R}\times\Sigma$ with $ds^2 = -\beta dt^2 + h_t$, where $\beta \in C^\infty(M;(0,\infty))$, h_t is a Riemannian metric on Σ depending smoothly on t and each locus $\{t = \text{const}\} \times \Sigma$ is a smooth spacelike Cauchy hypersurface embedded in M.

Each condition identifies M as a **globally hyperbolic spacetime**.

Remark 1.2.1. Equivalence of the three conditions is the arrival point of a long elaboration of the concept of global hyperbolicity. In particular, the first condition is due to [Ger70], who showed its equivalence with the original definition by [Ler52], the second is due to [HE73] and [BS07] (the latter having relaxed the stronger hypothesis of *strong causality*), while the last one is due to Bernal and Sánchez [BS03]. Other characterizations are possible, see for example [Min09].

A key point in the following analysis is that the (static) Casimir effect in general involves regions of Minkowski spacetime, $\Omega \subset \mathbb{R}^4$, with (timelike) boundaries which are *not* globally hyperbolic. Heuristically, the information propagating due to the dynamics from a set of initial data could be lost when hitting the boundary, as well as additional informations could emanate from the boundary itself. Fixing boundary conditions is, in this perspective a cure to restore well-posedness of the dynamical problem. Rigorously, the fact that globally hyperbolicity is lost can be read in the second characterization in Definition 1.2.5, since[2]one can always find a pair $x, x' \in \Omega$ such that $J^{\mathring{\Omega}}_-(x) \cap J^{\mathring{\Omega}}_+(x')$ is not compact, [MS08, Remark 3.72].

In the three specific examples we aim at dealing with, another criterion applies, due to [Kay78], which exploits an additional property of spacetimes, the ultrastatic property.

Definition 1.2.6. We say that a spacetime (M, g) is **ultrastatic** if there is a Riemannian manifold (N, h) such that $M = \mathbb{R} \times N$ and in local coordinates $(t, \underline{x}_i)$, with $(\underline{x}_i)$, $i = 1, 2, 3$, local coordinates on N, the metric is written

$$ds^2 = -dt^2 + h_{ij}(x) d\underline{x}^i d\underline{x}^j$$

for any $x \in M$

[2]Here we are implicitly considering that a manifold with boundary $\mathcal{M}$ is globally hyperbolic if and only if its interior $\mathring{\mathcal{M}}$ is globally hyperbolic. Such definition appears in Globally hyperbolic manifold – Wikipedia, without a clear reference in the literature. Up to our knowledge on this topic, no author refers to this notion for manifolds with boundaries. Nevertheless we assume its validity in this context, since it works for our purposes.

With reference to the notation in Definition 1.2.6, for an ultrastatic spacetime the following criterion applies, [Kay78],

Proposition 1.2.1. *An ultrastatic spacetime* (M, g) *is globally hyperbolic if and only if* (N, h) *is geodesically complete.*

1.2.2 Normally hyperbolic operators and Klein-Gordon operator

A key application of global hyperbolicity is the well-posedness of the Cauchy problem for Klein-Gordon equation. For definiteness, let us assume that a choice of a foliation $M \equiv \mathbb{R} \times \Sigma$ of M by Cauchy surfaces has been made. We then write:

$$\begin{cases} P\phi = (\Box - m^2 - \xi R)\phi = 0 \\ \phi|_\Sigma = \phi_0 \\ \nabla_N \phi|_\Sigma = \phi_1 \end{cases}, \tag{1.4}$$

where Σ is any Cauchy surface, ϕ_0 and $\phi_1 \in C^\infty(\Sigma)$ and N is the smooth vector field which is defined pointwisely as future directed normal vector to Σ, normalized so that $g_x(N, N) = -1$ for all $x \in M$. Existence and uniqueness of solutions of (1.2) are consequence of the general theory of normally hyperbolic operators,

Definition 1.2.7. Given a spacetime M, a linear operator $P : C^\infty(M) \to C^\infty(M)$ is **normally hyperbolic** if for any choice of a local chart it can be expressed locally as

$$P = g^{\mu\nu}(x)\partial_\mu\partial_\nu + A^\mu(x)\partial_\mu + B(x), \qquad \mu, \nu = 0, \dots, 3, \tag{1.5}$$

where B and A^μ are real smooth functions on M for all $\mu = 0, \dots, 3$.

Remark 1.2.2. The Klein-Gordon operator defined in (1.2) is normally hyperbolic.

We shall need the following spaces of functions:

- $C^\infty_{sc}(M) := \{f \in C^\infty(M) \mid \mathrm{supp}(f) \subseteq J^M_+(K) \cup J^M_-(K)$ for some compact $K \subset M\}$;
- $C^\infty_{fc}(M) := \{f \in C^\infty(M) \mid \mathrm{supp}(f) \cap J^M_+(x)$ is either compact or empty $\forall x \in M\}$;
- $C^\infty_{pc}(M) := \{f \in C^\infty(M) \mid \mathrm{supp}(f) \cap J^M_-(x)$ is either compact or empty $\forall x \in M\}$;
- $C^\infty_{tc}(M) := C^\infty_{fc}(M) \cap C^\infty_{pc}(M)$.

Elements of these spaces are called *spacelike*, *futurelike*, *pastlike* and *timelike compact*, respectively. Following [Fri75, Bär15],

Theorem 1.2.2. *Let (M, g) be a globally hyperbolic spacetime and let P be a normally hyperbolic operator. Then there exist unique* **advanced** $(-)$ **and retarded** $(+)$ **Green operators** $E_+ : C_0^\infty(M) \to C_{sc}^\infty(M)$ *and* $E_- : C_0^\infty(M) \to C_{sc}^\infty(M)$ *such that they are linear and*

(i) $P \circ E_\pm = id : C_0^\infty(M) \to C_0^\infty(M)$

(ii) $E_\pm \circ P = id : C_0^\infty(M) \to C_0^\infty(M)$

(iii) $\mathrm{supp}(E_\pm(f)) \subseteq J_\pm^M(\mathrm{supp}(f)) \quad f \in C_0^\infty(M)$

We define a pairing $\langle \cdot, \cdot \rangle : C^\infty(M) \times C_0^\infty(M) \to \mathbb{R}$ as

$$\langle u, f \rangle := \int_M d\mu_g(x) u(x) f(x), \qquad \forall u \in C^\infty(M),\, f \in C_0^\infty(M),$$

being $d\mu_g$ the metric induced volume form on M. With respect to the pairing, P is symmetric, that is, for all f, $g \in C_0^\infty(M)$,

$$\begin{aligned} \langle Pf, g \rangle &= \int_M d\mu_g(x)(Pf)(x)g(x) \\ &= \int_M d\mu_g(x) f(x) Pg(x) \\ &= \langle f, Pg \rangle, \end{aligned}$$

where we used integration by parts and the fact that f and g are compactly supported. Consequently, we have that $(E_\pm)^* = E_\mp$, in fact

$$\begin{aligned} \langle E_\pm f, g \rangle &= \int_M d\mu_g(x)(E_\pm f)(x)g(x) \\ &= \int_M d\mu_g(x)(E_\pm f)(x)(PE_\mp g)(x) \\ &= \int_M d\mu_g(x)(PE_\pm f)(x)(E_\mp g)(x) \\ &= \langle f, E_\mp g \rangle \end{aligned}$$

where we used the properties of $E_\pm$ and where the integration by parts in the second step gives vanishing boundary terms since $\mathrm{supp}(G_\pm f) \cap \mathrm{supp}(G_\mp g) \subset J_\pm^M(\mathrm{supp}(f)) \cap J_\mp^M(\mathrm{supp}(g))$ is compact in a globally hyperbolic spacetime.

Remark 1.2.3. By the Schwartz kernel theorem [Hör90, Th. 5.2.1], advanced and retarded operators define bidistributions, $E_\pm \in \mathcal{D}'(M \times M)$, with respect to the pairng,

$$\langle E_\pm f, f' \rangle := E_\pm(f, f'), \qquad \forall f,\, f' \in C_0^\infty(M).$$

In particular, they are solutions of the equation:

$$PE_\pm = \delta$$

where δ is the Dirac delta distribution and the equation is interpreted in a distributional sense.

The difference between the advanced and retarded operators is dubbed **causal propagator** (or Pauli-Jordan commutator function), $E = E_+ - E_-$. Let us list its main properties.

Proposition 1.2.3. *Given a normally hyperbolic operator P on a globally hyperbolic spacetime M, defined the causal propagator as*

$$E := E_+ - E_- : C_0^\infty(M) \to C_{sc}^\infty(M)$$

it holds that

(i) $P \circ E = E \circ P = 0$*;*

(ii) *Calling $\mathcal{S}_{sc}^{KG}(M)$ the space of spacelike compact solutions to* (1.4)*, there exists an isomorphism of topological vector spaces*

$$\frac{C_0^\infty(M)}{P(C_0^\infty(M))} \simeq \mathcal{S}_{sc}^{KG}(M), \tag{1.6}$$

the isomorphism being implemented by E.

The causal propagator induces a symplectic structure on $\mathcal{S}_{sc}^{KG}(M)$. Before stating this we recall that a (possibly infinite dimensional) vector space X is called *pre-symplectic* if it is endowed with a map $\sigma : X \times X \to \mathbb{R}$ which is bilinear and antisymmetric. Furthermore σ induces a map $\sigma_0 : X \to X^*$ such that $\sigma_0(v) = \sigma(v, \cdot)$ for all $v \in X$. If σ_0 is an injective map, than we say that σ is weakly non-degenerate, and (X, σ) is a symplectic space.

Proposition 1.2.4. *The map $\sigma : \frac{C_0^\infty(M)}{P(C_0^\infty(M))} \times \frac{C_0^\infty(M)}{P(C_0^\infty(M))} \to \mathbb{C}$ defined by*

$$\sigma(f, f') := \int_M d\mu_g(x) f(x) E f'(x), \tag{1.7}$$

for any representatives of $[f]$ and $[f'] \in \frac{C_0^\infty(M)}{P(C_0^\infty(M))}$, is a weakly non-degenerate symplectic form.

So far we characterized only spacelike solutions. Part of the above analysis can be applied to smooth solutions, provided to extending advanced and retarded Green operators [Bär15].

Proposition 1.2.5. *There are unique extensions of $E_\pm$*

$$\overline{E}_+ : C_{pc}^\infty(M) \to C_{pc}^\infty(M), \qquad \overline{E}_- : C_{fc}^\infty(M) \to C_{fc}^\infty(M), \tag{1.8}$$

such that, for all $f \in C_{pc}^\infty(M)$,

$$P(\overline{E}_+ f) = \overline{E}_+(Pf) = f,$$

and

$$\mathrm{supp}(\overline{E}_+ f) \subset J_+^M(\mathrm{supp}(f)).$$

and accordingly for E_- substituting pc with fc.

The extensions of the advanced and retarded Green operators yield in turn an extension of the causal propagator,

$$\overline{E} := \overline{E}_+ - \overline{E}_- : C^\infty_{tc}(M) \to C^\infty(M),$$

such that,

$$P \circ E = E \circ P = 0 \quad \text{on} \quad C^\infty_{tc}(M),$$

This provides us with a full characterization of the space of smooth solutions for (1.4).

Corollary 1.2.6. *The causal propagator $\overline{E}_\infty$ induces the following isomorphism,*

$$\frac{C^\infty_{tc}(M)}{P(C^\infty_{tc}(M))} \simeq \mathcal{S}^{KG}(M). \tag{1.9}$$

Remark 1.2.4 (Notation). From now on we will denote $E_\pm$ and E the extended propagators, with a slight abuse of notation.

1.3 Algebra of observables

We aim at modeling observables as numerical assignments to configurations of a physical system. In case of a classical or quantum field, we shall distinguish

- **dynamical configurations**, that is all suitable solutions to the Cauchy problem for the field equation,
- **kinematical configurations**, or kinematically allowed configurations, which fix the required regularity of dynamical configurations.

The space of kinematical configurations for a Klein-Gordon field is $\mathcal{C}^{KG}(M) \equiv C^\infty(M)$ and we consider it endowed it with the compact-open topology – see [Tre67]. We consider the topological dual of $\mathcal{C}'^{KG}(M) \equiv \mathcal{E}'(M)$, where $\mathcal{E}'(M)$ is the space of compactly supported distributions. We call the dual pairing $\langle \cdot, \cdot \rangle : \mathcal{E}'(M) \times C^\infty(M) \to \mathbb{C}$. The space of dynamical configurations $\mathcal{S}^{KG}(M)$ is the subspace of kinematical configurations which are solutions of (1.4).

1.3.1 Classical observables and quantization

Aiming at quantizing a scalar field theory, we shall provide preliminarily a notion of classical observable. With the minimal requirements of being a linear functional on the kinematical configurations, it is natural to identify classical observables with elements of $\mathcal{E}'(M)$, distributions with compact support. Following [Ben16], we refine our notion requiring observables to be **optimal** with respect to configurations, which means the two following properties:

- Observables are **separating** for configurations, that is, for every pair of different configurations $\phi, \phi' \in \mathcal{C}^{KG}(M)$, there exists a classical observable $F \in \mathcal{E}'(M)$ such that $F(\phi) \neq F(\phi')$. This means that classical observables are sufficiently rich to distinguish all possible classical field configurations;
- Observables are **non-redundant**, that is for every pair of classical observables $F, F' \in \mathcal{E}'(M)$, there exists at least one configuration $\phi \in \mathcal{C}^{KG}(M)$ such that $F(\phi) \neq F'(\phi)$. This amounts to require that no elements in this space provide the same outcomes upon evaluation on all possible configurations. If this is not the case, then one is over-counting observables.

On account of standard results in functional analysis we know that $C_0^\infty(M, \mathbb{C}) \subset \mathcal{E}'(M)$ is separating and non-redundant with respect to $\mathcal{C}^{KG}(M)$ via the dual pairing. In order to implement dynamics we shall restrict to dynamical configurations. In this way we loose optimality. In fact, being $\phi \in \mathcal{S}^{KG}(M)$ and $f \in C_0^\infty(M, \mathbb{C})$ and observing that $f + Pf' \in C_0^\infty(M, \mathbb{C})$ for any $f \in C_0^\infty(M, \mathbb{C})$, with P as in (1.2), we have

$$\langle \phi, f + Pf' \rangle = \langle \phi, f \rangle + \langle \phi, Pf' \rangle = \langle \phi, f \rangle + \langle P\phi, f' \rangle = \langle \phi, f \rangle,$$

where we used that P is symmetric. The way to restore optimality is to consider class of equivalences in $\frac{C_0^\infty(M,\mathbb{C})}{P(C_0^\infty(M,\mathbb{C}))}$. On account of Proposition 1.2.4, we see that classical observables comes endowed with a symplectic structure induced by causal propagator E. Let us define finally,

Definition 1.3.1. The complex span of all functionals $F_{[f]} : \mathcal{S}^{KG}(M) \to \mathbb{C}$, $[f] \in \frac{C_0^\infty(M,\mathbb{C})}{P(C_0^\infty(M,\mathbb{C}))}$ such that $F_{[f]}(\phi) = \int_M d\mu_g(x) f(x)\phi(x)$, denoted by $\mathcal{O}^{KG}(M)$, is the space of **classical observables**. This space is symplectic if endowed with the following weakly non-degenerate symplectic form:

$$\sigma : \mathcal{O}^{KG}(M) \times \mathcal{O}^{KG}(M) \to \mathbb{R},$$
$$\sigma(F_{[f]}, F_{[f']}) := \langle f, E(f') \rangle = \int_M d\mu_g(x)\; f(x)E(f')(x),$$

where f and f' are representatives of the respective equivalence classes $[f]$ and $[f']$.

Remark 1.3.1. Recalling[3] that, on account of Corollary 1.2.6, $\mathcal{S}^{KG}(M)$ is isomorphic to $\frac{C_{tc}^\infty(\mathbb{R}^4)}{P[C_{tc}^\infty(\mathbb{R}^4)]}$, we can redefine classical observables as

$$[\alpha] \mapsto F_{[f]}([\alpha]) := \int_M d\mu_g(x) f(x) E\alpha(x), \qquad \forall [\alpha] \in \mathcal{S}_{sc}^{KG}(M), \alpha \in [\alpha],$$

for all $[f] \in \frac{C_0^\infty(M)}{P[C_0^\infty(M)]}$.

[3] From now on, with a slight abuse of notation, we will denote $C_0^\infty(M, \mathbb{C}) \equiv C_0^\infty(M)$.

Given classical observables, we would like to implement the algebraic quantization. We gather the guiding idea from Dirac's canonical quantization: Look for a Hilbert space $\mathcal{H}$ on which classical observables $F_f \in \mathcal{O}^{KG}(M)$ are uniquely associated to operators $\widehat{F}_f \in \mathscr{L}(\mathcal{H})$ such that

$$[\widehat{F}_{[f]}, \widehat{F}_{[f']}] := i\sigma(F_{[f]}, F_{[f']})\mathbb{I} = iE(f, f')\mathbb{I},$$

that is such that *canonical commutation relations* (CCRs) are implemented. Accordingly we would like to encode classical observables in a $*$-algebra, endowed with a suitable product $\cdot$, such that

$$[F_{[f]}, F_{[f']}] := F_{[f]} \cdot F_{[f']} - F_{[f']} \cdot F_{[f]} = iE(f, f')\mathbb{I}.$$

There are different ways to implement this procedure. In what follows we will follow the functional approach, introduced by Brunetti, Fredenhagen and Dütsch in [BDF09], based on the identification of a suitable product which encodes directly CCRs. Other important approaches are for example Borchers-Uhlmann algebra and Weyl algebra. Both of these approaches rely on a quotient operation between a suitable algebra of classical observables and an ideal implementing the CCRs. They differ on the specific classical algebra. In particular Weyl algebra has the additional property of being a C*-algebra.

1.3.2 Regular functionals and algebra of observables

Following the same approach of [BDF09, BF09], we model observables as suitable functionals defined on $\mathcal{C}^{KG}(M)$. To better characterize, we need the following,

Definition 1.3.2. Let $F : \mathcal{C}^{KG}(M) \to \mathbb{C}$ be any functional and let $\mathcal{U} \subset \mathcal{C}^{KG}(M)$ be an open set. We say that F is *differentiable of order k* if, for all $m = 1, ..., k$, the following m-th order (Gâteaux) derivatives exist as jointly continuous maps from $\mathcal{U} \times \left(\mathcal{C}^{KG}(M)\right)^{\otimes m}$ to $\mathbb{C}$:

$$\begin{aligned} F^{(m)}[\phi](\phi_1, ..., \phi_m) &= \langle F^{(m)}[\phi], \phi_1 \otimes ... \otimes \phi_m \rangle \\ &\doteq \left.\frac{\partial^m}{\partial\lambda_1 ... \partial\lambda_m}\right|_{\lambda_1 = ... = \lambda_m = 0} F\left(\phi + \sum_{j=1}^{m} \lambda_j \phi_j\right). \end{aligned}$$

We say that a functional F is **smooth** if it is differentiable at all orders $k \in \mathbb{N}$.

Functionals come endowed with a notion of support,

Definition 1.3.3. Given a functional $F : \mathcal{C}^{KG}(M) \to \mathbb{C}$, we call **support** of F

$$\begin{aligned} \operatorname{supp}(F) \doteq \{x \in M \mid \forall \text{ neighbourhoods } U \ni x,\, \exists u, u' \in \mathcal{C}^{KG}(M),\, \operatorname{supp}(u) \subseteq U, \\ \text{such that } F[u + u'] \neq F[u]\}. \end{aligned}$$

A functional is *compactly supported* if its support is compact.

Functionals of paramount relevance are

- linear functionals, F_f, defined for any $f \in C_0^\infty(M)$ as

$$F_f(\phi) = \int_M d\mu_g(x)\phi(x)f(x) \qquad \forall \phi \in \mathcal{C}^{KG}(M), \tag{1.10}$$

  where $d\mu_g$ is the metric induced volume form;

- polynomial functionals, defined for any $N \in \mathbb{N}$ as

$$F(\phi) = \sum_{k=1}^{N<\infty} \int_M d\mu_g(x)\phi(x_1)\dots\phi(x_k)f_k(x_1,\dots,x_k) \qquad \forall \phi \in \mathcal{C}^{KG}(M), \tag{1.11}$$

  with symmetrical distributional densities f_k with compact support.

Functionals are a rather huge class of objects and we refine it imposing a regularity requirement.

Definition 1.3.4. A functional $F : \mathcal{C}^{KG}(M) \to \mathbb{R}$, is called **regular** if it is smooth and if, for all $k \geq 1$ and for all $\phi \in \mathcal{C}^{KG}(M)$, $F^{(k)}[\phi] \in C_0^\infty(M^k)$. In addition we require that only finitely many functional derivatives do not vanish. We indicate this set as $\mathcal{F}_0(M)$.

Since the dynamics is ruled by a Green-hyperbolic operator, we can endow $\mathcal{F}_0(M)$ with the structure of a $*$-algebra by means of the following product $\star : \mathcal{F}_0(M) \times \mathcal{F}_0(M) \to \mathcal{F}_0(M)$:

$$(F \star F')(\phi) = (\mathcal{M} \circ \exp(i\Gamma_E)(F \otimes F'))(\phi), \tag{1.12}$$

where $F, F' \in \mathcal{F}_0(M)$. Here $\mathcal{M}$ stands for the pointwise multiplication, *i.e.*, $\mathcal{M}(F \otimes F')(\phi) \doteq F(\phi)F'(\phi)$, whereas

$$\Gamma_E \doteq \frac{1}{2}\int_{M\times M} d\mu_g(x)d\mu_g(x')E(x,x')\frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(x')},$$

where $E(x,x')$ is the integral kernel of the causal propagator associated to P. The exponential in (1.12) is defined intrinsically in terms of the associated power series and, consequently, we can rewrite the product also as

$$(F \star F')(\phi) = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!}\langle F^{(n)}(\phi), E^{\otimes n}(F'^{(n)})(\phi)\rangle, \tag{1.13}$$

where the 0-th order is the pointwise multiplication, $\langle F^{(0)}(\phi), (F'^{(0)})(\phi)\rangle \doteq F(\phi)F'(\phi)$. The $*$-operation is complex conjugation, that is, for all $F \in \mathcal{F}_0(M)$ and for all $\phi \in \mathcal{C}^{KG}(M)$, $F^*(\phi) = \overline{F(\overline{\phi})}$. Since regular functionals are such that only a finite number of functional derivatives do not vanish, there is no issue concerning the convergence of (1.13).

Definition 1.3.5. We call $\mathcal{A}^{KG}(M) \doteq (\mathcal{F}_0(M), \star)$ the **off-shell algebra** of a Klein-Gordon field endowed with complex conjugation as $*$-operation.

We can realize $\mathcal{A}^{KG}(M)$ as being generated by linear functionals, barring a completion needed to account for the fact that $C_0^\infty(M) \times ... \times C_0^\infty(M)$ is dense in $C_0^\infty(M \times ... \times M)$, with respect to the topology of smooth and compactly supported functions. In this respect, compactly supported functions represent the labelling space of the off-shell algebra of functionals, building, thus, a bridge towards the more traditional approaches to a covariant quantization of a Klein-Gordon scalar field – see the remark at the end of this section. The $\star$-product implements CCRs, as it can be checked at the level of generators:

$$\begin{aligned}
[F_f, F_{f'}]_\star(\phi) &= (F_f \star F_{f'})(\phi) - (F_{f'} \star F_f)(\phi) \\
&= i\langle F_f^{(1)}(\phi), EF_{f'}^{(1)}(\phi)\rangle \\
&= iE(f, f')
\end{aligned}$$

where we used antisymmetry of E in the second step.

Remark 1.3.2. The off-shell algebra $(\mathcal{F}_0(M), \star)$ represents a deformation of the $*$-algebra of regular functionals endowed with the pointwise multiplication [FR15, §4-5]. In such approach one should assign a Poisson structure to the (classical) commutative algebra, in terms of Peierls brackets [Pei52], a functional counterpart of the symplectic form on linear functionals. For our purposes this is a rather technical aspect and we shall not dwell on details. An interested reader may refer to [BDF09].

Dynamics can be encoded by simply restricting functionals to $\mathcal{S}^{KG}(M)$. As a by-product, $\mathcal{F}_0(M)$ contains redundant functionals, that is those $F \in \mathcal{F}_0(M)$ such that $F(\phi) = 0$ for all $\phi \in \mathcal{S}^{KG}(M)$. At the level of $\mathcal{A}^{KG}(M)$, this restriction can be implemented considering the quotient between such algebra and the ideal $\mathcal{I}^{KG}(M)$ generated by those functionals of the form (1.10) with $f = P(h)$, $h \in C_0^\infty(M)$, P being the operator in (1.2). The $\star$-product descends to the quotient, as it can be checked at the level of generators, for any $\phi \in \mathcal{S}^{KG}(M)$, for any $f, f', h, h' \in C_0^\infty(M)$ such that $f, f + Ph \in [f] \in \frac{C_0^\infty}{P(C_0^\infty(M))}$ and $f', f' + Ph' \in [f'] \in \frac{C_0^\infty}{P(C_0^\infty(M))}$

$$\begin{aligned}
&(F_{f+Ph} \star F_{f'+Ph'})(\phi) = \\
&= \int_M d\mu_g\,(f + Ph)(f' + Ph')\phi - i\langle (f + Ph), E(f' + Ph')\rangle \\
&= \int_M d\mu_g\, f\, f'\phi - i\langle (f + Ph), E(f' + Ph')\rangle \\
&= \int_M d\mu_g\, f\, f'\phi - i\langle f, Ef'\rangle \\
&= (F_f \star F_{f'})(\phi),
\end{aligned} \tag{1.14}$$

where in the second step integrations of h and h' vanish integrating by part and being ϕ a solution to $P\phi = 0$, and in the third step we used the properties of the causal propagator E.

We can give the following definition,

Definition 1.3.6. We call on-shell algebra of Klein-Gordon field the quotient

$$\mathcal{A}^{KG}_{on}(M) \doteq \frac{\mathcal{A}^{KG}(M)}{\mathcal{I}^{KG}(M)}.$$

Remark 1.3.3. As we observed for the off-shell algebra, $\mathcal{A}^{KG}_{on}(M)$ is generated by linear functionals (up to completion of $C^\infty_0(M) \times \cdots \times C^\infty_0(M)$ in $C^\infty_0(M \times \cdots \times M)$ at the quotient), whose labelling space is constituted by the equivalence classes lying in $\frac{C^\infty_0(M)}{P[C^\infty_0(M)]}$. This allows to make contact with the Borchers-Uhlmann algebra,

Definition 1.3.7. The **Borchers-Uhlmann algebra $\mathcal{B}^{KG}(M)$** is defined as

$$\mathcal{B}^{KG}(M) := \frac{\mathcal{T}(M)}{\mathcal{J}^{KG}(M)},$$

where $\mathcal{T}(M)$ is the universal tensor algebra of compactly supported functions,

$$\mathcal{T}(M) \doteq \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} [C^\infty_0(M)^{\otimes n}],$$

and with a $*$-operation defined by the antilinear extension of $[f^*](x_1, \ldots, x_n) = \bar{f}(x_n, \ldots, x_1)$, whose elements are finite linear combinations of multi-component test functions. The set $\mathcal{J}^{KG}(M)$ is the closed $*$-ideal of $\mathcal{T}(M)$ generated by elements of the form $-iE(f, f') \oplus (f \otimes f' - f' \otimes f)$ for any $f, f' \in C^\infty_0(M)$, where E is the causal propagator and for all $f, f' \in C^\infty_0(M)$ $f \otimes f' \in C^\infty_0(M) \otimes C^\infty_0(M)$, and Pf for all $f \in C^\infty_0$.

Notice that having implemented CCRs by the quotient operation is tantamount to having deformed the point-wise product of a classical algebra of observables. This algebra admits a set of generators which is labelled by $\frac{C^\infty_0(M)}{P[C^\infty_0(M)]}$, the same labelling space of generators of $\mathcal{A}^{KG}_{on}(M)$. This is the guiding idea to prove that $\mathcal{A}^{KG}_{on}(M)$ is $*$-isomorphic to $\mathcal{B}^{KG}(M)$. Borchers-Uhlmann algebra is considered a rather standard construction – [Dim80, Hac10] – but its structure is less flexible compared to the algebra of functionals when dealing with extensions to less regular observables, as Wick polynomials.

1.3.3 Axiomatic structure

The algebra of a Klein-Gordon field constructed in terms of functionals fulfils a set of important properties, which are in agreement with the axiomatic structure of local quantum field theory. These properties are named after Haag and Kastler who first formulated them in [HK64]. They originate from the implementation of *principle of locality* in the general algebraic framework of quantum theory, presented in Section 1.1. A quantum field theory is modeled by assigning to any open region $\mathcal{O} \subset M$ of a spacetime M a $*$-algebra of observables $\mathcal{A}(\mathcal{O})$. The assignement should fulfil *isotony*: If $\mathcal{O}' \subset \mathcal{O}$, then $\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})$, that means $\mathcal{A}(\mathcal{O}')$ describes a subsystem of $\mathcal{A}(\mathcal{O})$. This condition guarantees that one does not loose observables when considering a larger region of spacetime. Locality is then implemented by isotony and the two properties:

- **Causality** – for any pair of open sets $\mathcal{O}, \mathcal{O}' \subset M$ such that they are spacelike separated, *i.e.*, $\mathcal{O} \cap (J^+(\mathcal{O}') \cup J^-(\mathcal{O}')) = \emptyset$,

$$[\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] = 0;$$

- **Covariance** – to any isometry $\varphi \colon \mathcal{O} \to \mathcal{O}'$ is associated a $*$-isomorphism $\alpha_\varphi \colon \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\mathcal{O}')$.

The algebra of functionals $\mathcal{A}^{KG}(M)$ fulfils isotony, causality and covariance. We define $\mathcal{A}^{KG}(\mathcal{O})$ for any $\mathcal{O} \subset M$ as the $*$-subalgebra of $\mathcal{A}^{KG}(M)$, of functionals supported in $\mathcal{O}$. Isotony follows naturally from such defition. For any open neighbourhood $\mathcal{O} \in M$, $\mathcal{A}^{KG}(\mathcal{O})$ can be constructed out of $f \in C_0^\infty(\mathcal{O})$, smooth functions such that $\mathrm{supp}(f) \subset \mathcal{O}$. The other two properties can thus be checked on generators. Causality is a consequence of the fact that for any $f, f' \in C_0^\infty(M)$ such that $\mathrm{supp}(f)$ and $\mathrm{supp}(f')$ are causally separated, it holds $E(f, f') = 0$. Finally for any isometry $\varphi \colon \mathcal{O} \to \mathcal{O}'$ the pullback $f \mapsto \varphi^*(\phi)(f) \doteq \phi(f \circ \varphi^{-1})$ preserves the symplectic form.

Haag-Kastler axioms include additional requirements, such as the spectrum condition, which is related to the positivity of energy. It depends strictly on Poincaré invariance in Minkowski spacetime. A formulation of the complete set of axioms can be found in [HK64, Haa92, Dim80]. It is important to stress that the Haag-Kastler axioms are required as necessary conditions that an algebra of observables should satisfy to be used in the quantization of a field theory. Our perspective is instead different: we start from a classical field theory, we construct the space of solutions and we associate to it an algebra of functionals endowed with suitable $\star$-product. Even if the axiomatic structure is proved *a posteriori*, this procedure is unambiguous, since it depends only on the global hyperbolicity of the spacetime M. In particular, it *results* that $\mathcal{A}^{KG}(M)$ coincides with the *universal algebra*, that is inductive limit of local algebras

$$\overline{\bigcup_{\mathcal{O}} \mathcal{A}^{KG}(\mathcal{O})}, \tag{1.15}$$

the completion being taken in the norm topology.

To conclude this section on the construction of the algebra of observables, we introduce the time-slice axiom,

- **Time-slice axiom** – Given a Cauchy surface $\Sigma \in M$ (M being globally hyperbolic) and $\mathcal{U}$ an open geodesically convex neighbourhood of Σ, $\mathcal{A}(\mathcal{O})$ is $*$-isomorphic to $\mathcal{A}(M)$, the quasi-local algebra in (1.15).

Time-slice axiom is intrinsically an on-shell property and relies on well-posedness of initial value problems. It can be proved that $\mathcal{A}^{KG}_{on}(M)$ fulfils this property – see [BD15].

Proposition 1.3.1. *The algebra $\mathcal{A}^{KG}_{on}(M)$ of on-shell observables of Klein-Gordon field fulfils the time-slice axiom.*

1.4 States for Klein-Gordon field

Having constructed the algebra of observables for a Klein-Gordon field, we can focus on discussing algebraic states thereon, namely any linear functional $\omega : \mathcal{A}^{KG}(M) \to \mathbb{C}$ for which

$$\omega(\mathbb{I}) = 1, \quad \omega(F^*F) \geq 0, \ \forall F \in \mathcal{A}^{KG}(M),$$

where $\mathbb{I}$ is the identity element. We stress that algebraic states make connection with Hilbert spaces, providing a cyclic representation through the GNS theorem.

Being the algebra of functionals $*$-isomorphic to an algebra of test functions, assigning a positive and normalized functional $\widetilde{\omega} : \mathcal{A}^{KG}(M) \to \mathbb{C}$ is tantamount to defining *n-point functions*, $\omega_n : [C^\infty_0(M)]^{\otimes n} \to \mathbb{C}$, that means elements of $\mathcal{D}'(M^n)$, subjected to suitable constraint induced by the structural properties of $\mathcal{A}^{KG}(M)$ and $\mathcal{A}^{KG}_{on}(M)$. It is customary to consider quasifree[4]states, which are determined only by the two-point function.

Definition 1.4.1. Given the $*$-algebra of functionals $\mathcal{A}^{KG}(M)$, a state $\omega : \mathcal{A}^{KG}(M) \to \mathbb{C}$ is **quasifree** if it has vanishing n-point functions for all odd $n \in \mathbb{N}$ and all even n-point can be built in terms of the 2-point function via the following expression:

$$\widetilde{\omega}_{2n}(f_1 \otimes ... \otimes f_{2n}) = \sum_{\pi_{2n} \in S'_{2n}} \prod_{i=1}^{n} \widetilde{\omega}_2 \left(f_{\pi_{2n}(i-1)} \otimes f_{\pi_{2n}(i)} \right),$$

where S'_{2n} stands for the set of ordered permutations of $2n$-elements.

Quasifree states are preferred partly for consistency with standard quantum field theory on Minkowski spacetime, where quasifree states are related to Fock space representations – Minkowski vacuum state is actually quasifree.

The two-point function is subjected to the following constraints $\omega_2 \in \mathcal{D}'(M \times M)$:

[4] We give here, for convenience, the definition of quasifree states for a Klein-Gordon field. Nonetheless, the definition is not tied to any field theoretical model, [KM15].

(i) $\omega(f, \overline{f}) \geq 0$ for any $f \in C_0^\infty(M)$;

(ii) $\omega_2(f, f') - \omega_2(f', f) = iE(f, f')$ for any $f, f' \in C_0^\infty(M)$ (CCRs);

(iii) $\omega_2(f, Pf') = \omega_2(Pf, f') = 0$ for any $[f], [f'] \in \frac{C_0^\infty(M)}{P(C_0^\infty(M))}$ and for any $f \in [f], f' \in [f']$ (on-shell property).

In particular the third condition entails that the state descends to the quotient $\frac{C_0^\infty(M)}{P(C_0^\infty(M))}$ and thus it restricts to a state on $\mathcal{A}_{on}^{KG}(M)$.

Another type of state which will be of interest in what follows is the KMS class of quasi-free states.

Definition 1.4.2. Let α_t denote a one-parameter group of $*$-automorphism on $\mathcal{A}(\mathcal{M})$. A state ω is α_t-invariant if

$$\omega(\alpha_t(a)) = \omega(a) \quad \forall a \in \mathcal{A}(\mathcal{M}).$$

In particular ω is a **KMS state** if it satisfies the *KMS condition*, *i.e.*, $\forall a, b \in \mathcal{A}(\mathcal{M})$:

- $\omega(\alpha_t(a)b)$ extends to the complex plane as an analytic function in the strip $-\beta < \text{ Im}z < 0$;
- $\omega(b\alpha_t(a))$ extends to the complex plane as an analytic function in the strip $0 < \text{ Im}z < \beta$;
- $\omega(b\alpha_{t+i\beta}(a)) = \omega(\alpha_t(a)b)$

The KMS condition gives a generalization in the algebraic language of the concept of thermal state. On suitable backgrounds, *e.g.* on Minkowski spacetime or ultrastatic spacetimes, α_t represents a time evolution. As a consequence, β is an inverse temperature, and the condition describes the thermal equilibrium. In this framework, **ground states** are KMS states associated to zero temperature.

