Instantonic effects in N=1 local models from magnetized branes

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This talk is mostly based on


It, of course, builds over a vast literature

- The few references scattered on the slides are of course not exhaustive. I apologize for the missing ones.
Foreword

This talk is mostly based on


Some very recent work dealing with very similar issues

Plan of the talk

1. Introduction
2. The set-up
3. The stringy instanton calculus
4. Instanton annuli and threshold corrections
5. Holomorphicity properties
Introduction
Wrapped brane scenarios

- Type IIB: magnetized D9 branes
- Type IIA (T-dual): intersecting D6 (easier to visualize)

Supersymmetric gauge theories on $\mathbb{R}^{1,3}$ with chiral matter and interesting phenomenology

- families from multiple intersections, tuning different coupling constants, ...
Wrapped brane scenarios

- Type IIB: magnetized D9 branes
- Type IIA (T-dual): intersecting D6 (easier to visualize)

- Low energy described by SUGRA with vector and matter multiplets
- Can be derived directly from string amplitudes (with different field normaliz.s)
- Novel stringy effects (pert. and non-pert.) in the eff. action?
Euclidean branes and instantons

E2 branes wrapped on the same cycle as some D6 branes are point-like in $\mathbb{R}^{1,3}$ and correspond to instantonic configurations of the gauge theory on the D6.

Analogous to the D3/D(-1) system:
- ADHM from strings attached to the instantonic branes
  (Witten, 1995; Douglas, 1995-1996; ...)
- non-trivial instanton profile of the gauge field

N.B. In type IIB, use D9/E5 branes

Billo et al, 2001
Euclidean branes and instantons

Exotic instantons

**E2 branes** wrapped differently from the **D6 branes** are still point-like in $\mathbb{R}^{1,3}$ but do not correspond to ordinary instantons config.s.

Still they can, in certain cases, give important **non-pert, stringy** contributions to the effective action, e.g. Majorana masses for neutrinos, moduli stabilizing terms, . . .

Blumenhagen et al 0609191; Ibanez and Uranga, 0609213; (long list)... ; Petersson 0711.1837

- Potentially crucial for **string** phenomenology
Perspective of this work

Clarify some aspects of the “stringy instanton calculus”, i.e., of computing the contributions of Euclidean branes

- Focus on ordinary instantons, but should be useful for exotic instantons as well
- Choose a toroidal compactification where string theory is calculable.
- Realize (locally) $\mathcal{N} = 1$ gauge SQCD in type IIB on a system of D9-branes and discuss contributions of E5 branes to the superpotential
- Analyze the rôle of annuli bounded by E5 and D9 branes in giving these terms suitable holomorphicity properties
The set-up
The Kähler param.s and complex structures determine the string frame metric and the $B$ field:

$$G^{(i)} = \frac{T_2^{(i)}}{U_2^{(i)}} \begin{pmatrix} 1 & U_1^{(i)} \\ U_1^{(i)} & |U^{(i)}|^2 \end{pmatrix} \quad \text{and} \quad B^{(i)} = \begin{pmatrix} 0 & -T_1^{(i)} \\ T_1^{(i)} & 0 \end{pmatrix}.$$
The background geometry
Complex coordinates

String fields: \( X^M \rightarrow (X^\mu, Z^i) \) and \( \psi^M \rightarrow (\psi^\mu, \psi^i) \), with

\[
Z^i = \sqrt{\frac{T_2^{(i)}}{2U_2^{(i)}}} \left( X^{2i+2} + U^{(i)} X^{2i+3} \right)
\]

10d spin fields decompose into space-time and internal parts:

\[
S^{\dot{A}} \rightarrow (S_\alpha S_{---}, S_\alpha S_{-++}, \ldots, S^{\dot{\alpha}} S^{+++}, \ldots)
\]
Action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold group elements:

$$h_1 : (Z^1, Z^2, Z^3) \rightarrow (Z^1, -Z^2, -Z^3),$$
$$h_2 : (Z^1, Z^2, Z^3) \rightarrow (-Z^1, Z^2, -Z^3),$$
$$h_3 : (Z^1, Z^2, Z^3) \rightarrow (-Z^1, -Z^2, Z^3).$$

The group has 4 irreducible representations:

$$R_0 \text{ (trivial)}, \, R_1, \, R_2, \, R_3.$$
The geometric moduli
Supergravity basis- tree level

Supergravity basis: $s, t^{(i)}, u^{(i)}$, with

$$\text{Im}(s) \equiv s_2 = \frac{1}{4\pi} e^{-\phi_{10}} T_2^{(1)} T_2^{(2)} T_2^{(3)},$$

$$\text{Im}(t^{(i)}) \equiv t_2^{(i)} = e^{-\phi_{10}} T_2^{(i)}, \quad u^{(i)} = u_1^{(i)} + i u_2^{(i)} = U^{(i)},$$

(real parts from suitable RR or B fields). N.B. $s_2 \sim 1/g_s$.

$\mathcal{N} = 1$ bulk Kähler potential:

$$K = -\log(s_2) - \sum_{i=1} \log(t_2^{(i)}) - \sum_{i=1} \log(u_2^{(i)})$$

Antoniadis et al, 9608012
The geometric moduli
Supergravity basis - corrections

At one-loop level, there are corrections to the bulk Kähler potential (and to the Einstein term)

- These lead to non-holomorphic redefinitions of the supergravity fields $s$ and $t_i$ w.r.t. the their tree-level expressions. In particular

$$s_2^{(0)} = s_2 + \frac{\delta}{8\pi^2}$$

- Differently from corresponding Heterotic constructions, $\delta$ in these models appears to be of order $g_s$ rather than 1.

