Instanton corrections in gauge theories realized via D-branes

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Stringy construction of instantons: why?

- Previous talks have convinced us that embedding gauge field theories into String Theory via D-brane constructions is a smart move:
  - known facts get well and (yes!) intuitively organized
  - connections, generalizations, new ideas (think of holographic correspondences!)
- Instantons are a particularly tractable class of non-perturbative configurations of gauge theories leading to many effects at strong coupling in QCD (e.g., U(1) puzzle) and SYM theories.
- They get reproduced in the string set-up by including D(-1)-branes (or other Euclidean branes).
  - Intuitive and efficient description
  - Leads to generalizations such as “exotic” instanton effects that can be important for string phenomenology
  - Instantonic branes are crucial for string dualities

Topologically non-trivial sectors in YM

The instanton number

- The path-integral over gauge fields decomposes into sectors characterized by an integer k:

$$\sum_{k\in\mathbb{Z}} \int DA_\mu^{(k)}\, \mathrm{e}^{-S_{\mathrm{YM}}[A^{(k)}]} \dots\, .$$

- k is the 2nd Chern number of the gauge bundle:

$$k = \frac{1}{8\pi} \int \mathrm{tr} F \wedge F = \frac{1}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}\ .$$

- $k = 0$ is the sector connected with the vacuum $A_\mu = 0$, where usual perturbation theory is carried out.

Instantonic solutions

- When $k \neq 0$ the gauge fields have a non-trivial winding on the $S_3^{(\infty)}$ boundary; in the Minkowskian regime this correspond to tunnelling.
- Instantons are (anti)self-dual configurations: $F = \pm {}^*F$. These have the lowest Euclidean action in a given sector:

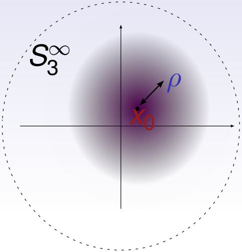
$$S_k = \frac{8\pi^2}{g^2}\,|k| - \mathrm{i}\theta k$$

(θ is the theta-angle).
- It is convenient to introduce $\tau = \frac{\theta}{\pi} + \mathrm{i}\frac{8\pi\mathrm{i}}{g^2}$ so that, for $k > 0$,

$$S_k = -\pi\mathrm{i}\tau k$$

For anti-instantons $S_{-k} = \bar{S}_k = \pi\mathrm{i}\bar{\tau}k$.

Example: one-instanton solution in SU(2)



- SU(2), $k = 1$ instanton solution:

$$A^i_\mu = \frac{2\eta^i_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2}$$

  It has a self-dual field-strength; for $r \to \infty$ winds once over $S_3^{(\infty)}$

- It depends on free parameters:
  - The center position x_0^μ (N.B. centered in space and time: instanton!)
  - The size ρ
  - A global SU(2) rotation (not explicitly visible)

Instanton calculus

Basic idea

- In each sector expanding around the instanton solution we end up with a path-integral of the form

$$\sum_k \int dM^{(k)} q^k \int D' \tilde{A}_\mu^{(k)} \mathrm{e}^{-S'[\tilde{A}_\mu^{(k)}]} \ldots$$

  - finite-dimensional integral over the moduli $M^{(k)}$ (moduli = parameters in the solution)
  - weight $q^k = \exp(-S_k)$, where $q = \exp(\pi \mathrm{i} \tau_0)$, non-perturbative in the coupling. Negligible in the UV, instantons become crucial at strong coupling.
  - fluctuation part, usually treated semi-classically. For the partition function yields fluctuation determinants

Instanton calculus

ADHM, susy, ...

- The construction of multi-instanton solutions ($|k| > 1$) and of their moduli space is very intricate
- ADHM construction: work in an enlarged moduli space $\mathcal{M}^{(k)}$, with an auxiliary $\mathrm{U}(k)$ symmetry and constraints imposed via Lagrange multipliers

$$\int dM^{(k)} q^k \longrightarrow \int d\mathcal{M}^{(k)} q^k \mathrm{e}^{-S_k^{ADHM}(\mathcal{M})}$$

- In SYM theories
  - bosonic and fermionic fluctuation determinants cancel in the partition functions
  - moduli include fermionic 0-modes. If unbalanced, they kill the path integral. Selection rules on non-perturbative contributions.