Remark 1.4.1. Ground states, as well as KMS states, can be further characterized as *passive states*. We do not go into details of this topic. Yet passivity allows to prove that positive frequency two-point functions induce ground states, following the argument in [SV00].

1.4.1 Hadamard states

We have sketched very minimal requirements allowing for an enormous set of possible states so that it is appropriate restricting to a suitable class under physically reasonable requirements. To this extent, a physically acceptable state should be such that

- it is consistent with the Minkowski vacuum, meaning that it shall mimic its UV behaviour;

- yields finite quantum fluctuations of the expectation value of the measured observables, such as the smeared components of the stress-energy tensor.

The two requirements entail consistency with the theory on flat spacetime and in particular they call for bridging the gap with normal ordering regularization and the definition Wick polynomials in Minkowski space. Wick polynomials are of paramount importance since they are the natural objects one uses to discuss perturbative interactions and – what is relevant for studying Casimir effect – to define the stress-energy tensor.

Let us make this clear by considering a massless free scalar field theory on Minkowski space. The integral kernel of the two-point function for the Poincaré invariant vacuum state is:

$$\omega_2^0(x,x') = \lim_{\varepsilon\to 0^+} \frac{1}{4\pi^2} \frac{1}{(x-x')^2 + i\varepsilon(x_0 - x_0') + \varepsilon^2} \tag{1.16}$$

where $(x-x')^2$ is meant as the product induced by the Minkowski metric and the limit is taken after smearing against two test functions. The limiting parameter is introduced to regularize the singularity at nulllike separated points, *i.e.*, for $(x-x')^2 = 0$. Vacuum state induces a Fock space representation, being $|0\rangle$ the cyclic vector of the GNS triple, and the expression (1.16) may be read formally as:

$$\omega_2^0(x,x') \doteq \langle 0|\Phi(x)\Phi(x')|0\rangle,$$

representing the mean value of a product of free field operators at separate points, $\Phi(x)\Phi(x')$. As far as the definition of $\Phi^2(x)$ is concerned, one is studying products of pairs of field operators evaluated at the coincidence limit $x \to x'$ – this procedure is usually known as *point-splitting*. Taking the limit to the point-separated product yields a strongly singular expression, as one could check by inspection of the above two point function, and a suitable regularization is needed to define a well-behaved observable. On Minkowski space, this role is played by *normal ordering*. This tool deals with unsmeared field operators expanded in momentum space in terms of creation and annihilation operators:

$$\Phi(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{k}}} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega_\mathbf{k} t} + a^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega_\mathbf{k} t} \right) ,$$

where $\omega_\mathbf{k}^2 := |\mathbf{k}|^\mathbf{2}$ and $\mathbf{k}\cdot\mathbf{x}$ denotes the Euclidean product. Normal ordering does mean replacing $a(\mathbf{k})a^*(\mathbf{k}')$ by $a^*(\mathbf{k}')a(\mathbf{k})$ in the product $\phi(x)\phi(x')$,

$$\begin{aligned}\Phi(t,\mathbf{x})\Phi(t',\mathbf{x}') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{1}{2\sqrt{\omega_\mathbf{k}\omega'_{\mathbf{k}'}}} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega_\mathbf{k} t} + a^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega_\mathbf{k} t} \right) \times \\ \times \left(a(\mathbf{k}') e^{i\mathbf{k}'\cdot\mathbf{x}'} e^{-i\omega_{\mathbf{k}'} t'} + a^*(\mathbf{k}') e^{-i\mathbf{k}'\cdot\mathbf{x}'} e^{i\omega_{\mathbf{k}'} t'} \right) .\end{aligned}$$

where $a(\mathbf{k})$ and $a^*(\mathbf{k})$ are the annihilation and creation operators, here fulfilling the commutation relations

$$\big[a(\mathbf{k}), a^*(\mathbf{k}')\big] = \delta(\mathbf{k}-\mathbf{k}')\mathbb{I}\,,$$

the substitution in the integral leads to

$$:\Phi(t,\mathbf{x})\Phi(t',\mathbf{x}'): = \Phi(t,\mathbf{x})\Phi(t',\mathbf{x}') - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega_{\mathbf{k}}t}\mathbb{I},$$

The integral term in the above expression is singular but, if we evaluate this expression on the Poincaré vacuum ω^0, we realize that the $\omega_2^0(:\Phi(t,\mathbf{x})\Phi(t',\mathbf{x}'):)$ becomes a meaningful expression also when $x \to x'$ since it is actually smooth. The message of the previous example is that, if one controls the singular structure of the two-point function it is possible to remove all the unwanted pathologies by means of a suitable subtraction (also known as regularization). It looks appropriate to require that physical reasonable states are those where such a procedure can be implemented.

The idea can be applied to a curved background, using suitable analytical tools to study singularities of distributions on manifolds. In this analysis the notion of wavefront set is crucial and we give a brief overview in Appendix A. On curved spacetimes, "squared fields" could be modelled as non-linear functionals of the form

$$F_f^2(\phi) = \int_M d\mu_g(x)\phi^2(x)f(x), \tag{1.17}$$

where $f \in C_0^\infty(M)$ while $\phi \in \mathcal{S}^{KG}(M)$, with expectation values given by

$$\omega(F_f^2(\phi))\text{“}=\text{”}\int_{M\times M} d\mu_g(x)d\mu_g(x')\omega_2(x,x')f(x)\delta(x-x'),$$

where "=" means that we would like to define the left hand side as the right hand side. Yet the product $\omega_2\delta$ is not necessarily a well defined distribution. Sufficient conditions for pointwise multiplication are provided by [Hör90, Th. 8.2.10] – see Appendix A – in terms of the wavefront set of distributions. Generalising normal ordering in curved spacetimes, means selecting states whose two-point functions are regular enough to allow pointwise multiplication after a suitable regularization procedure. The following paramount definition hits the target:

Definition 1.4.3. A quasi-free state $\omega : \mathcal{A}_{on}^{KG}(M) \to \mathbb{C}$ is called **Hadamard state** if its defining two-point function $\widetilde{\omega}_2 \in \mathcal{D}'(M\times M)$ is *of Hadamard form*, meaning that

$$WF(\widetilde{\omega}_2) = \{(x,x',k_x,-k_{x'}) \in T^*(M\times M)\setminus\{\mathbf{0}\} \mid (x,k_x)\sim(x',k_{x'}),\ k_x \rhd 0\}\,, \tag{1.18}$$

where $\sim$ entails that x and x' are connected via a lightlike geodesic γ such that k_x is coparallel and cotangent to γ at x and $k_{x'}$ is the parallel transport of k_x from x to x' along γ. The symbol $\rhd$ entails that k_x is a future pointing covector.

As partial justification of the above definition, let us make the following important remark. The wavefront set prescription has been introduced by Radzikowski [Rad96a, Rad96b], who has made a connection with a pre-existing definition of the Hadamard condition in terms of local parametrix [Wal77, Wal78a, KW91].

Definition 1.4.4. A state $\omega : \mathcal{A}^{KG}(M) \to \mathbb{C}$ is of *local Hadamard form* if, in any convex normal neighbourhood $\mathcal{O} \in M$, the kernel integral of its two-point function is of the form:

$$\begin{aligned}\omega_2(x,x') &= \lim_{\varepsilon\to 0^+}\left(\frac{U(x,x')}{4\pi^2\sigma_\varepsilon(x,x')} + V(x,x')\log\left(\frac{\sigma_\varepsilon(x,x')}{\lambda^2}\right)\right) + W(x,y)\\ &:= H(x,x') + W(x,x'),\end{aligned} \tag{1.19}$$

where $U, V \in C^\infty(\mathcal{O}\times\mathcal{O})$, being T any local time coordinate increasing toward the future

$$\sigma_\varepsilon(x,x') := \sigma(x,x') + 2i\varepsilon(T(x) - T(x')) + \varepsilon^2,$$

where $\sigma(x,x')$ is the halved square geodesic separation between x and x' (well-defined in a geodesically convex neighbourhood). Here $\lambda > 0$ is a renormalization length scale and $W \in C^\infty(\mathcal{O} \times \mathcal{O})$ is determined by the state up to redefinition of λ. The kernel $H(x,x')$ is the **Hadamard parametrix**.

Notice that on Minkowski space $\sigma_\varepsilon = -(t-t'-i\varepsilon)^2 + (\mathbf{x}-\mathbf{x}')^2$, $\mathbf{x}$ standing for spatial coordinates, and (1.19) has the same denominator of the Minkowski vacuum two-point function for the massless scalar field (1.16).

The functions $U(x,x')$ and $V(x,x')$ are defined in terms of geometric quantities and of the operator P in (1.2), ruling the dynamics. The length scale λ provides a residual freedom in the definition of the Hadamard parametrix $H(x,x')$, a multiplicative constant in the logarithmic term determined by the scalar curvature R, the mass m and the coupling constant ξ. This yields a renormalisation degree freedom in the definition of Wick polynomials [HW01] as we shall see in what follows.

1.5 Extended algebra of observables

The algebra $\mathcal{A}^{KG}(M)$ contains only basic observables and no Wick polynomials. We are left thus with two questions:

- How to extend the algebra of observables to a suitable notion of Wick polynomials?
- How to give a meaning to expectation values of such more singular observables?

Hadamard regularisation. The first question calls for a generalization to curved spacetime of normal ordering. To obtain this, we can combine Hadamard states with point-splitting regularisation. Consider the example of a quadratic observable (1.17). Following [HW01], we define the *regularised squared Wick polynomial* for any geodesic convex, open neighbourhood $\mathcal{O} \in M$ and $f \in C_0^\infty(\mathcal{O})$,

$$:\phi^2:_H(f) \doteq \int_{M\times M} d\mu_g(x)\, d\mu_g(x')\ (\phi(x)\phi(x') - H(x,x'))\, f(x)\delta(x,x'),$$

where $H(x,x')$ is the Hadamard parametrix in (1.19), $d\mu_g$ is the metric induced volume form and the integral is taken over the whole manifold on account of the support properties of f. Notice that the notation $:\phi^2:_H(f)$ is building a bridge with the Wick regularization in Minkowski space, but it does *no longer* mean "normal ordering" since we are not referring to any annihilation/creation operator. The expectation value

$$\omega(:\phi^2:_H(f)) = \int_M \int_M d\mu_g(x) d\mu_g(x')\, (\omega_2(x,x') - H(x,x'))\, f(x)\delta(x-x'),$$

is well defined on Hadamard states.

Remark 1.5.1. As already pointed out, the local parametrix is only determined up to λ. In particular, [HW01] observe that two definitions $:\phi^2:_H(f)$ and $:\phi^2:_{H'}(f)$ are related as

$$:\phi^2:_{H'}(f) = :\phi^2:_H(f) + \alpha R + \beta m^2,$$

where α and β are dimensionless constants, in which we have included the contribution due to ξ.

The composition of two of these functionals via the $\star$−product introduced in (1.13) is, however, ill-defined at a microlocal level. For any $f, f' \in C_0^\infty(M)$ in fact we have

$$\begin{gathered} :\phi^2:_H(f) \star :\phi^2:_H(f') = \\ \int_{M\times M} d\mu_g(x) d\mu_g(x') f(x) f'(x') \times \\ \times \left(:\phi^2:_H(x)\, :\phi^2:_H(x') + 4iE(x,x')\, :\phi(x)\phi(x'):_H - 2E(x,x')^2\right) \end{gathered}$$

where $E(x,x')$ is the integral kernel of the causal propagator. The problematic term in the above expression is the square of the distribution E. We see this considering its wavefront set,

$$WF(E) = \{(x,x',k_x,-k_{x'}) \in T^*(M\times M) \setminus \{\mathbf{0}\} \mid (x,k_x) \sim (x',k_{x'})\}\,, \quad (1.20)$$

where $\sim$ entails that x and x' are connected via a lightlike geodesic γ such that k_x is coparallel and cotangent to γ at x and $k_{x'}$ is the parallel transport of k_x from x to x' along γ. The square of $E(x,x')$ cannot be defined in terms of Hörmander's criterion for the multiplication of distribution [Hör90, Th. 8.2.10], since the sum of two vectors in the wave front set can yield zero. In order to overcome this difficulty, we follow [BDF09, BF09, FR15], modifying the composition rule in $\mathcal{A}^{KG}(M)$ and then extending the set ofobservables to include also additional regularized fields. The sought modification must preserve the commutation relations among the generators of $\mathcal{A}^{KG}(M)$. It can be written as in (1.12) with Γ_E replaced by

$$\Gamma_H = -i \int\limits_{M\times M} d\mu_g(x) d\mu_g(x') H(x,x') \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(x')}.$$

The product obtained in this way is denoted by $\star_H$ and on $\mathcal{A}^{KG}(M)$ it takes the same form given in (1.13) where the integral kernel $E(x,x')$ is replaced by $-2iH(x,x')$, up to multiplicative constants the (global) Hadamard parametrix. Notice that the antisymmetric part of $-2iH(x,x')$ coincides with $E(x,x')$ and hence the canonical commutation relations among the generators of $\mathcal{A}^{KG}(M)$ are left untouched. Furthermore, since the new $\star$-product is built only out of local structures, covariance of the scheme is guaranteed. In addition, the form (1.18) of the wavefront set of $H(x,x')$ entails that powers of $H(x,x')$ are meaningful since the Hörmander criterion for multiplication of distributions is satisfied – see [Hör90, Th. 8.2.10].

Equipping $\mathcal{F}_0(M)$ with the product $\star_H$ instead of the original $\star$ we obtain an algebra which is isomorphic to $\mathcal{A}^{KG}(M)$. Furthermore, following [BDF09], this isomorphism can be understood as a deformation of the original algebra $\mathcal{A}^{KG}(M)$ which is generated by

$$\alpha_H \doteq \sum_{n=0}^{\infty} \frac{\Gamma_H^n}{n!} : \mathcal{A}^{KG} \to \mathcal{A}^{KG} \tag{1.21}$$

via

$$(F \star_H F') = \alpha_H \left(\alpha_H^{-1}(F) \star \alpha_H^{-1}(F')\right).$$

After such deformation, the set of elements constituting the algebra can be enriched by adding also local non linear functionals like those of the form (1.17). For completeness, we recall the form of the set on which, after the deformation, the algebra of fields can be extended.

Definition 1.5.1. We call microcausal functionals for the Klein-Gordon field, $\mathcal{A}_\mu^{KG}(M)$, the collection of all smooth functionals $F : \mathcal{C}^{KG}(M) \to \mathbb{C}$ such for all $n \geq 1$ and for all $\phi \in \mathcal{C}^{KG}(M)$, $F^{(n)}[\phi] \in \mathcal{E}'(M)^{\otimes n}$. Only a finite number of functional derivatives do not vanish and $\mathrm{WF}(F^{(n)}) \subset \Xi_n$, where

$$\Xi_n \doteq T^*(M)^n \setminus \left\{(x_1,...,x_n,k_1,...,k_n) \,|\, (k_1,...,k_n) \in \left(\overline{V}_+^n \cup \overline{V}_-^n\right)\big|_{(x_1,...,x_n)}\right\},$$

where $\overline{V}_\pm$ are the subsets of T^*M formed by elements (x_i,k_i) where each covector k_i, $i = 1,...,n$ lies in the closed future (+) and in the closed past (-) light cone. The pair $(\mathcal{A}_\mu^{KG}(M), \star_H)$ is called extended algebra of Wick polynomials.

Notice that the expectation values of products of generators of $\mathcal{A}^{KG}(M)$ with respect to a state ω must be invariant under the deformation. In other words $\mathcal{A}^{KG}_\mu(M)$ contains a $*$-subalgebra isomorphic to $\mathcal{A}^{KG}(M)$.

As a last remark on this procedure we stress that there is a degree of freedom in the definition of Wick polynomials. This is known as **regularization freedom** and it is an intrinsic feature of the theory. As pointed out in [HW01, HW05], under minimal physical requirements, like covariance, scaling behaviour or commutation relations, Wick polynomials are determined up to local curvature terms, the mass m and the coupling to scalar curvature ξ. This can be seen to be a by-product of the free choice of the length parameter λ in $H(x, x')$ – see Remark 1.5.1. In principle, regularization freedom can only be fixed by comparison with experiments [DFP08, DHMP10].

1.5.1 The stress-energy tensor observable

The classical stress-energy is defined as the variation of the Klein-Gordon field action S_{KG} with respect to the metric tensor g, as follows

$$T_{\mu\nu} \doteq \frac{-2}{\sqrt{|\mathrm{det} g|}} \frac{\delta S_{\mathrm{KG}}}{\delta g^{\mu\nu}},$$

where the action of the Klein-Gordon field is defined by the Lagrangian (1.3)

$$\begin{aligned} S_{KG}[\phi] &= \int_M d\mu_g(x) \mathcal{L}_{KG}[\phi](x) \\ &= -\frac{1}{2} \int_M d\mu_g(x) \left(\nabla_\mu \phi(x) \nabla^\mu \phi(x) + m^2 \phi^2 + \xi R(x) \phi^2(x) \right) \end{aligned}$$

where g is the metric tensor and $d\mu_g$ the related element of volume, R is the scalar curvature, ξ is the coupling to scalar curvature. From computing the related stress-energy tensor it turns out

$$\begin{aligned} T_{\mu\nu}(x) &= \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} g_{\mu\nu}(x) \partial_\rho \phi(x) \partial^\rho \phi(x) + \\ &+ \left(\xi \left(g_{\mu\nu}(x) \nabla_\rho \partial^\rho - \nabla_\mu \partial_\nu + R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \frac{m^2}{2} \right) \phi^2(x) \end{aligned} \tag{1.22}$$

being $R_{\mu\nu}$ the Ricci tensor – notice that even when $R_{\mu\nu} = 0 = R$ the coupling constant ξ is contributing. The stress-energy tensor has the important property

$$\nabla^\nu T_{\mu\nu}(x) \equiv 0,$$

which should have a counterpart for any sensible quantum stress-energy observable – see remark below. As the classical expression for $T_{\mu\nu}$ is (at least) quadratic in the field, one can expect that a quantum stress-energy tensor has to be defined in terms of Wick squared observable $:\phi^2:_H(x)$, yielding a regularised observable $:T_{\mu\nu}(x):_H$ (Derivatives do not increase the wavefront set of a distribution, so we can safely restrict our considerations to squared fields. A rigorous treatment of such derivative terms can be found in [Mor03, HW05]). In order to define the regularized quantum stress-energy tensor we need a suitable "point-splitted" counterpart,

$$D^{(x,x')}_{\mu\nu}(x,x') = \partial_\mu\phi(x)\gamma^{\nu'}_{\nu}(x,x')\partial_{\nu'}\phi(x') - \frac{1}{2}g_{\mu\nu}(x)\nabla_\rho\phi(x)\gamma^{\rho'}_{\rho}(x,x')\partial^{\rho'}\phi(x') + \\ +\phi(x')\left(\xi\left(g_{\mu\nu}(x)\nabla_\rho\partial^\rho - \nabla_\mu\partial_\nu + R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{m^2}{2}\right)\phi(x) \tag{1.23}$$

where primed indexes means that the covariant derivative must be performed on the primed variable in the same point, while $\gamma^{\mu'}_{\mu}(x,x')\nabla_{\nu'}$ is parallel transport of $\nabla_{\nu'}$ from x to x'.

Various schemes to regularise the quantum stress-energy tensor have been proposed, including ζ-function regularisation [Haw77], which is often applied fruitfully in computations of the Casimir effect, [FP14a]. We remark that ζ-function regularization and point-splitting regularization yield consistent results, as discussed in [HM12]. It is natural to wonder which are the minimal requirements that any regularisation scheme should satisfy to give a physically sensible stress-energy tensor. The answer is provided by *Wald's axioms*, [Wal77] (developed in [Wal78b, Wal94, HW05]), which entail state-independence of regularization, covariance, covariant conservation (of the mean-value) and consistency with the theory in Minkowski space. Throughout the course of such analysis, it has been pointed out that for any regularization prescription there exists a family of different regularization prescriptions. This is the regularization freedom mentioned above. This leads to a four parameter family of stress-energy tensor observables, which differs by four divergence-free tensors, constructed out of the metric and the Riemann tensor, [Wal77, HW05]. In the following chapters, we will always assume the minimal choice yielding the normal ordered stress-energy tensor.

Remark 1.5.2. In [Mor03] it has been proposed to add a term proportional to the field operator P, $\frac{1}{3}g_{\mu\nu}\phi P\phi$. Such term accounts for the so-called *trace anomaly.* The quantum observable built with the bidifferential operator (1.23) is missing "semi-classical" covariant conservation,

$$\nabla^\nu\omega(:T_{\mu\nu}:_H(f)) \equiv 0 \qquad \forall f \in C^{KG}_0(M).$$

The term restores covariant conservation, but, as a by-product, produces a non-vanishing trace in case of conformally coupled theory (trace anomaly).

Chapter 2

Algebraic quantum field theory and the Casimir-Polder effect

In[1] this chapter we consider a prototypical example of spacetime with boundary, the half Minkowski spacetime. The aim is to have a first insight on the main features related to the introduction of a boundary in the theory outlined in Chapter 1. We exploit systematically the renowned method of images, which provides a useful tool to prove the main results by relating them to the theory on Minkowski spacetime. Despite we apply a constructive technique which make use of Minkowski spacetime, our approach provides an intrinsic theory, since all objects and results refers only to elements of the system. At the end of this chapter we discuss the construction of an extended algebra of regularized observables. We are going to show that a regularization in terms of a local parametrix, analogous to the Hadamard regularization presented in Chapter 1, provide well-defined obsevables only localized in globally hyperbolic subregion of the half-space. This does not prevent to give sense locally to the energy density as we shall discuss.

Let us consider the following region of Minkowski spacetime, $(\mathbb{H}^4, \eta)$, where $\mathbb{H}^4 = \mathbb{R}^3 \times [0, \infty)$ is the four dimensional upper half-plane endowed with the Lorentzian flat metric. We introduce cartesian coordinate such that the interval $[0, \infty)$, is described by the spatial coordinate z. For later convenience, we introduce also the notation $(\underline{x}, z) := (t, x, y, z)$. We consider a real scalar field vanishing on the boundary $\partial\mathbb{H}^4$ and whose dynamics is ruled by the Klein-Gordon equation (1.2). This scenario is often associated in the physics literature to the so-called Casimir-Polder effect [CP48], which describes originally the interaction between a neutral atom in an electromagnetic cavity and a perfectly conducting wall at a distance d. For that reason, from now on we shall refer to our setting as a *Casimir-Polder system*.

We recall a standard definition in analysis [Lee00, Chapter 1]:

[1]The content of this chapter is part of [DNP16, Section 2].

Definition 2.1. Let $\mathcal{O} \subseteq \mathbb{H}^4$. We say that $u \in C^\infty(\mathcal{O})$ if and only if there exist both an open subset $\widetilde{\mathcal{O}}$ of $\mathbb{R}^4$ such that $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$ and $\widetilde{u} \in C^\infty(\widetilde{\mathcal{O}})$ such that $\widetilde{u}|_{\mathcal{O}} = u$.

Notice that the existence of $\widetilde{u}$ is guaranteed if and only if u is continuous on the whole O, smooth on the interior $\mathring{O} \doteq O \setminus \partial O$ and each partial derivative of u on $\mathring{O}$ has a continuous extension to ∂O. With this last definition and in full analogy with the standard case of a real scalar field on Minkowski spacetime, we call **dynamical configurations** of a Casimir-Polder system the set $\mathcal{S}^{CP}(\mathbb{H}^4)$ of all $u \in C^\infty(\mathbb{H}^4)$ such that u satisfies the following boundary condition problem:

$$\begin{cases} Pu = (\Box - m^2 - \xi R)\, u = 0, & m \geq 0 \,\text{and}\, \xi \in \mathbb{R} \\ u(\underline{x}, 0) = 0 & \end{cases}, \tag{2.1}$$

where R is the scalar curvature. We recall that, although the scalar curvature on Minkowski spacetime or on any of its subsets vanishes identically, the coupling term ξR has a consequence on the form of the stress-energy tensor, which is proportional to the variation of the Lagrangian with respect to the metric.

2.1 Algebra of observables of a Casimir-Polder system

The half space is not a globally hyperbolic spacetime[2]. This can be argued by the criterion in Proposition 1.2.1, since $\mathbb{H}^4$ inherits the ultrastatic property from $\mathbb{R}^4$ and (referring to the notation in Defition 1.2.6) the spacelike section $N = \mathbb{R}^2 \times [0, \infty)$ is not geodesically complete at the boundary. The failure of globally hyperbolic property forces the introduction of boundary conditions in the dynamical problem. In order to tackle the boundary conditions, we construct dynamical configurations via the method of images. The analysis which will involve the remainder of the section can be divided in three parts and it will follows conceptually the one outlined for the Klein-Gordon scalar field on globally hyperbolic spacetime in Chapter 1.

Part 1 – Dynamical configurations: We introduce the isometry $\iota_z : \mathbb{R}^4 \to \mathbb{R}^4$ for which $(\underline{x}, z) \mapsto (\underline{x}, -z)$, bearing in mind the notation $(\underline{x}, z) := (t, x, y, z)$. With a slight abuse of notation, we adopt the same symbol also to indicate its natural action on $C^\infty(\mathbb{R}^4)$ and on generic distribution. We also recall, that, in view of Poincaré covariance, $\iota_z \circ E = E \circ \iota_z$, where E is the causal propagator of P.

Let us call $C^\infty_-(\mathbb{R}^4)$ the set of all smooth functions on Minkowski spacetime such that $\alpha(\underline{x}, z) = -\alpha(\underline{x}, -z)$. The elements lying in this set are said to be *odd* (under reflection along the hyperplane $z = 0$). Conversely we refer to $C^\infty_+(\mathbb{R}^4)$

[2] In the sense of footnote 2.

as the collection of smooth functions which are *even* under reflection along the hyperplane $z = 0$, that is $\alpha(\underline{x}, z) = \alpha(\underline{x}, -z)$. Notice the following splitting of vector spaces: $C^\infty(\mathbb{R}^4) = C^\infty_-(\mathbb{R}^4) \oplus C^\infty_+(\mathbb{R}^4)$. Furthermore, since the operator P contains only the second derivative along the z-direction, it holds that $P : C^\infty_\pm(\mathbb{R}^4) \to C^\infty_\pm(\mathbb{R}^4)$ and, thus, $P[C^\infty(\mathbb{R}^4)] = P[C^\infty_-(\mathbb{R}^4)] \oplus P[C^\infty_+(\mathbb{R}^4)]$ as well as

$$\frac{C^\infty(\mathbb{R}^4)}{P[C^\infty(\mathbb{R}^4)]} \simeq \frac{C^\infty_-(\mathbb{R}^4)}{P[C^\infty_-(\mathbb{R}^4)]} \oplus \frac{C^\infty_+(\mathbb{R}^4)}{P[C^\infty_+(\mathbb{R}^4)]}. \tag{2.2}$$

The isomorphism (2.2) holds true even restricted to $C^\infty_{tc}(\mathbb{R}^4)$, $C^\infty_{sc}(\mathbb{R}^4)$ and to $C^\infty_0(\mathbb{R}^4)$.

Proposition 2.1.1. *Let $\mathcal{S}^{CP}(\mathbb{H}^4)$ be the dynamical configurations of a Casimir-Polder system. It holds that $\rho_{\mathbb{H}^4} \circ (E - \iota_z \circ E) : \frac{C^\infty_{tc,-}(\mathbb{R}^4)}{P\left[C^\infty_{tc,-}(\mathbb{R}^4)\right]} \to \mathcal{S}^{CP}(\mathbb{H}^4)$ is a bijection. Here $\rho_{\mathbb{H}^4}$ stands for the restriction map on $\mathbb{H}^4$.*

Proof. The map $E - \iota_z \circ E$ implements the *method of images* on Minkowski spacetime. Hence its image is a solution to Klein-Gordon equation on $\mathbb{R}^4$ and, once restricted to $\mathbb{H}^4$ via ρ, it implements also the Dirichlet boundary condition.

Let us prove surjectivity. For any $u \in C^\infty(\mathbb{H}^4)$ fulfilling (2.1) we define the auxiliary function

$$\widetilde{u}(\underline{x}, z) = \begin{cases} u(\underline{x}, z), & \forall\, (\underline{x}, z) \in \mathbb{H}^4 \\ -u(\underline{x}, -z) & \forall\, (\underline{x}, z) \in \mathbb{R}^3 \times (-\infty, 0) \end{cases}.$$

Notice that $\widetilde{u} \in C^\infty(\mathbb{R}^4)$. To show it, it suffices to control the behaviour of the function at $\partial\mathbb{H}^4 = \mathbb{R}^3 \times \{0\}$. Since $u(\underline{x}, 0) = 0$ then $\widetilde{u}$ is continuous at $\partial\mathbb{H}^4$. Let us consider now the first order partial derivatives: Continuity at $\partial\mathbb{H}^4$ is guaranteed along any of the $\underline{x}$-directions since $u(\underline{x}, 0) = 0$ whereas that along the z-direction descends from the fact that $\widetilde{u}$ is odd along the z-directions and thus $\partial_z \widetilde{u}$ is even. A similar string of reasoning can be applied slavishly to the second derivative barring that along the z-direction for which we have first to recall that $\partial_z^2 u(\underline{x}, z) = \left(\partial_t^2 - \partial_x^2 - \partial_y^2 + m^2 + \xi R\right) u(\underline{x}, z)$. Consequently since u vanishes on $\partial\mathbb{H}^4$, so does $\partial_z^2 u$. Reiterating the procedure to all orders yields in combination with Schwarz theorem the sought result. Furthermore, since u is a solution of (2.1), it holds that $P\widetilde{u} = 0$. Consequently, in view of our discussion in Chapter 1, there exists $[\alpha] \in \frac{C^\infty_{tc}(\mathbb{R}^4)}{P[C^\infty_{tc}(\mathbb{R}^4)]}$ such that $\widetilde{u} = E(\alpha)$.

Since $\widetilde{u}$ is an odd function, $0 = \widetilde{u} + \iota_z \widetilde{u} = E(\alpha) + \iota_z E(\alpha) = E(\alpha + \iota_z \alpha) = 0$. Hence there exists $\lambda \in C^\infty_{tc,+}(\mathbb{R}^4)$, for which $\alpha + \iota_z \alpha = P\lambda$. If we add the information that $E \circ P = 0$ and that $P[C^\infty_{tc}(\mathbb{R}^4)] = P[C^\infty_{tc,+}(\mathbb{R}^4)] \oplus P[C^\infty_{tc,-}(\mathbb{R}^4)]$, to each $u \in \mathcal{S}^{CP}(\mathbb{H}^4)$, we can associate an equivalence class $[\alpha] \in \frac{C^\infty_{tc,-}(\mathbb{R}^4)}{P\left[C^\infty_{tc,-}(\mathbb{R}^4)\right]}$. This proves surjectivity. Notice that this map is per construction injective as, if $u = 0$, then $\widetilde{u} = 0$ and, thus we can write $\widetilde{u} = E(\alpha)$ with $\alpha \in P\left[C^\infty_{tc,-}(\mathbb{R}^4)\right]$. $\square$

Part 2 – The off-shell algebra: Following the scheme given in Chapter 1, we define at first a space of **kinematical configurations** for a Casimir-Polder system. Let us introduce the following map:

$$\begin{aligned} \eta : C^\infty(\mathbb{R}^4) &\to C^\infty(\mathbb{H}^4), \\ \phi(\underline{x}, z) \mapsto u(\underline{x}, z) &\doteq \frac{1}{\sqrt{2}} \left(\phi(\underline{x}, z) - \iota_z \phi(\underline{x}, z) \right) \Big|_{\mathbb{H}^4}, \end{aligned} \tag{2.3}$$

where the numerical pre-factor is a normalization.

Definition 2.1.1. We call *space of kinematical/off-shell configurations* for a Casimir-Polder system

$$\mathcal{C}^{CP}(\mathbb{H}^4) \doteq \left\{ u \in C^\infty(\mathbb{H}^4) \mid \; u|_{\partial\mathbb{H}^4} = 0 \text{ and } \exists \phi \in \mathcal{C}^{KG}(\mathbb{R}^4) \text{ such that } u = \eta(\phi) \right\},$$

where η is the map defined in (2.3). Since $\mathcal{C}^{CP}(\mathbb{H}^4) \subset C^\infty(\mathbb{H}^4)$ and since η is for construction surjective thereon, we endow $\mathcal{C}^{CP}(\mathbb{H}^4)$ with the quotient topology. In complete analogy we shall also consider $\mathcal{C}_0^{CP}(\mathbb{H}^4)$ where the subscript 0 stands for compact support.

For later convenience, we introduce $\eta^\dagger : C_0^\infty(\mathbb{H}^4) \to \mathcal{E}'(\mathbb{R}^4)$ defined via

$$\langle \eta^\dagger(h), \phi \rangle \doteq \langle h, \eta(\phi) \rangle_{\mathbb{H}^4} = \int_{\mathbb{H}^4} d^4x \, h(x) \eta(\phi)(x),$$

Notice both that it is possible to write the integral kernel of $\eta^\dagger(h) \in \mathcal{E}'(\mathbb{R}^4)$ as $\frac{1}{\sqrt{2}} (h(x)\Theta(z) - [\iota_z(h)](x)\Theta(-z))$, where Θ is the Heaviside step function and that

$$WF(\eta^\dagger(h)) \subset \{(x, k) \in T^*\mathbb{R}^4 \setminus \{0\} \mid x \in \partial\mathbb{H}^4 \text{ and } k^i = g^{ij} k_j = 0, \, \forall i \neq z\}, \tag{2.4}$$

where g stands here for the Minkowski metric written in Cartesian coordinates.

In analogy with Definition 1.3.2, we now introduce regular functionals on $\mathcal{C}^{CP}(\mathbb{H}^4)$.

Definition 2.1.2. Let $F : \mathcal{C}^{CP}(\mathbb{H}^4) \to \mathbb{C}$ be any smooth functional. We call it **regular** if for all $k \geq 1$ and for all $u \in \mathcal{C}^{CP}(\mathbb{H}^4)$, $F^{(k)}[u] \in C_0^\infty(\mathbb{H}^{4k})$ and if only finitely many functional derivatives do not vanish. We indicate this set as $\mathcal{F}_0(\mathbb{H}^4)$.

In order to introduce a suitable product in $\mathcal{F}_0(\mathbb{H}^4)$, analogous to (1.12), we define a map which plays the role of E in (2.6):

$$E_{\mathbb{H}^4} : C_0^\infty(\mathbb{H}^4) \to \mathcal{S}^{CP}, \quad E_{\mathbb{H}^4}(h) \doteq \eta \circ E \circ \eta^\dagger(h), \tag{2.5}$$

and we call it **CP-propagator**. Observe that $E \circ \eta^\dagger(h)$ is well-defined in view of (2.4) and of [Hör90, Th. 8.2.13]. The latter also ensures that, for all

$h \in C_0^\infty(\mathbb{H}^4)$, $E(\eta^\dagger(h)) \in C_{sc}^\infty(\mathbb{R}^4)$ and it solves the Klein-Gordon equation. As a consequence $E_{\mathbb{H}^4}$ is well-defined map into $\mathcal{S}^{CP}$.

Let us consider now:

$$\star_{\mathbb{H}^4} : \mathcal{F}_0(\mathbb{H}^4) \times \mathcal{F}_0(\mathbb{H}^4) \to \mathcal{F}_0(\mathbb{H}^4),$$

which associates to each $F, F' \in \mathcal{F}_0(\mathbb{H}^4)$

$$(F \star_{\mathbb{H}^4} F')(u) = \Big(\mathcal{M} \circ \exp(i\Gamma_{E^4_{\mathbb{H}}})(F \otimes F')\Big)(u). \tag{2.6}$$

Here $\mathcal{M}$ stands for the pointwise multiplication, *i.e.*, $\mathcal{M}(F \otimes F')(u) \doteq F(u)F'(u)$, whereas

$$\Gamma_{E_{\mathbb{H}^4}} \doteq \frac{1}{2}\int_{\mathbb{H}^4 \times \mathbb{H}^4} E_{\mathbb{H}^4}(x,x') \frac{\delta}{\delta u(x)} \otimes \frac{\delta}{\delta u(x')},$$

where $E_{\mathbb{H}^4}(x,x')$ is the integral kernel of (2.5). The exponential in (2.6) is defined intrinsically in terms of the associated power series and, consequently, we can rewrite the product also as

$$(F \star_{\mathbb{H}^4} F')(u) = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \langle F^{(n)}(u), E^{\otimes n}_{\mathbb{H}^4}(F'^{(n)})(u)\rangle_{\mathbb{H}^4}, \tag{2.7}$$

where $\langle , \rangle_{\mathbb{H}^4}$ stands for the pairing on $\mathbb{H}^4$ built of out integration. The 0-th order is defined as the pointwise multiplication, that is $\langle F^{(0)}(u), F'^{(0)}(u)\rangle \doteq F(u)F'(u)$. Notice that (2.7) defines a $\star$-product. In view of the definition of the CP-propagator, (2.5), $\langle F^{(n)}(u), E^{\otimes n}_{\mathbb{H}^4}(F'^{(n)})(u)\rangle_{\mathbb{H}^4}$ is well-defined for all $n \geq 0$ and the non-vanishing derivatives of $F \star_{\mathbb{H}^4} F'$ evaluated on any $u \in \mathcal{C}^{CP}(\mathbb{H}^4)$ are compactly suppported.