- It would be interesting [see later!] to clarify if any other mechanism can induce, in the models we consider, a shift $\delta^{(0)}$ of order 1.
\[ \mathcal{N} = 1 \text{ from magnetized branes} \]

The gauge sector

Place a stack of \( N_a \) fractional D9 branes ("color branes" 9a).

- Massless spectrum of 9a/9a strings gives rise, in \( \mathbb{R}^{1,3} \), to the \( \mathcal{N} = 1 \) vector multiplets for the gauge group \( U(N_a) \).

- The gauge coupling constant is given (at tree level) by

\[
\frac{1}{g_a^2} = \frac{1}{4\pi} e^{-\phi_{10}} T_2^{(1)} T_2^{(2)} T_2^{(3)} = s_2^{(0)}
\]
Place a stack of $N_a$ fractional D9 branes ("color branes" 9a).

- Massless spectrum of 9a/9a strings gives rise, in $\mathbb{R}^{1,3}$, to the $\mathcal{N} = 1$ vector multiplets for the gauge group $U(N_a)$.

- The Wilsonian coupling $1/\tilde{g}_a^2$ must correspond to (the imaginary part of) a chiral field, so it is corrected w.r.t. to the tree level:

\[
\frac{1}{g_a^2} = \frac{1}{\tilde{g}_a^2} + \frac{\delta}{8\pi^2}
\]
Add D9-branes ("flavor branes" $9b$) with quantized magnetic fluxes

$$f_b^{(i)} = \frac{m_b^{(i)}}{n_b^{(i)}}$$

and in a different orbifold rep.

► (Bulk) susy requires $\nu_b^{(1)} - \nu_b^{(2)} - \nu_b^{(3)} = 0$, where

$$f_b^{(i)} / T_2^{(i)} = \tan \pi \nu_b^{(i)} \quad \text{with} \quad 0 \leq \nu_b^{(i)} < 1,$$

(other possibilities by sign changes)
\( \mathcal{N} = 1 \) from magnetized branes

Add D9-branes ("flavor branes") \( 9b \) with quantized magnetic fluxes

\[ f_b^{(i)} = \frac{m_b^{(i)}}{n_b^{(i)}} \]

and in a different orbifold rep.

- \( 9a/9b \) strings are twisted by the relative angles

\[ \nu_{ba}^{(i)} = \nu_b^{(i)} - \nu_a^{(i)} \]

- If \( \nu_{ba}^{(1)} - \nu_{ba}^{(2)} - \nu_{ba}^{(3)} = 0 \), this sector is supersymmetric: massless modes fill up a chiral multiplet \( q_{ba} \) in the anti-fundamental rep \( \bar{N}_a \) of the color group.
$\mathcal{N} = 1$ from magnetized branes

Adding flavors

Add D9-branes (“flavor branes” $9b$) with quantized magnetic fluxes

$$f_b^{(i)} = \frac{m_b^{(i)}}{n_b^{(i)}}$$

and in a different orbifold rep.

- The degeneracy of this chiral multiplet is $N_b |l_{ab}|$, where $l_{ab}$ is the # of Landau levels for the $(a, b)$ “intersection”

$$l_{ab} = \prod_{i=1}^{N_b} (m_{a}^{(i)} n_{b}^{(i)} - m_{b}^{(i)} n_{a}^{(i)})$$
Local vs global realization

Introducing branes in a compact space requires the cancellation of the associated tadpoles. This can be achieved by a suitable orientifold projection in the string description, and severely constrains the set-up.

We take a “local” approach, and do not discuss the “global” requirement of tadpole cancellation (which is however to be assumed) and the contribution of orientifolds in these models:

- our goal is to understand certain mechanisms of the stringy instanton calculus rather than provide phenomenological models
- these aspects can be taken into account, and the picture goes through

see Akerblom et al, 0705.2366; Blumenhagen et al, 0711.0866
Introduce a third stack of $9c$ branes such that we get a chiral mult. $q_{ac}$ in the fundamental rep $N_a$ and that

$$N_b | I_{ab} | = N_c | I_{ac} | \equiv N_F$$

This gives a (local) realization of $\mathcal{N} = 1$ SQCD: same number $N_F$ of fundamental and anti-fundamental chiral multiplets, resp. denoted by $q_f$ and $\tilde{q}^f$
$\mathcal{N} = 1$ from magnetized branes

Introduce a third stack of $9c$ branes such that we get a chiral mult. $q_{ac}$ in the fundamental rep $N_a$ and that

$$N_b | l_{ab} | = N_c | l_{ac} | \equiv N_F$$

Kinetic terms of chiral mult. scalars from disks

$$\sum_{f=1}^{N_F} \left\{ D_\mu q^{\dagger f} D^\mu q_f + D_\mu \tilde{q}^f D^\mu \tilde{q}_f \right\}$$

Sugra Lagrangian: different field normaliz. s

$$\sum_{f=1}^{N_F} \left\{ K_Q D_\mu Q^{\dagger f} D^\mu Q_f + K_{\tilde{Q}} D_\mu \tilde{Q}^f D^\mu \tilde{Q}_f \right\}$$