D-branes of type IIB

- Dp-branes support $p+1$-dim SYM theories
- Dp action contains minimal coupling to RR $p+1$ forms:

$$S_{Dp} = -T_p \int_{p+1} e^{-\phi} \sqrt{\det(1 + 2\pi\alpha' F)} - T_p \int_{p+1} \sum_p C_{p+1} \wedge e^{2\pi\alpha' F}$$

- In type IIB we have C_0, C_2, C_4, (C_6, C_8) RR forms and thus D(-1), D1, D3, D5, D7, D9 branes.
- From F^2 terms, on D3-branes one identifies the 4d gauge couplings with the dilaton-axion: $e^{-\phi} \leftrightarrow \frac{4\pi}{g^2}$, $C_0 \leftrightarrow \frac{\theta}{2\pi}$

D-instantons (flat space)

- D(-1) branes impose DD b.c. on all directions, including time. They are points in space- time: D-instantons
- The (Euclidean) effective action is 0-dim and reduces to

$$S_{D(-1)} = 2\pi \mathrm{e}^{-\phi} - 2\pi \mathrm{i} C_0 = -\pi \mathrm{i}\,\tau \ ,$$

  where $\tau/2$ is the closed-string field $C_0 + \mathrm{i}\mathrm{e}^{-\phi}$, i.e., the complexified gauge coupling for the D3 gauge theory.
- The D-instanton action is thus just the classical instanton action!
- Conversely,from the WZ part of the D3 action, we see that a gauge field of instanton number k couples to C_0 exactly as k D(-1)'s do.

[Witten 1995, Douglas 1995, Dorey 1999, ...]

D-instantons vs. gauge instantons

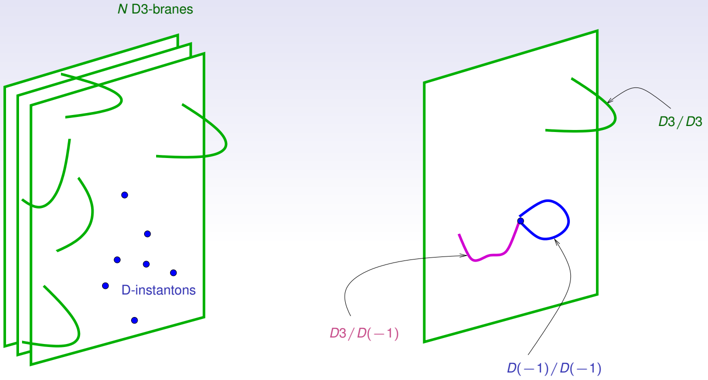
- The correspondence between D-instantons and gauge instantons goes well beyond the coincidence of the classical action
- The instanton moduli space, the profile of the instanton solution and the contributions of instanton sectors to correlation functions are all contained (and well organized!) in the brane description

  Polchinski 1994, Green-Gutperle 1997-1998, Billò et al 2002,...

- To see how this goes, we must study what happens to open strings when, beside D3 branes, they can end on D-instantons as well. B.c's are as follows:

	0	1	2	3	4	5	6	7	8	9
D3	—	—	—	—	*	*	*	*	*	*
D(−1)	*	*	*	*	*	*	*	*	*	*

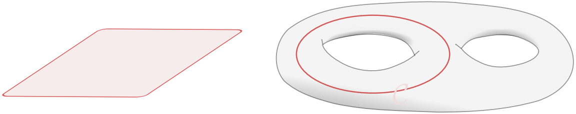
Excitations of open strings



- D3/D3 strings: NN or DD b.c.s, gauge theory fields
- D(-1)/D(-1) strings: DD $\rightarrow$ no momentum, instanton moduli
- D3/D(-1) strings: ND $\rightarrow$ no momentum, instanton moduli

(Gauge) instantons in brane-worlds

- The flat space case admits natural generalizations
- On a background $\mathbb{R}^4 \times X^6$, a D$(3+m)$-brane wrapped on an m-cycle $\mathcal{C} \subset X^6$ supports a 4d SYM theory
- A Dm-brane wrapped on the same cycle $\mathcal{C}$ is point-like in space-time. Its w.v. action equals that of gauge instantons on the D$(3+m)$ brane. This "euclidean brane" indeed represents the gauge instantons.
- Euclidean branes wrapped on different cycles can produce novel, stringy, non- perturbative effects (neutrino Majorana masses, moduli stabilizing terms,...): bonus of the string construction! [Blumenhagen et al, 2006; Ibanez and Uranga, 2006;...]