Definition 2.1.3. We call $\mathcal{A}^{CP}(\mathbb{H}^4) \equiv (\mathcal{F}_0(\mathbb{H}^4), \star_{\mathbb{H}^4})$ the *off-shell $*$-algebra* of a Casimir-Polder system endowed with complex conjugation as $*$-operation. It is generated by the functionals $F_h(u) = \int\limits_{\mathbb{H}^4} d^4x\, u(x)h(x)$ where $h \in C_0^\infty(\mathbb{H}^4)$ while $u \in \mathcal{C}^{CP}(\mathbb{H}^4)$.

Part 3 – The on-shell algebra: To conclude our investigation on the algebra of observables for a Casimir-Polder system, we want to investigate how $\mathcal{A}^{CP}(\mathbb{H}^4)$ should be modified if we restrict the allowed configurations from $\mathcal{C}^{CP}(\mathbb{H}^4)$ to $\mathcal{S}^{CP}(\mathbb{H}^4)$. At this stage it is more advantageous to work on the counterpart of $\mathcal{S}^{CP}(\mathbb{H}^4)$ on Minkowski spacetime specified by Proposition 2.1.1.

Proposition 2.1.2. *Let $\mathcal{O}^{KG}_{-}(\mathbb{R}^4)$ be the span of all functionals $F_{[\zeta]} : \frac{C^\infty_{tc,-}(\mathbb{R}^4)}{P[C^\infty_{tc,-}(\mathbb{R}^4)]} \to \mathbb{C}$, $[\zeta] \in \frac{C^\infty_{0,-}(\mathbb{R}^4)}{P[C^\infty_{0,-}(\mathbb{R}^4)]}$ such that $F_{[\zeta]}([\alpha]) = \int\limits_{\mathbb{R}^4} \zeta E(\alpha)$. This space is:*

1. **separating***, that is for every pair of different configurations $[\alpha], [\alpha'] \in \frac{C^\infty_{tc,-}(\mathbb{R}^4)}{P[C^\infty_{tc,-}(\mathbb{R}^4)]}$, there exists a classical observable $[\zeta] \in \frac{C^\infty_{0,-}(\mathbb{R}^4)}{P[C^\infty_{0,-}(\mathbb{R}^4)]}$ such that $F_{[\zeta]}([\alpha]) \neq F_{[\zeta]}([\alpha'])$.*

2. **optimal**, *that is, for every pair of classical observables* $[\zeta], [\zeta'] \in \frac{C^\infty_{0,-}(\mathbb{R}^4)}{P[C^\infty_{0,-}(\mathbb{R}^4)]}$ *there exists at least one configuration* $[\alpha] \in \frac{C^\infty_{tc,-}(\mathbb{R}^4)}{P[C^\infty_{tc,-}(\mathbb{R}^4)]}$ *such that* $F_{[\zeta]}([\alpha]) = F_{[\zeta']}([\alpha])$

3. **symplectic** *if endowed with the following weakly non-degenerate symplectic form*[3]*:*

$$\sigma : \mathcal{O}^{KG}_-(\mathbb{R}^4) \times \mathcal{O}^{KG}_-(\mathbb{R}^4) \to \mathbb{R},$$
$$\sigma(F_{[\zeta]}, F_{[\zeta']}) = \langle \zeta, E(\zeta')\rangle = \int_{\mathbb{R}^4} d^4x\, \zeta(x)E(\zeta')(x).$$

Proof. Let us prove *1.* Consider any pair $[\alpha], [\alpha'] \in \mathcal{S}^{CP}(\mathbb{H}^4)$, $[\alpha] \neq [\alpha']$, and two representatives $\alpha, \alpha' \in C^\infty_{tc,-}(\mathbb{R}^4)$. On account of standard arguments in analysis we know that $C^\infty_0(\mathbb{R}^4)$ is separating for $C^\infty(\mathbb{R}^4)$ with respect to the pairing (1.3.2). Hence, since $E(\alpha - \alpha') \in C^\infty(\mathbb{R}^4)$ is not vanishing, there must exist $\beta \in C^\infty_0(\mathbb{R}^4)$ such that $(\beta, E(\alpha - \alpha')) \neq 0$. Since $E(\alpha - \alpha') \in C^\infty_-(\mathbb{R}^4)$, it holds that $(\beta, E(\alpha-\alpha')) = (\zeta, E(\alpha-\alpha'))$ where $\zeta(\underline{x}, z) \doteq \beta(\underline{x}, z) - \beta(\underline{x}, -z) \in C^\infty_{0,-}(\mathbb{R}^4)$. ζ identifies a non trivial element in $\mathcal{O}^{KG}_-(\mathbb{R}^4)$, hence the statement is proven.

We focus on *2.* Let $[\zeta], [\zeta'] \in \mathcal{O}^{KG}_-(\mathbb{R}^4)$ and let ζ, ζ' be two arbitrary representatives. For the same reason as in the previous point, since $E(\zeta - \zeta') \in C^\infty(\mathbb{R}^4)$ is non vanishing there must exist $\gamma \in C^\infty_0(\mathbb{R}^4)$ such that $\mathrm{supp}(\gamma) \cap (\mathrm{supp}(E(\zeta)) \cup \mathrm{supp}(E(\zeta'))) \neq \emptyset$ and that $(\gamma, E(\zeta - \zeta')) \neq 0$. Let $\alpha(\underline{x}, z) \doteq \gamma(\underline{x}, z) - \gamma(\underline{x}, -z) \in C^\infty_{0,-}(\mathbb{R}^4) \subset C^\infty_{tc,-}(\mathbb{R}^4)$ individuate an element in $\mathcal{S}^{CP}(\mathbb{H}^4)$ via the action of the causal propagator E. It holds that $F_{[\zeta]-[\zeta']}([\alpha]) = (\zeta - \zeta', E(\alpha)) = -(E(\zeta - \zeta'), \alpha)) = 2(E(\zeta - \zeta'), \gamma) \neq 0$, which entails the sought result.

At last we prove *3.* Notice that, per construction, σ is bilinear and antisymmetric. Suppose that, per absurd, there exists a non trivial $F_{[\zeta]} \in \mathcal{O}^{KG}_-(\mathbb{R}^4)$ such that $\sigma(F_{[\zeta]}, F_{[\zeta']}) = 0$ for every $F_{[\zeta']} \in \mathcal{O}^{KG}_-(\mathbb{R}^4)$. Since every representative of $[\zeta]$ is odd, the same statement holds true for every $[\zeta'] \in \frac{C^\infty_0(\mathbb{R}^4)}{P[C^\infty_0(\mathbb{R}^4)]}$ since $\frac{C^\infty_{0,-}(\mathbb{R}^4)}{P[C^\infty_{0,-}(\mathbb{R}^4)]}$ and $\frac{C^\infty_{0,+}(\mathbb{R}^4)}{P[C^\infty_{0,+}(\mathbb{R}^4)]}$ are orthogonal to each other with respect to σ. □

Corollary 2.1.3. *Let* $\mathcal{O}^{CP}(\mathbb{H}^4)$ *be the span of all functionals* $F_{[h]} : \mathcal{S}^{CP}(\mathbb{H}^4) \to \mathbb{C}$ *with* $[h] \in \frac{\mathcal{C}^{CP}_0(\mathbb{H}^4)}{P[\mathcal{C}^{CP}_0(\mathbb{H}^4)]}$ *such that* $F_{[h]}(u) = \int_{\mathbb{H}^4} d^4x\, h(x)u(x)$*, endowed with the symplectic form:*

$$\sigma_{\mathbb{H}^4} : \mathcal{O}^{CP}(\mathbb{H}^4) \times \mathcal{O}^{CP}(\mathbb{H}^4) \to \mathbb{R},$$
$$\sigma_{\mathbb{H}^4}(F_{[h]}, F_{[h']}) \doteq \langle h, E_{\mathbb{H}^4} h'\rangle_{\mathbb{H}^4} = \int_{\mathbb{H}^4} d^4x\, h(x)E_{\mathbb{H}^4}(h')(x).$$

[3]Notice that, from a geometrical point of view, it would be more appropriate to refer to σ as a Poisson structure. We stick to the more traditional codification used in algebraic quantum field theory.

There exists an isomorphism of symplectic spaces between $\mathcal{O}^{CP}(\mathbb{H}^4)$ and $\mathcal{O}^{KG}_{-}(\mathbb{R}^4)$.

Proof. First of all we notice that $F_{[h]}(u)$ with $u \in \mathcal{S}^{CP}(\mathbb{H}^4)$ and $[h] \in \frac{\mathcal{C}_0^{CP}(\mathbb{H}^4)}{P[\mathcal{C}_0^{CP}(\mathbb{H}^4)]}$ is well-defined as the choice of the representative of $[h]$ is not relevant. As a matter of fact, on account of the boundary conditions of all elements involved, we can still integrate by parts canceling all boundary terms so that, for all Ph', $h' \in \mathcal{C}_0^{CP}(\mathbb{H}^4)$, $\int\limits_{\mathbb{H}^4} d^4x\, P(h')u = \int\limits_{\mathbb{H}^4} d^4x\, h'Pu = 0$ since $u \in \mathcal{S}^{CP}(\mathbb{H}^4)$. To prove the isomorphism we construct explicitly a bijective map from $\mathcal{O}^{CP}(\mathbb{H}^4)$ to $\mathcal{O}^{KG}_{-}(\mathbb{R}^4)$, preserving the symplectic structure. We observe that the map η of (2.3) is injective on $C^\infty_{0,-}(\mathbb{R}^4)$ thus it admits an inverse map η^{-1} defined on $\eta[C^\infty_{0,-}(\mathbb{R}^4)] \equiv \mathcal{C}_0^{CP}(\mathbb{H}^4)$. Since both η and η^{-1} descend to the quotients $\frac{C^\infty_{0,-}(\mathbb{R}^4)}{P[C^\infty_{0,-}(\mathbb{R}^4)]}$ and $\frac{\mathcal{C}_0^{CP}(\mathbb{H}^4)}{P[\mathcal{C}_0^{CP}(\mathbb{H}^4)]}$, we can define with a slight abuse of notation the pull-back:

$$\eta^* : \mathcal{O}^{CP}(\mathbb{H}^4) \to \mathcal{O}^{KG}_{-}(\mathbb{R}^4), \quad \eta^*(F_{[h]})([\alpha]) \doteq F_{[h]}(\eta(\phi)), \tag{2.8}$$

where $\phi = E([\alpha]) \in \mathcal{S}^{KG}$. Since any $u \in \mathcal{S}^{CP}$ is the unique image of a $[\alpha] \in \frac{C^\infty_{tc,-}(\mathbb{R}^4)}{P[C^\infty_{tc,-}(\mathbb{R}^4)]}$ via the bijection *2.* of Proposition 2.1.1, η^* is an isomorphism of vector spaces. It also preserving the symplectic structure $\sigma_{\mathbb{H}^4}$. To prove it, let us observe that $\eta^*(F_{[h]}) = F_{\eta^\dagger([h])}$. We thus can write:

$$\begin{aligned}\sigma(\eta^*(F_{[h]}), \eta^*(F_{[h']})) &= \sigma(F_{\eta^\dagger([h])}, F_{\eta^\dagger([h'])}) \\ &= \langle \eta^\dagger(h), E(\eta^\dagger(h'))\rangle = \langle h, \eta E \eta^\dagger(h')\rangle_{\mathbb{H}^4} = \\ &= \sigma_{\mathbb{H}^4}(F_{[h]}, F_{[h']}),\end{aligned}$$

which is valid for all $F_{[h]}, F_{[h']} \in \mathcal{O}^{CP}(\mathbb{H}^4)$. □

By a computation analogous to (1.14), it is possible to prove that the $\star$-product of Definition 1.12 descends to $\mathcal{O}^{CP}(\mathbb{H}^4)$. Corollary 2.1.3 thus ensures that $\star_{\mathbb{H}^4}$ defined in (2.6) is well defined on $\mathcal{O}^{CP}(\mathbb{H}^4)$. Consequently, we have finally,

Definition 2.1.4. We call **on-shell $*$-algebra of observables for a Casimir-Polder system** the algebra $\left(\mathcal{A}^{CP}_{on}(\mathbb{H}^4), \star_{\mathbb{H}^4}\right)$ generated by the functionals defined in Corollary 2.1.3, $\mathcal{O}^{CP}(\mathbb{H}^4)$, where $\star_{\mathbb{H}^4}$ is defined in (2.6).

Before proving several important properties of $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$, we want to investigate how it relates with the Minkowski counterpart $\mathcal{A}^{KG}(\mathbb{R}^4)$. This will give us the chance to prove the already mentioned properties.

Proposition 2.1.4. *Let $\widetilde{\eta}^* : \mathcal{A}^{CP}_{on}(\mathbb{H}^4) \to \mathcal{A}^{KG}_{on}(\mathbb{R}^4)$ be the natural extension of the pull-back map η^* defined on $\mathcal{O}^{CP}(\mathbb{H}^4)$. This is an injective $*$-homomorphism of algebras which becomes an isomorphism onto $\mathcal{A}^{KG}_{on,-}(\mathbb{R}^4)$, the $*$-subalgebra of $\mathcal{A}^{KG}(\mathbb{R}^4)$ generated by functionals $\mathcal{O}^{KG}_{-}(\mathbb{R}^4)$.*

Proof. Let us prove that $\widetilde{\eta}^*$ is injective. Suppose that there exists $F_{[h]} \in \mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ such that $\eta^*(F_{[h]})$ is the vanishing functional. Then, for all $\phi \in \mathcal{S}^{KG}(\mathbb{R}^4)$ one has $0 = \eta^*(F_{[h]})(\phi) = F_{[h]}(\eta(\phi)) = F_{\eta^\dagger([h])}(\phi)$. Since ϕ is arbitrary, the only possible solution is $\eta^\dagger([h]) = 0$ and, thus, $[h] = 0$, which entails the sought injectivity. In order to prove that $\widetilde{\eta}^*$ is also a $*$-homomorphism it suffices to focus again only on the generators. Let $F_{[h]}, F_{[h']} \in \mathcal{O}^{CP}(\mathbb{H}^4)$ and $\phi \in \mathcal{S}^{KG}(\mathbb{R}^4)$. Then, on account of (2.6) the following holds true:

$$\begin{aligned}\left(\eta^*(F_{[h]}) \star \eta^*(F_{[h']})\right)(\phi) &= (F_{\eta^\dagger([h])} \star F_{\eta^\dagger([h'])})(\phi) \\ &= F_{\eta^\dagger([h])}(\phi)F_{\eta^\dagger([h'])}(\phi) + \frac{i}{2}\langle \eta^\dagger(h), E(\eta^\dagger(h'))\rangle = \\ &= F_{[h]}(\eta(\phi))F_{h'}(\eta(\phi)) + \frac{i}{2}\langle h, \eta E \eta^\dagger(h')\rangle_{\mathbb{H}^4} = \\ &= \left(F_{[h]} \star_{\mathbb{H}^4} F_{[h']}\right)(\phi).\end{aligned}$$

Since the $*$-operation is complex conjugation, it is left untouched by all the operations above and, as a consequence, we can infer that $\widetilde{\eta}^*$ is a $*$-homomorphism. The isomorphism $\mathcal{A}^{CP}_{on}(\mathbb{H}^4) \simeq \mathcal{A}^{KG}_{on,-}(\mathbb{R}^4)$ descends directly from Corollary 2.1.3. □

In the following proposition, we investigate the structural properties of $\mathcal{A}^{CP}(\mathbb{H}^4)$, in particular causality and the time-slice axiom [BFV01, Dim80]. The latter property needs a few comments. Recall that $\mathcal{A}^{KG}_{on}(\mathbb{R}^4)$ fulfills the time-slice axiom, namely, given any open neighbourhood $\mathcal{N}$ of a Cauchy surface Σ in Minkowski spacetime, containing all causal curves whose endpoints lie in $\mathcal{N}$, then $\mathcal{A}^{KG}_{on}(\mathbb{R}^4)$ is $*$-isomorphic to $\mathcal{A}^{KG}_{on}(\mathcal{N})$. Since $\mathbb{H}^4$ is not globally hyperbolic there is no notion of a Cauchy surface. Yet, if we consider the extension of the isomorphism of Proposition 2.1.4 to $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$, this is $*$-isomorphic to a $*$-subalgebra of $\mathcal{A}^{KG}_{on}(\mathbb{R}^4)$ for which the time-slice axiom is a well-defined concept. The validity of the time-slice axiom carries the idea that the boundary is never acting as an absorber or emitter of dynamical information. This is ultimately a consequence of having fixed a boundary condition. In addition to these two properties, we show that $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ satisfies the F-locality condition [FH95, Kay92], a requirement which should be met by the algebra of observables of a quantum field theory on a non globally-hyperbolic spacetime. In a few words and in the case at hand, it requires that $\mathcal{A}^{CP}(\mathbb{H}^4)$ and $\mathcal{A}^{KG}(\mathbb{R}^4)$, restricted to any globally hyperbolic subregion of $\mathbb{H}^4$ must be $*$-isomorphic. Such condition can be seen as an extension of the locality paradigm, according to which, from local algebras, one should not be able to extract information on the global structure of the background. Let us denote $\mathcal{A}^{CP}_{on}(\mathcal{O})$ the localization of the algebra of observables $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ on any subregion $\mathcal{O} \subset \mathbb{H}^4$, that is the $*$-subalgebra whose generators are supported in $\mathcal{O}$.

Proposition 2.1.5. *The algebra $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ is causal, it fulfills the time-slice axiom and it satisfies the F-locality property, namely $\mathcal{A}^{CP}_{on}(\mathcal{O})$ is isomorphic to $\mathcal{A}^{KG}_{on}(\mathcal{O})$ where $\mathcal{O}$ is any globally hyperbolic subregion of $\mathbb{H}^4$. The isomorphism is implemented by the identity.*

Proof. $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ is causal, since, for any two generators $F_{[h]}, F_{[h']}$, $[h], [h'] \in \frac{\mathcal{C}^{CP}_0(\mathbb{H}^4)}{P[\mathcal{C}^{CP}_0(\mathbb{H}^4)]}$ such that there exists two representatives $h, h' \in C^\infty_{0,-}(\mathbb{H}^4)$ which are spacelike separated, $F_{[h]} \star_{\mathbb{H}^4} F_{[h']} - F_{[h']} \star_{\mathbb{H}^4} F_{[h]} = i\langle h, E_{\mathbb{H}^4} h'\rangle = 0$. This descends from $\mathrm{supp}(h) \cap (\mathrm{supp}(E(h')) \cup \mathrm{supp}(E(\iota_z(h')))) = \emptyset$.

In order to prove the time-slice axiom, we need to show that, given any geodesically convex neighbourhood $\mathcal{N}$ of a Cauchy surface Σ in Minkowski spacetime, then $\mathcal{A}^{CP}_{on}(\mathcal{N}) = \mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ where $\mathcal{A}^{CP}_{on}(\mathcal{N})$ is the subalgebra of $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ obtained by considering only those $h \in \mathcal{C}^{CP}_0(\mathbb{H}^4)$ such that $\mathrm{supp}(h) \subset \mathcal{N}$. In view of Corollary 2.1.3 and of Proposition 2.1.2, this is equivalent to considering any $F_{[\zeta]} \in \mathcal{O}^{KG}_-(\mathbb{R}^4)$ and showing that there exists at least a representative of the label $[\zeta]$ whose support is contained in $\mathcal{N}$. Let us thus fix any Σ and $\mathcal{N}$ as above and let us consider two Cauchy surfaces $\Sigma^\pm$ such that $\Sigma \subset J^+(\Sigma^-) \cap J^-(\Sigma^+) \subset \mathcal{N}$. Choose $\chi \in C^\infty(\mathbb{R}^4)$ such that χ is z-independent and $\chi = 1$ for all points in $J^+(\Sigma^+)$ while it vanishes on $J^-(\Sigma^-)$. Let us consider any $[\zeta] \in \mathcal{O}^{KG}_-(\mathbb{R}^4)$ and any of its representatives which we indicate with ζ. Define the new function

$$\widetilde{\zeta} \doteq \zeta - P\left(E^-(\zeta) + \chi E(\zeta)\right), \tag{2.9}$$

where E^- is the retarded fundamental solution of P. Notice that, per construction and on account of the support properties of both $E^\pm$ and χ, $\widetilde{\zeta} \in C^\infty_{0,-}(\mathcal{N})$ and it is a representative of $[\zeta]$.

We are left to prove that $\mathcal{A}^{KG}_{on}(\mathcal{O})$ is isomorphic to $\mathcal{A}^{CP}_{on}(\mathcal{O})$. Each of these algebras is generated by those functionals whose labeling space is $C^\infty_0(\mathcal{O})$ and the identity map represents an isomorphism of topological vector spaces. Since the $*$-operation is complex conjugation, which is not affected by the identity map, to conclude the proof, it suffices to exhibit the following chain of identities: Let $h, h' \in C^\infty_0(\mathcal{O})$ and let $F_{[h]}$ and $F_{[h']}$ be the associated generators in $\mathcal{A}^{CP}_{on}(\mathcal{O})$, then, for any $u \in \mathcal{C}^{CP}(\mathbb{H}^4)$

$$\left(F_{[h]} \star_{\mathbb{H}^4} F_{[h']}\right)[u] = F_{[h]}(u)F_{[h']}(u) + \frac{i}{2}\langle h, E_{\mathbb{H}^4}(h')\rangle = \left(F_{[h]} \star F_{[h']}\right)[u]. \tag{2.10}$$

The last equality descends from

$$\langle h, E_{\mathbb{H}^4}(h')\rangle = \langle \eta^\dagger h, E(\eta^\dagger h')\rangle = \langle h, E(h')\rangle,$$

which holds true since $\iota_z(\mathcal{O})$ is causally disjoint from $\mathcal{O}$. Notice that (2.10) entails that the isomorphism between $\mathcal{A}^{CP}_{on}(\mathcal{O})$ and $\mathcal{A}^{KG}_{on}(\mathcal{O})$ is implemented by the identity map. □

2.2 Hadamard states for a Casimir-Polder system

Having constructed the algebra of observables for a Casimir-Polder system, we can focus on discussing algebraic states thereon, namely, as in Definition 1.1.4, any linear functional $\omega : \mathcal{A}^{CP}(\mathbb{H}^4) \to \mathbb{C}$ for which

$$\omega(\mathbb{I}) = 1, \quad \omega(a^* a) \geq 0, \ \forall a \in \mathcal{A}^{CP}(\mathbb{H}^4),$$

where $\mathbb{I}$ is the identity element. As for the usual free field theories on any globally hyperbolic spacetime, the key question is under which conditions ω is physically acceptable. We recall that the answer for $\mathcal{A}^{KG}(\mathbb{R}^4)$, the algebra of observables for a Klein-Gordon field on the whole Minkowski spacetime, goes under the name of *Hadamard states*, Definition 1.4.3. In view of the structure of $\mathbb{H}^4$, extending such notion to $\mathcal{A}^{CP}(\mathbb{H}^4)$ is not straightforward. A similar problem appeared in Abelian gauge theories [FP03] or in linearized gravity [BDM14, FH12]. The way out that we propose is partly inspired by these papers, partly by F-locality: We require that a physically acceptable, quasi-free state on $\mathcal{A}^{CP}(\mathbb{H}^4)$ is such that its restriction to any globally hyperbolic subregion of $\mathbb{H}^4$ descends from a bi-distribution, thereon of Hadamard form.

Definition 2.2.1. We call a linear map $\omega : \mathcal{A}^{CP}(\mathbb{H}^4) \to \mathbb{C}$ a quasi-free **Hadamard state for a Casimir-Polder system** if it is normalized, positive, quasi-free and if, for all globally hyperbolic submanifolds $\mathcal{O} \subset \mathbb{H}^4$, the restriction of ω to $\mathcal{A}^{CP}(\mathcal{O})$ is such that there exists $\omega_2 \in \mathcal{D}'(\mathcal{O} \times \mathcal{O})$ whose wavefront set is of Hadamard form

$$WF(\omega_2) = \{(x, x', k_x, -k_{x'}) \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \{\mathbf{0}\} \mid (x, k_x) \sim (x', k_{x'}),\ k_x \triangleright 0\},$$

and, for all $F_h, F_{h'} \in \mathcal{A}^{CP}(\mathcal{O})$

$$\omega\left(F_h \star_{\mathbb{H}^4} F_{h'}\right) = \omega_2(h, h').$$

In order for ω to descend to a state on $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ a compatibility condition with the equations of motion must be required[4]. In other words, the two-point function defining a Hadamard state on the on-shell algebra is also a bisolution to the boundary value problem (2.1).

In view of the last definition, the first question to answer is whether one can build a connection between Hadamard states for the on-shell algebra of the Klein-Gordon field on Minkowski spacetime and that of a Casimir-Polder system.

Proposition 2.2.1. *Let $\widetilde{\eta}^* : \mathcal{A}^{CP}_{on}(\mathbb{H}^4) \to \mathcal{A}^{KG}_{on}(\mathbb{R}^4)$ be the map defined in Proposition 2.1.4. Then, for every quasi-free state $\widetilde{\omega} : \mathcal{A}^{KG}_{on}(\mathbb{R}^4) \to \mathbb{C}$, there exists a quasi-free state ω on $\mathcal{A}^{CP}_{on}(\mathbb{H}^4)$ such that for all $a \in \mathcal{A}^{CP}_{on}(\mathbb{H}^4)$, $\omega(a) \doteq \widetilde{\omega}(\widetilde{\eta}^*(a))$. In particular, if $\widetilde{\omega}$ is of Hadamard form, so is ω.*

Proof. As a starting point, notice that ω inherits the normalization, positivity and the property of being quasi-free directly from $\widetilde{\omega}$. We need only to check

[4]The descended state will be called Hadamard state as well.

the Hadamard property. Let $\mathcal{O} \subset \mathbb{H}^4$ be any globally hyperbolic submanifold; for every $F_h, F_{h'} \in \mathcal{A}^{CP}(\mathcal{O})$

$$\omega(F_h \star_{\mathbb{H}^4} F_{h'}) = \widetilde{\omega}(\eta^*(F_h \star_{\mathbb{H}^4} F_{h'})) = \widetilde{\omega}(F_h \star F_{h'}) = \frac{1}{2}\widetilde{\omega}_2(h - \iota_z(h), h' - \iota_z(h')).$$

In other words the bi-distribution associated to ω can be built out of $\widetilde{\omega}_2$ itself. Since the latter has per hypothesis the Hadamard wavefront set and since, if $\mathrm{supp}(h), \mathrm{supp}(h') \subset \mathcal{O} \subset \mathbb{H}^4$, then neither $\iota_z(h)$ nor $\iota_z(h')$ can be entirely supported therein, the only singular term in the above identity is $\widetilde{\omega}_2(h, h')$. Hence ω is of Hadamard form. □

As a last step, we wish to compare our approach with the ***method of images*** which is commonly used on Minkowski spacetime.

Lemma 2.2.2. *Let $\widetilde{\omega}$ be any quasi-free Hadamard state for $\mathcal{A}^{KG}(\mathbb{R}^4)$ whose associated two-point function $\widetilde{\omega}_2 \in \mathcal{D}'(\mathbb{R}^4 \times \mathbb{R}^4)$ has an integral kernel which is invariant under reflection in both entries along the z-direction, that is such that $\widetilde{\omega}_2(\underline{x}, z, \underline{x}', z') = \widetilde{\omega}_2(\underline{x}, -z, \underline{x}', -z')$. Then the state ω on $\mathcal{A}^{CP}(\mathbb{H}^4)$ built as per Proposition 2.2.1 is a quasi-free Hadamard state whose associated two-point function $\omega_2 \in \mathcal{D}'(\mathbb{H}^4 \times \mathbb{H}^4)$ has the following integral kernel*

$$\omega_2(\underline{x}, z, \underline{x}', z') = \widetilde{\omega}_2(\underline{x}, z, \underline{x}', z') - \widetilde{\omega}_2(\underline{x}, -z, \underline{x}', z'). \tag{2.11}$$

Proof. On account of Proposition 2.2.1, we can conclude that ω is a Hadamard state on $\mathcal{A}^{CP}(\mathbb{H}^4)$ and it is quasi-free per construction. In order to show the last statement, it suffices instead an explicit calculation. Let ω be as per hypothesis and let ω_2 be the associated bi-distribution. For all $h, h' \in \mathcal{C}_0^{CP}(\mathbb{H}^4)]$, seen as labels for two generators of $\mathcal{A}^{CP}(\mathbb{H}^4)$, it holds in view of Proposition 2.2.1

$$\omega(F_h \star_{\mathbb{H}^4} F_{h'}) = \widetilde{\omega}(\eta^* (F_h \star_{\mathbb{H}^4} F_{h'})) =$$
$$\frac{1}{2}\widetilde{\omega}_2(h - \iota_z(h), h' - \iota_z(h')) = \widetilde{\omega}_2(h - \iota_z(h), h'),$$

where, in the last equality, we used the symmetry hypothesis of the two-point function to conclude that $\omega_2(h, h') = \omega_2(\iota_z(h), \iota_z(h'))$ and $\omega_2(h, \iota_z(h')) = \omega_2(\iota_z(h), h')$. The above chain of equalities entails the sought identity at a level of integral kernels. □

Remark 2.2.1. The statement of Lemma 2.2.2 applies to the Poincaré vacuum and the KMS state for a massive or massless Klein-Gordon field on Minkowski spacetime, for which $\widetilde{\omega}_2$ induces the same quasi-free state which one obtains via the method of images.

For completeness, we want now to discuss the form of the singular structure of the two-point function of the states obtained in Lemma 2.2.2. In view of (2.11) we have that

$$WF(\omega_2) = WF(\widetilde{\omega}_2|_{\mathbb{H}^4}) \cup WF((\widetilde{\omega}_2 \circ (\iota_z \otimes \mathbb{I})) \,|_{\mathbb{H}^4}) \tag{2.12}$$

where the restriction map refers to the points of the singular support. Furthermore

$$\begin{gathered}WF(\widetilde{\omega}_2 \circ (\iota_z \otimes \mathbb{I})|_{\mathbb{H}^4}) = \\ \left\{(x, x', k_x, -k'_{x'}) \in T^* \left(\mathbb{H}^4 \times \mathbb{H}^4\right) \setminus \{\mathbf{0}\} \mid (x, k_x) \sim (\iota_z x', (\iota_z)_* k'_{x'}), k_x \triangleright 0\right\} \\ = (\iota_z \otimes \mathbb{I}) WF(\widetilde{\omega}_2|_{\mathbb{H}^4}).\end{gathered}$$

In the previous expression, ι_z acts on covectors inverting the sign of the $z-$component. Heuristically, we might say that $(x, x'; k_x, k'_{x'})$ are contained in $WF((\widetilde{\omega}_2\,(\iota_z \otimes \mathbb{I}))\,|_{\mathbb{H}^4})$ if and only if x and x' are connected by a null geodesic *reflected* at the surface $\partial\mathbb{H}$ and if $\eta^{-1}(k_x)$ and $\eta^{-1}(-k'_{x'})$ are tangent vectors at the end points of this reflected geodesic. Notice that, whenever ω_2 is restricted to a globally hyperbolic region $\mathcal{O} \subset \mathbb{H}$, its wave front set enjoys the microlocal spectrum condition because $WF((\widetilde{\omega}_2 \circ (\iota_z \otimes \mathbb{I}))\,|_{\mathcal{O}})$ is the empty set. No lightlike geodesic starting from $\mathcal{O}$ can re-enter after reflection.

2.2.1 Wick ordering in a Casimir-Polder system

To conclude the section, we show how to make contact between the previous analysis and the standard results in the literature concerning the Casimir-Polder energy. To this end, we need first of all to introduce the (local) Wick polynomials for a Casimir-Polder system. From a conceptual point of view, this question is the same as for a Klein-Gordon field on a globally hyperbolic spacetime. We shall see, however, that, on every globally hyperbolic submanifolds of $\mathbb{H}^4$, the local Wick monomials generate an algebra of observables which is isomorphic to the restriction thereon of the Klein-Gordon one. Hence it is well-defined. Yet, in order to build a global algebra of Wick polynomials, one has to take into account that, on account of the presence of the boundary conditions, it is not possible to define a global Hadamard function which depends only on local properties of the spacetime. We shall show that such obstacle can be circumvented, though at the price that the the embedding of the local algebras into the global one involves a non-local deformation.

We start the construction Wick polynomials restricting to any globally hyperbolic submanifold $\mathcal{O} \subset \mathbb{H}^4$ and extending $*$-algebra $\mathcal{A}^{CP}(\mathcal{O})$ as in Section 1.5. To this end we recall the definition of support for functionals as introduced in [FR15] and adapted to our case.

Definition 2.2.2. Let $F : \mathcal{C}^{CP}(\mathbb{H}^4) \to \mathbb{C}$ be any functional on the space of off-shell configurations for a Casimir-Polder system as per Definition 2.1.1. We call *support* of F

$$\begin{gathered}\mathrm{supp}(F) \doteq \{x \in \mathbb{H}^4 \mid \forall\,\text{neighbourhoods}\, U \ni x,\, \exists u, u' \in \mathcal{C}^{CP}(\mathbb{H}^4),\, \mathrm{supp}(u) \subseteq U, \\ \text{such that } F[u + u'] \neq F[u]\}.\end{gathered}$$

Let $\mathcal{O} \subset \mathbb{H}^4$ be any globally hyperbolic submanifold, to which we associate $\mathcal{A}^{CP}(\mathcal{O})$ as per Proposition 2.1.5. In view of Definition 2.2.1 we follow the same procedure, used for $\mathcal{A}_\mu^{KG}(\mathbb{R}^4)$, to obtain $\mathcal{A}_\mu^{CP}(\mathcal{O})$ an extended algebra of Wick polynomials. Furthermore, in view of Proposition 2.1.5, $\mathcal{A}_\mu^{CP}(\mathcal{O})$ is $*$-isomorphic to $\mathcal{A}_\mu^{KG}(\mathcal{O})$, the restriction of $\mathcal{A}_\mu^{KG}(\mathbb{R}^4)$ to $\mathcal{O}$.

The next step consists of gluing together all $\mathcal{A}_\mu^{CP}(\mathcal{O})$, so to obtain a global extend algebra of Wick polynomials for a Casimir-Polder system. The following remark shows that an obstruction arises in considering $\star_H$ as the product for the global extended algebra. It turns out that the gluing becomes possible only after a suitable deformation of $\star_H$.

Remark 2.2.2. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two globally hyperbolic submanifolds of $\mathbb{H}^4$ whose union is not contained in a third globally hyperbolic submanifold of $\mathbb{H}^4$. Consider now $F_h^{(2)} \in \mathcal{A}_\mu^{CP}(\mathcal{O}_1, \star_H)$ and $F_{h'}^{(2)} \in \mathcal{A}_\mu^{CP}(\mathcal{O}_2, \star_H)$ such that

$$F_h^{(2)}(u) \doteq \int_{\mathbb{H}^4} d\mu_g(x) h(x) u^2(x), \qquad F_{h'}^{(2)}(u) \doteq \int_{\mathbb{H}^4} h'(x) d\mu_g(x) u^2(x),$$

where $u \in \mathcal{C}^{CP}(\mathbb{H}^4)$ while $\text{supp} h \subset \mathcal{O}_1$ and $\text{supp} h' \subset \mathcal{O}_2$. In view of Proposition 2.1.5, we choose the Hadamard parametrix $H(x, x')$ to be the same one as for a Klein-Gordon scalar field on Minkowski spacetime, though restricted to the region(s) of interest. In order to compute the correlations between the above two elements, we need to recognize them as being part of a larger extended algebra. Yet, if we try to follow the same procedure used in (1.5) for $\mathcal{A}_\mu^{KG}(\mathbb{R}^4)$, we notice that the local $\star$-products for the non-deformed algebra are defined replacing $\frac{i}{2} E_{\mathbb{H}^4}(x, x')$ with

$$H(x, x') + \frac{i}{2} \left(E_{\mathbb{H}^4}(x, x') - E(x, x') \right).$$

In the computation of $F_h^{(2)} \star_H F_{h'}^{(2)}$ some pathologies occur, due to terms including $(E_{\mathbb{H}^4}(x, x') - E(x, x'))$ multiplied with itself, which are ill-defined.