- Related via the Kähler metrics: $q = \sqrt{K_Q} Q$, $\tilde{q} = \sqrt{K_{\tilde{Q}}} \tilde{Q}$
Non-perturbative sectors from $E_5$

Adding “ordinary” instantons

Add a stack of $k$ $E_5$ branes whose internal part coincides with the $D9a$:

- **ordinary instantons** for the $D9a$ gauge theory
- **would be exotic** for the $D9b, c$ gauge theories
- New types of open strings: $E_{5a}/E_{5a}$ (neutral sector), $D9_{a}/E_{5a}$ (charged sector), $D9_{b}/E_{5a}$ or $E_{5a}/D9_{c}$ (flavored sectors, twisted)
- These states carry no momentum in space-time: moduli, not fields. [Collective name: $\mathcal{M}_k$]
- charged or neutral moduli can have KK momentum
Non-perturbative sectors from $E5$

The spectrum of moduli

<table>
<thead>
<tr>
<th>Sector</th>
<th>ADHM</th>
<th>Meaning</th>
<th>Chan-Paton</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5a/5a$ NS</td>
<td>$a_\mu$</td>
<td>centers</td>
<td>adj. $U(k)$</td>
<td>(length)</td>
</tr>
<tr>
<td></td>
<td>$D_c$</td>
<td>Lagrange mult.</td>
<td>:</td>
<td>$(\text{length})^{-2}$</td>
</tr>
<tr>
<td>$R$</td>
<td>$M^\alpha$</td>
<td>partners</td>
<td>:</td>
<td>$(\text{length})^{\frac{1}{2}}$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_{\dot{\alpha}}$</td>
<td>Lagrange mult.</td>
<td>:</td>
<td>$(\text{length})^{-\frac{3}{2}}$</td>
</tr>
<tr>
<td>$9a/5a$ NS</td>
<td>$w_{\dot{\alpha}}$</td>
<td>sizes</td>
<td>$N_a \times \bar{k}$</td>
<td>(length)</td>
</tr>
<tr>
<td>$5a/9a$</td>
<td>$\bar{w}_{\dot{\alpha}}$</td>
<td>:</td>
<td>$k \times \bar{N}_a$</td>
<td>:</td>
</tr>
<tr>
<td>$9a/5a$ R</td>
<td>$\mu$</td>
<td>partners</td>
<td>$N_a \times \bar{k}$</td>
<td>$(\text{length})^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>$5a/9a$</td>
<td>$\bar{\mu}$</td>
<td>:</td>
<td>$k \times \bar{N}_a$</td>
<td>:</td>
</tr>
<tr>
<td>$9_b/5a$ R</td>
<td>$\mu'$</td>
<td>flavored</td>
<td>$N_F \times \bar{k}$</td>
<td>$(\text{length})^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>$5a/9c$</td>
<td>$\tilde{\mu}'$</td>
<td>:</td>
<td>$k \times \bar{N}_F$</td>
<td>:</td>
</tr>
</tbody>
</table>
Among the neutral moduli we have the center of mass position $x_0^\mu$ and its fermionic partner $\theta^\alpha$ (related to susy broken by the E5a):

$$a^\mu = x_0^\mu \mathbb{1}_{k \times k} + y_c^\mu T^c, \quad M^\alpha = \theta^\alpha \mathbb{1}_{k \times k} + \zeta_c^\alpha T^c,$$

In the flavored sectors only fermionic zero-modes:

- $\mu'_f$ (D9b/E5a sector)
- $\tilde{\mu}'^f$ (E5a/D9c sector)
The stringy instanton calculus
Instantonic correlators
The stringy way

In presence of Euclidean branes, dominant contribution to correlators of gauge/matter fields from one-point functions.

Polchinski, 1994; Green and Gutperle, 1997-2000; Billo et al, 2002; Blumenhagen et al, 2006

E.g., a correlator of chiral fields $\langle q\tilde{q} \ldots \rangle$ is given by

$$\langle \qquad \tilde{q} \quad \rangle = \left( 1 + \frac{1}{2} \right) + \ldots$$

Disks: $\equiv -\frac{8\pi^2}{g_a^2} k + S_{mod}(M_k)$ (with moduli insertions)

Annuli: $\equiv \mathcal{A}_{5a}$ (no moduli insert.s, otherwise suppressed)
The effective action in an instantonic sector

The various instantonic correlators can be obtained shifting the moduli action by terms dependent on the gauge/matter fields. In the case at hand,

\[ S_{\text{mod}}(q, \tilde{q}; \mathcal{M}_k) = \quad + \quad + \quad + \ldots \]

\[ = \text{tr}_k \left\{ iD_c \left( \bar{w}_{\dot{\alpha}} (\tau^c)^{\dot{\alpha}}_{\dot{\beta}} w^{\dagger}_{\dot{\beta}} \right) + i\bar{\eta}^c_{\mu\nu} [a^\mu, a^\nu] \right\} \]

\[ - i\lambda \bar{\mu} (\bar{w}_{\dot{\alpha}} + \tilde{\bar{w}}_{\dot{\alpha}}) \mu + [a_\mu, M^{\dot{\alpha}}] \sigma_{\dot{\alpha} \dot{\alpha}} \}

\[ + \text{tr}_k \sum_f \left\{ \bar{w}_{\dot{\alpha}} \left[ q^{\dagger f} q_f + \tilde{q}^{\dagger f} \tilde{q}_f \right] w^{\dagger}_{\dot{\alpha}} - \frac{i}{2} \bar{\mu} \left[ q^{\dagger f} \mu'_f + \tilde{q}^{\dagger f} \tilde{q}_f \right] \mu \right\} . \]
The effective action
in an instantonic sector

There are other relevant diagrams involving the superpartners of $q$ and $\tilde{q}$, related to the above by susy Ward identities. Complete result:

$$q(x_0), \quad \tilde{q}(x_0) \rightarrow q(x_0, \theta), \quad \tilde{q}(x_0, \theta)$$

in $S_{mod}(q, \tilde{q}; M_k)$.

