Instanton Calculus In R-R Background And The Topological String

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Paris, September 28, 2006

INSTITUT HENRI POINCARE

Foreword

This talk is based on

M. Billo, M. Frau, F. Fucito and A. Lerda, arXiv:hep-th/0606013.

It of course builds over a vast literature. The few references scattered on the slides are by no means intended to be exhaustive. I apologize for the many relevant ones which will be missing.

Plan of the talk

1. Introduction
2. Microscopic string description of $\mathcal{N} = 2$ SYM
3. Instanton calculus by mixed string diagrams
4. Deformation from a graviphoton background
5. Relation to topological strings on CY

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Introduction

$\mathcal{N} = 2$ SYM and Seiberg-Witten solution

$\mathcal{N} = 2$ SYM theories in $d = 4$: an important test-bed for non-perturbative physics

- Seiberg-Witten: exact expression of the prepotential $\mathcal{F}(a)$ governing the low energy dynamics on the Coulomb branch using duality and monodromy properties; this involves an auxiliary Riemann surface
- "Geometrical engineering" construction [Kachru et al 1995, Katz et al 1996]: SW solution $\hookrightarrow$ Type IIB string theory on a "local" CY manifold $\mathfrak{M}$ whose geometric moduli are suitably related to the gauge theory quantities (Λ, a,...)

The quest for the multi-instanton contributions

Semi-classical limit: 1-loop plus instanton contributions ▸ Back

$$\mathcal{F}(a) = \frac{\mathrm{i}}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(a)$$

- Important task: compute the multi-instanton contributions $\mathcal{F}^{(k)}(a)$ within the "microscopic" description of the non-abelian gauge theory to check them against the SW solution
- Only recently fully accomplished using localization techniques to perform the integration over the moduli space of the ADHM construction of the super-instantons

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

The localizing deformation

Introduce a deformation of the ADHM measure on the moduli spaces exploiting the 4d chiral rotations symmetry of ADHM constraints.

- The deformed instanton partition function

$$Z(a,\varepsilon) = \sum_k Z^{(k)}(a,\varepsilon) = \sum_k \int d\widehat{\mathcal{M}}_{(k)} \mathrm{e}^{-\mathcal{S}_{\mathrm{mod}}(a,\varepsilon;\mathcal{M}_{(k)})}$$

  can then be computed using localization techniques using the topological twist of its supersymmetries. One has

$$Z(a,\varepsilon) = \exp\left(\frac{\mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon)}{\varepsilon^2}\right)$$

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon) = \mathcal{F}_{\mathrm{n.p.}}(a) = \text{non-pert. part of SW prepotential}$$

Multi-instanton calculus and topological strings

What about higher orders in the deformation parameter ε?

- Nekrasov's proposal: terms of order $\varepsilon^{2h} \leftrightarrow$ gravitational F-terms in the $\mathcal{N} = 2$ eff. action involving metric and graviphoton curvatures

  [Nekrasov 2002, Losev et al 2003, Nekrasov 2005]

$$\int d^4x (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

- When the effective $\mathcal{N} = 2$ theory is obtained from type II strings on CY via geometrical engineering, such terms
  - arise from world-sheets of genus h
  - are computed by the topological string [Bershadsky et al 1993, Antoniadis et al 1993]
- For the local CY describing the SU(2) theory the proposal has been tested [Klemm et al, 2002]

"Microscopic" string description of the gauge theory

The "semiclassical" approach to the low energy $\mathcal{N} = 2$ effective action can be given a simple (i.e., calculable) string theory realization

- The non-abelian gauge theory d.o.f. are realized by open strings attached to (fractional) D3 branes. In the $\mathcal{N} = 2$ case, the perturbative 1-loop contributions to the prepotential are easily retrieved. [Douglas-Li 1996, Lawrence et al 1998, ...]
- The instantonic sectors of gauge theories can be realized by including D(-1) branes (a.k.a. D-instantons)
  - The spectrum of the D3/D(-1) systems encodes the quantities of the mathematical ADHM construction of the instanton moduli spaces

    [Witten 1995, Douglas 1995, ... ; Dorey et al, 2002 (review)]

Instantonic effects and gravitational backgrounds

The description of gauge instantons via D(-1)-branes is more than a book-keeping device for the ADHM construction.