Such obstructions can be removed exploiting the fact that algebras whose $\star$-products are constructed with different Hadamard functions are $*$-isomorphic [BDF09]. Mimicking the construction of $E_{\mathbb{H}^4}$ starting from E, and in view of (2.11), let us consider the bidistribution $H_{\mathbb{H}^4}$ whose integral kernel is

$$H_{\mathbb{H}^4}(\underline{x}, z, \underline{x}', z') \doteq H(\underline{x}, z, \underline{x}', z') - H(\underline{x}, -z, \underline{x}', z').$$

Notice that $H_{\mathbb{H}^4}$ yields $\mathcal{A}_\mu^{CP}(\mathbb{H}^4, \star_{H_{\mathbb{H}^4}})$, a well defined global algebra. Hence, the correlations among elements of $\mathcal{A}_\mu^{CP}(\mathcal{O}_1, \star_H)$ and of $\mathcal{A}_\mu^{CP}(\mathcal{O}_2, \star_H)$ are meaningful only if we embed them in $\mathcal{A}_\mu^{CP}(\mathbb{H}^4, \star_{H_{\mathbb{H}^4}})$. Such embedding is realized by $\alpha_{H_{\mathbb{H}^4} - H}$ as in (1.21) and it is an injective $*$-isomorphism.

Despite this hurdle, concepts like smeared energy density are still well-defined within each $\mathcal{A}_\mu^{CP}(\mathcal{O}, \star_H)$. In particular the finite vacuum expectation values of the stress-energy components agrees with the non-vanishing quantities found in literature - [BD84, DeW75, DeW79]. Furthermore, regardless of the existence of an extended algebra of observables, well-known blows-up in computing quantities, such as the energy density, still remain due to additional divergences present in observables supported on the boundaries. The regularized vacuum expectation values of the stress-energy components in $\mathcal{A}_\mu^{CP}(\mathcal{O}, \star_{H_{\mathbb{H}^4}})$ are vanishing in the ground state. Nonetheless it would be inappropriate claiming that no Casimir-Polder effect occurs, rather it should be addressed the question what is the physical meaning of those quantities. In fact, the quantum counterpart of the energy density should be considered the observable regularized with the local parametrix H, for compatibility reasons with the usual notion in Minkowski spacetime.

We can make finally a correspondence to the standard results in the literature, in particular recovering the dependence of the energy density on the forth power of the distance along the z-axis between a point in the bulk and one on the boundary. Before stating the result, we recall that, on Minkowski spacetime, the so-called *improved stress-energy tensor* of a massless conformally coupled scalar field ϕ is on-shell [CCJ70, Mor03]

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\phi\partial_\rho\phi + \xi(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\phi^2, \tag{2.13}$$

where ξ is the coupling constant with the scalar curvature R introduced in (2.1).

Lemma 2.2.3. *Let us consider a massless, arbitrarily coupled to scalar curvature, scalar field and let ω^0 be the Hadamard state for $\mathcal{A}^{CP}(\mathbb{H}^4)$ induced from the Poincaré vacuum $\widetilde{\omega}^0$ via Lemma 2.2.2. Let $\mathcal{O}$ be a globally hyperbolic subregion of $\mathbb{H}^4$ and $\mathcal{A}_\mu^{CP}(\mathcal{O})$ the extended algebra defined on $\mathcal{O}$ (with the $\star_H$ product). Then, for all $h \in C_0^\infty(\mathcal{O})$,*

$$\omega^0(:\phi^2:_H(h)) = -\frac{1}{32\pi^2}\int\limits_{\mathbb{R}^4} d^4x \frac{h(\underline{x}, z)}{z^2},$$

and

$$\omega^0(:T_{\mu\nu}:_H(h)) = A_{\mu\nu}\frac{6\xi - 1}{32\pi^2}\int\limits_{\mathbb{R}^4} d^4x \frac{h(\underline{x}, z)}{z^4},$$

where $\{T_{\mu\nu}\}$ are the components of the stress-energy tensor (2.13) *while* $A = \mathrm{diag}(-1, 1, 1, 0)$.

Proof. We need only to collect what already proven together with the explicit form of

$$\widetilde{\omega}_2^0(x, x') = \lim_{\varepsilon\to 0^+}\frac{1}{4\pi^2}\frac{1}{\left(\eta_3^{\mu\nu}(\underline{x}_\mu - \underline{x}'_\mu)(\underline{x}_\nu - \underline{x}'_\nu) + (z - z')^2\right) + i\varepsilon(\underline{x}_0 - \underline{x}'_0) + \varepsilon^2}, \tag{2.14}$$

where $\eta_3 = \mathrm{diag}(-1, 1, 1)$. On account of both Proposition 2.2.1 and Lemma 2.2.2, we know that ω^0 is a Hadamard state for $\mathcal{A}^{CP}(\mathbb{H}^4)$. The definition of

$\mathcal{A}^{CP}_\mu(\mathbb{H}^4)$ together with both $\widetilde{\omega}^0_2(x,x') = H(\underline{x},z,\underline{x}',z') = H(\underline{x},-z,\underline{x}',-z')$ and Proposition 2.2.1 entail that, calling $\omega^0_2(x,x')$ the two-point function of ω^0

$$\omega^0(:\phi^2:_H(\zeta)) \doteq$$
$$\int\limits_{\mathbb{H}^4\times\mathbb{H}^4} d^4x\, d^4x' \left(\omega^0_2(x,x') - H(\underline{x},z,\underline{x}',z')\right) h(\underline{x},z)\delta(x-x') =$$
$$= -\int\limits_{\mathbb{H}^4\times\mathbb{H}^4} d^4x\, d^4x' H(\underline{x},-z,\underline{x}',z')h(\underline{x},z)\delta(x-x') = -\frac{1}{16\pi^2}\int\limits_{\mathbb{H}^4} d^4x \frac{h(\underline{x},z)}{z^2}$$

In order to compute $\omega^0(:T_{\mu\nu}:_H(\zeta))$ it suffices to apply the point-splitting scheme as introduced in [Mor03]. All results obtained in this cited paper apply without modifications to the case at hand. In particular it holds that

$$\omega^0(:T_{\mu\nu}:_H(\zeta)) =$$
$$\int\limits_{\mathbb{R}^4\times\mathbb{R}^4} d^4x\, d^4x' \left(D^{(x,x')}_{\mu\nu}\left(\omega^0_2(x,x') - H(\underline{x},z,\underline{x}',z')\right)\right) h(\underline{x},z)\delta(x-x'),$$

where – see [Hac10, §4]

$$\begin{aligned} D^{(x,x')}_{\mu\nu} &= \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta} \\ &+\xi\left(\eta_{\mu\nu}\eta^{\rho\lambda}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\lambda} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right). \end{aligned} \tag{2.15}$$

Thesought resultis then achived by inserting this expression in the above integral and replacing $\omega^0_2(x,x') - H(\underline{x},z,\underline{x}',z')$ with $H(\underline{x},-z,\underline{x}',z')$. □

Let us make some observation on the above result.

1. We have defined the Wick polynomials only for smooth and compactly supported functions whose support does not intersect the boundary of the region of interest. The reason can be seen explicitly by looking at the last lemma: If we inspect the integral kernels $\widetilde{\omega}_2(\underline{x},-z,\underline{x}',z')$ and $\widetilde{\omega}_2(\underline{x},z,\underline{x}',-z')$, they become singular at $z=z'=0$ so that they cannot be tested with $\delta(z-z')$. This is no surprise and it is at the heart of the often mentioned problem that, in a Casimir or in a Casimir-Polder system, the total energy, computed out of the integral of the time-component of the stress-energy tensor diverges.

2. By Lemma 2.2.3 we have shown that our construction reproduces the known results for a mass-less scalar field in presence of a boundary – [BD84, DeW75, DeW79, BKMM09, Mil01]. At the same time, as a new feature, it frames them in an intrinsic axiomatic theory. So we are inclined to regard the divergence of the integrated energy density, mentioned in the above remark, as not a consequence of a bad or incomplete

regularization. Often in literature this is regarded as consequence of the fact that the Dirichlet boundary condition is not suitable to model a real confinement. Physically oriented models of boundaries are offered by the delta-potential approach, where the boundary condition is replaced by a background potential, peaked in correspondence of the boundary itself – [GJK02]. The analysis of such models, however, goes beyond the aim of this thesis.

3. In case of conformal coupling ($\xi = \frac{1}{6}$), all the component of the stress-energy tensor are vanishing. We postpone any comment at this regard to the next section, where an analogous cancellation occurs.

Chapter 3

Algebraic quantum field theory and the Casimir effect

In[1] this section we shall focus on the second scenario, we are interested in, namely the one of a massless scalar field defined between two parallel infinite spacelike surfaces. It is related to the Casimir effect, namely the attraction force between two parallel, perfectly conducting, plates as discussed for the first time in [Cas48]. We shall refer to it as the *Casimir system.* To carry out the analysis we exploit the method of images as in the previous chapter. Despite we apply a constructive technique which make use of Minkowski spacetime, our approach provides an intrinsic theory, since all objects and results refers only to elements of the system. In the present situation time the application of the method of images presents several complications, since the presence of two boundaries implies to consider a series of images which could be not converging. This has relevance in the definition of states induced by the method of images, since one has to pay attention to some convergence requirements. As for a Casimir-Polder system, the construction of the extended algebra via the local Hadamard parametrix H works only on globally hyperbolic subregions.

At a geometric level, the model consists of the region $Z \doteq \mathbb{R}^3 \times [0,d] \subset \mathbb{R}^4$ endowed with the (restriction of the) Minkowski metric. In analogy to the previous section, the interval $[0,d]$ runs along the spacelike z-direction. At a field theoretical level, our starting point are are all $u \in C^\infty(Z)$, where smoothness is meant as in Definition 2.1 since $Z \subset \mathbb{H}^4$. *Dynamical configurations* are instead the elements of the vector space $\mathcal{S}^C(Z)$ built out of the smooth solutions of

$$\begin{cases} Pu = (\Box - \xi R - m^2)u = 0, \quad m^2 \geq 0, \quad \xi \in \mathbb{R} \\ u(\underline{x},0) = u(\underline{x},d) = 0 \end{cases}, \tag{3.1}$$

where R is the scalar curvature and we recall the notation $(\underline{x},z) := (t,x,y,z)$. Since the scalar curvature vanishes, the term ξR plays no role at a dynamical

[1]The content of this chapter is part of [DNP16, Section 3].

level, but it affects the structure of the stress-energy tensor which we will consider later.

3.1 Algebra of observable of a Casimir system

We follow the same path as in a Casimir-Polder system, proceeding in three main steps.

Part 1 – Dynamical configurations:
Notice that, in full analogy with the previous section, neither is (Z, η) a globally hyperbolic spacetime, nor is (3.1) an initial value problem, rather it is a boundary value problem. Hence, in order to characterize $\mathcal{S}^C(Z)$, we follow the same strategy used in a Casimir-Polder system, namely we identify each smooth solution of (3.1) with a specific counterpart for a Klein-Gordon field on the whole Minkowski spacetime. Before outlining the details, we introduce the auxiliary regions

$$\begin{aligned} Y_0 &\doteq \mathbb{R}^3 \times [-d, d], \\ Y_n &\doteq \{x \in \mathbb{R}^4 \mid \exists(\underline{x}, z) \in Y_0 \text{ for which } x = (\underline{x}, z + 2nd)\},\ n \in \mathbb{Z}\}. \end{aligned} \tag{3.2}$$

As a consequence $\mathbb{R}^4 = \bigcup_{n\in\mathbb{Z}} Y_n$.

Proposition 3.1.1. *There exists a vector space isomorphism between $\mathcal{S}^C(Z)$ and the quotient $\frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$ where $C^\infty_{tc,C}(\mathbb{R}^4)$ is the collection of all $\alpha \in C^\infty_{tc}(\mathbb{R}^4)$ such that the following conditions are met:*

1. *$\alpha \in C^\infty_{tc,-}(\mathbb{R}^4)$, that is $\alpha(\underline{x}, z) = -\alpha(\underline{x}, -z)$*
2. *$\alpha(\underline{x}, z) = -\alpha(\underline{x}, 2d - z)$*

Proof. As a first step we show that there exists an isomorphism between $\mathcal{S}^C(Z)$ and a vector subspace of $\mathcal{S}^{KG}(\mathbb{R}^4) \doteq \{\phi \in C^\infty(\mathbb{R}^4) \mid P\phi = 0\}$. Let $u \in \mathcal{S}^C(Z)$ and let

$$v(x) \doteq \begin{cases} u(x), & x \in Z \\ -u(-x), & x \in Y_0 \setminus Z \end{cases}.$$

Following the same argument as in the proof of Proposition 2.1.1, we can conclude that $v \in C^\infty(Y_0)$ and $v(\underline{x}, 0) = v(\underline{x}, d) = v(\underline{x}, -d) = 0$. Define $\phi(x) = \phi(\underline{x}, z) \doteq v(\underline{x}, z - 2nd)$, for any $x \in Y_n$. By a similar argument as for $v(x)$, it descends that $\phi \in C^\infty(\mathbb{R}^4)$ and that, moreover, $P\phi = 0$, as this property is traded from that of u. In other words we have found a linear map

$$\begin{aligned} F : \mathcal{S}^C(Z) &\to \mathcal{S}^C(\mathbb{R}^4) \subset \mathcal{S}^{KG}(\mathbb{R}^4) \\ \mathcal{S}^C(\mathbb{R}^4) &= \left\{\phi \in C^\infty_-(\mathbb{R}^4) \mid P\phi = 0 \text{ and } \phi(\underline{x}, 2d - z) = -\phi(\underline{x}, z)\right\}. \end{aligned} \tag{3.3}$$

The map is per construction surjective, since for every $\phi \in \mathcal{S}^C(\mathbb{R}^4)$, $\phi|_Z \in \mathcal{S}^C(Z)$ and $F(\phi|_Z) = \phi$. Furthermore F is also injective since $F(u) = 0 \in$

$\mathcal{S}^C(\mathbb{R}^4)$ implies $\phi = 0$ and, thus $u = \phi|_Z = 0$. In other words F is an isomorphism of vector spaces. To prove the statement of the proposition we need to show that $\mathcal{S}^C(\mathbb{R}^4)$ is isomorphic to $\frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$. As a first step we show that the map induced by E is surjective. Let thus $\phi \in \mathcal{S}^C(\mathbb{R}^4)$. Since $P\phi = 0$, there must exist $\alpha \in C^\infty_{tc}(\mathbb{R}^4)$ such that $\phi = E(\alpha)$. Since ϕ is odd per reflection along the hyperplane $z = 0$, we know from Proposition 2.1.1 that α must lie in $C^\infty_{tc,-}(\mathbb{R}^4)$. Repeating slavishly the proof of Proposition 2.1.1 with respect to the condition $\phi(\underline{x}, 2d - z) = -\phi(\underline{x}, z)$ we obtain that $\alpha \in C^\infty_{tc,-,d}(\mathbb{R}^4)$ where $C^\infty_{tc,-,d}(\mathbb{R}^4) = \{\alpha \in C^\infty_{tc} \mid \alpha(\underline{x}, 2d - z) = -\alpha(\underline{x}, z)\}$. Putting all together $\alpha \in C^\infty_{tc,-}(\mathbb{R}^4) \cap C^\infty_{tc,-,d}(\mathbb{R}^4) = C^\infty_{tc,C}(\mathbb{R}^4)$. Taking into account that $E \circ P = 0$, we have associated to each element in $\mathcal{S}^C(\mathbb{R}^4)$ an equivalence class in $\frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$. We focus now on injectivity. Let $\alpha \in C^\infty_{tc,C}(\mathbb{R}^4)$ and let $\phi_\alpha \doteq E(\alpha)$ where E is the causal propagator of P on Minkowski spacetime. Per construction $P\phi_\alpha = 0$. Furthermore since both the map $\iota_z : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\iota_z(\underline{x}, z) = (\underline{x}, -z)$ and $\iota_s : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\iota_s(\underline{x}, z) = (\underline{x}, z + s)$, $s \in \mathbb{R}$, are isometries of $(\mathbb{R}^4, \eta)$ it holds that $E \circ \iota_z = \iota_z \circ E$ and $E \circ \iota_s = \iota_s \circ E$. Consequently $\phi = E(\alpha) = E(-\iota_z\alpha) = -\iota_z E(\alpha) = -\iota_z\phi$ which entails $\phi(\underline{x}, 0) = 0$. At the same time, replacing ι_z with $\iota_s \circ \iota_z$, $s = 2d$, we obtain that $\phi(\underline{x}, 2d - z) = -\phi(\underline{x}, z)$ which implies $\phi(\underline{x}, d) = 0$. Since $E \circ P = 0$, the map which associates to each $[\alpha] \in \frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$, $E(\alpha) \in \mathcal{S}^C(\mathbb{R}^4)$ does not depend on the choice of the representative in $[\alpha]$ and it is, moreover, injective. As a matter of facts, suppose $E(\alpha) = 0$. This entails that there exists $\rho \in C^\infty_{tc}(\mathbb{R}^4)$ such that $\alpha = P\rho$. Yet, since $\alpha(\underline{x}, z) = -\alpha(\underline{x}, -z) = -\alpha(\underline{x}, 2d - z)$ and since P is invariant both under the map $(\underline{x}, z) \mapsto (\underline{x}, -z)$ and $(\underline{x}, z) \mapsto (\underline{x}, z + 2d)$, $\rho \in C^\infty_{tc,C}(\mathbb{R}^4)$. As a consequence $P\rho$ lies in the trivial equivalence class of $\frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$. □

Remark 3.1.1. It is noteworthy that the two conditions defining the elements of $\mathcal{S}^C(\mathbb{R}^4)$ in (3.3) are actually already implementing the method of images at a level of dynamical configurations. As a matter of facts, consider any $\phi \in \mathcal{S}^C(\mathbb{R}^4)$: For any $n \in \mathbb{Z}$, first applying the reflection along the hyperplane $(\underline{x}, d)$ and then the one along $(\underline{x}, 0)$, the following chain of identities holds true:

$$\phi(\underline{x}, z + 2nd) = -\phi(\underline{x}, -z - 2(n-1)d) = \phi(\underline{x}, z + 2(n-1)d),$$

and equivalently $\phi(\underline{x}, z + 2nd) = \phi(\underline{x}, z + 2(n+1)d)$. In other words every element in $\mathcal{S}^C(\mathbb{R}^4)$ is both odd with respect to the reflection along the hyperplane $z = 0$ and $2d$-periodic.

Our next goal is to expand cohesively the content of the above remark. Therein our philosophy was to show that, to each dynamical configuration for a Casimir system, we can associate a solution of the equation of motion of a Klein-Gordon scalar field, which is periodic along the z-direction. From the quantum field theory point of view, especially when constructing algebraic states, we will be interested in a complementary problem, namely we would like to start from an element of $\mathcal{S}^{KG}(\mathbb{R}^4) \doteq \{\phi \in C^\infty(\mathbb{R}^4) \mid P\phi = 0\}$ and associate to it one in $\mathcal{S}^C(\mathbb{R}^4)$. Following an argument almost identical to that of Proposition 3.1.1, this problem can be translated to associating to an element of $C^\infty_{tc}(\mathbb{R}^4)$ one of $C^\infty_{tc,C}(\mathbb{R}^4)$. Barring the reflection along the plane $z = 0$, the key procedure consists of making a smooth function on $\mathbb{R}^4$ periodic. This operation, which is strongly tied to the Poisson's summation formula – see [Hör90, §7.2], does not yield in general a well-defined result on the whole $C^\infty_{tc}(\mathbb{R}^4)$. Yet we can individuate a notable subset which suffices to reach our goal. More precisely

Proposition 3.1.2. *Let* $C^\infty_{0,C}(\mathbb{R}^4) \doteq \big\{\alpha \in C^\infty_{tc,C}(\mathbb{R}^4) \mid \mathrm{supp}(\alpha) \cap (\mathbb{R}^3 \times \{z\})$ *is compact* $\forall z \in \mathbb{R}\big\}$ *and let* $N : C^\infty_0(\mathbb{R}^4) \to C^\infty_{0,C}(\mathbb{R}^4)$ *be defined as*

$$N(f)(\underline{x}, z) = \sum_{n=-\infty}^{\infty} \left(f(\underline{x}, z + 2nd) - f(\underline{x}, -z + 2nd)\right). \tag{3.4}$$

The following statements hold true:

1. *The map* N *is surjective, but not injective.*
2. N *is an isomorphism between* $C^\infty_0(\mathring{Z}) \subset C^\infty_0(\mathbb{R}^4)$ *and* $C^\infty_{0,I}(\mathbb{R}^4) \subset C^\infty_{0,C}(\mathbb{R}^4)$, *where*

$$C^\infty_{0,I}(\mathbb{R}^4) \doteq \{\alpha \in C^\infty_{tc,C}(\mathbb{R}^4) \mid \mathrm{supp}(\alpha) \cap \mathring{Z} \text{ is compact}\}.$$

Proof. We remark, that, per construction $N(f)$ is a smooth function which is $2d$-periodic and odd for reflection along the z-axis for any $f \in C^\infty_0(\mathbb{R}^4)$. The compact support ensures the convergence of the series.

Let us focus on *1.*: To show that N is surjective, let $\zeta \in C^\infty_{0,C}(\mathbb{R}^4)$ and let $\chi \in C^\infty(\mathbb{R}^4)$ be a function constructed as follows. It depends only on z and, at fixed value of $\underline{x}$, $\chi(z) \in C^\infty_0(\mathbb{R}^4)$ in such a way that χ vanishes for all $|z| \geq 2d - \alpha$, $\alpha \in (0, d)$. Furthermore $\chi(z) = 1$ if $z \in (-\alpha, \alpha]$ and for all other values of z it is such to satisfy the identity $\chi(z) + \chi(z + 2d) = 1$ for all $z \in [-2d, 0]$. Consequently $\chi\zeta \in C^\infty_0(\mathbb{R}^4)$ and a direct calculation shows that $N(\chi\zeta) = \zeta$. Hence N is surjective. To show that N is not injective it suffices to exhibit an explicit example: Consider any $\beta \in C^\infty_0\left((0, d) \times \mathbb{R}^3\right)$ and $f(\underline{x}, z)$ as $\beta(\underline{x}, z)$ if $z > 0$ and as $-\beta(\underline{x}, -z)$ if $z < 0$. At the same time define

$$\beta'(\underline{x}, z) = \begin{cases} \frac{1}{2}\beta(\underline{x}, z) & z \in (0, d) \\ \frac{1}{2}\beta(\underline{x}, z - d) & z \in (d, 2d) \end{cases}.$$

If we consider $f'(\underline{x}, z)$ as $\beta'(\underline{x}, z)$ if $z > 0$ and as $-\beta'(\underline{x}, -z)$ if $z < 0$, using (3.4), it turns out that $N(f) = N(f')$.

Let us now focus on *2.*: Let $\zeta \in C^\infty_{0,I}(\mathbb{R}^4)$; per definition $f \doteq \zeta|_Z \in C^\infty_0(Z)$. On account of (3.4) $N(f) = \zeta$, that is N is surjective on $C^\infty_{0,C}(\mathbb{R}^4)$. Let us assume that there exists $f' \in C^\infty_0(Z)$ such that $N(f') = 0$. Formula (3.4) entails that $N(f')|_Z = f' = 0$, which proves that N is injective. $\square$

According to our overall strategy, the next step calls for the identification of a counterpart for a Casimir system of $E_{\mathbb{H}^4}$ which played a key role in studying a Casimir-Polder system. Notice that the key role of $E_{\mathbb{H}^4}$ was on the one hand to generate all smooth solutions with the wanted boundary conditions, while on the other hand, it yielded a symplectic form on the space of classical observables. We have emphasized this second aspect since it is easy to grasp that identifying eventually a symplectic form in Casimir system, is more difficult on account of the periodicity of the elements in $\mathcal{S}^C(Z)$. A solution to this problem lies in this proposition:

Proposition 3.1.3. *We call* $\mathcal{S}^C_{sc}(Z)$ *the collection of all solutions* $u \in C^\infty(Z)$ *of* (3.1) *such that* $\mathrm{supp}(u) \cap (\{t\} \times \mathbb{R}^2 \times [0,d])$ *is compact for all* $t \in \mathbb{R}$. *This is*

1. *a vector space isomorphic to* $\frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$,

2. *a symplectic space if endowed with the following weakly non-degenerate symplectic form:*

$$\sigma_C(u,u') = \sigma_C([\zeta],[\zeta']) = (\zeta, E(\zeta'))_C = -\left(E(\zeta),\zeta'\right)_C, \tag{3.5}$$

where ζ *and* ζ' *are representatives of* $[\zeta],[\zeta'] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$ *so that* $u = E(\zeta)$ *and* $u' = E(\zeta')$ *and where*

$$(\zeta, E(\zeta'))_C \doteq \int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz\, \zeta E(\zeta') = -\int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz\, E(\zeta)\zeta'. \tag{3.6}$$

Proof. On account of Proposition 3.1.1, to every element $u \in \mathcal{S}^C_{sc}(Z) \subset \mathcal{S}^C(Z)$, we can associate via the map F in (3.3) a function $\phi \in C^\infty_-(\mathbb{R}^4)$, solution of $P\phi = 0$, so that $u = \phi|_Z$. Furthermore, there exists $[\alpha] \in \frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P\left[C^\infty_{tc,C}(\mathbb{R}^4)\right]}$ such that $\phi = E(\alpha)$. Since ϕ is per hypothesis compactly supported along the x, y-directions, but neither in time nor along z, the standard support properties of the causal propagator E entail, in turn, that α must be smooth and compactly supported along the t, x, y-directions without additional constraints imposed along the z-direction. Repeating slavishly the proof of Proposition 3.1.1, *1.* descends.

Let us focus on *2.*: As a first step, we show that (3.6) is well-posed. Since for any $u, u' \in \mathcal{S}^C(Z)$, there exists $[\zeta],[\zeta'] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$, such that $u^{(\prime)} = E(\zeta^{(\prime)})$, well-posedness descends from showing that for any $\zeta' \in C^\infty_{0,C}(\mathbb{R}^4)$ the integral $\int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz\, P(\zeta')E(\zeta)$ vanishes. Define $P_{(3)} = P - \frac{\partial^2}{\partial z^2}$ and rewrite the integral as

$$\int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz \left(P_{(3)} + \frac{\partial^2\zeta'}{\partial z^2}\right) E(\zeta) = \int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz \left(\frac{\partial^2\zeta'}{\partial z^2} E(\zeta) + \zeta' P_{(3)} E(\zeta)\right),$$

where we used both that $P_{(3)}$ is a formally self-adjoint operator which does not depend on z and that we are integrating along the whole $\mathbb{R}^3$. If we use the identity $P_{(3)}E(\zeta) = PE(\zeta) - \frac{\partial^2 E(\zeta)}{\partial z^2}$ and we integrate by parts, it holds

$$\int_{\mathbb{R}^3} d^3\underline{x} \int_0^d dz \left(\frac{\partial^2 \zeta'}{\partial z^2} E(\zeta) + \zeta' P_{(3)} E(\zeta) \right) = \left(\frac{\partial \zeta'}{\partial z} E(\zeta) - \zeta' \frac{\partial E(\zeta)}{\partial z} \right) \Bigg|_0^d = 0,$$

where we used that both ζ' and $E(\zeta)$ vanish both at $z = 0$ and at $z = d$. From this computation it also descends that, for any $\zeta, \zeta' \in C^\infty_{0,C}(\mathbb{R}^4)$

$$(\zeta, E^+(\zeta'))_C = (PE^-\zeta, E^+(\zeta'))_C = (E^-\zeta, \zeta')_C,$$

where $E^\pm$ are the advanced and the retarded fundamental solutions of P on the whole Minkowski spacetime. From this last identity it descends that $(\zeta, E(\zeta'))_C = -(E(\zeta), \zeta')_C$. In other words, σ_C is both bilinear and anti-symmetric. To prove non-degenerateness, suppose there exists $[\zeta] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$ such that $(\zeta, E(\zeta'))_C = 0$ for all $[\zeta'] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$. In particular this entails that $(E(\zeta), \zeta')_C = 0$. If we choose ζ' so that $(\text{supp}(\zeta') \cap Z) \subset \mathring{Z}$, calling $\zeta_0 \doteq \zeta'|_Z$ the following identity holds true:

$$(E(\zeta), \zeta')_C = \int_{\mathbb{R}^4} d^4x\, E(\zeta)\zeta_0.$$

Notice that $\zeta \in C^\infty_{0,C}(\mathbb{R}^4) \subset C^\infty_{tc}(\mathbb{R}^4)$ and that the right hand side coincides with the standard pairing between $\frac{C^\infty_{tc}(\mathbb{R}^4)}{P[C^\infty_{tc}(\mathbb{R}^4)]}$ and $\frac{C^\infty_0(\mathbb{R}^4)}{P[C^\infty_0 \mathbb{R}^4)]}$ on the whole Minkowski spacetime, which is non degenerate – see for example [Ben16]. Hence there must exist $\alpha \in C^\infty_{tc}(\mathbb{R}^4)$ such that $\zeta = P\alpha$. Notice that, since (3.4) guarantees us that N is built out of isometries of the standard Minkowski metric, it holds that $E^\pm \circ N = N \circ E^\pm$. Since $\alpha = E^+(\zeta) = E^-(\zeta)$ and $\zeta \in C^\infty_{0,C}(\mathbb{R}^4)$ per hypothesis, α lies in $C^\infty_{0,C}(\mathbb{R}^4)$, concluding the proof that σ_C is weakly non-degenerate. □

Notice that restricting the domain of integration in (3.6) is necessary to obtain finite quantities and it encodes the physical idea that only the information contained between the boundaries at $z = 0$ and at $z = d$ are physically relevant. Before concluding this part of our investigation of a Casimir system, we elaborate from Proposition 3.1.3 the following Definition

Definition 3.1.1. We call **Casimir causal propagator** the map

$$E_Z : C^\infty_0(\mathbb{R}^4) \to \mathcal{S}^C_{sc}(Z),$$
$$E_Z \doteq \rho_Z \circ E \circ N,$$

where N is defined in (3.4), E is the causal propagator of the Klein-Gordon scalar field on Minkowski spacetime, while ρ_Z is the restriction map to Z.

Remark 3.1.2. Notice that there is no symplectic isomorphism between $\mathcal{S}^C_{sc}(\mathbb{R}^4)$ and the space of spacelike compact solutions of the Klein-Gordon equation on Minkowski spacetime . The reason is that N does not preserve the symplectic form, since for arbitrary $f, f' \in C^\infty_0(\mathbb{R}^4)$,

$$E(f, f') \neq \sigma_C\left((E \circ N)(f), (E \circ N)(f')\right) = (\zeta, E(\zeta'))_C, \tag{3.7}$$

where σ_C is the one introduced in (3.5) and, setting $\zeta = N(f)$ and $\zeta' = N(f')$, the last equality holds on account of (3.6). The main consequence of this failure will be the impossibility at a later stage to construct states for the algebra of observables for a Casimir system as the pull-back of states for the counterpart on the whole Minkowski spacetime.

Part 2 – The off-shell algebra: Having characterized all possible dynamical configurations for a Casimir system, we can address the question on how to build an algebra of observables following the construction given in Chapter 1. Our guiding principle will be the same as in section 2 and, in particular, we shall use the functional formalism. We stress that there will be several modifications in comparison to our analysis of the previous section. These can be ultimately ascribed to the more complicated underlying geometry and to the fact that we have well under control the convergence of the series (3.4) only with respect to compactly supported functions.

Definition 3.1.2. We call *space of kinematical/off-shell configurations* for a Casimir system

$$\mathcal{C}^C(Z) \doteq \left\{u \in C^\infty(Z) \mid \; u|_{\partial Z} = 0 \text{ and } \exists \phi \in \mathcal{C}^{KG}(\mathbb{R}^4) \text{ such that } u = \phi|_Z\right\},$$

We consider $\mathcal{C}^C(Z)$ endowed with the compact-open topology.

Notice that $\mathcal{S}^C(Z) \subset \mathcal{C}^C(Z)$. As next step, we want to construct a space of functionals measuring off-shell configurations and we want to endow it with the structure of a $*$-algebra. In this respect Definition 3.1.1 plays a key role.

Definition 3.1.3. Let $F : \mathcal{C}^C(Z) \to \mathbb{C}$ be any smooth functional. We call it **regular** if for all $k \geq 1$ and for all $u \in \mathcal{C}^C(Z)$, $F^{(k)}[u] \in C^\infty_0(Z^k)$, and if only finitely many functional derivatives do not vanish. We indicate this set as $\mathcal{F}^C_0(Z)$.

Let us define, analagously to (1.12):

$$\star_Z : \mathcal{F}^C_0(Z) \times \mathcal{F}^C_0(Z) \to \mathcal{F}^C_0(Z),$$

which associates to each $F, F' \in \mathcal{F}^C_0(Z)$

$$\left(F \star_Z F'\right)(u) = \left(\mathcal{M} \circ \exp(i\Gamma_{E_Z})(F \otimes F')\right)(u) \tag{3.8}$$

Here $\mathcal{M}$ stands for the pointwise multiplication, *i.e.*, $\mathcal{M}(F \otimes F')(u) \doteq F(u)F'(u)$, whereas

$$\Gamma_{E_Z} \doteq \frac{1}{2}\int_{Z\times Z} E_Z(x, x')\frac{\delta}{\delta u(x)} \otimes \frac{\delta}{\delta u(x')},$$

where $E_Z(x,x')$ is the integral kernel of (2.5). The exponential in (3.8) is defined intrinsically in terms of the associated power series and, consequently, we can rewrite the product also as

$$\left(F \star_Z F'\right)(u) = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \langle F^{(n)}(u), E_Z^{\otimes n}(F'^{(n)})(u)\rangle, \tag{3.9}$$

where the 0-th order is the pointwise multiplication, $\langle F^{(0)}(u), F'^{(0)}(u)\rangle \doteq F(u)F'(u)$. Notice that (3.9) is well-defined, since $E_Z = E \circ N$ and thus elements in $C^\infty_{0,C}(Z)$ are per definition such that their image under the action of N lies in $C^\infty_{tc,C}(Z)$. To summarize

Definition 3.1.4. We call $\mathcal{A}^C(Z) \equiv \left(\mathcal{F}^C_0(Z), \star_Z\right)$ the *off-shell $*$-algebra* of a Casimir system endowed with complex conjugation as $*$-operation.

Remark 3.1.3. Notice that, in complete analogy with $\mathcal{A}^{CP}(\mathbb{H}^4)$, $\mathcal{A}^C(Z)$ can be seen as being generated by the functionals $F_h(u) = \int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz\, u(\underline{x},z)h(\underline{x},z)$ where $h \in C^\infty_0(Z)$, while $u \in \mathcal{C}^C(Z)$. At the same time, if we consider as generating functionals only those whose labeling space is $C^\infty_0(\mathring{Z})$, we obtain the *extensible $*$-algebra* $\mathcal{A}^C_{ext}(Z)$, which is a $*$-subalgebra of both $\mathcal{A}^C(Z)$. Notice that $\mathcal{A}^C_{ext}(Z)$ plays a distinguished role as we will be able to define Hadamard states only for such algebra.

The causal propagator for the Casimir system is constructed modifying the causal propagator of the Minkowski spacetime with the operator N. Thanks to the causal properties of E, when employed on test functions supported in a globally hyperbolic set $\mathcal{O}$ strictly contained in Z it holds that

$$E_Z(f,f') = E(f,f'), \qquad f,f' \in C^\infty_0(\mathcal{O})$$

because the reflections and the translations used in N map the support of f' in regions which are causally disjoint from $\mathcal{O}$. The following proposition states that the local algebra of observables of a Casimir system cannot be distinguished from Klein-Gordon counterpart, in full agreement with the paradigm of locality.

Proposition 3.1.4. *Let $\mathcal{O}$ be any globally hyperbolic open region strictly contained in Z. There exists a $*$-isomorphism between $\mathcal{A}^{KG}(\mathcal{O}) \doteq \mathcal{A}^{KG}(\mathbb{R}^4)\big|_{\mathcal{O}}$ and $\mathcal{A}^C(\mathcal{O}) \doteq \mathcal{A}^C(Z)\big|_{\mathcal{O}}$. The isomorphism is implemented by the identity map.*

The proof of this proposition can be obtained along the guidelines of that of Proposition 2.1.5 together with the property of E_Z stated above.

Part 3 – The on-shell algebra: Having investigated the algebra probing kinematical configurations, we want to conclude our analysis by constructing the counterpart on the solutions to the equation of motion. This is tantamount to restricting the allowed configurations from $\mathcal{C}^C(Z)$ to $\mathcal{S}^C(Z)$. As outlined in Chapter 1 and in Chapter 2 for a Casimir-Polder system, this entails that several functionals become redundant as they are automatically vanishing when evaluated on any solution. This calls for the identification and for the elimination of these observables via a suitable quotient. At a level of algebras the solution of this problem is contained in Proposition 3.1.1 and in the isomorphism between $\mathcal{S}^C(Z)$ and $\frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$. This suggests to consider the functionals $F_{[\zeta]} : \frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]} \to \mathbb{R}$ so that $F_{[\zeta]}([\alpha]) = (\zeta, E(\alpha))_C$, where the right hand side is defined in (3.6).