The moduli have to be integrated over
The instanton partition function
as an integral over moduli space

Summarizing, the effective action has the form (Higgs branch)

$$S_k = C_k e^{-\frac{8\pi^2}{g^2}} e^{A_{5a}} \int dM_k e^{-S_{mod}(q, \tilde{q}; M_k)}$$
The instanton partition function as an integral over moduli space

Summarizing, the effective action has the form (Higgs branch)

$$S_k = C_k e^{-\frac{8\pi^2}{g_a^2} k} e^{A'_{5a}} \int dM_k e^{-S_{mod}(q, \tilde{q}; M_k)}$$

- In $A'_{5a}$ the contribution of zero-modes running in the loop is suppressed because they’re already explicitly integrated over.

Blumenhagen et al, 2006
The instanton partition function
as an integral over moduli space

Summarizing, the effective action has the form (Higgs branch)

\[ S_k = C_k e^{-\frac{8\pi^2}{g_a^2} k} e^{\mathcal{A}' a} \int dM_k e^{-S_{\text{mod}}(q, \tilde{q}; M_k)} \]

\[ C_k \] is a normalization factor, determined (up to numerical constants) counting the dimensions of the moduli \( M_k \):

\[ C_k = \left( \sqrt{\alpha'} \right)^{-(3N_a - N_F)k} (g_a)^{-2N_a k}. \]

Notice the appearing of the \( \beta \)-function coeff. \( b_1 \).
Instanton induced superpotential

In $S_{\text{mod}}(q, \tilde{q}; \mathcal{M}_k)$, the superspace coordinates $x_0^\mu$ and $\theta^\alpha$ appear only through superfields $q(x_0, \theta), \tilde{q}(x_0, \theta), \ldots$

We can separate $x, \theta$ from the other moduli $\widehat{\mathcal{M}}_k$ writing

$$S_k = \int d^4 x_0 \, d^2 \theta \, W_k(q, \tilde{q}),$$

with the effective superpotential

$$W_k(q, \tilde{q}) = C_k \, e^{-\frac{8\pi^2}{g_a^2} k} \, e^{A_5^k} \int d\widehat{\mathcal{M}}_k \, e^{-S_{\text{mod}}(q, \tilde{q}; \widehat{\mathcal{M}}_k)}$$
A superpotential is expected to be holomorphic. We found

\[ W_k(q, \tilde{q}) = C_k e^{-\frac{8\pi^2}{g^2_a} k} e^{A'_a} \int d\hat{M}_k e^{-S_{mod}(q, \tilde{q}; \hat{M}_k)} \]
A superpotential is expected to be holomorphic. We found

\[ W_k(q, \tilde{q}) = C_k e^{-\frac{8\pi^2}{g_a^2} k} e^{A'_5 a} \int d\hat{M}_k e^{-S_{\text{mod}}(q, \tilde{q}; \hat{M}_k)} \]

- \( S_{\text{mod}}(q, \tilde{q}; \hat{M}_k) \) explicitly depends on \( q^\dagger \) and \( \tilde{q}^\dagger \). This dependence disappears upon integrating over \( \hat{M}_k \) as a consequence of the cohomology properties of the integration measure.

- However, we have to re-express the result in terms of the SUGRA fields \( Q \) and \( \tilde{Q} \)
A superpotential is expected to be holomorphic. We found

\[ W_k(q, \tilde{q}) = C_k \, e^{-\frac{8\pi^2}{g_a^2} k} \, e^{A_5^a} \int d\tilde{M}_k \, e^{-S_{\text{mod}}(q, \tilde{q}; \tilde{M}_k)} \]

The prefactors should combine into a dynamically generated holomorphic scale \( \Lambda_{\text{hol}} \), obtained by integrating the Wilsonian \( \beta \)-function of the \( \mathcal{N} = 1 \) SQCD

Novikov et al, 1983; Dorey et al, 2002; ...

To this effect, it is crucial that \( A_5^a \) can introduce a non-holomorphic dependence on the complex and Kähler structure moduli of the compactification space.

Our aim is to consider the interplay of all these observations. For this we need the explicit expression of the mixed annuli term \( A_5^a \)
The ADS/TVY superpotential

To be concrete, let's focus on the single instanton case, $k = 1$. In this case, the integral over the moduli can be carried out explicitly.

- Balancing the fermionic zero-modes requires $N_F = N_a - 1$
- The end result is

\[
W_{k=1}(q, \tilde{q}) = C_k e^{-\frac{8\pi^2}{g_a^2}k} e^{A'_{5a}} \frac{1}{\det(\tilde{q}q)}
\]

Dorey et al, 2002; Akerblom et al, 2006; Argurio et al, 2007
The ADS/TVY superpotential

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- Same form as the ADS/TVY superpotential

Dorey et al, 2002; Akerblom et al, 2006; Argurio et al, 2007

Affleck et al, 1984; Taylor et al, 1983;
To be concrete, let’s focus on the single instanton case, $k = 1$. In this case, the integral over the moduli can be carried out explicitly.