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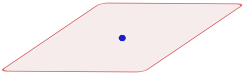

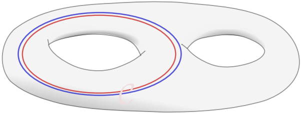

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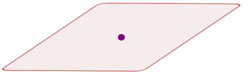
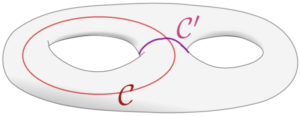
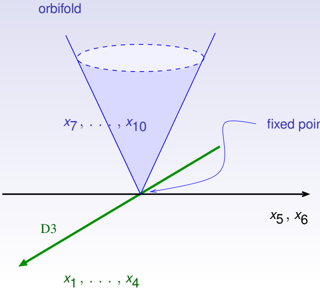




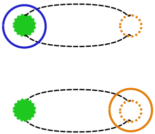
Different set-ups

- Choice of internal manifold, type of branes, background fields, etc allows to construct gauge theories with different gauge groups, amount of susy and matter content
- Gauge instantons lead then to different effects, e.g.
  - in $\mathcal{N} = 1$ SQCD they induce the ADS superpotential for $N_f = N_c - 1$ ($\longrightarrow$ SUSY breaking);
  - in $\mathcal{N} = 2$ SYM they contribute to the SW Prepotential ($\longrightarrow$ exact strong/weak coupling duality).
- We will now consider the same example used in Alberto Lerda's talk, namely an orbifold model with fractional D3's supporting $\mathcal{N} = 2$ SU(N) SYM

A specific model: the $\mathbb{Z}_2$, $\mathcal{N} = 2$ quiver



- Two kinds of fractional branes, even or odd w.r.t. to the $\mathbb{Z}_2$ orbifold group
- $\mathcal{N} = 2$ SU(N) SYM lives on N fD3's of one type, say the even one.



- Even fD(-1) correspond to gauge instantons (we focus on these);
- odd ones to "exotic" instantons

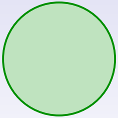
- The complexified gauge coupling of fD3's is in this case related to twisted closed string fields: $\tau = \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g^2} = c + \mathrm{i}e^{-\phi}b$ and the even fD(-1) classical action reads $-\pi\mathrm{i}\tau$

Moduli spectrum

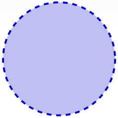
	ADHM	Meaning	Vertex	Chan-Paton
-1/-1 (NS)	a'_μ	*centers*	$\psi^\mu(z)\mathrm{e}^{-\varphi(z)}$	adj. $\mathrm{U}(k)$
	χ	*aux.*	$\overline{\Psi}(z)\mathrm{e}^{-\varphi(z)}$	$\vdots$
(aux. vert.)	D_c	*Lagrange mult.*	$\bar{\eta}^c_{\mu\nu}\psi^\nu(z)\psi^\mu(z)$	$\vdots$
(R)	$M^{\alpha A}$	*partners*	$S_\alpha(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$
	$\lambda_{\dot{\alpha}A}$	*Lagrange mult.*	$S^{\dot{\alpha}}(z)S^A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$
-1/3 (NS)	$w_{\dot{\alpha}}$	*sizes*	$\Delta(z)S^{\dot{\alpha}}(z)\,\mathrm{e}^{-\varphi(z)}$	$k \times \overline{N}$
	$\bar{w}_{\dot{\alpha}}$	$\vdots$	$\overline{\Delta}(z)S^{\dot{\alpha}}(z)\mathrm{e}^{-\varphi(z)}$	$\vdots$
(R)	μ^A	*partners*	$\Delta(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$
	$\bar{\mu}^A$	$\vdots$	$\overline{\Delta}(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$