Notice that, still in view of Proposition 3.1.1, we can rewrite each of these functionals also as $F_{[\zeta]} : \mathcal{S}^C(Z) \to \mathbb{C}$, thus as a genuine classical observable on the dynamical configurations of a Casimir system. The underlying philosophy is to single out via the labeling space $\frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$ the generators of an on-shell algebra of observables for a Casimir system. As a preliminary step, we exhibit some relevant properties of these generating functionals, which justify their choice:

Proposition 3.1.5. *We call* **classical observable for a Casimir system** *the linear functional* $F_{[\zeta]} : \frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]} \to \mathbb{C}$, $[\zeta] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$, *defined as*

$$F_{[\zeta]}([\alpha]) = (\zeta, E(\alpha))_C, \tag{3.10}$$

where ζ *and* α *are arbitrary representatives of* $[\zeta]$ *and* $[\alpha]$ *respectively. The collection of all classical observables* $\mathcal{O}^C(Z)$ *is a vector space which is both separating and optimal in the sense of Proposition 2.1.2. Furthermore* $(\mathcal{O}^C(Z), \sigma_C)$ *is a symplectic space,* σ_C *being defined in* (3.5).

Proof. We notice that (3.10) is a well-defined quantity whose right hand side does not depend on the representatives chosen, as one can infer by repeating slavishly the same reasoning as in Proposition 3.1.3 using additionally that $(\mathrm{supp}(\zeta) \cap \mathrm{supp}(E(\alpha)) \cap Z$ is compact.

Since $\mathcal{O}_{[\zeta]}$ is linear in $[\zeta]$, $\mathcal{O}^C(Z)$ is a vector space which is isomorphic to $\mathcal{S}^C_{sc}(Z)$. Hence, since the latter is a symplectic space as proven in Proposition 3.1.3, so is $\mathcal{O}^C(Z)$ endowed with σ_C. We need only to show that the collection of classical observables is separating and optimal. The first descends from the following remark:$\frac{C^\infty_{tc,C}(\mathbb{R}^4)}{P[C^\infty_{tc,C}(\mathbb{R}^4)]}$ is isomorphic via E to $\mathcal{S}^C(Z)$ which in turn identifies a vector subspace of $\frac{C^\infty(Z)}{P[C^\infty(Z)]}$. With respect to the pairing we have introduced, standard arguments in functional analysis guarantee that $\frac{C^\infty_0(Z)}{P[C^\infty_0(Z)]}$ separates $\frac{C^\infty(Z)}{P[C^\infty(Z)]}$. Since $C^\infty_0(Z) \subset C^\infty_{0,C}(Z)$ the sought statement holds true.

To conclude we show that our choice is optimal. Suppose that there exists a classical observable generated by $\zeta \in C^\infty_{0,C}(\mathbb{R}^4)$ such that $(\zeta, E(\alpha))_C = 0$ for all $\alpha \in C^\infty_{tc,C}(\mathbb{R}^4)$. Equivalently this entails that $(E(\zeta), \alpha)_C = 0$. Since α is an arbitrary timelike compact function in Z, the same reasoning as for the scalar field on the whole Minkowski spacetime entails that $E(\zeta)$ must vanish thereon. In other words $\zeta \in P[C^\infty_{0,C}(\mathbb{R}^4)]$, that is it generates the trivial class in $\frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$. □

By a computation analogous to (1.14), it is possible to prove that the $\star$-product of Definition 1.12 descends to $\mathcal{O}^C(Z)$. Consequently, we have finally,

Definition 3.1.5. We call **on-shell $*$-algebra of observables for a Casimir system** the algebra $\left(\mathcal{A}^C_{on}(Z), \star_Z\right)$ generated by the functionals $F_{[\zeta]} : \mathcal{S}^C(Z) \to \mathbb{C}$ with $[\zeta] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$ such that $F_{[\zeta]}(u) = \int\limits_{\mathbb{R}^3} d^3\underline{x} \int\limits_0^d dz\, \zeta(\underline{x}, z) u(\underline{x}, z)$, $u \in \mathcal{S}^C(Z)$.

Let us show that our choice for the algebra of observables enjoys causality and the time-slice property. The validity of the time-slice axiom carries the idea that the boundary is never acting as an absorber or emitter of dynamical information. This is ultimately a consequence of having fixed a boundary condition.

Lemma 3.1.6. *The algebra $\mathcal{A}^C_{on}(Z)$ is causal and it satisfies the time-slice axiom.*

Proof. The property of an algebra being causal is tantamount to showing that spacelike separated observables do commute. It suffices to check it for all generators and it is equivalent to proving that, for all $[\zeta], [\zeta'] \in \frac{C^\infty_{0,C}(\mathbb{R}^4)}{P[C^\infty_{0,C}(\mathbb{R}^4)]}$, it holds $\sigma_C([\zeta], [\zeta']) = 0$ if there exists two representative ζ, ζ' which are spacelike separated. On account of Proposition 3.1.3 this is a consequence of the support properties of the causal propagator.

With respect to the time-slice axiom, mutatis mutandis, the procedure is identical to the one outlined in the proof of Lemma 2.1.5 and we shall thus not repeat it. □

To conclude we remark that $\mathcal{A}^C_{on}(Z)$ could have been realized also as the quotient between $\mathcal{A}^C(Z)$ and the $*$-ideal generated by elements of the form Ph, where P is the Klein-Gordon operator and $h \in C^\infty_{0,C}(Z)$.

3.2 Hadamard states for a Casimir system

In this section we discuss a possible way to construct a certain class of states for the Casimir system. We shall restrict our attention to those which are quasi-free and have suitable regularity. In particular we follow the same philosophy used in the previous section, namely we will focus our attention on those states from which stems a prescription to construct Wick polynomials which coincides with the standard one if we restrict our attention to any globally hyperbolic submanifold $\mathcal{O} \subset Z$. Well-posedness of this line of thought is a by-product of Proposition 3.1.4, which guarantees that $\mathcal{A}^C(\mathcal{O})$ is $*$-isomorphic to $\mathcal{A}^{KG}(\mathcal{O})$. Accordingly,

Definition 3.2.1. A state $\omega : \mathcal{A}^C(Z) \to \mathbb{C}$ is of **Hadamard** form if it is normalized, positive, quasi-free and, if, for any globally hyperbolic submanifold $\mathcal{O} \subset Z$, the restriction of ω to $\mathcal{A}^C(\mathcal{O})$ is such that there exists $\omega_2 \in \mathcal{D}'(\mathcal{O} \times \mathcal{O})$ whose wavefront set is

$$WF(\omega_2) = \{(x, x', k_x, -k_{x'}) \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \{\mathbf{0}\} \mid (x, k_x) \sim (x', k_{x'}),\ k_x \triangleright 0\},$$

and, for all $F_h, F_{h'} \in \mathcal{A}^C(\mathcal{O})$

$$\omega\left(F_h \star_Z F_{h'}\right) = \omega_2(h, h'), \quad h, h' \in C_0^\infty(\mathcal{O}).$$

As for a Casimir-Polder system we want to exhibit explicit examples of Hadmard states for a Casimir system and our initial plan is to build them starting from a quasi-free counterpart $\widetilde{\omega} : \mathcal{A}^{KG}(\mathbb{R}^4) \to \mathbb{C}$, which is of Hadamard form itself. In other words we would like to mimic the content of Proposition 2.2.1. Alas, there does not exist a $*-$homomorphism between $\mathcal{A}^C(Z)$ and $\mathcal{A}^{KG}(\mathbb{R}^4)$ and hence no corresponding pull-back of states. We shall avoid such hurdle by working directly at the level of the two-point function adapting the **image method** used previously in Definition 3.1.1 for the causal propagator. Notice that, with this procedure, we will be constructing actually a state for $\mathcal{A}^C_{ext}(Z)$.

More precisely our starting point is any Hadamard state $\widetilde{\omega} : \mathcal{A}^{KG}(\mathbb{R}^4) \to \mathbb{C}$, whose associated two-point function $\widetilde{\omega}_2 \in \mathcal{D}'(\mathbb{R}^4 \times \mathbb{R}^4)$. In view of Definition 3.1.1, applying the image method to $\widetilde{\omega}_2$ is tantamount to proving that $\widetilde{\omega}_2 \circ (\mathbb{I} \otimes N) \in \mathcal{D}'(\mathring{Z} \times \mathring{Z})$. Notice that the outcome does not define an *image state* for $\mathcal{A}^C(Z)$ but only for $\mathcal{A}^C_{ext}(Z)$.

Since our goal is to exhibit explicit cases where this procedure works, we restrict the attention only to quasi-free states for $\mathcal{A}^{KG}(\mathbb{R}^4)$ whose associated two-point function has an integral kernel which is invariant under the simultaneous action on both entries of both ι_z, the reflection along the hyperplane $z = 0$ and of ι_s, the translation of step s along the z-direction, $s \in \mathbb{R}$:

$$\widetilde{\omega}_2(\iota_z(f), \iota_z(f')) = \widetilde{\omega}_2(\iota_s(f), \iota_s(f')) = \widetilde{\omega}_2(f, f'), \tag{3.11}$$

where $f, f' \in C_0^\infty(\mathbb{R}^4)$. As an additional ingredient we recall, that all two-points functions of Hadamard form differ only by a smooth integral kernel. Hence, since in this section we are interested in a massless real scalar field, we can split

$$\widetilde{\omega}_2(x, x') = \widetilde{\omega}_2^0(x, x') + W(x, x') \tag{3.12}$$

where $W \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4)$, while $\widetilde{\omega}_2^0(x, x')$ is the integral-kernel of the two-point function of the Poincaré vacuum. Therefore, we will analyze separately

$W(x,x')$ and $\widetilde{\omega}_2^0(x,x')$ starting from the latter, which fulfills the requirements of (3.11). Recall that

$$\widetilde{\omega}_2^0(f,f') \doteq \lim_{\varepsilon\to 0^+} \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^4\times\mathbb{R}^4} d^4x\, d^4x' \frac{f(x)f'(x')}{-(t-t'-i\varepsilon)^2 + (\mathbf{x}-\mathbf{x}')^2}. \tag{3.13}$$

Upon Fourier transform, we can rewrite the last expression as

$$\widetilde{\omega}_2^0(f,f') = \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} d^3\mathbf{k} \frac{1}{2|\mathbf{k}|} e^{-i(t-t')|\mathbf{k}|} \widehat{f}(t,\mathbf{k}) \overline{\widehat{f'}(t',\mathbf{k})} \tag{3.14}$$

where $\widehat{f}(t,\mathbf{k})$ is the three dimensional spatial Fourier transform[2] of $f(t,\mathbf{x})$.

Proposition 3.2.1. *Let $\widetilde{\omega}_2^0$ be the two-point function of the Poincaré vacuum for a real, massless scalar field on Minkowski spacetime. Then $\omega_2^0 \doteq \widetilde{\omega}_2^0 \circ (\mathbb{I}\otimes N) = \widetilde{\omega}_2^0 \circ (N\otimes \mathbb{I}) \in \mathcal{D}'(\mathring{Z}\times\mathring{Z})$. Furthermore the integral kernel of ω_2^0 can be written as the $\varepsilon \to 0$ limit of the following ε-regularized integral kernel:*

$$\frac{1}{8\pi d \chi_\varepsilon}\left(\frac{\sinh\frac{\pi\chi_\varepsilon}{d}}{\cosh\frac{\pi\chi_\varepsilon}{d} - \cos\left(\frac{\pi}{d}(z-z')\right)} - \frac{\sinh\frac{\pi\chi_\varepsilon}{d}}{\cosh\frac{\pi\chi_\varepsilon}{d} - \cos\left(\frac{\pi}{d}(z+z')\right)} \right) \tag{3.15}$$

where $\chi_\varepsilon \doteq -(\underline{x}^0 - \underline{x}'^0 - i\varepsilon)^2 + (\underline{x}^1 - \underline{x}'^1)^2 + (\underline{x}^2 - \underline{x}'^2)^2$.

Proof. With respect to the standard Cartesian coordinates (but keeping the notation $x = (\underline{x}, z)$) and fixing $e^0 = (1,0,0,0)$ and $e^3 = (0,0,0,1)$, we can write the formal expression

$$\widetilde{\omega}_2^0(f, Nf') =$$
$$\lim_{\varepsilon\to 0^+} \int d^4x d^4x' \left[\widetilde{\omega}_2^0(x+i\varepsilon e^0, x') - \widetilde{\omega}_2^0(x+i\varepsilon e^0, \iota_z x')\right] f(x) \sum_n f'(x'+2nde^3).$$

Up to a change of variables of integration for every element of the sum, we obtain

$$\lim_{\varepsilon\to 0^+} \int d^4x d^4x' \sum_n \left[\widetilde{\omega}_2^0(x+i\varepsilon e^0, x'+2dne^3) - \widetilde{\omega}_2^0(x+i\varepsilon e^0, \iota_z x'+2dne^3)\right] f(x) f'(x').$$

For every $\varepsilon > 0$,

$$\left(\widetilde{\omega}_2^0 \circ (\mathbb{I}\otimes N)\right)(x+i\varepsilon e^0, x') =$$
$$\lim_{m\to\infty} \sum_{|n|<m} \left[\widetilde{\omega}_2^0(x+i\varepsilon e^0, x'+2dne^3) - \widetilde{\omega}_2^0(x+i\varepsilon e^0, \iota_z x'+2dne^3)\right].$$

[2]Our convention for the spatial Fourier transform is the following: $\widehat{f}(t,\mathbf{k}) \doteq \frac{1}{(\sqrt{2\pi})^3}\int d^3\mathbf{x} e^{i\mathbf{k}\cdot x} f(t,\mathbf{x})$.

If we recall that for complex variables $a, b \in \mathbb{C}$, it holds – see [GR07, §1.445]

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2 + (b+n)^2} = \frac{\pi}{a} \frac{\sinh(2\pi a)}{\cosh(2\pi a) - \cos(2\pi b)},$$

and if we recall the form of $\widetilde{\omega}_2^0(x, x')$ given in (3.13) we can show that the $\widetilde{\omega}_2^0 \circ (\mathbb{I} \otimes N)$ converges to

$$\omega_2^0(x, x') =$$
$$\lim_{\varepsilon \to 0^+} \frac{1}{8\pi d \chi_\varepsilon} \left(\frac{\sinh \frac{\pi \chi_\varepsilon}{d}}{\cosh \frac{\pi \chi_\varepsilon}{d} - \cos\left(\frac{\pi}{d}(x^3 - y^3)\right)} - \frac{\sinh \frac{\pi \chi_\varepsilon}{d}}{\cosh \frac{\pi \chi_\varepsilon}{d} - \cos\left(\frac{\pi}{d}(x^3 + y^3)\right)} \right).$$

in the limit of $n \to \infty$.

We interpret $\underline{x}^0 - \underline{x}'^0 + i\varepsilon$ as an extension of $\underline{x}^0 - \underline{x}'^0$ to the complex plane and we investigate the properties of (3.15) as an analytic function. Notice that $\sinh(\xi)/\xi$ is entire analytic as a function of ξ^2. Hence its composition with $\xi^2 = (\pi/d)^2 \chi^2(\underline{x}, \underline{x}') = (\pi/d)^2(-(\underline{x}^0 - \underline{x}'^0)^2 + (\underline{x}^1 - \underline{x}'^1)^2 + (\underline{x}^2 - \underline{x}'^2)^2)$ is in turn entire analytic itself on $\mathbb{C}^8$. Furthermore, since the function $1/(\cosh(\alpha) - \cos(\beta))$ can be expanded in Laurent series in terms of α^2 and β^2 whenever $\cosh(\alpha) \neq \cos(\beta)$, this result applies to our scenario whenever $\underline{x}^0 - \underline{x}'^0 + i\varepsilon$ has a sufficiently large imaginary component while the other coordinates have a small imaginary part. Under these conditions we can conlcude the existence of a domain of analyticity for (3.15). Notice that a boundary component of such domain is obtained constraining all spatial coordinates to be real and taking the limit $\varepsilon = \operatorname{Im}(\underline{x}^0 - \underline{x}'^0)$ to 0^+. Furthermore, by direct inspection, (3.15) is bounded up to a multiplicative constant by ε^{-2}, close to the mentioned boundary component. Hence we can apply Theorem 3.1.15 of Hörmander [Hör90] to conclude that the boundary value of (3.15) at $\varepsilon = 0$ is itself a distribution. □

To conclude that $\omega_2^0(x, x')$ defines a state on $\mathcal{A}^C_{ext}(Z)$ we prove the following:

Proposition 3.2.2. *The distribution $\omega_2^0 \in \mathcal{D}'(\mathring{Z} \times \mathring{Z})$ built in Proposition 3.2.1 is the two-point function of a quasi-free state $\omega^0 : \mathcal{A}^C_{ext}(Z) \to \mathbb{C}$.*

Proof. In view of the previous proposition and of the properties of the Poincaré vacuum, it remains to be shown that ω is positive. We shall check it for test functions f and f' that can be factorized in the $z-$direction, namely of the form $f^{(\prime)}(\underline{x}, z) = f_\perp^{(\prime)}(\underline{x}) f_z^{(\prime)}(z)$ where $f_\perp^{(\prime)} \in C_0^\infty(\mathbb{R}^3)$ and where $f_z^{(\prime)} \in C_0^\infty((0, d))$. Notice that, although we are not exhausting all possible elements of $C_0^\infty(\mathring{Z})$, we are still considering a dense subset, which suffices as far as positivity is concerned. With respect to this kind of functions we can introduce the following distribution on $C_0^\infty((0, d) \times (0, d))$

$$w^{f'_\perp, f_\perp}(f'_z, f_z) \doteq \widetilde{\omega}_2^0(f'_\perp f'_z, f_\perp f_z) = \lim_{\varepsilon \to 0^+} \int_0^d dz \int_0^d dz' w_{2,\varepsilon}^{f'_\perp, f_\perp}(z - z') f_z(z) f'_z(z'),$$

where as usual the limits are meant in the weak sense. Since $w^{f'_\perp, f_\perp}$ is a Schwartz distribution, see *e.g.* (3.14), we might rewrite it in the Fourier domain

$$\widetilde{\omega}_2^0(\overline{f'_\perp} f'_z, f_\perp f_z) = w^{f'_\perp, f_\perp}(f'_z, f_z) = \int_{\mathbb{R}} d\xi \, \widehat{w}_2^{\overline{f'_\perp}, f_\perp}(\xi) \widehat{f}'_z(\xi) \widehat{f}_z(\xi). \tag{3.16}$$

Notice that, since the two-point function $\widetilde{\omega}_2$ of the Poincaré vacuum is itself a quadratic form, we have that $\widehat{w}_2^{\overline{f_\perp}, f_\perp}(\xi)$ is a positive function which is continuous almost everywhere. In particular, from the expression of the spectrum built in (3.14), we can infer that continuity could fail only at $\xi = 0$, although $\widehat{w}_2^{\overline{f_\perp}, f_\perp}(\xi)$ is a locally integrable function, also in a neighbourhood of 0.

Let us now consider $w^{f_\perp, f_\perp}$ applied to $(\overline{f_z}, N f_z)$. By Poisson summation formula it holds $\sum_l f_z(z + 2dl) = \sum_n f_n e^{inz\pi/d}$ where f_n are the Fourier coefficients of f_z computed in the interval $[-d, d]$ and they coincide with the ordinary Fourier transform evaluated at $\xi = n\pi/d$, namely $f_n = \widehat{f}_z(n\pi/d)$. Hence, taking into account the anti-symmetrization present in N, $N f_z = \sum_n (f_n - f_{-n}) e^{inz\pi/d}$. Furthermore, its Fourier transform can be computed in a distributional sense as

$$\widehat{N f_z} := \left(\widehat{f}_z(\xi) - \widehat{f}_z(-\xi)\right) \sum_n \delta\left(\xi - n\frac{\pi}{d}\right).$$

Dropping the superscripts $\overline{f_\perp}, f_\perp$ from both w and $\widehat{w}$ it holds

$$w(\overline{f_z}, N f_z) = \int d\xi \; \widehat{w}_2(\xi) \; \left(\widehat{f}_z(\xi) - \widehat{f}_z(-\xi)\right) \sum_n \delta\left(\xi - n\frac{\pi}{d}\right) \overline{\widehat{f}_z(\xi)}.$$

Notice that, despite of the presence of an infinite sum of Dirac delta functions, the previous expression is well defined because $\widehat{w}_2(\xi)$ is continuous for $\xi \neq 0$, it grows at most polynomially for large $|\xi|$ and it is bounded close to zero[3]. The only delta function in the sum which could give a divergent contribution is the one supported at 0. Since $\widehat{f}_z$ is a Schwartz function, $\left(\widehat{f}_z(\xi) - \widehat{f}_z(-\xi)\right)$ vanishes, however, at zero and hence, thanks to the boundedness of $\widehat{w}_2(\xi)$ near that point, the contribution of the delta function supported at 0 vanishes. We

[3] In order to check boundedness of $\widehat{w}_2(\xi)$, notice from (3.14) that for some positive constant C

$$\begin{aligned} |\widehat{w}(\xi)| &\leq C \sup_{t,t' \in I} \int_{\mathbb{R}^2} dk_\perp \frac{1}{\sqrt{k_\perp^2 + \xi^2}} |\widehat{f}_\perp(t, k_\perp)| |\widehat{f'}_\perp(t', k_\perp)| \\ &\leq C \sup_{t,t' \in I} \int_0^\infty d|k_\perp| \int_0^{2\pi} d\theta |\widehat{f}_\perp(t, k_\perp)| |\widehat{f'}_\perp(t', k_\perp)| \end{aligned}$$

where the supremum is taken in some interval I chosen in such a way that $I \times \mathbb{R}^2$ contains the supports of both $f_\perp$ and $f'_\perp$. Furthermore, $\widehat{f}_\perp$ and $\widehat{f}'_\perp$ are the spatial Fourier transform of $f_\perp$ and $f'_\perp$ and hence they decay rapidly for large values of $|k_\perp|$. The result of the two integrals can thus be bounded by some positive constant.

have

$$w(\overline{f_z}, Nf_z) = \sum_n \widehat{w}_2(n\pi/d)(f_n - f_{-n})\overline{f_n} = \sum_{n\geq 1} \widehat{w}_2(n\pi/d)|f_n - f_{-n}|^2$$

where, in the last equality, we use the fact that ω_2 is symmetric under $z-$reflections and hence $\widehat{w}_2(n\pi/d) = \widehat{w}_2(-n\pi/d)$. The last term of the above chain of equalities is positive because it is a sum of positive quantities, since we have started from the two-point function of a state and, hence, $\widehat{w}_2(n\pi/d)$ is a quadratic form for every n. □

In order to generalize the result obtained for another quasi-free Hadamard state $\widetilde{\omega}$ whose two-point function integral kernel enjoys the symmetries stated in (3.11), we recall that the two-point function of such state differs from the vacuum one by a smooth function $W(x, x')$. We have now to make sure that $\mathbb{I} \otimes N$ can be applied also to $W(x, x')$. To this end, we need to impose technical restrictions on the admissible class of smooth functions.

Proposition 3.2.3. *Let $\widetilde{\omega}$ be a quasi-free state of Hadamard form for $\mathcal{A}^{KG}(\mathbb{R}^4)$. Suppose that the integral kernel of its two-point function $\widetilde{\omega}_2(x, x') = \widetilde{\omega}_2^0(x, x') + W(x, x')$ is invariant under* (3.11)*. Suppose that the following conditions hold for the smooth part $W \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4)$:*

(i) the function $W^{f_\perp, h_\perp}(z, z') := \int_{\mathbb{R}^3\times\mathbb{R}^3} d^3\underline{x} d^3\underline{x}'\, W(\underline{x}, z, \underline{x}', z') f_\perp(\underline{x}) f'_\perp(\underline{x}')$ lies in $\mathcal{S}'(\mathbb{R}^2)$ for every $f_\perp, h_\perp \in \mathcal{D}(\mathbb{R}^3)$.

(ii) for every value of x^3 and x'^3, $W^{f_\perp, f'_\perp}(z, z')$ generates a distribution in $\mathcal{D}'(\mathbb{R}^6)$, hence it is continuous on $\mathcal{D}(\mathbb{R}^6)$.

(iii) Let $w(z - z') \doteq W^{f_\perp, f'_\perp}(z, z')$ and let $\widehat{w}(\xi)$ be its Fourier transform. It is a continuous function for $\xi \geq \frac{\pi}{d}$,

(iv) $\xi \mapsto \widehat{w}(\xi)\xi$ is a continuous function in a neighbourhood of $\xi = 0$ and it vanishes for $\xi = 0$.

Hence, in view of Proposition 3.1.2 we can extend $\widetilde{\omega}_2$ to a map on $C_0^\infty(\mathring{Z}) \times N[C_0^\infty(\mathring{Z})]$ and

$$\omega_2(f', f) = \widetilde{\omega}_2(f', Nf).$$

gives rise to a quasi-free state $\omega : \mathcal{A}^C_{ext}(Z) \to \mathbb{C}$.

Proof. Consider a compactly supported smooth function $f \in \mathcal{D}(\mathring{Z})$ which can be factorized in the following way $f(\underline{x}, z) = f_\perp(\underline{x}) f_z(z)$. Let us study Nf and notice that N acts only on f_z. Furthermore, by the Poisson summation formula (see [Hör90, §7.2]), we know that $Nf_z(z) = \sum_n (f_n - f_{-n}) e^{inz\pi/d}$ and, as discussed in the proof of the previous proposition, the Fourier transform can be computed in the distributional sense yielding

$$\widehat{Nf_z}(\xi) := \left(\widehat{f}_z(\xi) - \widehat{f}_z(-\xi)\right) \sum_n \delta\left(\xi - n\frac{\pi}{d}\right)$$

where δ is the Dirac delta function. For every other $f' \in \mathcal{D}(\mathring{Z})$ which can also be factorized, we analyze

$$
\begin{aligned}
W(f', Nf) &:= \int_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4x'\, f'(x) W(x,x') Nf(x') \\
&= \int_{\mathbb{R}} d\xi\, \widehat{W}^{f'_\perp, f_\perp}(\xi) \overline{\widehat{f'_z}(\xi)} \widehat{N f_z}(\xi) = \\
&= \int_{\mathbb{R}} d\xi\, \widehat{w}(\xi) \overline{\widehat{f'_z}(\xi)} \left(\widehat{f_z}(\xi) - \widehat{f_z}(-\xi)\right) \sum_n \delta\left(\xi - n\frac{\pi}{d}\right).
\end{aligned}
$$

The previous expression is well defined for the following reasons:

a) conditions (iii) implies that $\widehat{w}(\xi)$ is continuous for $|\xi| \geq \pi/d$,

b) thanks to hypothesis (i), $w(z)$ is a Schwartz distribution, hence its Fourier transform, grows at most polynomially for large ξ and

c) requirement (iv) implies that $\widehat{w}(\xi)\xi$ is continuous near zero and vanishes for $\xi = 0$.

The Dirac delta supported in 0 gives a vanishing contribution to the sum because $\left(\widehat{f_z}(\xi) - \widehat{f_z}(-\xi)\right)/\xi$ is a continuous function near zero and hence $\widehat{w}(\xi)\xi \cdot \left(\widehat{f_z}(\xi) - \widehat{f_z}(-\xi)\right)/\xi$ is continuous in 0 and there it vanishes. Furthermore, what remains is

$$
W(f', Nf) = \sum_n \widehat{w}\left(\frac{n\pi}{d}\right) \overline{f'_n} \left(f_n - f_{-n}\right) = \sum_{n\geq 1} \widehat{w}\left(\frac{n\pi}{d}\right) \overline{\left(f'_n - f'_{-n}\right)} \left(f_n - f_{-n}\right)
$$

which is continuous with respect to the topology of $\mathcal{D}'((0,d)\times(0,d))$. Hence, taking into account hypothesis (ii), $W(f', Nf)$ is separately continuous on $\mathcal{D}((0,d)\times(0,d)) \otimes \mathcal{D}(\mathbb{R}^6)$ and thus it is a distribution in $\mathcal{D}'(\mathring{Z}\times\mathring{Z})$.

For this reason, ω_2 is also a well-defined distribution being the sum of ω_2^0 and $W \circ (\mathbb{I}\otimes N)$. Positivity remains to be shown, but it can be checked following a proof similar to the proof of Proposition 3.2.2, hence we shall omit it. $\square$

The requirements of the previous proposition are quite involved to check. For this reason, in the following lemma we give an alternative sufficient condition which implies the four points assumed in the previous proposition.

Lemma 3.2.4. *Let $\widetilde{\omega}$ be a quasi-free state of Hadamard form for $\mathcal{A}^{KG}(\mathbb{R}^4)$. Suppose that its two-point function $\widetilde{\omega}_2 = \widetilde{\omega}_2^0$ is invariant under $z-$reflections and under $z-$translations as in* (3.11). *Consider the smooth function $W := \widetilde{\omega}_2 - \widetilde{\omega}_2^0$. Suppose that $W \in L^\infty(Z)$ and that the integral over z of $\frac{\partial}{\partial z} W(\underline{x}, z, \underline{x}', z')$ is bounded uniformly in $\underline{x}$ and $\underline{x}'$. Then the hypotheses of the previous proposition are satisfied and thus the following expression*

$$
\omega_2(f', f) = \widetilde{\omega}_2(f', Nf).
$$

is a well defined two-point function of a quasi-free state $\omega : \mathcal{A}^C_{ext}(Z) \to \mathbb{C}$.

Proof. Since W is bounded, it is the integral kernel of a Schwarz distribution. Hence, by the Schwartz kernel theorem W can be seen as a map between smooth functions over $\mathbb{R}^6$ and Schwartz distributions over $\mathbb{R}^2$. The first three requirements of Proposition 3.2.3 descend immediately. The forth one requires a few words. Since the derivative along z of W is in L^1, by the Riemann-Lebesgue lemma, its Fourier transform along the $z-$direction $\widehat{w}(\xi)$ is equal to a continuous function $u(\xi)$ divided by ξ. Furthermore, since W is symmetric under reflections generated by ι_z, $\widehat{w}$ must be invariant under mapping of $\xi \to -\xi$, and thus $u(\xi) = \xi\widehat{w}(\xi)$ is an odd continuous function, hence it must vanish for $\xi = 0$. $\square$

Before concluding this section we analyze the singular structure of Hadamard states obtained by the image method described so far. We already know that these states are of Hadamard form when restricted on globally hyperbolic sub regions of $\mathbb{H}$, hence therein the singular structure is known, however we expect further singularities when states for the full algebra $\mathcal{A}^C(Z)$ is considered. Actually, the following proposition holds.

Proposition 3.2.5. *Consider the two-point function of a quasi-free state ω for $\mathcal{A}^C(Z)$ obtained by the image method starting from a quasi-free Hadamard state $\widetilde{\omega}$ of $\mathcal{A}^{KG}(\mathbb{R}^4)$. The wave front set of its two-point function ω_2 has the following form*

$$WF(\omega_2) = \left\{(x, x', k_x, -k_{x'}) \in T^*\left(\mathring{Z} \times \mathring{Z}\right) \setminus \{0\} | (x, k_x) \sim_Z (x', k_{x'}), k_x \triangleright 0\right\}$$

where $(x, k_x) \sim_Z (x', k_{x'})$ whenever there exists a null geodesic γ reflected at the boundaries a countable number of times, such that x, y are its end points, k_x is the cotangent vector to γ at x while k_y is the parallel transport of k_x along γ.

Proof. We recall that

$$\omega_2(x, x') = \sum_{n\in\mathbb{N}} \left[\widetilde{\omega}_2(x, (\underline{x}', z' + 2nd)) - \widetilde{\omega}_2(x, (\underline{x}', -z' + 2nd))\right],$$

Hence, $WF(\omega_2)$ is contained in the union of the wavefront sets of $\widetilde{\omega}_2(x, (\underline{x}', z' + 2nd))$ and of $\widetilde{\omega}_2(x, (\underline{x}', -z' + 2nd))$.

Let us analyze $WF(\widetilde{\omega}_2(x, (\underline{x}', z' + 2nd)))$. Notice that $\widetilde{\omega}_2(x, (\underline{x}', z' + 2nd))$ is nothing but as $\widetilde{\omega}_2$ in Minkowski with a translation applied to x'. Hence we just need to apply the corresponding transformation on its wavefront set to obtain the wavefront set of $WF(\widetilde{\omega}_2(x, (\underline{x}', z' + 2nd)))$. Furthermore, if the points (x, x') are contained in its singular support, this means that x and $\iota_{2nd}(x')$ are connected by a null geodesic in Minkowski spacetime. This geodesic in Minkowski passes trough the points z where z^3 is a multiple of d, $|2n|$ times. Hence, in the Casimir region, it is like a null geodesic reflected $2n$ times at the boundaries. We can treat in a similar way $WF(\widetilde{\omega}(x, (\underline{x}, -z' +$

$2nd)))$ and it coincides with the wave front set of $\widetilde{\omega}$ where the second entry of that distribution is reflected and translated $2n$ times. Hence, (x, x') are in its singular support only if they are connected by a null geodesic reflected $|2n-1|$ times at the boundaries.

Finally, we notice that the wave front set of $\widetilde{\omega}(x, (\underline{x}', z' + 2nd))$ and of $\widetilde{\omega}(x, (\underline{x}', -z' + 2nd))$ are all disjoined, (their singular support might overlap only when both $z = z' = d/2$ but in this case the corresponding covectors have opposite $z-$direction). Hence, in the sum defining ω_2 no cancellation of singularity might occur. We thus conclude that $WF(\omega_2)$ coincides with the union of the wave front sets of the distributions in the sum written above. $\square$

3.2.1 The vacuum and the KMS states for the Casimir system

In this subsection, we shall construct states $\omega^T : \mathcal{A}^C_{ext}(Z) \to \mathbb{C}$ at finite temperature T for the Casimir system. We shall show that these states are obtained applying the image method to a KMS state for a Klein-Gordon field on Minkowski spacetime. As a corollary, we obtain that ω^0 is the vacuum state of the theory and it coincides with $\lim_{T\to 0} \omega^T$. Our computations are consistent with the literature on the topic, see for example [BM69, FR87, KCD79] for the thermal case and [Ful89] for the vacuum.

As before, we work at the level of two-point function. Hence, let us suppose that the hypotheses of Proposition 3.2.3 are met. If so, we can apply the image method to a state $\widetilde{\omega}$ on $\mathcal{A}^{KG}(\mathbb{R}^4)$ to obtain a quasi-free Hadamard state ω for $\mathcal{A}^C_{ext}(Z)$, such that $\omega_2(f, f') = \widetilde{\omega}_2(f, Nf')$, $f, f' \in C^\infty_{0,C}(\mathbb{R}^4)$. Suppose also that the state $\widetilde{\omega}$ is invariant under the natural action induced on it by the time translation t_ξ of step $\xi \in \mathbb{R}$. Since N commutes with t_ξ, also the state ω must be invariant under time translations.