- Balancing the fermionic zero-modes requires $N_F = N_a - 1$
- The end result is

$$W_{k=1}(q, \tilde{q}) = C_k e^{-\frac{8\pi^2}{g_a^2} k} e^{A_{5a}'} \frac{1}{\det(\tilde{q}q)}$$

- We’ll see how these factors conspire to give an holomorphic expression in the sugra variables $Q$ and $\tilde{Q}$
Instanton annuli and threshold corrections
The mixed annuli

The amplitude $\mathcal{A}_{5a}$ is a sum of cylinder amplitudes with a boundary on the $E5a$ (both orientations)

$$\mathcal{A}_{5a} = \mathcal{A}_{5a;9_a} + \mathcal{A}_{5a;9_b} + \mathcal{A}_{5a;9_c}$$
The mixed annuli

The amplitude $A_{5a}$ is a sum of cylinder amplitudes with a boundary on the $E_{5a}$ (both orientations)

$$A_{5a} = A_{5a;9_a} + A_{5a;9_b} + A_{5a;9_c}$$

- Both UV and IR divergent. The UV divergences (IR in the closed string channel) cancel if tadpole cancellation assumed. Regulate the IR with a scale $\mu$
The mixed annuli

The amplitude $\mathcal{A}_{5a}$ is a sum of cylinder amplitudes with a boundary on the $E_{5a}$ (both orientations)

\[ \mathcal{A}_{5a} = \mathcal{A}_{5a;9_a} + \mathcal{A}_{5a;9_b} + \mathcal{A}_{5a;9_c} \]

- There is a relation between these instantonic annuli and the running gauge coupling constant.

Back

Abel and Goodsell, 2006; Akerblom et al, 2006

\[ \mathcal{A}_{5a} = -\frac{8\pi^2k}{g_a^2(\mu)|_{1\text{-loop}}} . \]

- Indeed, in susy theories, mixed annuli compute the running coupling by expanding around the instanton bkg.

Billo et al, 2007
Computing the YM effective action using different backgrounds

There are two gauge backgrounds on which string theory is computable and yields the effective action for the gauge fields

- **Constant gauge field** $f$ (turned on a color D9-brane)
  - At tree level, the YM action $\frac{1}{g_a^2} \int d^4 x \ Tr \frac{1}{2} F_{\mu \nu}^2$ evaluates to
    \[ S(f) = \frac{\text{Vol}_4 f^2}{2 g_a^2} \]
  - At loop level, we have ($\Delta$ are threshold corrections)
    \[ S(f)\big|_{1-\text{loop}} = \left( \frac{b_1}{16\pi^2} \log \alpha' \mu^2 + \Delta \right) \text{Vol}_4 f^2 \]
    \[ = \frac{\text{Vol}_4 f^2}{2 g_a^2(\mu)|_{1-\text{loop}}} \]
Computing the YM effective action using different backgrounds

There are two gauge backgrounds on which string theory is computable and yields the effective action for the gauge fields:

- **Instanton background** (realized by $k$ E5 branes)
  - At tree level, the YM action evaluates to
    \[ S_{\text{inst}} = \frac{8\pi^2 k}{g_a^2} \]
  - At loop level, we have the analogous relation:
    \[ S_{\text{inst}}|_{1\text{-loop}} = \frac{8\pi^2 k}{g_{a(\mu)}^2|_{1\text{-loop}}} = A_{5a} \]

With susy, the 1-loop determinants of the non-zero-modes cancel out: the only effect is the renormalization of the gauge coupling constant.

Dadda et al, 1977; ...
Expression of the annuli

The explicit computation of the annuli confirms the relation of these annuli to the running coupling. Imposing the appropriate b.c.'s and GSO one starts from

\[ \int_0^{\infty} \frac{d\tau}{2\tau} \left[ \text{Tr}_{\text{NS}} \left( P_{\text{GSO}} P_{\text{orb.}} q^{L_0} \right) - \text{Tr}_{\text{R}} \left( P_{\text{GSO}} P_{\text{orb.}} q^{L_0} \right) \right] . \]

- For \( A_{5a;9a} \), KK copies of zero-modes on internal tori \( T_2^{(i)} \) give a (non-holomorphic) dependence on the Kähler and complex moduli.

- For \( A_{5a;9b} \) and \( A'_{5a;9c} \), the modes are twisted and the result depends from the angles \( \nu^{(i)}_{ba} \) and \( \nu^{(i)}_{ac} \).

Lüst and Stieberger, 2003
\[ A_{5a;9a} = -8\pi^2 k \left[ \frac{3N_a}{16\pi^2} \log(\alpha'\mu^2) \right. \]
\[ + \frac{N_a}{16\pi^2} \sum_i \log \left( U^{(i)}_2 T^{(i)}_2 (\eta(U^{(i)})^4) \right) \],

\[ A_{5a;9b} + A_{5a;9c} = 8\pi^2 k \left( \frac{N_F}{16\pi^2} \log(\alpha'\mu^2) \right. \]
\[ + \frac{N_F}{32\pi^2} \log \left( \Gamma_{ba} \Gamma_{ac} \right) \],
Expression of the annuli