Disk amplitudes and brane actions

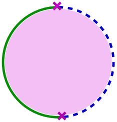
D3 disks

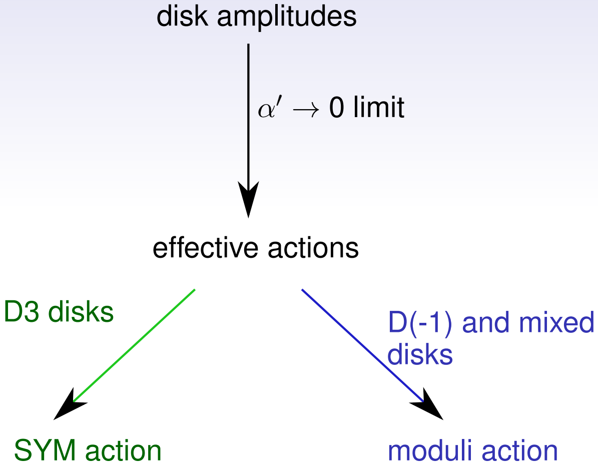


D(–1) disks



D3/D(-1)
mixed disks





Moduli action

From disk diagrams with insertion of moduli vertices, in the field theory limit we extract the ADHM moduli action (at fixed k)

$$\mathcal{S}^{(k)}_{\text{mod}} = \mathcal{S}^{(k)}_{\text{bos}} + \mathcal{S}^{(k)}_{\text{fer}} + \mathcal{S}^{(k)}_{\text{c}}$$

with

$$\begin{aligned}
\mathcal{S}^{(k)}_{\text{bos}} &= \mathrm{tr}_k\Big\{ -2\,[\chi^\dagger, a'_\mu][\chi, a'^\mu] + \chi^\dagger \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi + \chi \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi^\dagger \Big\} \\
\mathcal{S}^{(k)}_{\text{fer}} &= \mathrm{tr}_k\Big\{ \mathrm{i}\,\frac{\sqrt{2}}{2}\,\bar\mu^A \epsilon_{AB}\mu^B \chi^\dagger - \mathrm{i}\,\frac{\sqrt{2}}{4}\, M^{\alpha A}\epsilon_{AB}[\chi^\dagger, M^B_\alpha] \Big\} \\
\mathcal{S}^{(k)}_{\text{c}} &= \mathrm{tr}_k\Big\{ -\mathrm{i}D_c\big(W^c + \mathrm{i}\bar\eta^c_{\mu\nu}\,[a'^\mu, a'^\nu]\big) \\
&\qquad -\mathrm{i}\lambda^{\dot\alpha}_A\big(\bar\mu^A w_{\dot\alpha} + \bar{w}_{\dot\alpha}\mu^A + [a'_{\alpha\dot\alpha}, M'^{\alpha A}]\big)\Big\}
\end{aligned}$$

- $\mathcal{S}^{(k)}_{\text{c}}$: bosonic and fermionic ADHM constraints

Field-dependent moduli action

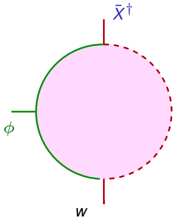
- We want to reproduce the instanton corrections to the effective action

$$S_{\text{eff}}[\Phi] = \int d^4x\, d^4\theta\, \mathcal{F}(\Phi) + \text{c.c}$$

for the chiral multiplet in the Cartan direction

$$\Phi(x, \theta) = \phi + \theta\Lambda + (\theta\gamma^{\mu\nu}\theta)F^{+}_{\mu\nu} + \ldots$$

- The D-instantons modify correlators of ϕ, Λ, F, hence the effective action, through disk interactions among Φ and the moduli
- Such interactions make the moduli action field-dependent: $S^{(k)}_{\text{mod}}(\Phi, \mathcal{M})$



The effective action and the pre-potential

- The combinatorics of boundaries [Polchinski, 1994] is such that D-instanton diagrams exponentiate
- Integrating over the moduli one gets the effective action

$$S^{(k)}_{\text{eff}}[\Phi] = \sum_k \Lambda^{2Nk} \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\, q^k \mathrm{e}^{-\mathcal{S}_{\text{mod}}(\Phi(x,\theta),\widehat{\mathcal{M}}_{(k)})}$$