Consider now the quasi-free KMS state $\widetilde{\omega}^T : \mathcal{A}^{KG}(\mathbb{R}^4) \to \mathbb{C}$ at temperature T which is invariant under the action induced by t_ξ. For every $f, f' \in C^\infty_0(\mathbb{R}^4)$ the function $\xi \mapsto \widetilde{\omega}_2(t_\xi f, g)$ is analytic in the strip $\operatorname{Im}(\xi) \in [0, \beta]$ where $\beta = (k_B T)^{-1}$ is the inverse temperature and k_B is the Boltzmann constant. Furthermore, the KMS condition holds, namely

$$\widetilde{\omega}^T_2(t_{i\beta} f, f') = \widetilde{\omega}^T_2(f', f).$$

We recall also that,

$$\widetilde{\omega}^T_2(x, x') = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi\beta|\mathbf{x} - \mathbf{x}'|} \frac{\sinh\left(2\pi \frac{|\mathbf{x}-\mathbf{x}'|}{\beta}\right)}{\cosh\left(2\pi \frac{|\mathbf{x}-\mathbf{x}'|}{\beta}\right) - \cosh\left(2\pi \frac{(t-t'-i\varepsilon)}{\beta}\right)},$$

for $\operatorname{Im}(t - t') \in (-\beta + \varepsilon, 0]$, where we use t for the time coordinate and $\mathbf{x}$ for the space coordinates. Furthermore

$$\widetilde{\omega}^T_2(x, x') = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3} d^3\mathbf{k} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2|\mathbf{k}|} \left(\frac{e^{-i|\mathbf{k}|(t-t')}}{1 - e^{-\beta|\mathbf{k}|}} + \frac{e^{i|\mathbf{k}|(t-t')}}{e^{\beta|\mathbf{k}|} - 1} \right) e^{-\varepsilon|\mathbf{k}|}$$

We shall check that, it is possible to apply the image method to this state by analyzing the behavior of $W := \widetilde{\omega}_2^T - \widetilde{\omega}_2^0$ and verifying that the hypotheses of Lemma 3.2.4 is satisfied and thus Proposition 3.2.3 holds. First of all, we notice that W is a Schwartz distribution, which has the desired symmetry properties (3.11). The spatial Fourier transform of its integral kernel has the following form

$$\widehat{W}(t,t';\mathbf{k}) = C\frac{1}{|\mathbf{k}|}\left(\frac{\cos(|\mathbf{k}|(t-t'))}{e^{\beta|\mathbf{k}|}-1}\right).$$

It is a smooth function except when $|\mathbf{k}| = 0$ and it decays rapidly for large $|\mathbf{k}|$. From this observation conditions (i),(ii) and (iii) of Proposition 3.2.3 are met. It remains to prove the (iv). In order to check it we proceed analyzing

$$\widehat{w}^T(\xi) = \int_{\mathbb{R}} dt \int_{\mathbb{R}} dt' \int_{\mathbb{R}^2} dk_\perp \widehat{W}(t,t';k_\perp,\xi)\widehat{f}_\perp(t,k_\perp)\overline{\widehat{f'_\perp}(t',k_\perp)}.$$

for a pair of compactly supported function $f_\perp, f'_\perp \in \mathcal{D}(\mathbb{R}^3)$. Above, $\widehat{f}_\perp(t,k_\perp)$ is the spatial (two-dimensional) Fourier transform of $f_\perp(t,\underline{x}^1,\underline{x}^2)$. Notice that there exists a positive constant C such that $|\widehat{W}(t,t';\mathbf{k})| \leq C/|\mathbf{k}|^2$. Hence

$$|\widehat{w}^T(\xi)| \leq C \sup_{t,t'\in I} \int_{\mathbb{R}^2} dk_\perp \frac{1}{k_\perp^2+\xi^2}|\widehat{f}_\perp(t,k_\perp)||\widehat{h}_\perp(t',k_\perp)|,$$

where the supremum is taken in an interval I chosen in accordance to the supports of both $f_\perp$ and $f'_\perp$. Since, $\widehat{f}_\perp$ and $\widehat{f}'_\perp$ are two Schwartz functions it holds that

$$|\widehat{w}(\xi)| \leq \sup_{t,t'\in I} C'(t,t') \int_0^\infty dk \frac{k}{k^2+\xi^2}\frac{1}{1+k^2}$$

for some positive set of constants $C'(t,t')$ bounded in I^2. The $k-$integral can be computed and it yields a function of ξ which is logarithmically divergent near 0, and hence, also requirement (iv) of Proposition 3.2.3 is met.

For completeness we check the applicability of the image method directly on the two-point function. We obtain

$$\begin{aligned}\omega_2^T(x,x') &\doteq \left(\widetilde{\omega}_2^T\left(\mathbb{I}\otimes N\right)\right)(x,x') = \\ -\sum_{n=-\infty}^{\infty}&\left(\frac{1}{2\pi\beta r_n}\frac{\sinh\frac{2\pi r_n}{\beta}}{\cosh\frac{2\pi r_n}{\beta}-\cos\frac{2\pi i}{\beta}(\underline{x}^0-\underline{x}'^0+i\varepsilon)} + \right. \\ &\left. -\frac{1}{2\pi\beta\widetilde{r}_n}\frac{\sinh\frac{2\pi\widetilde{r}_n}{\beta}}{\cosh\frac{2\pi\widetilde{r}_n}{\beta}-\cos\frac{2\pi i}{\beta}(\underline{x}^0-\underline{x}'^0+i\varepsilon)}\right),\end{aligned} \tag{3.17}$$

where $r_n^2 \doteq (\underline{x}^1-\underline{x}'^1)^2+(\underline{x}^2-\underline{x}'^2)^2+(z-z'+2nd)^2$ while $\widetilde{r}_n^2 \doteq (\underline{x}^1-\underline{x}'^1)^2+(\underline{x}^2-\underline{x}'^2)^2+(-z-z'+2nd)^2$. Notice that, for every $\varepsilon>0$ and for every x,y in Z we have the the sum is absolutely convergent. As a matter of fact, for large n, both r_n and $\widetilde{r}_n$ grow like $2nd$ hence, the asymptotic behavior of the $n-$th element of series is governed by

$$\frac{1}{2\pi\beta r_n} - \frac{1}{2\pi\beta\widetilde{r}_n} = \frac{1}{2\pi\beta}\frac{\widetilde{r}_n - r_n}{r_n\widetilde{r}_n} = \frac{1}{2\pi\beta}\frac{\widetilde{r}_n^2 - r_n^2}{r_n\widetilde{r}_n(r_n + \widetilde{r}_n)}$$

and the right hand side of the previous expression is majored by C/n^2 hence it can be summed.

We conclude this section with a proposition which ensures that the image method preserves the thermal properties of states.

Proposition 3.2.6. *The quasi-free state $\omega^T : \mathcal{A}^C_{ext}(Z) \to \mathbb{C}$, whose two-point function ω_2^T is obtained applying the image method to the two-point function $\widetilde{\omega}_2^T$ of the KMS state $\widetilde{\omega}^T$ as in* (3.17) *is a KMS state. The limit of ω^T as $T \to 0$ is a vacuum state.*

Proof. In order to prove the proposition, we want now to show that $\omega_2^T(f, f') = \widetilde{\omega}_2^T(f, Nf')$ for $f, f' \in C_0^\infty(\mathring{Z})$, enjoys the KMS condition in $\mathcal{A}^C_{ext}(Z)$. To this end we recall that the KMS condition can alternatively be written as

$$\widetilde{\omega}_2^T(t_{i\beta}(f), f') - \widetilde{\omega}_2^T(f, f') = -iE(f, f')$$

where E is the causal propagator of the theory. Hence, let us analyze it for ω^T

$$\begin{gathered}\omega_2^T(t_{i\beta}(f), f') - \omega_2^T(f, f') = \\ \widetilde{\omega}_2^T(t_{i\beta}(f), Nf') - \widetilde{\omega}_2^T(f, Nf') = -iE(f, Nf') = -iE_Z(f, f'),\end{gathered}$$

where E_Z is constructed in Definition 3.1.1 Since in the limit $\beta \to 0$ we recover ω_2^0 we might safely say that ω^0 is the ground state of the Casimir system. □

Notice that the very same conclusion could have been drawn using instead a more general argument following the analysis of [SV00]. It is noteworthy that the analysis of this section could have been performed for the Hadamard states of a massive real scalar field on the whole Minkowski spacetime. Yet, in such case, on account of the fall-off properties at infinity of the Poincaré vacuum, we would have obtained far better convergence results of the image method.

3.2.2 Wick ordering in a Casimir system

To conclude the section, as for a Casimir-Polder system we want to make contact with the standard results in the literature concerning the expectation value of the regularized two-point function and stress-energy tensor. To this end we need first of all to define the extended algebra of Wick polynomials. The procedure is identical to the one discussed in Section 2.2.1 and, thus, we will not repeat it here. In particular it is possible to introduce an algebra of extended observables on any globally hyperbolic submanifolds $\mathcal{O} \subset Z$. Furthermore, thereon, $\mathcal{A}^C_\mu(\mathcal{O})$ is $*$-isomorphic (actually it coincides) to $\mathcal{A}^{KG}_\mu(\mathcal{O})$. For the same reasons discussed in the Casimir-Polder case, however, the extended algebras $\mathcal{A}^C_\mu(\mathcal{O})$ can be realized as part of a global extended algebra $\mathcal{A}^C_\mu(Z)$ only after a deformation of the $\star_Z$ product into a globally defined one. This can be built for example by replacing H with the two-point function of a Hadamard state.

Despite of this difficulty, we can locally make sense of observables like the stress tensor or the Wick square. In particular the finite vacuum expectation values of the stress-energy components agrees with the non-vanishing quantities found in literature - [BD84, Ful89, SF02, BKMM09, Mil01]. This is proved by the following:

Lemma 3.2.7. *Let us consider a massless, real scalar field and let* $\omega : \mathcal{A}^C_{ext}(Z) \to \mathbb{C}$ *be the quasi-free state whose two-point function* $\omega_2 = (N \otimes \mathbb{I})\,\widetilde{\omega}^0_2$ *is built with the image method from the Poincaré vacuum. Let* $\mathcal{O}$ *be a globally hyperbolic subregion of* $\mathbb{H}^4$ *and* $\mathcal{A}^{CP}_\mu(\mathcal{O})$ *the extended algebra defined on* $\mathcal{O}$ *(with the* $\star_H$ *product). Then, the expectation value of the unsmeared squared field (localized in* $\mathcal{O}$*) turns out to be*

$$\omega^0(:\phi^2:_H(x)) = \frac{1}{48d^2}\left(1 - \frac{3}{\sin^2\frac{\pi z}{d}}\right),$$

and the expectation values of the unsmeared components of the stress-energy tensor (localized in $\mathcal{O}$*) turn out,*

$$\omega^0(:T_{\mu\nu}:_H(x)) =$$
$$\frac{\pi^2}{1440d^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \left(\xi - \frac{1}{6}\right)\frac{\pi^2}{8d^4}\frac{3 - 2\sin^2\left(\frac{\pi}{d}z\right)}{\sin^4\left(\frac{\pi}{d}z\right)}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Recall that, according to Proposition 3.2.2, ω is a Hadamard state as per Definition 3.2.1. In order to compute the Wick squared scalar field, we recall result of the image method and we obtain, for any $\zeta \in C^\infty_{0,C}(\mathcal{O})$,

$$\begin{aligned}\omega^0(:\phi^2:_H(\zeta)) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \int\limits_{\mathbb{R}^4\times\mathbb{R}^4} d^4x d^4x' \zeta(\underline{x}, z)\delta(x - x')\times \\ \times\left(\widetilde{\omega}^0_2(\underline{x} - \underline{x}', z - z' + 2nd) - \widetilde{\omega}^0_2(\underline{x} - \underline{x}', -z - z' + 2nd)\right) = \\ = \frac{1}{4\pi^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\int\limits_{\mathbb{R}^4} d^4x \left(\frac{1}{(2nd)^2} - \frac{1}{(2z + 2nd)^2}\right)\zeta(\underline{x}, z) \\ = \int\limits_{\mathbb{R}^4} d^4x \left(\frac{1}{96d^4} - \frac{1}{32d^4}\frac{1}{\sin^2\frac{\pi z}{d}}\right)\zeta(\underline{x}, z),\end{aligned}$$

where we used the smoothness property of the sum of the integral kernels in the region of interest, first to deduce that the result of the integrals is finite and then to exchange the sum with the integrals. In the last equality we have computed the sum by using for the first term the definition of the Riemann zeta function and in the second still [GR07, §1.445]. In order to compute the expectation value of the smeared Wick ordered time-diagonal component of the stress-energy tensor, we follow the same procedure as in the proof of Lemma 2.2.3, that is, for any $\zeta \in C^\infty_{0,C}(\mathcal{O})$,

$$\omega(:T_{\mu\nu}:_H(\zeta)) = \int\limits_{\mathbb{R}^4\times\mathbb{R}^4} d^4x\, d^4x' D^{(x,x')}_{\mu\nu} \left(\omega_2(x,x') - H(x,x')\right) \zeta(x)\delta(x-x'),$$

where $D^{(x,x')}_{\mu\nu}$ is the same as in (2.15). Following the same procedure as for $\omega(:\phi^2:_H(\zeta))$ the sought result descends. $\square$

Let us make some observation on the above result.

1. As in the previous chapter, we have defined the Wick polynomials only for smooth and compactly supported functions whose support does not intersect the boundary of the region of interest. The reason is analogous to the case of Casimir-Polder system, and it can be seen explicitly by looking at the smeared quantities of the last lemma, which diverges as z^{-4} and $(d-z)^{-4}$. As a consequence the total energy, computed out of the integral of the time-component of the stress-energy tensor diverges.

2. By Lemma 3.2.7 we have shown that our construction reproduces the known results for a mass-less scalar field confined between two parallel plates – [BD84, Ful89, SF02, BKMM09, Mil01]. At the same time, as a new feature, it frames them in an intrinsic axiomatic theory. So we are inclined to regard the divergence of the integrated energy density, mentioned in the above remark, as not a consequence of a bad or incomplete regularization.

3. In the limit $d \to \infty$ we recover the case of the Casimir-Polder effect. This is desirable of course, for consistency. At the same time it suggests the idea that there are two kind of contributions: one constant, depending on the length of the cavity d, and one depending on the coordinate z, related saparately to the presence of the boundary in $z=0$ and in $z=d$. Following Fulling, [Ful89], we interpret the first contribution as due to resonances of the ground state in the cavity. The second one could be understood as a pure effect of the boundary (condition).

As we underlined at the beginning of this chapter, the double parallel plates scenario is related to the Casimir effect. We recall that here the Casimir effect is meant to be the occurrence of an attractive or repulsive force acting on the boundaries[4], computed as the gradient of a suitably defined energy in the ground state. This is in complete analogy with the case of an electromagnetic field, where the Dirichlet walls are modelling perfectly conducting plates, as depicted in the original paper [Cas48].

The total energy at time t – in the ground state – is usually defined as the following quantity:

$$E(t) \overset{*}{=} \int_{\Sigma_t} \omega^0(:T_{00}(x):_H) d^4x, \tag{3.18}$$

where the integration is performed on the timeslice $\Sigma_t \subset Z$ – see [BKMM09, Mil01]. Of course the equality is ill-defined since the domain of integration is infinitely extended. This problem could circumvented by considering a limited domain of integration, or rather an energy per unit area. This solution is often preferred since it allows to call it *total* energy, even if it is, as a matter of fact, an abuse of language. The integral could be still divergent due to the non integrability of $\omega^0(:T_{00}(x):_H)$ at the boundary for any $\xi \neq \frac{1}{6}$. Often this call for a further regularization of the integral. We comment on this below.

Aside this, one should notice that in (3.18) the expectation value of the stress energy tensor component is *unsmeared.* In the framework we have just introduced, this would suggest to define the *energy in the spatial volume V_{t_0} at time t_0* as

$$E(V_{t_0}) := \omega^0(:T_{00}(\chi[V_{t_0}]\delta_{t_0}):_H), \tag{3.19}$$

where $\chi[V_{t_0}]$ is the characteristic function of the volume V_{t_0} and δ_{t_0} the Dirac delta centered in t_0. The left hand side is consistently an energy, by dimensional analysis. Moreover, any question about the convergence of the total energy is a matter of asking whether $\chi[V_{t_0}]\delta_{t_0}$ belongs or not to a local extended algebra $\mathcal{A}_\mu^{CP}(\mathcal{O})$, for some $\mathcal{O} \subset Z$. In this way any further regularization should be understood as a redefinition of a classical quantity (mean values of quantum observables). In this sense we argue in complete analogy with [FP14c], concerning the definition of the Casimir force as the integration of pressure on the boundary.

To conclude we reproduce the calculation of the Casimir force in the very special case of the conformal coupling $\xi = \frac{1}{6}$. In this case we can integrate $\omega(:T_{00}:_H(z))$ at a fixed time on the cavity. We consider a spatial volume $V_t = [0, d] \times A_t$, for a time t, such that A_t is a unit area in the plane $x - y$. The energy is thus, for $\xi = \frac{1}{6}$,

$$\begin{aligned} E(V_t) &= \omega^0(:T_{00}(\chi[V_{t_0}]\delta_{t_0}):_H) \\ &= \int_0^d \omega^0(:T_{00}(z):_H) dz \\ &= -\frac{\pi^2}{1440 d^3}. \end{aligned}$$

[4]The boundaries have to be treated themselves as physical objects, capable of motion. This sounds slightly illegal since the boundaries have not an assigned mass. We will not comment on this.

The attractive force between the two boundaries – or their physical counterparts – turns out to be,

$$F = -\frac{\partial E}{\partial d} = -\frac{\pi^2}{480d^4},$$

consistently with literature – [BKMM09, Mil01].

Chapter 4

Algebraic quantum field theory and the Casimir effect in a wedge

Before starting our analysis of wedge-shaped geometry for the Casimir effect, let us outline a brief history of the problem. The first analyses of the Casimir effect between two intercepting planes for a quantum scalar field are due to Dowker and Kennedy [Dow77, DK78] and to Deutsch and Candelas [DC79] (who investigated this scenario also for an electromagnetic field). The early interest was related mostly to the analysis of quantum field theory in presence of a conical singularity [Dow78, Dow87a, Dow90] and for the close analogy with the model of a straight and ultra tight cosmic string [Dow87b]. Nevertheless many computations of two-point functions and Green operators apply in either wedge, cones and cosmic string senarios. The interest in wedge-shaped regions traces back to the older and more classical problem of diffraction around cones and edges. The first rigorous example is due to Sommerfeld who analyzed the diffraction into the shadow region behind a straight edge in two dimensions. In a series of papers [Som96, Som97, Som01] he developed the use of the method of images to solve the problem, advocating Riemann surfaces as a necessary tool to apply the method in a more general situation. He only sketched a treatment of the most general situation by means of an infinitely sheeted surface. Carslaw [Car99, Car10, Car20] gave credit to such idea developing a theory of "Diffraction of waves by a wedge of any angle" (1920). The idea of Sommerfeld and the work of Carslaw fixed a standard in the approach to the wedge problem, as we find in [Obe54, Sta67]. As far as the more general problem of diffraction by sharp obstacles is concerned, many more examples were studied by Friedlander [Fri58]. A cornerstone in such analysis is represented by the work of Cheeger, who generalized it to elliptic differential operators in presence of conical singularities using techniques based on functional calculus, involving the Hankel transform [Che79, Che83]. This work led to a collaboration with

Taylor on the diffraction of waves by conical singularities [CT82a], which covers the example of wedges [CT82b]. This approach is not related with the method of images, yet it represents a reference point in the analysis of singularities for the causal propagator. A last historical remark has to be done. Until [Dow77] the infinitely sheeted Riemannian surface was considered just an artificial tool. Related to its cosmological interest, Dowker regarded it as a Lorentzian manifold, paving the way to its study in the context of quantum field theory on curved backgrounds. Many approaches followed the one of Dowker, leading to many complementary and equivalent results for Green functions, propagators and stress-energy tensor – we remand the reader to [FTTW12] for an overview. We postpone to the last section of this chapter the main references dealing with wedges and Casimir effect, when we will draw comparisons with our results. The Casimir effect in a wedge-shaped region found its place also in standard textbooks of quantum field theory on curved backgrounds [BD84, Ful89].

A wedge-shaped region consists of a portion of Minkowski space bounded by two infinite plates intercepting along an axis γ with an opening angle $0 < \alpha \leq \pi$. It manifests a cylindrical symmetry, so that it appears natural adopting cylindrical coordinates along the axis,

$$(t, z, r, \theta), \quad t \in \mathbb{R},\, z \in \mathbb{R},\, r \in [0, \infty),\, \theta \in (0, \alpha], \quad \text{such that} \quad \{\gamma\} \equiv \{r = 0\}, \tag{4.1}$$

yielding the following local expression of the metric,

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2. \tag{4.2}$$

We then define the following region, which has a non empty boundary,

$$\begin{aligned} \mathcal{W}_\alpha &:= \{x \in \mathbb{R}^4 \mid r \in [0, \infty)\,,\, \theta \in [0, \alpha]\}, \\ \partial \mathcal{W}_\alpha &= \{x \in \mathbb{R}^4 \mid r = 0\,,\, \theta = 0 \text{ or } \alpha\}. \end{aligned}$$

If we allow $\alpha = \pi$ or 2π, we reconstruct $\mathbb{H}^4$ and $\mathbb{R}^4$, modulo the identification of $\theta = 0$ with $\theta = 2\pi$ in the second case. We thus define the *wedge-shaped region* as

$$W_\alpha := \mathcal{W}_\alpha \setminus \{\gamma\},$$

where we are excluding the edge of the corner, at $r = 0$. This choice is better motivated later in this section. Notice that its boundary is $\partial W_\alpha = \partial \mathcal{W}_\alpha \setminus \{\gamma\}$

We consider a real scalar field vanishing at the boundary ∂W_α whose dynamics is ruled by the Klein-Gordon equation. This scenario has several analogies with a Casimir system, so we call it **wedge-shaped Casimir system** (a shortened version is "wedge Casimir system"). As in the previous cases, we call dynamical configurations for a wedge-shaped Casimir system the set $\mathcal{S}^{wed}(W_\alpha) \subset C^\infty(W_\alpha)$ of all smooth solutions of the boundary value problem:

$$\begin{cases} Pu = (\Box - \xi R - m^2)u, & \xi \in \mathbb{R},\, m \geq 0 \\ u(\underline{z}, 0) = 0 = u(\underline{z}, \alpha) \end{cases}, \tag{4.3}$$

where we used the notation $(t, z, r, \theta) := (\underline{z}, \theta)$ – not to be confused with $(\underline{x}, z)$ of the previous section. The scalar curvature on Minkowski spacetime or on any of its subsets vanishes identically, but, as already stressed, the coupling term ξR contributes to the variation of the Lagrangian with respect to the metric, in the definition of the stress-energy tensor. The $\Box$ operator in cylindrical coordinates reads

$$-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{4.4}$$

which is defined on W_α, having excluded the singular point. Analogously to $\mathbb{H}^4$ and Z, wedge-shaped regions, W_α, are *not globally hyperbolic* spacetimes and we ask whether it is possible to adapt the previous approaches based on the method of images to the study of wedge Casimir systems.

Method of images and Dowker space For particular values of the opening angle, $\alpha = \frac{\pi}{N}$ for any integer $N \in \mathbb{N}$, a solution to the Dirichlet problem (4.3) is given in terms of method of images applied along the angular coordinate. Heuristically, if we consider perpendicular reflections through each boundary plane, we realize that the series of reflections closes consistently and the application of the method consists of a finite summation. For generic angles α, the series of reflections closes in more than one turns around the axis, or even it could not close at all, therefore the method of images does not apply. Nevertheless, as pointed out throughout the work of Sommerfeld and Carslaw, it is possible to circumvent such obstacle by considering a different spacetime embedding a wedge-shaped region, such that the series of images is no longer periodic.

We consider Minkowski spacetime deprived of $\{\gamma\}$ and the cylindrical coordinates given in (4.1), and we observe that the local form of the metric is not influenced by the range of the angular coordinate θ. We regard thus the angular coordinate as the restriction to $(0, 2\pi]$ of a Cartesian coordinate $s \in \mathbb{R}$. We call **Dowker space**, W, the Lorentzian manifold with global coordinates, (t, z, r, s), yielding a metric

$$\eta_W = -dt \otimes dt + dz \otimes dz + dr \otimes dr + r^2 ds \otimes ds.$$

The nomenclature is historically motivated, after Dowker's [Dow77], and appeared recently in [FTTW12]. This space is connected and time-oriented, diffeomorphic to $\mathbb{R}^3 \times (0, \infty)$ and it is ultrastatic, being $N = \mathbb{R}^2 \times (0, \infty) \ni (z, r, s)$ the leaf of the timelike foliation (we refer to the notation in Defition 1.2.6).

The wedge region, in which we are interested, is defined as the region for which $s \in [0, \alpha]$ and it presents close analogies with the parallel plates of Casimir systems, in particular it is possible to apply the method of images because θ is no longer periodic. To adapt the analysis of Chapter 3 to wedge-shaped Casimir systems, however, we shall start from the theory of a Klein-Gordon field on Dowker space. This is the goal of the next section.

Remark 4.0.1. So defined, Dowker space is an open manifold. The choice of excluding the boundary $\partial W \equiv \{r = 0\}$ from the definition of W is physically motivated choice. The closure $\overline{W}$ is a conic spacetime, since it has a conical singularity at the boundary. In other words, at $r = 0$ it produces a delta-like singularity in the curvatures invariants of the manifolds [FS95]. Regarding W_α as a subregion of Dowker space, thus, the choice of removing $r = 0$ is tantamount to avoiding spacetime singularities, which appear unphysical in the context of Casimir effect, where subregions of Minkowski spacetime are considered. A different comment would be made in the context of cosmological applications, as cosmic strings – see the historical overview.

4.1 Scalar field theory on Dowker space

Aiming at studying Klein-Gordon equation on Dowker space, we shall notice that, as in the previous cases, a Cauchy problem is not well posed since W is not globally hyperbolic. This can be argued by the criterion in Proposition 1.2.1, since W ultrastatic and $\mathbb{R}^2 \times (0, \infty)$ is not geodesically complete at the boundary. A boundary condition is needed and, in the present context, it appears natural to prescribe Dirichlet boundary conditions. This time the method of images will not help at all, since we miss a larger space where reflections would take place. The way we follow is then the most direct: We shall adapt the main results of normally hyperbolic operators on globally hyperbolic spacetimes to the case at hand. We will see how boundary conditions make up for the missing global hyperbolicity of the background.

Let us, thus, consider the following

$$\begin{cases} Pu = (\Box - \xi R - m^2)u, \quad \xi \in \mathbb{R},\, m \geq 0 \\ u|_{x \to \partial W} = 0 \end{cases}, \tag{4.5}$$

where, having excluded the singular boundary, the Ricci scalar R is everywhere vanishing, but it plays a role in the definition of the stress-energy tensor, as already remarked. The D'Alambert operator $\Box$ is of the form (4.4),

$$-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial s^2}, \tag{4.6}$$

As next step we identify kinematical and dynamical configurations[1].

Definition 4.1.1. We call space of **kinematical configurations** for the Klein-Gordon field on Dowker spacetime the space

$$\mathcal{C}^D(W) := \{f \in C^\infty(\overline{W}) \mid \lim_{x \to \partial W} f(x) = 0, \text{ where } x = (t, z, r, s)\}.$$

[1] From now on, a D superscript or subscript will mean "Dowker".

As for the kinematical configurations in Minkowski spacetime, we consider $\mathcal{C}^D(W)$ endowed with the compact-open topology and we consider its topological dual $\mathcal{C}'^D(W)$. We call the dual pairing $\langle\cdot,\cdot\rangle_W : \mathcal{C}'^D(W)\times\mathcal{C}^D(W)\to\mathbb{C}$.

In order to characterize support prescriptions for kinematical configurations, let us denote causal structures of $\overline{W}$ as $J_\pm^{\overline{W}}(x)$ and $I_\pm^{\overline{W}}(x)$ for any $x\in\overline{W}$. We define the space of kinematical configurations *with compact support* as follows,

$$\mathcal{C}_0^D(W) := \{f\in\mathcal{C}^D(W)\mid \mathrm{supp}(f)\subset K \text{ for some compact } K\subset\overline{W}\}.$$

Accordingly, we can give the notion of *spacelike compact* configurations,

$$\mathcal{C}_{sc}^D(W) := \{f\in\mathcal{C}^D(W)\mid \mathrm{supp}(f)\subset J_+^{\overline{W}}(K)\cup J_-^{\overline{W}}(K) \text{ for some compact } K\subset\overline{W}\}.$$

The definition of timelike compact functions requires a further refinement.

Definition 4.1.2. A function $f\in\mathcal{C}^D(W)$ is *pastlike (futurelike) compact* if $\mathrm{supp}(f)\cap J_-^{\overline{W}}(x)$ $(\mathrm{supp}(f)\cap J_+^{\overline{W}}(x))$ is either compact or empty for all $x\in\overline{W}$. We denote the set of such functions $\mathcal{C}_{pc}^D(W)$ $(\mathcal{C}_{fc}^D(W))$. *Timelike compact* functions are defined to be elements of $\mathcal{C}_{tc}^D(W)=\mathcal{C}_{pc}^D(W)\cap\mathcal{C}_{fc}^D(W)$.

Solutions to (4.5), *i.e.*, **dynamical configurations**, are elements of $\mathcal{S}^D(W)\subset\mathcal{C}^D(W)$. We call consistently $\mathcal{S}_{sc}^D(W)\subset\mathcal{C}_{sc}^D(W)$ the space of spacelike compact configurations.

Having completed the zoo of functions, let us start our analysis. Recall that neither the spacetime is globally hyperbolic, nor we can rely on a constructive procedure such as the method of images. The first goal is thus to adapt the main results of normally hyperbolic operators on globally hyperbolic backgrounds to the present situation.

4.1.1 Klein-Gordon operator on Dowker spacetime

We start constructing advanced and retarded operators for (4.5). According to Remark 1.2.3, we look for $E_D^\pm\in\mathcal{D}'(\overline{W}\times\overline{W})$ such that

$$PE_D^\pm=\delta \tag{4.7}$$

defined respectively for $t>t'$ and $t'<t$, where δ is the Dirac delta. This non-homogeneous equation can be solved by standard methods of partial differential equations, based on the Fourier and Hankel tranforms. For the remainder, we will restrict our attention to the massless case, but all considerations extend to the massive case modulo more complicated expressions. We are thus left with

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial z^2}\right)E_D^\pm=\frac{1}{r}\delta(t-t')\delta(z-z')\delta(r-r')\delta(s-s').$$

Applying Fourier transform along t, z and s, it turns out

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+(\omega^2-k^2)-\frac{\nu^2}{r^2}\right)\widehat{E}_D^\pm(r,r';\underline{\xi})=\frac{\delta(r-r')}{r} \tag{4.8}$$

where $\underline{\xi} = (\omega, k, \nu)$ and where $\widehat{E}_D^{\pm}$ is given by

$$\widehat{E}_D^{\pm}(r, r'; \underline{\xi}) = \int_{\mathbb{R}^3} d^3\underline{x}\, E_D^{\pm}(r, \underline{x}, r', \underline{x}')\, e^{-i\underline{\xi}\cdot(\underline{x}-\underline{x}')},$$

with $\underline{x} = (t, z, s)$ and $\underline{\xi}\cdot\underline{x} := -\omega t + kz + \nu s$. Notice that the two expressions of $E_D^{\pm}$ are translational symmetric in $\underline{x}$ and $\underline{x}'$, *i.e.* they depend on the difference $\underline{x} - \underline{x}'$. Equation (4.8) can be further expanded via an Hankel transform,

$$H_\nu(g)(\lambda) = \int_0^\infty dr\, r\, g(r) J_\nu(\lambda r) \qquad \forall g \in L^2(0, +\infty), \quad \lambda \in \mathbb{R}$$

which is defined in $L^2((0,\infty), \lambda\, d\lambda)$ for any $\nu \geq -1/2$. We can exploit the completness relation, [Wat22],

$$\int_0^\infty d\lambda\, \lambda\, J_\nu(\lambda r) J_\nu(\lambda r') = \frac{\delta(r - r')}{r},$$

and

$$\begin{aligned} H_\nu[P_\nu g](\lambda) &= \int_0^\infty dr\, r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu^2}{r^2} \right) g(r) J_\nu(\lambda r) \\ &= -\lambda^2 H_\nu[g](\lambda), \end{aligned}$$

to write

$$E_D^{\pm}(r, \underline{x}, r', \underline{x}') = \int_{\mathbb{R}^3} d^3\underline{\xi} \int_0^\infty d\lambda\, \lambda\, \widetilde{E}_D^{\pm}(\lambda, \underline{\xi}) J_{|\nu|}(\lambda r) J_{|\nu|}(\lambda r') e^{-i\underline{\xi}\cdot(\underline{x}-\underline{x}')}, \tag{4.9}$$

$\widetilde{E}_D^{\pm}(\lambda, \underline{\xi})$ being

$$\widetilde{E}_D^{\pm}(\lambda, \underline{\xi}) = \frac{1}{\omega^2 - (\lambda^2 + k^2)}. \tag{4.10}$$

The absolute value in the order of the Bessel functions implements Dirichlet boundary condition $E_D^{\pm} = 0$ both at $r = 0$ and $r' = 0$, since for any $\nu > 0$, $J_\nu(0) = 0$. We first integrate (4.9) with respect to the temporal momentum, and we observe that $\widetilde{E}_D^{\pm}(\lambda, \underline{\xi})$ has two poles. The integral can be performed using standard complex analysis techniques. We need thus to fix a suitable of integration in the complex plane, the choice of which leads to different forms for the bisolution to (4.7). Advanced ($t' > t$) and retarded ($t' < t$) operators are defined by the two choices indicated in Figure 4.1 – [Ful89, §4].

We thus have:

$$E_D^{\pm}(r, \underline{x}, r', \underline{x}') = \tag{4.11}$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{4\pi^2} \int_{\mathbb{R}^3} d^3\underline{\xi} \int_0^\infty d\lambda\, \lambda\, \frac{e^{-i\underline{\xi}\cdot(\underline{x}-\underline{x}')}}{(\omega \pm i\varepsilon)^2 - \omega'^2} J_{|\nu|}(\lambda r) J_{|\nu|}(\lambda r')$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} d\nu\, dk \int_0^\infty d\lambda\, \lambda\, \frac{\sin(\omega'(t - t' \mp i\varepsilon))}{\omega'} \times \tag{4.12}$$

$$\times J_{|\nu|}(\lambda r) J_{|\nu|}(\lambda r') e^{-i(k(z-z')+\nu(s-s'))} \tag{4.13}$$

where $\omega'^2 = \lambda^2 + k^2$.

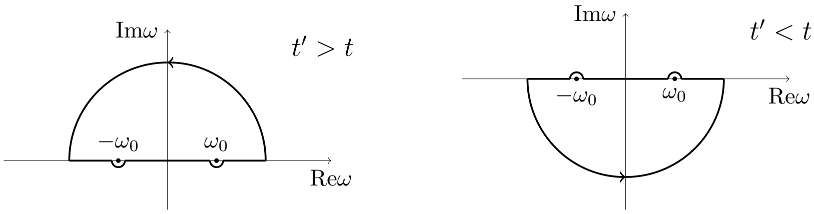

Figure 4.1: Contours of integration; $\omega_0^2 = \lambda^2 + k^2$.

Lemma 4.1.1. *The two integrals,* $(+)$ *and* $(-)$*, in* (4.13) *defines respectively two operators*

$$E_D^{\pm} : \mathcal{C}_0^D(W) \to \mathcal{C}_{sc}^D(W)$$

which enjoy the following properties:

(*i*) $P(E_D^{\pm} f) = E_D^{\pm}(Pf) = f$, *for all* $f \in \mathcal{C}_0^D(W)$,

(*ii*) $\mathrm{supp}(E_D^{\pm}(f)) \subseteq J_{\pm}^{\overline{W}}(\mathrm{supp}(f))$, *for all* $f \in \mathcal{C}_0^D(W)$.

Proof. Property (*i*) is given by construction, by (4.7) and the regularization prescriptions in the integral kernels. Thus, we need to prove that, for all $f \in \mathcal{C}_0^D(W)$, $E_D^{\pm}(f) \in \mathcal{C}^D(W)$. We apply Hörmander's theorem on partial evaluation of bidistributions, [Hör90, Th. 8.2.12]. By the argument in Appendix B, we can write that

$$WF(E_D^{\pm}) = \\ \{(x, x', k_x, -k_{x'}) \in T^*(W \times W) \setminus \{\mathbf{0}\} \mid (x, k_x) \sim_D (x', k_{x'}),\ k_x \text{ is lightlike}\},$$

where $\sim_D$ entails that x and x' are connected via a lightlike geodesic γ, possibly reflected at the boundary, such that k_x is coparallel and cotangent to γ at x and $k_{x'}$ is the parallel transport of k_x from x to x' along γ. Since $WF(E_D^{\pm})$ does not contain any point of the form $(x, x', k_x, 0) \in T^*(W \times W) \setminus \{\mathbf{0}\}$, by [Hör90, Th. 8.2.12] we can conclude that $WF(E_D^{\pm}(f)) = \emptyset$ for any $f \in C_0^{\infty}(W)$, which entails $E_D^{\pm}(f) \in \mathcal{C}^D(W)$. Condition (*ii*) can be inferred by the general theory of the Klein-Gordon operator P on globally hyperbolic spacetimes. By explicit computations – see Appendix B – it turns out that, fixing x', $E_D^{\pm}$ are supported only in $J_{\pm}^{W}(x')$, and the same holds true inverting the role of x and of x'. This proves the support properties in (*ii*). $\square$

Let $\langle \cdot, \cdot \rangle_W$ denote the integral pairing with respect to the measure $d\mu_D = r dr\, dz\, ds\, dt$ on Dowker space. From now on we exploit selfadjointness of the operator P on $\mathcal{C}_0^D(W)$,

$$\langle Pf, f' \rangle_W = \langle f, Pf' \rangle_W, \qquad \text{for any } f,\ f' \in \mathcal{C}_0^D(W),$$

which stems from the Dirichlet boundary conditions in (4.5).