Explicit result

\[ \mathcal{A}_{5a;9a} = -8\pi^2 k \left[ \frac{3N_a}{16\pi^2} \log(\alpha' \mu^2) \right. \]
\[ \left. + \frac{N_a}{16\pi^2} \sum_i \log \left( U_2^{(i)} T_2^{(i)} (\eta (U^{(i)})^4) \right) \right], \]

\[ \mathcal{A}_{5a;9b} + \mathcal{A}_{5a;9c} = 8\pi^2 k \left( \frac{N_F}{16\pi^2} \log(\alpha' \mu^2) \right. \]
\[ \left. + \frac{N_F}{32\pi^2} \log (\Gamma_{ba} \Gamma_{ac}) \right), \]

\[ \beta \text{-function coefficient of SQCD: } 3N_a - N_F \]
Expression of the annuli

Explicit result

\[ A_{5a;9a} = -8\pi^2 k \left[ \frac{3N_a}{16\pi^2} \log(\alpha' \mu^2) \right. \]
\[ + \frac{N_a}{16\pi^2} \sum_i \log \left( U_2^{(i)} T_2^{(i)} (\eta (U^{(i)})^4) \right) \],

\[ A_{5a;9b} + A_{5a;9c} = 8\pi^2 k \left( \frac{N_F}{16\pi^2} \log(\alpha' \mu^2) \right. \]
\[ + \frac{N_F}{32\pi^2} \log (\Gamma_{ba} \Gamma_{ac}) \],

Non-holomorphic threshold corrections
Expression of the annuli

Explicit result

\[ A_{5a;9a} = -8\pi^2 k \left[ \frac{3Na}{16\pi^2} \log(\alpha' \mu^2) \right. \]
\[ + \frac{Na}{16\pi^2} \sum_i \log \left( U_2^{(i)} T_2^{(i)} (\eta (U^{(i)})^4) \right) \],

\[ A_{5a;9b} + A_{5a;9c} = 8\pi^2 k \left( \frac{N_F}{16\pi^2} \log(\alpha' \mu^2) \right. \]
\[ + \frac{N_F}{32\pi^2} \log \left( \Gamma_{ba} \Gamma_{ac} \right) \],

\[ \Gamma_{ba} = \frac{\Gamma(1 - \nu_{ba}^{(1)})}{\Gamma(\nu_{ba}^{(1)})} \frac{\Gamma(\nu_{ba}^{(2)})}{\Gamma(1 - \nu_{ba}^{(2)})} \frac{\Gamma(\nu_{ba}^{(3)})}{\Gamma(1 - \nu_{ba}^{(3)})} \]
Holomorphicity properties
The holomorphic gauge coupling

Computing the pure instantonic disks and annuli yields the gauge coupling up to 1 loop in the form

\[ A_{1\text{-loop}} = - \frac{8\pi^2 k}{g_a^2(\mu)} = - \frac{8\pi^2 k}{g_a^2} + A_{5a} \]

The very general Kaplunovsky-Louis formula expresses the one-loop gauge coupling in terms of the wilsonian coupling \( 1/\bar{g}_a^2 = s_2 \) and of other tree-level quantities in the effective action.
Kaplunovsky-Louis relation at one loop

Dixon et al, 1991; Kaplunovsky and Louis, 1994-95; ...

\[
\frac{1}{g^2(\mu)} = \frac{1}{\tilde{g}^2} + \frac{1}{8\pi^2} \left[ \frac{b}{2} \log \frac{\mu^2}{M^2_P} - f^{(1)} - \frac{c}{2} K + T(G) \log \frac{1}{\tilde{g}^2} \right.
\]

\[- \sum_r n_r T(r) \log K_r \]

▶ Here \( T_A \) = generators of the gauge group, \( n_r \) = # chiral mult. in rep. \( r \) and

\[
T(r) \delta_{AB} = \text{Tr}_r(T_A T_B) \quad , \quad T(G) = T(\text{adj}) \]

\[
b = 3 \ T(G) - \sum_r n_r \ T(r) \quad , \quad c = T(G) - \sum_r n_r \ T(r) \]
Kaplunovsky-Louis relation

at one loop

Dixon et al, 1991; Kaplunovsky and Louis, 1994-95; ...

\[
\frac{1}{g^2(\mu)} = \frac{1}{\tilde{g}^2} + \frac{1}{8\pi^2} \left[ \frac{b}{2} \log \frac{\mu^2}{M_P^2} - f^{(1)} - \frac{c}{2} K + T(G) \log \frac{1}{\tilde{g}^2} \right. \\
\left. - \sum_r n_r T(r) \log K_r \right]
\]

▶ Holomorphic function
Kaplunovksy-Louis relation at one loop

\[
\frac{1}{g^2(\mu)} = \frac{1}{\bar{g}^2} + \frac{1}{8\pi^2} \left[ \frac{b}{2} \log \frac{\mu^2}{M_P^2} - f^{(1)} - \frac{c}{2} K + T(G) \log \frac{1}{\bar{g}^2} \right. \\
\left. - \sum_{r} n_r T(r) \log K_r \right]
\]

▶ Non-holomorphic corrections

Dixon et al, 1991; Kaplunovsky and Louis, 1994-95; ...
Kaplunovsky-Louis relation at one loop

Dixon et al, 1991; Kaplunovsky and Louis, 1994-95; ...

\[
\frac{1}{g^2(\mu)} = \frac{1}{\tilde{g}^2} + \frac{1}{8\pi^2} \left[ \frac{b}{2} \log \frac{\mu^2}{M_P^2} - f^{(1)} - \frac{c}{2} K + T(G) \log \frac{1}{\tilde{g}^2} - \sum_r n_r T(r) \log K_r \right]
\]