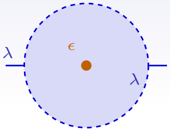
  - The moduli x (center of mass position) and θ (susies broken by the D(-1)) appear in S_{mod} only through $\Phi(x, \theta)$
  - The factor Λ^{2Nk} compensates the dimensionality of $d\widehat{\mathcal{M}}_{(k)}$
- The prepotential is thus given by

$$\mathcal{F}(\Phi) = \sum_k \Lambda^{2Nk} \int d\widehat{\mathcal{M}}_{(k)}\; \mathrm{e}^{-\,\mathcal{S}_{\text{mod}}(\Phi;\widehat{\mathcal{M}}_{(k)})}$$

- $\Phi(x, \theta)$ is constant w.r.t. $\widehat{\mathcal{M}}_{(k)}$; we can freeze it to a constant value a (some care needed, see later!)

BRST structure and localization

- Take a component Q of the susy charge as BRST charge
  - "Lorentz" symmetry restricted to $SU(2)^3$ preserving Q
  - moduli organize in BRST-doublets
  - the moduli action is Q-exact: $S_{\text{mod}} = Q\,\Xi$
- Deformations arise from the interactions of a closed string RR 3-form ϵ with the moduli: $S_{\text{mod}}(\Phi, \epsilon; \widehat{\mathcal{M}}_{(k)})$



- Q is equivariantly closed w.r.t. to the action of $U(k)$ and $U(N)$ CP groups and of the $\mathrm{SU}(2)^3$ symmetry:

$$Q^2\mathcal{M} = T_{U(k)}(\chi)\mathcal{M} + T_{U(N)}(\phi)\mathcal{M} + T_{SU(2)^3}(\epsilon)\mathcal{M}$$

  where $T_{U(k)}(\chi)$ = U(k) rotation parametrized by χ, ...
- This structure leads to localization of the moduli integrals

Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...

Integration

- The (deformed) BRST structure allows to suitably rescale the integration variables and show that the semiclassical approximation is exact

  Moore+Nekrasov+Shatashvili, 1998; ...; Nekrasov, 2002; Flume+Poghossian, 2002; Bruzzo et al, 2003; ...

- The integrals over all moduli except χ become quadratic and yield in the end

$$\prod_{\mathcal{M}_0} det_{\mathcal{M}_0}^{\pm\frac{1}{2}}(Q^2)$$

  where $\mathcal{M}_0$ = first components of BRST doublets. Entirely determined by symmetry properties

- The χ integrals can be done as contour integrals and the final result for the partition function

$$Z_k(a,\epsilon) = \int d\mathcal{M}_{(k)} \mathrm{e}^{-S_{\mathrm{mod}}(a,\epsilon;\mathcal{M}_{(k)})}$$

  comes from a sum over residues

Final expression

- Removing the ϵ deformation needs some care
  - The ϵ's regulate the integral over x and θ: there's a divergence $1/(\epsilon_1\epsilon_2)$ that gets re-interpreted as the supervolume $\int d^4x\, d^4\theta$
  - In the deformed theory, at instanton # k, we get disconnected contributions from instantons $\{k_i\}$ (with $\sum_i k_i = k$). Take the log to single out connected terms
- Altogether, the final expression for the prepotential reads

$$\mathcal{F}_{n.p.}(a) = \lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \log\left(\sum_k \Lambda^{2Nk} Z_k(a,\epsilon)\right)$$

- For instance, in the SU(2) case one gets

$$\mathcal{F}_{n.p.}(a) = \frac{1}{2}\frac{\Lambda^4}{a^2} + \frac{5}{64}\frac{\Lambda^8}{a^6} + \frac{3}{64}\frac{\Lambda^{12}}{a^{10}} + \dots$$

in agreement with Seiberg-Witten solution.

Conclusions

- Instanton configurations and instanton calculus à la Nekrasov are naturally and efficiently embedded in brane constructions of gauge theories via instantonic branes
- The brane realizations offer natural generalizations and extensions, for instance
  - keeping ϵ terms leads to "gravitational" non-perturbative terms containing graviphoton interactions, connected to topological string amplitudes;
  - exotic instantonic effects;
  - instantonic branes account for the non-perturbative part of the gravitational profiles in gauge/gravity pairs;
  - higher dimensional analogues, e.g. in 8d, often crucial for string dualities
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