Lemma 4.1.2. *For any $f,\, g \in \mathcal{C}_0^D(W)$, it holds that*

$$\langle E_D^{\pm} f, g\rangle_W = \langle f, E_D^{\mp} g\rangle_W$$

Proof. On account of the properties of $E_D^{\pm}$, in Lemma 4.1.1, we can write

$$\begin{aligned}\langle E_D^{\pm} f, g\rangle_W &= \int_W d\mu_D\, (E_D^{\pm} f)\, P(E_D^{\mp} g)\\ &= \int_W d\mu_D\, P(E_D^{\pm} f)\, (E_D^{\mp} g)\\ &= \langle f, E_D^{\mp} g\rangle_W\end{aligned}$$

where the partial integration in the second equation is justified by the fact that $\mathrm{supp}(E_D^{\pm} f) \cap \mathrm{supp}(E_D^{\mp} g)$ is compact and $E_D^{\pm} f$ and $E_D^{\mp} g$ vanish for $r, r' = 0$, according to Dirichlet boundary conditions. □

The **causal propagator** of the Klein-Gordon operator in (4.5) on Dowker spacetime is $E_D := E_D^+ - E_D^-$, such that

(*i*) $E_D : \mathcal{C}_0^D(W) \to \mathcal{S}_{sc}^D(W)$,

(*ii*) $P(E_D f) = E_D(Pf) = 0$, for all $f \in \mathcal{C}_0^D(W)$.

We are now ready to prove the main result of this section, which provides a first characterization of spacelike compact dynamical configurations.

Proposition 4.1.3. *The sequence of linear maps*

$$0 \longrightarrow \mathcal{C}_0^D(W) \xrightarrow{P} \mathcal{C}_0^D(W) \xrightarrow{E_D} \mathcal{C}_{sc}^D(W) \xrightarrow{P} \mathcal{C}_{sc}^D(W)$$

is exact. The causal propagator E_D induces the following isomorphism,

$$\frac{\mathcal{C}_0^D(W)}{P(\mathcal{C}_0^D(W))} \simeq \mathcal{S}_{sc}^D(W). \tag{4.14}$$

Proof. To prove exactness at the first step, we need to prove that $P : \mathcal{C}_0^D(W) \to \mathcal{C}_0^D(W)$ is injective. Equivalently, we prove that $\mathrm{ker}P = \{0\}$. Consider an $f \in \mathcal{C}_0^D$ such that $Pf = 0$. Then, by the properties of E_D^+, $f = E_D^+(Pf) = E_D^+ 0 = 0$. This proves injectivity of P. Exactness at the second arrow means to prove that $\mathrm{Im}P = \mathrm{Ker}E_D$. Let $f \in \mathcal{C}_0^D(W)$ such that $E_D f = 0$. Thus, $E_D^+ f = E_D^- f := g$. We shall now show that $g \in \mathcal{C}_0^D(W)$. Observe that $g \in \mathcal{C}^D(W)$, by Lemma 4.1.1, and

$$\mathrm{supp}(g) = \big(\mathrm{supp}(E_D^+ f) \cap \mathrm{supp}(E_D^- f)\big) \subset J_+^{\overline{W}}(\mathrm{supp}(f)) \cap J_-^{\overline{W}}(\mathrm{supp}(f)),$$

which is compact everywhere, and vanishing at the boundary ∂W. We can conclude therefore that $g \in \mathcal{C}_0^D(W)$. Thus, it holds that $Pg = P(E_D^{\pm} f) = f$, that means $f \in P(\mathcal{C}_0^D(W))$. Exactness at the third arrow entails that

$\mathrm{Im}E_D = \mathrm{Ker}P$. Consider $u \in \mathcal{C}^D_{sc}(W)$ such that $Pu = 0$. We assume that there exists $K \subset \overline{W}$, compact but at the boundary, such that $\mathrm{supp}(u) \subset I^W_+(K) \cup I^W_-(K)$. We thus define the partition of unity $\chi \in C^\infty(W)$, such that $\chi = 1$ on $I^W_+(K) \cup I^W_-(K) \setminus I^W_-(K)$ while $\chi = 0$ on $I^W_+(K) \cup I^W_-(K) \setminus I^W_+(K)$. Hence it holds $u = \chi u + (1-\chi)u$ and it follows that $P(\chi u) = -P((1-\chi)u) := h$. By construction $\mathrm{supp}(h) \subset J^{\overline{W}}_+(K) \cap J^{\overline{W}}_-(K)$ and we see that $h \in C^\infty_0(W)$. On account of Lemma 4.1.2 and of the self-adjointness of P, we have that for any $\varphi \in \mathcal{C}^D_0(W)$

$$\begin{aligned}\int_W d\mu_D\, \varphi\, E^+_D(P(\chi u)) &= \int_W d\mu_D\, (E^-_D\varphi)\, (P(\chi u))\\ &= \int_W d\mu_D\, P(E^-_D\varphi)\, \chi u\\ &= \int_W d\mu_D\, \varphi\, \chi u,\end{aligned}$$

thus, by Riesz theorem, we can conclude that $E^+_D(h) = \chi u$. Similarly we can prove that $E^-_D(h) = (1-\chi)u$. In conclusion, $E_D h = E^+_D h - E^-_D h = u$ and u is proved to be the image of h. To summarize, we have that

$$0 \longrightarrow \mathcal{C}^D_0(W) \xrightarrow{P} \mathcal{C}^D_0(W) \xrightarrow{E_D} \mathcal{S}^D_{sc}(W) \longrightarrow 0,$$

which yields the isomorphism in (4.14). □

Remark 4.1.1. The proofs of Lemma 4.1.2 and Theorem 4.1.3 follows slavishly those of [BGP07, Lem. 3.4.4] and [BGP07, Th. 3.4.7], with the unique difference that, whenever in the original proof it is used compactness of double cones, here we use the Dirichlet boundary conditions imposed at ∂W.

The previous proposition provides a characterization of spacelike configurations. In order to have control of all smooth solutions we need to prove an extension to timelike compact initial data. It descends from the following proposition.

Proposition 4.1.4. *There exist unique extensions of E^+_D and E^-_D,*

$$\overline{E}^+_D : \mathcal{C}^D_{pc}(W) \to \mathcal{C}^D_{pc}(W), \qquad \overline{E}^-_D : \mathcal{C}^D_{fc}(W) \to \mathcal{C}^D_{fc}(W), \tag{4.15}$$

such that, for all $f \in \mathcal{C}^D_{pc}(W)$,

$$P(\overline{E}^+_D f) = \overline{E}^+_D(Pf) = f, \quad \textit{and} \quad \mathrm{supp}(\overline{E}^+_D f) \subset J^{\overline{W}}_+(\mathrm{supp}(f)),$$

and such that, respectively, for all $f \in \mathcal{C}^D_{fc}(W)$,

$$P(\overline{E}^-_D f) = \overline{E}^-_D(Pf) = f, \quad \textit{and} \quad \mathrm{supp}(\overline{E}^-_D f) \subset J^{\overline{W}}_-(\mathrm{supp}(f)).$$

Proof. The proof proceeds as in [Bär15, Th. 3.8], adapting to the present situation the definition of the pointwise action of $\overline{E}^{\pm}_D$. Fix a point $x \in W$ and define $\chi \in C^\infty_0(\overline{W})$, such that $\chi \equiv 1$ in $J^{\overline{W}}_{\mp}(x) \cap \mathrm{supp}(f)$ and vanishes on the complement. We define

$$(\overline{E}^{\pm}_D f)(x) := (E^{\pm}_D(\chi f))(x).$$

Observe that it is well-defined since $\chi f \in \mathcal{C}^D_0(W)$. The remainder of the proof follows on account of self-adjointness of P. □

The extensions of the advanced and retarded operators yields

$$\overline{E}_D := \overline{E}^+_D - \overline{E}^-_D : \mathcal{C}^D_{tc}(W) \to \mathcal{C}^D(W),$$

such that, for all $f \in \mathcal{C}^D_{tc}(W)$,

$$P(\overline{E}_D f) = \overline{E}_D(Pf) = 0.$$

As a by-product we obtain a full characterization of the space of smooth solutions for (4.5).

Corollary 4.1.5. *The causal propagator $\overline{E}_D$ induces the following isomorphism of topological vector spaces,*

$$\frac{\mathcal{C}^D_{tc}(W)}{P(\mathcal{C}^D_{tc}(W))} \simeq \mathcal{S}^D(W). \tag{4.16}$$

Remark 4.1.2 (Notation). Being interested in the characterization of the observables for all smooth configurations, from now on, with a slight abuse of notation, we will denote the extended propagators $E^{\pm}_D$ and E_D.

4.1.2 Algebraic quantization of a scalar field on Dowker space

In complete analogy with the theory on Minkowski spacetime, we construct the off-shell algebra of observables associated to the (massless) Klein-Gordon scalar field by introducing regular functionals on $\mathcal{C}^D(W)$ as well as suitable product.

Definition 4.1.3. We call **regular functional** on $\mathcal{C}^D(W)$ any $F : \mathcal{C}^D(W) \to \mathbb{C}$ such that for all $k \geq 1$ and for all $u \in \mathcal{C}^D(W)$, $F^{(k)}[u] \in C^\infty_0(W)$ and if only finitely many functional derivatives do not vanish. We indicate this set as $\mathcal{F}^D_0(W)$.

We endow $\mathcal{F}^D_0(W)$ with a $\star$-product (algebra structure), $\star_D : \mathcal{F}^D_0(W) \times \mathcal{F}^D_0(W) \to \mathcal{F}^D_0(W)$, defined by:

$$(F \star_D F')(u) = (\mathcal{M} \circ \exp(i\Gamma_{E_D})(F \otimes F'))(u). \tag{4.17}$$

Here $\mathcal{M}$ stands for the pointwise multiplication, *i.e.*, $\mathcal{M}(F \otimes F')(u) \doteq F(u)F'(u)$, whereas

$$\Gamma_{E_D} \doteq \frac{1}{2} \int_{W \times W} d\mu_D(x)\, d\mu_D(x')\, E_D(x, x') \frac{\delta}{\delta u(x)} \otimes \frac{\delta}{\delta u(x')},$$

where $E_D(x, x')$ is the integral kernel of the causal propagator. The exponential in (4.17) is defined intrinsically in terms of the associated power series and, consequently, we can rewrite the product also as

$$(F \star_D F')\,(u) = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \langle F^{(n)}(u), E_D^{\otimes n}(F'^{(n)})(u) \rangle_W, \tag{4.18}$$

where $\langle , \rangle_W$ stands for the pairing on W built of out integration. The 0-th order is defined as the pointwise multiplication, that is $\langle F^{(0)}(u), F'^{(0)}(u) \rangle \doteq F(u)F'(u)$. In view of the properties of E_D, (4.18) defines a $\star$-product on regular functionals. Endowing $\mathcal{F}_0^D(W)$ with the usual $*$-operation induced by complex conjugation, we can eventually define

Definition 4.1.4. We call $\mathcal{A}^D(W) := \left(\mathcal{F}_0^D(W), \star_D\right)$ **the off-shell $*$-algebra** of a scalar field on Dowker space endowed with complex conjugation as $*$-operation.

The off-shell algebra of observables is generated by linear functionals $F_f : \mathcal{C}^D(W) \to \mathbb{C}$ such that

$$u \mapsto F_f(u) = \int_W d\mu_D\, u(x) f(x) \qquad \forall u \in \mathcal{C}^D(W) \tag{4.19}$$

for any $f \in \mathcal{C}_0^D(W)$ (up to the completion of $\mathcal{C}_0^D(W) \times \mathcal{C}_0^D(W)$ as $\mathcal{C}_0^D(W \times W)$). With the similar techniques mentioned in 1.3.1 for the Minkowskian case, it turns out that:

- $\mathcal{O}^D(W)$ *separates* $\mathcal{C}^D(W)$, that is, for every pair of different configurations $u, u' \in \mathcal{C}^D(W)$, there exists a linear observable $F_f \in \mathcal{A}^D(W)$ such that $F_f(u) \neq F_{f'}(u')$;

- $\mathcal{O}^D(W)$ is *non redundant* $\mathcal{C}^D(W)$, that is for every pair of linear observables $F_f, F_{f'} \in \mathcal{A}^D(W)$, there exists at least one configuration $u \in \mathcal{C}^D(W)$ such that $F_f(u) \neq F_{f'}(u)$.

In order to encode dynamics at the level of observables, we proceed by quotienting the off-shell functionals with the ideal $\mathcal{I}^D(W)$ generated by linear functionals of the form (4.19) with $f = P(h)$, for $h \in \mathcal{C}^D(W)$. The $\star$-product descends to the quotient, as it can be proved at the level of generators: For any $[f]$, $[f'] \in \frac{\mathcal{C}_0^D(W)}{P(\mathcal{C}_0^D(W))}$, consider two representatives, respectively f and f'.

Being $h \in \mathcal{C}_0^D(W)$, for any $u \in \mathcal{S}^D(W)$ we write

$$\begin{aligned}(F_f \star_D F_{f'+Ph})(u) &= F_f(u)F_{f'+Ph}(u) + \frac{i}{2}\langle f, E(f'+Ph)\rangle_W \\ &= F_f(u) \cdot \int_W d\mu_D\, u(x)\,(f'(x) + Ph(x)) + \frac{i}{2}\langle f, Ef'\rangle_W \\ &= F_f(u) \cdot \int_W d\mu_D\, u(x) f'(x) + \frac{i}{2}\langle f, Ef'\rangle_W \\ &= (F_f \star_D F_{f'})(u),\end{aligned}$$

where the third equality holds since P is self-adjoint due to boundary conditions.

We can therefore define, in its full glory, the following

Definition 4.1.5. The **on-shell $*$-algebra** of a scalar field on Dowker space is the algebra

$$\mathcal{A}^D_{on}(W) \doteq \frac{\mathcal{A}^D(W)}{\mathcal{I}^D(W)}.$$

On generators, this yields the following functionals,

$$u \mapsto F_{[f]}(u) = \int_W d\mu_D\, u(x) f(x), \quad \forall u \in \mathcal{C}^D(W), \tag{4.20}$$

for any $f \in \frac{\mathcal{C}_0^D(W)}{P(\mathcal{C}_0^D(W))}$ and for any $f \in [f]$. On account of Corollary 4.1.5, we can redefine them as $F_{[f]} : \frac{\mathcal{C}_{tc}^D(W)}{P(\mathcal{C}_{tc}^D(W))} \to \mathbb{C}$ such that

$$[\alpha] \mapsto F_{[f]}(u) = \int_W d\mu_D\, (E\alpha)(x) f(x), \qquad \forall [\alpha] \in \frac{\mathcal{C}_{tc}^D(W)}{P(\mathcal{C}_{tc}^D(W))} \text{ and } \alpha \in [\alpha]$$

We can introduce the space of **classical observables** $\mathcal{O}^D(W)$, spanned by the functionals (4.20) and endow it with a bilinear form,

$$\sigma_D : \mathcal{O}^D(W) \times \mathcal{O}^D(W) \to \mathbb{R}, \quad \sigma_D(F_{[h]}, F_{[h']}) = \langle f, E_D(f')\rangle_W, \tag{4.21}$$

which is proved to be symplectic analogously to Proposition 1.2.4. It is also true that $\mathcal{O}^D(W)$ *separating and non-redundant* for dynamical configurations, $\mathcal{S}^D(W)$.

Before concluding the analysis of the algebra of observables for the scalar field on Dowker space, we test its structural properties, in order to check its compatibility with the axiomatic framework of local quantum field theory. As in the previous cases, the question arises from the observation that we are working in a different framework then the usual one of Cauchy problems on globally hyperbolic spacetimes. On Dowker spacetime there is a further complication with the timeslice axiom, since we miss in principle the possibility to define a timeslice in terms of restriction of a Cauchy surface of Minkowski spacetime.

Nonetheless we can bypass such obstruction. As a first step, we define a times-lice on Dowker space as an open and geodesically convex neighbourhood of any spacelike hypersurface of codimension one which restricts to Cauchy surface of Minkowski spacetime on the region $\mathbb{R}^4 \subset W$, *i.e.*, for $s \in [0, 2\pi)$.

Proposition 4.1.6. *The algebra $\mathcal{A}^D_{on}(W)$ is causal and fulfils the timeslice axiom.*

Proof. The property of an algebra being causal is a consequence of the support properties of the causal propagator, in particular its action on $\mathcal{C}^D_0(W)$. To check the time-slice axiom we shall adapt the procedure employed in the proof of Lemma 2.1.5 and the proof of Proposition 4.1.3. Let us consider a spacelike hypersurface of codimension one $\widetilde{\Sigma} \subset W$, such that

$$\widetilde{\Sigma} := \{(t, z, r, s) \in W \mid t \equiv t_0,\ \forall t_0 \in \mathbb{R}\},$$

and observe that it restricts to a Cauchy surface for Minkowski for $s \in [0, 2\pi)$. Consider thus $\mathcal{V}$, any neighbourhood of $\widetilde{\Sigma}$ and define a smooth cutoff function $\chi \in C^\infty(\overline{W})$ such that $\chi = 1$ on $J^{\overline{W}}_+(\mathcal{V}) \setminus \mathcal{V}$ and $\chi = 0$ on $J^{\overline{W}}_-(\mathcal{V}) \setminus \mathcal{V}$. Let us consider any $[f] \in \mathcal{O}^D(W)$ and any of its representatives which we indicate with f. Define the new function

$$\widetilde{f} \doteq f - P\left(E^-_D(f) + \chi E_D(f)\right),$$

and notice that, per construction and on account of the support properties of both $E^\pm_D$ and χ, $\widetilde{f} \in \mathcal{C}^D_0(W)$ and it is a representative of $[f]$. □

4.1.3 Hadamard states and regularization

Having constructed the algebra of observables, we focus on states. An algebraic state is than defined, as in Section 1.4, to be any continuous linear functional $\omega : \mathcal{A}^D(W) \to \mathbb{C}$ which is positive and normalized. As discussed in Section 1.4, we must select a class of states which is suitable to describe a physical state. To this avail, in analogy with the Definitions 2.2.1 and 3.2.1, we adapt the notion of Hadamard states to Dowker space.

Definition 4.1.6. A state $\omega : \mathcal{A}^D(W) \to \mathbb{C}$ is of Hadamard form if it is normalized, positive, quasi-free and, if, for any globally hyperbolic region $\mathcal{O} \subset W$, the restriction of ω to $\mathcal{A}^D(\mathcal{O})$ is such that there exists $\omega_2 \in \mathcal{D}'(\mathcal{O} \times \mathcal{O})$ whose wavefront set is

$$WF(\omega_2) = \{(x, x', k_x, -k_{x'}) \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \{\mathbf{0}\} \mid (x, k_x) \sim (x', k_{x'}),\ k_x \triangleright 0\},$$

where $\sim$ entails that x and x' are connected via a lightlike geodesic γ such that k_x is coparallel and cotangent to γ at x and $k_{x'}$ is the parallel transport of k_x from x to x' along γ. The symbol $\triangleright$ entails that k_x is a future pointing covector. We also require that, for all $F_h, F_{h'} \in \mathcal{A}^D(\mathcal{O})$,

$$\omega\left(F_h \star_D F_{h'}\right) = \omega_2(h, h'), \quad h, h' \in C^\infty_0(\mathcal{O}).$$

It turns out straightforwardly that states on $\mathcal{A}^D(W)$ descend to counterparts on the on-shell algebra $\mathcal{A}^D_{on}(W)$, provided that $\omega_2 \in \mathcal{D}'(W \times W)$ is a bisolution of (4.5),

$$\omega_2(f, Pf') = \omega_2(Pf, f') = 0, \qquad \forall f, f' \in \mathcal{C}^D_0(W), \tag{4.22}$$

with the antisymmetric part fixed by the causal propagator,

$$\omega_2(h, h') - \omega_2(h', h) = E_D(h, h'), \qquad \forall f, f' \in \mathcal{C}^D_0(W),$$

where $E_D(x, x')$ is the integral kernel of the causal propagator.

It is not guaranteed in general that a two-point function $\omega_2 \in \mathcal{D}'(\mathcal{O} \times \mathcal{O})$ being of Hadamard form for any $\mathcal{O} \subset W$, extends to a state on $\mathcal{A}^D(W)$. Such issue in particular prevents us from inducing Hadamard states for Dowker system (in the sense of Definition 4.1.6) from Hadamard states on Minkowski (in the sense of Definition 1.4.3), even though $\mathbb{R}^4 \subset W$. In this thesis we do not provide any criterion for the extendibility of local two-point functions of Hadamard. Yet we provide the explicit example of the ground state for a massless scalar field. We construct it directly from the general definition in Section 4.1.1, as the positive frequency bisolution (4.22). We start from

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial z^2}\right)\omega_2(r, \underline{x}, r', \underline{x}') = 0,$$

where we use the same notation on Section 4.1.1, $\underline{x} := (t, z, s)$. By means of separation of variables, we have

$$\omega_2(r, \underline{x}, r', \underline{x}') = \int_{\mathbb{R}^3} d^3\underline{\xi} \int_0^\infty d\lambda\, \lambda\, \widetilde{\omega}_2(\lambda, \underline{\xi}) J_{|\nu|}(\lambda r) J_{|\nu|}(\lambda r') e^{-i\underline{\xi}\cdot(\underline{x}-\underline{x}')},$$

which can be integrated in the complex plane. Selecting positive frequencies is tantamount at choosing a contour of integration as in Figure 4.2 – the two-point function defined by this choice is often dubbed as Wightman function, see [Ful89, §4]. The integration yields,

$$\omega_2(r, \underline{x}, r', \underline{x}') =$$
$$\lim_{\varepsilon \to 0^+} \frac{1}{4\pi^2} \int_{\mathbb{R}^2} d\nu\, dk \int_0^\infty d\lambda\, \lambda\, \frac{e^{-i\omega'(t-t'-i\varepsilon)}}{\omega'} J_{|\nu|}(\lambda r) J_{|\nu|}(\lambda r') e^{-i(k(z-z')+\nu(s-s'))}$$

with $\omega'^2 = \lambda^2 + k^2$, which can be evaluated explicitly – full calculations in Appendix B,

$$\omega_2(x, x') = -\lim_{\varepsilon \to 0^+} \frac{1}{2\pi^2 r r' \sinh \chi_\varepsilon} \frac{\chi_\varepsilon}{\chi_\varepsilon^2 + \theta^2}, \tag{4.23}$$

where, by means of analytic continuation,

$$2rr' \cosh \chi_\varepsilon \doteq \left(-(t - t' - i\varepsilon)^2 + r^2 + r'^2 + (z - z')^2\right). \tag{4.24}$$

These computations agree with [FTTW12] and are consistent with the conical case treated in [Dow77]. We recall that the state defined by (4.23) is a ground state thanks to the argument in [SV00]. In addition, by the argument reported in Appendix B, we can reasonably conjecture that it is of Hadamard form in the sense of Definition 4.1.6.

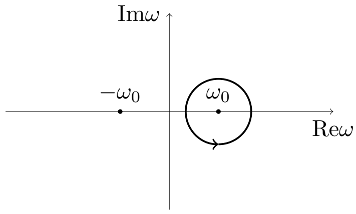

Figure 4.2: Contour of integration; $\omega_0^2 = \lambda^2 + k^2$.

Remark 4.1.3. As pointed out in [FTTW12], it is possible to obtain the Dowker two-point function in an alternative way. We find (the integral kernel of) the Fourier expansion in θ of the two point function of the Poincaré vacuum (3.13), in cylindrical coordinates:

$$\omega_2^0(x,x') = \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi)^2}\frac{1}{2rr'}\frac{1}{\cosh\chi_\varepsilon - \cos(\theta-\theta')} \tag{4.25}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi)^2}\frac{1}{2rr'\sinh\chi_\varepsilon}\sum_{n=-\infty}^{+\infty} e^{-\chi_\varepsilon|n|}e^{i\theta n}, \tag{4.26}$$

where χ_ε is defined as in (4.24). It holds the identity

$$\frac{\sinh\chi_\varepsilon}{\cosh\chi_\varepsilon - \cos\theta} = \sum_{n=-\infty}^{+\infty} e^{-\chi_\varepsilon|n|-in\theta}$$

being a geometric series. The two-point function (4.23) agrees with (4.25) provided that we substitute summation (in n) with an integration in $d\nu$ over $\mathbb{R}$. Apparently this seems to be a mere *escamotage*, but it is motivated by the definition of gruond state to be invariant under all isometries and by the observation that by the extension of the angular coordinate we are promoting a rotational degree of freedom (related to the Fourier *series*) to a translational one (related to the Fourier *transform*). Furthermore, this simple approach could suggest a general way to induce states from Minkowski spacetime to states for Dowker. Yet, even though the extension of modes does preserve the Hadamard form of any restriction of the two point function of globally hyperbolic regions of W, one should already prove that the states extends to the whole algebra $\mathcal{A}^D(W)$. This aspect should be object of further investigations.

4.2 Method of images and the wedge-shaped geometry

Having at hand the algebra of observables for the massless scalar field on Dowker space, we can apply the method of images to construct the quantum theory of wedge-shaped Casimir system. In complete analogy with the case of Casimir system, we call

$$N^{wed}: \mathcal{C}_0^D(W) \to \mathcal{C}^D(W)$$
$$N^{wed}(f)(\underline{z}, s) = \sum_{n=-\infty}^{\infty} \left(f(\underline{z}, s + 2n\alpha) - f(\underline{z}, -s + 2n\alpha)\right),$$

which is the counterpart of N defined by (3.4). Accordingly it is surjective on a subspace of $\mathcal{C}^D(W)$, of functions 2α-periodic in s. Actually we can adapt the entire construction of Chapter 3 to the present case just by replacing all spaces of functions on $\mathbb{R}^4$ as well as then support prescriptions with the respective counterparts on W:

$$C^\infty(\mathbb{R}^4) \mapsto \mathcal{C}^D(W), \qquad C_0^\infty(\mathbb{R}^4) \mapsto \mathcal{C}_0^D(W), \tag{4.27}$$
$$C_{sc}^\infty(\mathbb{R}^4) \mapsto \mathcal{C}_{sc}^D(W), \qquad C_{tc}^\infty(\mathbb{R}^4) \mapsto \mathcal{C}_{tc}^D(W). \tag{4.28}$$

In this way we can construct off-shell and on-shell algebra of observables, $\mathcal{A}^{wed}(W_\alpha)$ and $\mathcal{A}^{wed}_{on}(W_\alpha)$, define Hadamard states with the distinguished class of image states and eventually discuss Wick polynomials and regularization of stress energy tensor. We do not present the entire construction, since it would follow slavishly from Section 3.1.

As already discussed, it is reasonable to require consistency between any localization of $\mathcal{A}^{wed}(W_\alpha)$ and $\mathcal{A}^{KG}(\mathbb{R}^4)$ in globally hyperbolic open regions of W_α. Proposition 3.1.4 can be proved equivalently for wedge Casimir systems, but it does not represent the sought property, since we aim at relating such systems with their physical embedding space, that is Minkowski spacetime. We thus need to prove the following

Proposition 4.2.1. *For any globally hyperbolic open region $\mathcal{O} \subset W_\alpha$, there exists a $*$-isomorphism implemented by the identity map between $\mathcal{A}^{KG}(\mathcal{O})$ and $\mathcal{A}^{wed}(\mathcal{O}) \doteq \mathcal{A}^{wed}(W_\alpha)\big|_{\mathcal{O}}$.*

Proof. We show first that the following chain of isomorphisms is implemented by the identity map at any step, for any globally hyperbolic $\mathcal{O} \subset W_\alpha$,

$$\mathcal{A}^{wed}(W_\alpha)\big|_{\mathcal{O}} \simeq \mathcal{A}^D(\mathcal{O}) \simeq \mathcal{A}^{wed}(W_\pi)\big|_{\mathcal{O}} \simeq \mathcal{A}^{CP}(\mathcal{O}). \tag{4.29}$$

The statement, thus, follows by Proposition 2.1.5. The proof of the first isomorphism is analogous to Proposition 3.1.4, *mutatis mutandis*, as well as the second one, since $\mathcal{O} \subset W_\alpha \subset W_\pi$. The last isomorphism follows straightforwardly by the observing that $\mathcal{A}^{wed}(W_\pi) \simeq \mathcal{A}^{CP}(\mathbb{H}^4)$, since they are generated by the same labelling space and the two causal propagators $E_{\mathbb{H}^4}$ and E_{wed} coincide, as can be directly checked by integral kernels. □

Turning our attention to states, we shall give a definition of Hadamard states which adapts to the wedge-shaped geometry, combining Definitions 4.1.6

and 3.2.1. We can use the method of images to induces explicit examples, adapting to the present case the analysis of Section 3.2 and in particular Proposition 3.2.3. In analogy with Casimir systems, the method of images induces states defined on a $*$-subalgebra of observables localized out of the boundary, *i.e.*, on $\mathring{W}_\alpha$.

In order to build a connection with the existing literature on the Casimir effect on wedge geometries, we report the example of the ground state for the massless case. The application of the method of images to the ground state for Dowker space, (4.23), yields the following,

$$\omega_2(x,x') = -\lim_{\varepsilon\to 0^+} \frac{1}{2\pi\alpha r r' \sinh\chi_\varepsilon} \times \\ \times \left(\frac{\sinh(\frac{2\pi}{\alpha}\chi_\varepsilon)}{\cosh(\frac{2\pi}{\alpha}\chi_\varepsilon) - \cos\frac{2\pi}{\alpha}(\theta-\theta')} - \frac{\sinh(\frac{2\pi}{\alpha}\chi_\varepsilon)}{\cosh(\frac{2\pi}{\alpha}\chi_\varepsilon) - \cos\frac{2\pi}{\alpha}(\theta+\theta')} \right), \tag{4.30}$$

recalling that $2rr'\cosh\chi = -(t-t'-i0)^2 + (z-z')^2 + r^2 + r'^2$. Formula (4.30) is the integral kernel of $\omega_2 \in \mathcal{D}'(\mathring{W}_\alpha \times \mathring{W}_\alpha)$, the two-point function of a state on $\mathcal{A}^{wed}_{on}(\mathring{W}_\alpha)$. This state is of Hadamard form, as a consequence of what conjectured in Proposition 3.2.5. It is also a ground state, since the method of images does not modify positivity of frequencies in (4.23). Equation (4.30) agrees with previous results, in particular [DC79, FTTW12, FP14c].

Remark 4.2.1. As it has been discussed in Section 4.1.3, we have not a criterion for constructing Hadamard states for Klein-Gordon field on Dowker spacetime. Moreover, in an operative perspective, we rather ought to induce states for a wedge-shaped Casimir system from states for Klein-Gordon field on Minkowski spacetime. At this regard, Remark 4.1.3 offers an interesting suggestion for a two-step procedure hitting the target. Yet, untill now, this is still in the realm of congectures.

4.2.1 Extended algebra and stress-energy tensor

Having a notion of Hadamard states, we aim at defining an extended algebra of Wick polynomials. For any globally hyperbolic region $\mathcal{O} \subset W_\alpha$ an algebra of extended observables, $\mathcal{A}^{wed}_\mu(\mathcal{O})$, is given and it is $*$-isomorphic to $\mathcal{A}^{KG}_\mu(\mathcal{O})$, analogously to Casimir-Polder and Casimir systems. Accordingly, the extended algebras $\mathcal{A}^{wed}_\mu(\mathcal{O})$ can be realized as part of a global extended algebra $\mathcal{A}^{wed}_\mu(W_\alpha)$ only after a suitable deformation, which yields a different definition of the regularized Wick polynomials. Notwithstanding, we are interested to local extensions, in order to preserve the usual regularization and make contact with standard results in literature. In particular, we make sense of the stress-energy tensor observable, bridging the gap with the previous analyses of the Casimir effect.

The classical stress-energy tensor for a massless scalar field in a wedge-shaped region of Minkowski spacetime (coupled by ξ with the scalar curvature

R) is

$$T_{\mu\nu}(x) = \partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}g_{\mu\nu}(x)\partial_\rho\phi(x)\partial^\rho\phi(x) + \xi\left(g_{\mu\nu}(x)\nabla_\rho\partial^\rho - \nabla_\mu\partial_\nu\right)\phi^2(x),$$

where the covariant derivatives are induced by the cylindrical metric (4.2), such that,

$$\nabla_t\partial_t = \partial_t^2, \quad \nabla_z\partial_z = \partial_z^2, \quad \nabla_r\partial_r = \partial_r^2,$$
$$\nabla_\theta\partial_\theta = \partial_\theta^2 + r\partial_r, \quad \nabla_r\partial_\theta = \nabla_\theta\partial_r = \partial_r\partial_\theta - \frac{1}{r}\partial_\theta.$$

After a long and tedious computation, we have the following,

Proposition 4.2.2. *Let us consider a massless, real scalar field and let $\omega^0 : \mathcal{A}^{wed}(\mathring{W}_\alpha) \to \mathbb{C}$ be the quasi-free state whose two-point function is given by the integral kernel* (4.30) *and it is built with the image method from the Poincaré vacuum. Then, the expectation value of the unsmeared squared field turns out to be*

$$\omega^0(:\phi^2:_H(x)) = \frac{1}{48\alpha^2 r^2}\left(1 - \frac{\alpha^2}{\pi^2}\right) - \frac{1}{16\alpha^2 r^2 \sin^2\left(\frac{\pi\theta}{\alpha}\right)}, \tag{4.31}$$

where $x = (t, z, r, \theta)$, and the expectation values of the unsmeared components of the stress-energy tensor turn out,

$$\begin{aligned}
\omega^0(:T_{tt}:_H(x)) &= -\omega^0(:T_{zz}:_H(x)) = -\frac{1}{1440\alpha^2 r^4}\left(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2}\right) \\
&\quad + \frac{\left(\xi - \frac{1}{6}\right)}{32\alpha^2 r^4}\left\{\frac{8}{3}\left(1 - \frac{\alpha^2}{\pi^2}\right) - \frac{12}{\sin^2(\frac{\pi\theta}{\alpha})}\left[\frac{\frac{\pi^2}{\alpha^2}}{\sin^2(\frac{\pi\theta}{\alpha})} - \frac{2}{3}\frac{\pi^2}{\alpha^2} + \frac{2}{3}\right]\right\}, \\
\omega^0(:T_{rr}:_H(x)) &= \frac{1}{1440\alpha^2 r^4}\left(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2}\right) \\
&\quad + \frac{\left(\xi - \frac{1}{6}\right)}{32\alpha^2 r^4}\left\{\frac{4}{3}\left(1 - \frac{\alpha^2}{\pi^2}\right) + \frac{12}{\sin^2\left(\frac{\pi\theta}{\alpha}\right)}\left[\frac{\frac{\pi^2}{\alpha^2}}{\sin^2\left(\frac{\pi\theta}{\alpha}\right)} - \frac{2}{3}\frac{\pi^2}{\alpha^2} - \frac{1}{3}\right]\right\}, \\
\omega^0(:T_{r\theta}:_H(x)) &= \omega^0(:T_{\theta r}:_H(x)) = -\frac{\left(\xi - \frac{1}{6}\right)}{8\alpha^2 r^2}\frac{3}{\sin\left(\frac{\pi\theta}{\alpha}\right)}\frac{d}{d\theta}\frac{1}{\sin\left(\frac{\pi\theta}{\alpha}\right)}, \\
\omega^0(:T_{\theta\theta}:_H(x)) &= -\frac{3}{1440\alpha^2 r^4}\left(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2}\right) - \frac{\left(\xi - \frac{1}{6}\right)}{8\alpha^2 r^2}\left(1 - \frac{\alpha^2}{\pi^2} - \frac{3}{\sin^2\left(\frac{\pi\theta}{\alpha}\right)}\right).
\end{aligned}$$

These results agree with those one derived by Saharian and Tarloyan by means of the point-splitting regularization in [ST05]. It agrees also with former results for the case of conformal coupling $\xi = \frac{1}{6}$, in [DC79, BD84, BL96]. A computation of the general coupling limited to the tt-component is also available in [RS02]. A last remarkable result is that of [FP14c, NLS02], which has the same result for the full tensor with generic coupling in a different regularization scheme, namely that of the local ζ-function.

Let us comment about the structure of the rather complicated stress energy tensor.

1. The tensor contains a term which depends on the arc length $r\alpha$ and which is analogous to the conformally coupled Casimir stress-energy tensor of Lemma 3.2.7:
$$\frac{\pi^2}{1440(\alpha r)^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

2. A term depending on r^{-4} has the same tensorial structure of the previous one. Such dependence on the radial coordinate is not surprising, on account of the underlying geometry. In particular we observe that for very small values of the angle parameter α the term depending on the arc length $r\alpha$ is leading, which correspond to the situation in which the wedge-shaped region "looks like" the geometry of two parallel plates.