- Inside the square bracket the bulk Kähler potential $K$ and the Kähler metrics for the matter multiplets $K_r$ are at tree level
Kaplunovsky-Louis relation

at one loop

Dixon et al, 1991; Kaplunovsky and Louis, 1994-95; ...

\[
\frac{1}{g^2(\mu)} = \frac{1}{\tilde{g}^2} + \frac{1}{8\pi^2} \left[ \frac{b}{2} \log \frac{\mu^2}{M^2_P} - f^{(1)} - \frac{c}{2} K + T(G) \log \frac{1}{\tilde{g}^2} - \sum_r n_r T(r) \log K_r \right]
\]

The only place where the shift $\delta$ in the holomorphic coupling matters is the tree-level term. Moreover only a shift $\delta^{(0)}$ of order 1 in $g_s$ is relevant at this level!
Instantonic annuli
in Kaplunovsky-Louis form

The result for the instantonic annuli can be recast in the following form:

\[
\mathcal{A}_{5a} = -\frac{8\pi^2 k}{\tilde{g}_a^2} + k\left[ -\frac{3N_a - N_F}{2} \log \frac{\mu^2}{M_P^2} - N_a \sum_{i=1}^{3} \log \left( \eta(u^{(i)})^2 \right) \\
+ \frac{N_a - N_F}{2} K + N_a \log g_a^2 - \delta^{(0)} + \frac{N_F}{2} \log (Z_{ba} Z_{ac}) \right]
\]

with (similarly for \( Z_{ac} \))

\[
Z_{ba} = (4\pi s_2)^{-\frac{1}{4}} \left( t_2^{(1)} t_2^{(2)} t_2^{(3)} \right)^{-\frac{1}{4}} \left( u_2^{(1)} u_2^{(2)} u_2^{(3)} \right)^{-\frac{1}{2}} \left( \Gamma_{ba} \right)^{\frac{1}{2}}
\]

If \( \delta^{(0)} = 0 \), \( Z_{ba} \) coincides with the Kähler metric \( K_{ab} \) of the twisted matter
Instantonic annuli
in Kaplunovsky-Louis form

The result for the instantonic annuli can be recast in the following form:

\[ \mathcal{A}_{5a} = -\frac{8\pi^2 k}{g_a^2} + k \left[ -\frac{3N_a - N_F}{2} \log \frac{\mu^2}{M_P^2} - N_a \sum_{i=1}^{3} \log \left( \eta(u^{(i)})^2 \right) ight. \\
+ \frac{N_a - N_F}{2} K + N_a \log g_a^2 - \delta^{(0)} + \left. \frac{N_F}{2} \log(\mathcal{Z}_{ba} \mathcal{Z}_{ac}) \right] \]

If there is some one-loop shift of \( s_2 \) of order 1, i.e., \( \delta^{(0)} \neq 0 \), then we have

\[ K_{ab} = \chi_{ab} \mathcal{Z}_{ba} \]

with

\[ \delta^{(0)} + \frac{N_F}{2} \log \chi_{ab} \chi_{bc} = 0 \]
The Kähler metric for twisted matter

Thus, up to possible factors $\chi$ due to one-loop shifts $\delta^{(0)}$, the Kähler metric of chiral multiplets $Q$ arising from twisted $D9_a/D9_b$ strings is given by

$$K_Q = \left(4\pi s_2\right)^{-\frac{1}{4}} \left(t_2^{(1)} t_2^{(2)} t_2^{(3)}\right)^{-\frac{1}{4}} \left(u_2^{(1)} u_2^{(2)} u_2^{(3)}\right)^{-\frac{1}{2}} \left(\Gamma_{ba}\right)^{\frac{1}{2}}$$

with

$$\Gamma_{ba} = \frac{\Gamma(1 - \nu_{ba}^{(1)}) \Gamma(\nu_{ba}^{(2)}) \Gamma(\nu_{ba}^{(3)})}{\Gamma(\nu_{ba}^{(1)}) \Gamma(1 - \nu_{ba}^{(2)}) \Gamma(1 - \nu_{ba}^{(3)})}$$

This is very interesting because:

▸ for twisted fields, the Kähler metric cannot be derived from compactification of DBI
The Kähler metric for twisted matter

Thus, up to possible factors $\chi$ due to one-loop shifts $\delta^{(0)}$, the Kähler metric of chiral multiplets $Q$ arising from twisted $D9_a/D9_b$ strings is given by

$$K_Q = (4\pi s_2)^{-\frac{1}{4}} (t^{(1)}_2 t^{(2)}_2 t^{(3)}_2)^{-\frac{1}{4}} (u^{(1)}_2 u^{(2)}_2 u^{(3)}_2)^{-\frac{1}{2}} (\Gamma_{ba})^{\frac{1}{2}}$$

with

$$\Gamma_{ba} = \frac{\Gamma(1 - \nu^{(1)}_{ba}) \Gamma(\nu^{(2)}_{ba}) \Gamma(\nu^{(3)}_{ba})}{\Gamma(\nu^{(1)}_{ba}) \Gamma(1 - \nu^{(2)}_{ba}) \Gamma(1 - \nu^{(3)}_{ba})}$$

This is very interesting because:

- the part dependent on the twists, namely $\Gamma_{ba}$, is reproduced by a direct string computation