3. Any dependence on the angular coordinate is cancelled by the introduction of the conformal coupling ($\xi = \frac{1}{6}$), in complete analogy with the previous situations. In particular the usual divergences at the boundary occur, at $\theta = 0$ and $\theta = \alpha$. For comments on those infinities we refer to comments at the end of Chapter 2 and 3

4. Off diagonal terms occurs, mixing the radial and the angular components. They could be ascribed to pressures contributions along the arc coordinate.

5. As in the the case of parallel plates, if we consider the conformal coupling $\xi = \frac{1}{6}$ we are able to compute the Casimir effect. In this case we will deal with a torque rather than a force.

Appendix A

The wavefront set

To carry out a singularity analysis, there exist two approaches in the literature. One consists of looking at the concrete realization of two-point function, dealing hence with the construction of a *local Hadamard parametrix*. That is an explicit expression for the ultraviolet behavior of $\omega_2^{\mathrm{vac}}(x, y)$ which must hold true for every state. Although this method is well-suited for actual calculations, it is quite unwieldy and leaves no clear evidence of conceptual problems. The other way is rather abstract and it is based on *microlocal analysis*. The motivating idea of this subject is that the decay properties of the Fourier transform of a distribution are related to its smoothness. Even if in general a global Fourier transform is not defined if the spacetime metric g is not covariant under the action of some translation, one may point-wisely restricts to the tangent space, which is isomorphic to $\mathbb{R}^4$. Turning back to the manifold is a matter of finding a suitable local chart which defines a local isomorphism between open neighborhoods, like the exponential map. The reader interested in a more comprehensive treating of microlocal analysis might not ignore the capital reference of Hormander, [Hör90, §8]. Two examples are introduced to clarify the interaction between the Fourier transform and the underling singularities.

1. Consider $u \in C_0^\infty(\mathbb{R}^n)$ and write

   $$\big(1 + |k|^{2m}\big)|\hat{f}(k)| = \big|\big[1 + \widehat{(-\Delta)^m f}\big](k)\big| \leq \int d^n x \big|\big[1 + (-\Delta)^m f\big](x)\big| < \infty$$

   In particular for each $N \in \mathbb{N}$, there exists a constant C_N such that

   $$|\hat{f}(k)| \leq \frac{C_N}{1 + |k|^N} \qquad \text{as } |k| \to \infty \qquad \forall k$$

   which implies that the Fourier transform is rapidly decaying for $|k| \to \infty$.

2. Consider the Dirac δ-distribution. After testing with an arbitrary $f \in C_0^\infty(\mathbb{R}^n)$, it turns out

   $$\hat{\delta}(f) = \int d^n x \int \frac{d^n k}{(2\pi)^{\frac{n}{2}}} e^{ikx} \delta(x) f(k) = \int \frac{d^n k}{(2\pi)^{\frac{n}{2}}} f(k)$$

which implies that $\hat{\delta} = 1$ which exhibits no decay at ∞.

These two examples show the connection between smoothness and Fourier transform decay properties, cited above. Such observation suggests the definition of a regular direction. Before introducing it its definition, the following ancillary notion is needed:

Definition A.0.1 (Conic Neighborhood)**.** A neighborhood $\mathcal{O}$ of $k_0 \in \mathbb{R}^n$ is called conic *if* $k \in \mathcal{O}$ *implies* $\lambda k \in \mathcal{O}$ *for all* $\lambda \in (0, \infty)$.

Definition A.0.2 (Regular Direction)**.** Given $u \in \mathcal{D}'(\mathbb{R}^n)$, a pair $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is a regular direction *for* u *if there exists* $\phi \in C_0^\infty(\mathbb{R}^n)$ *with* $\phi(x) \neq 0$, *a conic neighborhood* V *of* k *and constants* C_N,$N \in \mathbb{N}$ *such that*

$$|\widehat{\phi u}(k)| < \frac{C_N}{1 + |k|^N} \qquad \forall\, k \in V, N \in \mathbb{N}$$

$\widehat{\phi u}$ *is said to be rapidly decreasing as* $|k| \to \infty$

The main object of microlocal analysis is the *wavefront set*. The idea of the wave front set is to investigate both the points and the directions in which a distribution takes singular values.

Definition A.0.3 (Wavefront Set)**.** Given a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, the wavefront set *of* u *is defined to be*

$$WF(u) \doteq \big\{(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid (x, k) \,\text{is not a regular direction for}\, u\big\}$$

Glancing at the previous examples may be explicative.

1. Being $u \in C_0^\infty(\mathbb{R}^n)$, every pair $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ must be regular, so that
$$WF(u) = \emptyset$$
2. Note that $\forall \phi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \notin supp(\phi)$, $\langle \delta, \phi \rangle = \phi(0)$ so (x, k) is a regular direction $\forall x \neq 0$; otherwise $\langle \delta, \phi \rangle = \phi(0)$, the Fourier transform is never rapidly decreasing. According to the definition:
$$WF(\delta) = \big\{(0, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid k \neq 0\big\}$$

The above definitions have to be extended to manifolds. To this avail it is fundamental to make precise how the wavefront set transforms under a diffeomorphism.

Theorem A.0.3 (Transformation Properties under Diffeomorphism)**.** *Given* V *and* $U \subset \mathbb{R}^n$, *a distribution* $u \in \mathcal{D}'(V)$ *and a diffeomorphism* $\varphi\colon V \to U$, *define* $\varphi^* u$ *by* $(\varphi^* u)(f) \doteq u(f \circ \varphi^{-1})$. *Then*

$$WF(\varphi_* u) = \varphi^* WF(u) \doteq \big\{(\varphi(x), \varphi^* k) \mid (x, k) \in WF(u)\big\}$$

Thus under coordinate changes, the wavefront set transforms as a subset of the cotangent bundle. Having characterized its geometrical nature it is now possible to extend it to distributions on general curved manifolds M by patching together wavefront sets in different coordinate maps of M:

$$WF(u) = \bigcup_{\mathcal{O}} WF(u|_{\mathcal{O}})$$

and, as a consequence, given a distribution $u \in \mathcal{D}'(M)$, $WF(u) \subset T^*M \setminus 0$.

The great advantages, one gains in using wavefront set, are due to its properties.

Theorem A.0.4 (Wavefront set properties-I). *Given a distribution* $u \in \mathcal{D}'(\mathbb{R}^n)$

(i) if u *is smooth, it holds* $WF(u) = \emptyset$

(ii) for every $u, v \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\forall \alpha, \beta \in \mathbb{C}$ *it holds*

$$WF(\alpha u + \beta v) \subseteq WF(u) \cup WF(v)$$

The importance of these first two properties is clear if one recalls the first regularity requirement carried out in Minkowski case analysis: The difference $\omega_2 - \omega_2^{\text{vac}}$ must be smooth for every ω_2 in order to take a well-defined coincidence limit. In the light of *(i)* it must be

$$WF(\omega_2 - \omega_2^{\text{vac}}) = \emptyset \tag{A.1}$$

The *(ii)* tells that it is possible to control wavefront set of such a subtraction.

The following properties of WF give a precise characterization of the singularity structure only in terms of field equations and the surrounding spacetime geometry. Few definitions are needed

Definition A.0.4 (Principal Symbol and Characteristic Set). Given $m \in \mathbb{N} \setminus \{0\}$, α a multiindex and $D^\alpha \doteq \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} \ldots \partial^{\alpha_k}}$, let $P = \sum_{|\alpha|=m} a_\alpha(x)(iD)^\alpha$ be an m-order partial differential operator, the principal symbol *is*

$$p_m(x, k) \doteq \sum_{|\alpha|=m} a_\alpha(x) k^\alpha$$

and the characteristic set *is*

$$\text{Char}\, P = \big\{(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \mid p_m(x, k) = 0\big\}$$

The following holds true:

Theorem A.0.5 (Wavefront set properties-II). *Given a distribution* $u \in \mathcal{D}'(\mathbb{R}^n)$

(iii) Let P be a differential operator on a manifold M. It holds

$$WF(Pu) \subseteq WF(u) \subseteq WF(Pu) \cup \mathrm{Char}\, P$$

(iv) Given a differential operator P on a manifold M, with real valued principal symbol p_m, and $f \in C^\infty(M)$, let $Pu = f$. Then $WF(u) \setminus WF(f)$ is invariant under the local hamiltonian flow generated by p_m on $T^*M \setminus WF(f)$.

Properties *(iii)* and *(iv)* correlate wavefront set of a solution of a certain differential equation with the properties of the operator itself. What is of great interest is the fact that in particular one gets the capability to derive singular structure of solutions by only looking at the principal symbol of the differential operator and at the integral curves of an Hamiltonian flow. To understand the importance of this theorem it is convenient to apply these last properties directly to the case of interest, i.e. Klein-Gordon.

Let P be the Klein-Gordon operator in (1.2) for the massless field. Its principal symbol is

$$p_2(x,k) = -g^{\mu\nu}k_\mu k_\nu$$

and so the characteristic set is

$$\mathrm{Char}\, P = \mathcal{N}_0$$

which is a bundle of non-zero null covectors in T^*_xM, $\mathcal{N}_0 \doteq \{(x,y) \in T^*M \mid g^{\mu\nu}k_\mu k_\nu = 0\}$. Hence the wavefront set of any distributional solution to $Pu = 0$ obeys

$$WF(u) \subseteq \mathcal{N}_0$$

Moreover, $WF(u)$ is invariant under the Hamiltonian flow $\lambda \mapsto \big(x(\lambda), k(\lambda)\big) \in T^*M$ given by $p_2(x,k)$, namely

$$\begin{cases} \dot{x}^\mu = -g^{\mu\nu}k_\nu & x(0) = p \\ \dot{k}_\rho = -\nabla_\rho g^{\mu\nu}k_\mu k_\nu = 0 & k(0) = \overline{k} \end{cases}$$

which entails that $\dot{x}^\mu$ is parallel transported along the curve $\lambda \mapsto x(\lambda)$ and that $\dot{k}_\rho$ is cotangent to this curve. Thus if $(x,k) \in WF(u)$, the wavefront set contains every point $\big(x(\lambda), k(\lambda)\big)$ for $\lambda \in \mathbb{R}$, where $x(\lambda)$ is the null geodesic through x with $g^{\mu\nu}k_\nu$ as tangent vector and $k(\lambda)$ is the parallel transport of k along $x(\lambda)$.

Since the present interest lies in the analysis of two-point function, it is correct to consider bisolutions to Klein-Gordon equation. Bisolutions are as follows: $G \in \mathcal{D}'(M \times M)$ such that

$$P_xG(x,y) = P_yG(x,y) = 0$$

The operator P_x has principal symbol

$$p_2(x,k;x',k') = -g^{\mu\nu}(x)k_\mu k_\nu$$

and characteristic set

$$\mathrm{Char}\, P_x = \mathcal{N}_0 \times T^*M \setminus \{0\}$$

where $\mathcal{N}_0$ coincides with $\mathcal{N}$, though without the request of non-zero covectors and $T^*M \setminus \{0\}$ is the cotangent bundle of M with the zero section deleted. Similarly, $\mathbb{I} \otimes P$ has the principal symbol

$$p_2'(x, k; x', k') = -g^{\mu\nu}(x')k_\mu' k_\nu'$$

and characteristic set

$$\mathbb{I} \otimes \mathrm{Char}\, P = T^*M \setminus \{0\} \times \mathcal{N}_0 \tag{A.2}$$

The bisolution G therefore has wavefront set with upper buond

$$WF(G) \subseteq \big(\mathcal{N}_0 \times T^*M \setminus \{0\}\big) \cap \big(T^*M \setminus \{0\} \times \mathcal{N}_0\big) \subseteq \mathcal{N}_0 \times \mathcal{N}_0 \tag{A.3}$$

Appendix B

Computation of Green operators on Dower spacetime

In this appendix we aim at providing explicit computations for the following results:

1. A closed form for the integral kernel of the advanced and retarded operators on Dowker spacetime, $E^{\pm}_{D}(x,x')$;
2. A closed form for the integral kernel of the two-point function of the ground state on Dowker spacetime, $\omega^0_2(x,x')$;
3. Make considerations on the wavefront sets of the two mentioned bidistributions.

In the first two computations, we follow [FTTW12].

Let us start with the computation for advanced and retarded operators. The starting point is (4.9), which we repeat conveniently here,

$$\lim_{\varepsilon\to 0^+}\frac{1}{4\pi^2}\int_{\mathbb{R}^2} d\nu\, dk\int_0^\infty d\lambda\,\lambda\,\frac{\sin(\omega'(t\mp i\varepsilon))}{\omega'}J_{|\nu|}(\lambda r)J_{|\nu|}(\lambda r')e^{-i(k(z)+\nu(s))},$$

where $\omega'^2=\lambda^2+k^2$ and we set $t'=z'=s'=0$ for notational convenience. The sine can be suitably rewritten such that

$$\operatorname{Im}\lim_{\varepsilon\to 0^+}\frac{1}{8\pi^2}\int_{\mathbb{R}^2} d\nu\, dk\int_0^\infty d\lambda\,\lambda\,\frac{e^{-i\omega'(t-t'\mp i\varepsilon)}}{\omega'}J_{|\nu|}(\lambda r)J_{|\nu|}(\lambda r')e^{-i(k(z-z')+\nu(s-s'))},$$

where the imaginary part and the limit cannot be exchanged, due to the distributional nature of the integral kernel. Integration in dk can be performed on account of [GR07, 3.961.2], reading

$$\int_{-\infty}^{\infty} dk\frac{e^{-i\sqrt{\lambda^2+k^2}(t-t'\mp i\varepsilon)}}{\sqrt{\lambda^2+k^2}}e^{ikz}=\lim_{\varepsilon\to 0^+}\operatorname{Im}K_0(\lambda\sqrt{-(t-i\varepsilon)^2+z^2}).\qquad\text{(B.1)}$$

We then focus on integration in $d\lambda$, which can be computed by means of [GR07, 6.522.3],

$$\int_0^\infty d\lambda\,\lambda\, J_{|\nu|}(\lambda r) J_{|\nu|}(\lambda r') K_0(\lambda\zeta) = \frac{1}{r_1 r_2}\left(\frac{r_2 - r_1}{r_2 + r_1}\right)^{|\nu|}, \tag{B.2}$$

where $\zeta_\varepsilon := \sqrt{-(t - i\varepsilon)^2 + z^2}$,

$$r_1 \equiv \sqrt{(r - r')^2 + \zeta_\varepsilon^2} \quad \text{and} \quad r_2 \equiv \sqrt{(r + r')^2 + \zeta_\varepsilon^2}. \tag{B.3}$$

Finally, to perform the last integration, we define (at least by analytic continuation)

$$\frac{r_2 - r_1}{r_2 + r_1} \equiv e^{-\chi_\varepsilon},$$

such that the integral becomes

$$\begin{aligned} E_D^\pm(x, x') &= \operatorname{Im} \lim_{\varepsilon \to 0^+} -\frac{1}{4\pi^2 r' r \sinh \chi_\varepsilon} \int_{-\infty}^{\infty} d\nu\, e^{-|\nu|\chi_\varepsilon + i\nu(s - s')} \\ &= \operatorname{Im} \lim_{\varepsilon \to 0^+} -\frac{1}{4\pi^2 r' r \sinh \chi_\varepsilon} \frac{2\chi_\varepsilon}{(s - s')^2 + \chi_\varepsilon^2}\,. \end{aligned} \tag{B.4}$$

The computation for the two-point function $\omega_2^0(x, x')$ follows exactly from above by dropping the Im operation (this is actually the computation performed in [FTTW12]).

From (B.3) we see that all null-like separated pairs $x, x' \in W$ are in the singular support of $E_D^\pm$ and ω_2. By direct comparison with the case on Minkowski spacetime and by checking the decaying properties of the Fourier transform in s, (B.4), we can conjecture that the wavefront set is given by $WF(E_\pm)$ and $WF(\omega_2^0)$ on Minkowski by adding all light rays reflected at the boundary.

Acknowledgements

First of all I would like to thank my supervisor Claudio Dappiaggi. Without his guidance and support this work would have never be possible. I am particularly grateful to him for being always present as a mentor and as a friend.

My thanks goes also to the University of Pavia, which provided me the possibility to carry my Ph.D. program.

I am grateful with the University of York for hosting my activity for six months (September 2015 – March 2016). I am especially grateful to Chris Fewster for his collaboration, which enriched me with many enlightening discussions, comments and a precious suggestion.

I owe Nicola Pinamonti my thankfulness for his collaboration and in particular for his support. It would not be inappropriate to say that he has been for me like a second supervisor.

I would like to thank Karl-Henning Rehren and Gerardo Morsella for agreeing to be the referees of my thesis. Their careful reading has been fundamental for bringing this work to its final stage. My gratitude goes also to Livio Pizzocchero for his interest in my work.

Furthermore, I would like to thank my friends and colleagues Simone Murro, Nicolò Drago, Samuel Rutili, Hugo Ferreira, Francesco Bussola, Federico Faldino, Marco Benini and a number of other people from the LQP community, for their friendship, which has brought an infinite amount of inspiring conversations, amazing journeys and other *memorabilia* (and beer, nonetheless).

A crowd of people I should now thank, whose contribution goes far beyond the limited experience of this work. I hope to be able to thank properly at least some of them, somewhere else, somehow, sooner or later. Maybe some page later.

To conclude, my special gratitude goes to my family, who always support me and who have always the right advise, even when they do not understand what I am doing.

Bibliography

[Bär15] C. Bär, *Green-hyperbolic operators on globally hyperbolic spacetimes*, Comm. Math. Phys. , **333.3** (2015), 1585-1615.

[BD15] M. Benini and C. Dappiaggi *"Models of Free Quantum Field Theories on Curved Backgrounds"*, in *"Advances in Algebraic Quantum Field Theory"* (pp. 31-74), Springer (2015).

[BD84] N. D. Birrell and P. C. W. Davies, *"Quantum fields in curved space,"* Cambridge university press (1984).

[BDF87] D. Buchholz, C. D'Antoni and K. Fredenhagen, *"The universal structure of local algebras,"* Comm. Math. Phys. **111**, 1, (1987).

[BDF09] R. Brunetti, M. Duetsch and K. Fredenhagen, *"Perturbative Algebraic Quantum Field Theory and the Renormalization Groups,"* Adv. Theor. Math. Phys. **13** (2009) 1541 [arXiv:0901.2038 [math-ph]].

[BDFY15] R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason, *"Advances in algebraic quantum field theory,"* Springer (2015).

[BDH13] M. Benini, C. Dappiaggi and T. -P. Hack, *"Quantum Field Theory on Curved Backgrounds – A Primer"*, Int. J. Mod. Phys. A **28** 1330023 (2013) [arXiv:1306.0527 [gr-qc]].

[BDM14] M. Benini, C. Dappiaggi and S. Murro, *"Radiative observables for linearized gravity on asymptotically flat spacetimes and their boundary induced states"*, J. Math. Phys. **55**, 082301 (2014), arXiv:1404.4551 [gr-qc].

[BDS12] M. Benini, C. Dappiaggi and A. Schenkel, *"Quantum field theory on affine bundles"*, Annales Henri Poincare **15** 171 (2014) [arXiv:1210.3457 [math-ph]].

[BDS13] M. Benini, C. Dappiaggi and A. Schenkel, *"Quantized Abelian principal connections on Lorentzian manifolds"*, Commun. Math. Phys. **330** 123 (2014) [arXiv:1303.2515 [math-ph]].

[Ben16] M. Benini, *"Optimal space of linear classical observables for Maxwell k-forms via spacelike and timelike compact de Rham cohomologies"*, Jour. Math. Phys. , **57.5** (2016) 053502, arXiv:1401.7563 [math-ph].

[BF09] R. Brunetti and K. Fredenhagen, *"Quantum Field Theory on Curved Backgrounds,"* arXiv:0901.2063 [gr-qc].

[BFM09] R. Brunetti, L. Franceschini and V. Moretti, *"Topological features of massive bosons on two dimensional Einstein space-time. I: Spatial approach,"* Annales Henri Poincare **10** (2009) 1027 doi:10.1007/s00023-009-0007-x [arXiv:0812.0533 [gr-qc]].

[BFR12] R. Brunetti, K. Fredenhagen and P. L. Ribeiro, *"Algebraic Structure of Classical Field Theory I: Kinematics and Linearized Dynamics for Real Scalar Fields,"* arXiv:1209.2148 [math-ph].

[BFV01] R. Brunetti, K. Fredenhagen and R. Verch, *"The Generally covariant locality principle: A New paradigm for local quantum field theory"*, Commun. Math. Phys. **237**, 31 (2003), arXiv:math-ph/0112041.

[BGP07] C. Bär, N. Ginoux and F. Pfäffle, *Wave Equations on Lorentzian Manifolds and Quantization*, 1st edn. (Eur. Math. Soc., Zürich, 2007).

[BKMM09] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *"Advances in the Casimir effect,"* OUP Oxford (2009).

[BL96] I. Brevik and M. Lygren, *"Casimir effect for a perfectly conducting wedge,"* Ann. Phys. **251** (1996).

[BM69] L. S. Brown and G. J. Maclay, *"Vacuum stress between conducting plates: An Image solution,"* Phys. Rev. **184** (1969) 1272.

[BR87] O. Bratteli, D. W. Robinson, *"Operator algebras and quantum statistical mechanics,"* Vol. 1, Springer-Verlag, (1987)

[BS03] A. N. Bernal and M. Sánchez, *"On smooth Cauchy hypersurfaces and Geroch's splitting theorem,"* Class. Quant. Grav. **24**, 3 (2007).

[BS07] A. N. Bernal and M. Sánchez, *"Globally hyperbolic spacetimes can be defined as 'causal' instead of 'strongly causal',"* Commun. Math. Phys. **243**, 3 (2003).

[Car99] H. S. Carslaw, *"Some multiform solutions of the partial differential equations of physical mathematics and their applications,"* Proc. Lon. Math. Soc. **30** (1899).

[Car10] H. S. Carslaw, *"The Green's funcion for a wedge of any angle, and other problems in the conduction of heat,"* Proc. Lon. Math. Soc. **8** (1910).

[Car20] H. S. Carslaw, *"Diffraction of waves by a wedge of any angle,"* Proc. Lon. Math. Soc. **18** (1920).

[Cas48] H. B. G. Casimir, *"On the Attraction Between Two Perfectly Conducting Plates"*, Indag. Math. **10** 261 (1948).

[CCJ70] C. G. Callan, Jr., S. R. Coleman and R. Jackiw, *"A New improved energy - momentum tensor"*, Annals Phys. **59** 42 (1970).

[Che79] J. Cheeger, *"On the spectral geometry of spaces with cone-like singularities,"* Proc. Nat. Ac. Scien. **76**, 5 (1979).

[Che83] J. Cheeger, *"Spectral geometry of singular Riemannian spaces,"* Jour. Diff. Geom., **18**, 4 (1983).

[CP48] H. B. G. Casimir and D. Polder, *"The Influence of retardation on the London-van der Waals forces"*, Phys. Rev. **73** 360 (1948).

[CT82a] J. Cheeger and M. Taylor, *"On the diffraction of waves by conical singularities. I,"* Comm. Pure Appl. Math., **25** (1982)

[CT82b] J. Cheeger and M. Taylor, *"On the diffraction of waves by conical singularities. II,"* Comm. Pure Appl. Math., **35**, 4 (1982)

[DC79] D. Deutsch and P. Candelas, *"Boundary Effects in Quantum Field Theory"*, Phys. Rev. D **20** 3063 (1979).

[Dim80] J. Dimock, *"Algebras of Local Observables on a Manifold"*, Commun. Math. Phys. **77**, 219 (1980).

[DeW75] B. S. DeWitt, *"Quantum field theory in curved spacetime,"* Phys. Rep. **19**, 6 (1975).

[DeW79] B. S. DeWitt, *" Quantum gravity: the new synthesis"*, In *General relativity: an Einstein centenary survey*, ed. S. W. Hawking and W. Israel, (1979).

[DFP08] C. Dappiaggi, K. Fredenhagen and N. Pinamonti, *"Stable cosmological models driven by a free quantum scalar field,"* Phys. Rev. D **77** (2008) 104015, arXiv:0801.2850 [gr-qc]

[DHMP10] C. Dappiaggi, T.-P. Hack, J. Moller and N. Pinamonti, "Dark Energy from Quantum Matter", arXiv:1007.5009 [astro-ph.CO]

[DK78] J. S. Dowker and G. Kennedy, *"Finite temperature and boundary effects in static space-times,"* Jour. Phys. A **11**, 5 (1978).

[DNP16] C. Dappiaggi, G. Nosari and N. Pinamonti, *"The Casimir effect from the point of view of algebraic quantum field theory.,"* Math. Phys. Anal. Geom. **19**, 2 (2016), arXiv:1412.1409 [math-ph]

[Dow77] J. S. Dowker, *"Quantum field theory on a cone,"* J. Phys. A **10** (1977).

[Dow78] J. S. Dowker, *"Thermal properties of Green's functions in Rindler, de Sitter, and Schwarzschild spaces,"* Phys. Rev. D **18** (1978).

[Dow87a] J. S. Dowker, *"Casimir effect around a cone,"* Phys. Rev. D **36** (1987).

[Dow87b] J. S. Dowker, *"Vacuum averages for arbitrary spin around a cosmic string,"* Phys. Rev. D **36** (1987).

[Dow90] J. S. Dowker, *"Quantum field theory around conical defects,"* in *The Formation and Evolution of Cosmic Strings* ed. G. Gibbons, S. Hawking and T. Vachaspati Cambridge (1990).

[FH95] C. J. Fewster and A. Higuchi, *"Quantum field theory on certain non-globally hyperbolic space-times,"* Class. Quant. Grav. **13** (1996) 51 [gr-qc/9508051].

[FH12] C. J. Fewster and D. S. Hunt, *"Quantization of linearized gravity in cosmological vacuum spacetimes"*, Rev. Math. Phys. **25** 1330003 (2013), arXiv:1203.0261 [math-ph].

[FP03] C. J. Fewster and M. J. Pfenning, *"A Quantum weak energy inequality for spin one fields in curved space-time"*, J. Math. Phys. **44**, 4480 (2003), arXiv:gr-qc/0303106.

[FP14a] D. Fermi and L. Pizzocchero, *"Local zeta regularization and the Casimir effect,"* Prog. Theor. Phys. **126.3** (2011), 419-434 arXiv:1104.4330 [math-ph]

[FP14b] D. Fermi and L. Pizzocchero, *"Local zeta regularization and the scalar Casimir effect I. A general approach based on integral kernels,"* arXiv:1505.00711 [math-ph]

[FP14c] D. Fermi and L. Pizzocchero, *"Local zeta regularization and the scalar Casimir effect II. Some explicitly solvable cases,"* arXiv:1505.01044 [math-ph]

[FR87] S. A. Fulling and S. N. M. Reijsenaars, *"Temperature, periodicity and horizons"* Phys. Rep. **152**, 135 (1987).

[FR15] K. Fredenhagen and K. Rejzner, *"Perturbative construction of models of algebraic quantum field theory"*, in *"Advances in Algebraic Quantum Field Theory"* (pp. 31-74), Springer (2015).

[Fri58] F. G. Friedlander, *"Sound Pulses,"* Cambridge Univ. Press (1958)

[Fri75] F. G. Friedlander, *"The Wave Equation on a Curved Space-Time,"* Cambridge University Press (1975)

[FS95] D. V. Fursaev and S. N. Solodukhin, *" On the Description of the Riemannian Geometry in the Presence of Conical Defects,"* Phys. Rev. D **52** (1995), arXiv:9501.127 [hep-th].

[FTTW12] S. A. Fulling, C. S. Trendafilova, P. N. Truong and J. Wagner, *"Wedges, cones, cosmic strings and their vacuum energy,"* Jour. Phys. A, **45**, 37 (2012).

[Ful89] S. A. Fulling, *"Aspects of Quantum Field Theory in Curved Spacetime,"* London Math. Soc. Student Texts **17** (1989) 1.

[Ger70] R. P. Geroch, *"Domain of dependence,"* Journal of Mathematical Physics **11**, (1970) 437.

[GR07] I. S. Gradshteyn and I. M. Ryzhik *Tables of Integrals, Series and Products*, 7th edition, Academic Press (2007).

[Haa92] R. Haag, *"Local Quantum Physics: Fields, Particles, Algebras"*, Springer-Verlag, Berlin (1992).

[Hac10] T.-P. Hack, *On the backreaction of scalar and spinor quantum fields in curved spacetimes*, arXiv:1008.1776 [gr-qc], PhD thesis, Universität Hamburg (2010).

[Haw77] S. Hawking, *"Zeta function regularization of path integrals in curved space-time,"* Commun. Math. Phys. **55.2** (1977), 133-148.

[HE73] S. Hawking and G. Ellis, *"The Large Scale Structure of Space-Time,"* Cambridge University Press (1973)

[Her04] A. Herdegen, *"Quantum backreaction (Casimir) effect. I. What are admissible idealizations?"*, Annales Henri Poincare **6** 657 (2005) [hep-th/0412132].

[Her05] A. Herdegen, *"Quantum backreaction (Casimir) effect. II. Scalar and electromagnetic fields"*, Annales Henri Poincare **7** 253 (2006) [hep-th/0507023].

[Her10] A. Herdegen and M. Stopa, *"Global versus local Casimir effect"*, Annales Henri Poincare **11** 1171 (2010) [arXiv:1007.2139 [hep-th]].

[HK64] R. Haag and D. Kastler, *"An Algebraic approach to quantum field theory"*, J. Math. Phys. **5**, 848 (1964).

[HM12] T.-P. Hack and V. Moretti, *"On the Stress-Energy Tensor of Quantum Fields in Curved Spacetimes - Comparison of Different Regularization Schemes and Symmetry of the Hadamard/Seeley-DeWitt Coefficients,"* Jour. Phys. A, **45.37** (2012) 374019 arXiv:1202.5107 [gr-qc].

[Hör90] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd edn. Springer, Berlin (1990).

[HW01] S. Hollands and R. M. Wald, *"Local Wick polynomials and time ordered products of quantum fields in curved space-time"*, Commun. Math. Phys. **223**, 289 (2001), arXiv:0103.074 [gr-qc].

[HW05] S. Hollands and R. M. Wald, *"Conservation of the stress tensor in interacting quantum field theory in curved spacetimes,"* Rev. Math. Phys. **17** (2005) 227 arXiv:0404.074 [gr-qc].

[HW14] S. Hollands and R. M. Wald, *"Quantum fields in curved spacetime"*, arXiv:1401.2026 [gr-qc].

[GJK02] R. L. Jaffe N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, *"Calculating vacuum energies in renormalizable quantum field theories:: A new approach to the Casimir problem"*, Nucl. Phys. B, **645.1** (2002).

[Kay78] B. S. Kay, *"Linear spin-zero quantum fields in external gravitational and scalar fields"*, Comm. Math. Phys. **62** (1978).

[Kay79] B. S. Kay, *"Casimir effect in quantum field theory"*, Phys. Rev. D **20** (1979) 3052.

[Kay92] B. S. Kay, *"The Principle of locality and quantum field theory on (non-globally hyperbolic) curved space-times,"* Rev. Math. Phys. SI **1** (1992) 167.

[KCD79] G. Kennedy, R. Critchley and J. S. Dowker, *"Finite Temperature Field Theory with Boundaries: Stress Tensor and Surface Action Renormalization,"* Annals Phys. **125** (1980) 346.

[KM15] I. Khavkine and V. Moretti *"Algebraic QFT in Curved Spacetime and Quasifree Hadamard States: An Introduction"*, in *"Advances in Algebraic Quantum Field Theory"* (pp. 31-74), Springer (2015).

[Küh05] H. Kühn, *Thermische Observablen gekoppelter Felder in Casimir-Effekt,* Diploma Thesis in German, Universität Hamburg (2005), available at `www.desy.de/uni-th/lqp/psfiles/dipl-kuehn.ps.gz`

[KW91] B. S. Kay and R. M. Wald, *"Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon,"* Phys. Rept. **207**, 49 (1991).

[Lee00] J. M. Lee, *Introduction to smooth manifolds*, Springer (2000).

[Ler52] J. Leray, *"Lectures on hyperbolic equations with variable coefficients."* The Institute for Advanced Study, Princeton (1952).

[Mil01] K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-point Energy.* World Scientific, (2001).

[Min09] E. Minguzzi, *"Characterization of some causality conditions through the continuity of the Lorentzian distance,"* Jour. Geom. Phys. **59** (2009) 827–833.

[Mor03] V. Moretti, *"Comments on the stress energy tensor operator in curved space-time"*, Commun. Math. Phys. **232**, 189 (2003), arXiv:gr-qc/0109048.

[MS08] E. Minguzzi, and M. Sánchez, *"The causal hierarchy of spacetimes. Recent developments in pseudo-Riemannian geometry,"* ESI Lect. Math. Phys, (2008).

[Nie09] O. Niekerken *Quantum and Classical Vacuum Forces at Zero and Finite Temperature*, Diploma Thesis in German (2009), Universität Hamburg, available at `http://www-library.desy.de/preparch/desy/thesis/desy-thesis-09-019.pdf`

[NLS02] V. V. Nesterenko, G. Lambiase and G. Scarpetta, *"Casimir effect for a perfectly conducting wedge in terms of local zeta function,"* Ann. Phys. **298** (2002)

[Nos15] G. Nosari, *"Towards quantum field theory on spacetimes with boundaries"*, to appear in *Proceeding of Marcel Grossmann workshop on Algebraic Quantum Field Theory - XIV* (2015) arXiv:1512.07114 [math-ph]

[Obe54] F. Oberhettinger *"Diffraction of waves by a wedge,"* Commun. Pure Appl. Math. **7** (1954).

[Pei52] R. E. Peierls, *"The Commutation Laws of Relativistic Field Theory,"* Proc. Roy. Soc. London **A214** (1952) 143-157.

[Rad96a] M. J. Radzikowski, *"Micro-local approach to the Hadamard condition in quantum field theory on curved space-time"*, Commun. Math. Phys. **179**, 529 (1996).

[Rad96b] M. J. Radzikowski, *"A Local to global singularity theorem for quantum field theory on curved space-time"*, Commun. Math. Phys. **180**, 1 (1996).

[RS02] A. H. Rezaeian, A. A. Saharian, *"Local Casimir energy for a wedge with a circular outer boundary,"* Class. Quant. Grav. **19** (2002).

[SDH12] K. Sanders, C. Dappiaggi and T. -P. Hack, *"Electromagnetism, Local Covariance, the Aharonov-Bohm Effect and Gauss' Law"*, Commun. Math. Phys. **328** 625 (2014) [arXiv:1211.6420 [math-ph]].

[SF02] V. Sopova and L. H. Ford, *"Energy density in the Casimir effect"*, Phys. Rev. D **66**, 045026 (2002).

[Som96] A. Sommerfeld, *"Mathematische Theorie der Diffraction,"* Math. Ann. **47** (1896).

[Som97] A. Sommerfeld, *"Über verzweigte Potentiale im Raum,"* Proc. Lon. Math. Soc. **28** (1897).

[Som01] A. Sommerfeld, *"Theoretisches über die Beugung der Röntgenstralen,"* Zeitschr. Mat. Physik **46** (1901).

[Som06] C. Sommer, *Algebraische Charakterisierung von Randbedingungen in der Quantenfeldtheorie* , Diploma Thesis in German, Universität Hamburg (2006), available at `http://www.desy.de/uni-th/lqp/psfiles/dipl-sommer.ps.gz`

[ST05] A. A. Saharian and A. S. Tarloyan, *"Whightman function and scalar Casimir densities for a wedge with a cylindrical boundary,"* Jour. Phys. A **38** (2005).

[Sta67] I. Stakgold, *"Boundary Value Problems of Mathematical Physics - Vol. 2,"* MacMillan, New York (1967).

[SV00] H. Sahlmann and R. Verch, *"Passivity and microlocal spectrum condition,"* Commun. Math. Phys. **214** (2000) 705 [math-ph/0002021].

[Tre67] F. Trèves, *"Topological vector spaces, distributions and kernels,"* Courier Corporation (1967).

[Wal77] R. M. Wald, *"The Back Reaction Effect In Particle Creation In Curved Space-Time,"* Commun. Math. Phys. **54** (1977) 1.

[Wal78a] R. M. Wald, *"Trace Anomaly Of A Conformally Invariant Quantum Field In Curved Space-Time,"* Phys. Rev. D **17** (1978) 1477.

[Wal78b] R. M. Wald, *"Axiomatic Renormalization Of The Stress Tensor Of A Conformally Invariant Field In Conformally Flat Space-Times,"* Annals Phys. **110**, 472 (1978).

[Wal84] R. M. Wald, *General Relativity*, 1st edn. The University of Chicago Press, Chicago, (1984).

[Wal94] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, 1st edn. The University of Chicago Press Chicago, (1994).

[Wat22] G. N. Watson, *"A treatise on the theory of Bessel functions,"* Cambridge university press (1995).