Lüst et al, 2004; Bertolini et al, 2005

- the prefactors, depending on the geometric moduli, are more difficult to get directly: the present suggestion is welcome!
The Kähler metric for twisted matter

Thus, up to possible factors $\chi$ due to one-loop shifts $\delta^{(0)}$, the Kähler metric of chiral multiplets $Q$ arising from twisted $D9_a/D9_b$ strings is given by

$$
\Gamma_{ba} = \frac{\Gamma(1 - \nu_{ba}^{(1)}) \Gamma(\nu_{ba}^{(2)}) \Gamma(\nu_{ba}^{(3)})}{\Gamma(\nu_{ba}^{(1)}) \Gamma(1 - \nu_{ba}^{(2)}) \Gamma(1 - \nu_{ba}^{(3)})}
$$

This is very interesting because:

- We have checked this expression against the known results for Yukawa couplings of magnetized branes: perfect consistency!

Cremades et al, 2004

N.B. This check also severely constrains the possible extra pre-factors $\chi_{ba}$, ....
Beside being related to the gauge thresholds, the instantonic annuli $A_{5a}$ are relevant because they enter the stringy instanton calculus.

In particular, the form of the $A_{5a}$ annuli is crucial for the holomorphicity properties of E5 non-perturbative contributions.

We consider for definiteness the ADS/TVY case.
We found (recall that $N_F = N_a - 1$ in this case)

$$W_{k=1}(q, \tilde{q}) = C_k e^{-\frac{8\pi^2}{g_a^2} k} e^{A'_5 a} \frac{1}{\text{det}(\tilde{q}q)}$$

- Insert the expression of the annuli, from which we must subtract the contrib. of the zero-modes running in the loop, which are responsible for the IR divergences.

- Use the natural UV cut-off of the low-energy theory, the Planck mass $M_P^2 = \frac{1}{\alpha'} e^{-\phi_{10}} s_2$ and write

$$A_{5a} = -k \frac{b_1}{2} \log \frac{\mu^2}{M_P^2} + A'_5 a$$
Back to the ADS/TVY superpotential

Making it holomorphic

\[ W_{k=1}(q, \tilde{q}) = C_k \ e^{-\frac{8\pi^2}{g_a^2} k} \ e^{A'_{5a}} \ \frac{1}{\det(\tilde{q} q)} \]

- Make explicit the prefactor \( C_k \)
- Allow for a possible shift in the gauge coupling:

\[ \frac{1}{g_a^2} = \frac{1}{\tilde{g}_a^2} + \frac{\delta}{8\pi^2} \]
Back to the ADS/TVY superpotential

Making it holomorphic

In this way we obtain

\[ W_{k=1} = e^{K/2} \prod_{i=1}^{3} \left( \eta(u^{(i)})^{-2N_a} \right) \left( \sqrt{\alpha'} \right)^{-b_1} e^{-\frac{8\pi^2}{g_a^2}} \]

\[ (K_{\tilde{Q}Q})^{\frac{N_a-1}{2}} \frac{1}{\det(\tilde{q}q)} \]

- Rescale the chiral multiplet to their sugra counterparts assuming \( K_{\tilde{Q}}, K_Q \) are the matter Kähler metrics
- Introduce the invariant scale in the Wilsonian scheme

\[ \Lambda_{\text{hol}}^{b_1} = \left( \sqrt{\alpha'} \right)^{-b_1} e^{-\frac{8\pi^2}{g_a^2}} \]
Back to the ADS/TVY superpotential

Making it holomorphic

We get thus

\[ W_{k=1} = e^{K/2} \prod_{i=1}^{3} \left( \eta(u^{(i)})^{-2N_a} \right) \Lambda_{\text{hol}}^{2N_a+1} \frac{1}{\text{det}(\tilde{Q} Q)} \]

\[ \equiv e^{K/2} \Lambda_{\text{hol}}^{2N_a+1} \frac{1}{\text{det}(\tilde{Q} Q)} \]

- In the second step the moduli dependent factors of \( \eta(u^{(i)}) \) are readorsed by a holomorphic redefinition of the scale.
- A part from the prefactor \( e^{K/2} \), the final expression is holomorphic in the variables of the Wilsonian scheme.
Back to the ADS/TVY superpotential

Making it holomorphic

We get thus

\[ W_{k=1} = e^{K/2} \prod_{i=1}^{3} (\eta(u^{(i)})^{-2N_a}) \Lambda_{\text{hol}}^{2N_a+1} \frac{1}{\det(\tilde{Q} Q)} \]

\[ \equiv e^{K/2} \Lambda_{\text{hol}}^{2N_a+1} \frac{1}{\det(\tilde{Q} Q)} \]

The rôle of the annuli in these non-perturbative considerations leads to equivalent information on the Kähler metric of the twisted matter as the comparison with the perturbative KL formula.
Remarks and conclusions

Also in $\mathcal{N} = 2$ toroidal models the instanton-induced superpotential is in fact holomorphic in the appropriate sugra variables if one includes the mixed annuli in the stringy instanton calculus.

W.r.t. to the “color” D9$_a$ branes, the E5$_a$ branes are ordinary instantons. For the gauge theories on the D9$_b$ or the D9$_c$, they would be exotic (less clear from the field theory viewpoint).

The study of the mixed annuli and their relation to holomorphicity can be relevant for exotic, new stringy effects as well.

Akerblom et al, 2007; Billo et al, 2007