- The D(-1)'s act as sources that produce the actual profile of the gauge instanton solution [Billò et al 2002]
- The prescriptions of the "instantonic calculus" of correlators arise naturally [Polchinski 1994, Green-Gutperle 1997-1998, Billò et al 2002]

One can include closed string backgrounds producing interesting deformations of the gauge theory. For instance

- non-commutative theories from NSNS background $B_{\mu\nu}$
- non-anticommutative theories from specific RR backgrounds

[Billò et al 2004-2005,...]

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[Billò et al 2004-2005,...]

The aim of this work

- Reproduce the semiclassical instanton expansion of the low energy effective action for the $\mathcal{N} = 2$ SYM theory in the microscopic string realization via (fractional) D3/D(-1) branes

- Show that the inclusion of the graviphoton of the $\mathcal{N} = 2$ bulk sugra, which comes from the RR sector,
  - leads exactly to the localization deformation on the instanton moduli space which allows to perform the integration
  - produces in the effective action the gravitational F-terms which are computed by the topological string on local CY

The aim of this work

- Reproduce the semiclassical instanton expansion of the low energy effective action for the $\mathcal{N} = 2$ SYM theory in the microscopic string realization via (fractional) D3/D(-1) branes

- The situation is therefore as follows:

Microscopic string description

Geometrically engineered string description of l.e.e.t on local CY

deformed multi-instanton computations

topological string amplitudes at genus h

Gravitational F-term interactions

- The two ways to compute the same F-terms must coincide if the two descriptions are equivalent

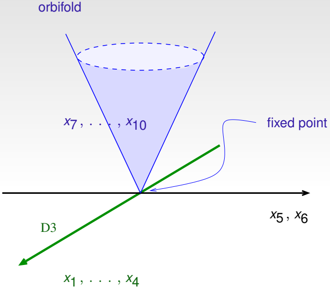
Microscopic string description of $\mathcal{N} = 2$ SYM

SYM from fractional branes

Consider pure SU(N) Yang-Mills in 4 dimensions with $\mathcal{N} = 2$ susy.

- It is realized by the massless d.o.f. of open strings attached to fractional D3-branes in the orbifold background

$$\mathbb{R}^4 \times \mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$$



- The orbifold breaks 1/2 SUSY in the bulk, the D3 branes a further 1/2:

$$32 \times \frac{1}{2} \times \frac{1}{2} = 8 \text{ real supercharges}$$

Fields and string vertices

- Field content: $\mathcal{N} = 2$ chiral superfield

$$\Phi(x,\theta) = \phi(x) + \theta\Lambda(x) + \frac{1}{2}\,\theta\sigma^{\mu\nu}\theta\, F^{+}_{\mu\nu}(x) + \cdots$$

- String vertices:

$$\begin{aligned} V_A(z) &= \frac{A_\mu(p)}{\sqrt{2}}\,\psi^\mu(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)} \\ V_\Lambda(z) &= \Lambda^{\alpha A}(p)\,S_\alpha(z)S_A(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z)} \\ V_\phi(z) &= \frac{\phi(p)}{\sqrt{2}}\,\overline{\Psi}(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)} \end{aligned}$$

with all polariz.s in the adjoint of $U(N)$

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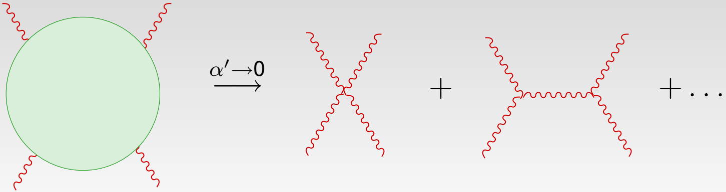
Gauge action from disks on fD3's



- String amplitudes on disks attached to the D3 branes in the limit

  $\alpha' \to 0$ with gauge quantities fixed.

  give rise to the tree level (microscopic) $\mathcal{N} = 2$ action

$$S_{\rm SYM} = \int d^4x \, \mathrm{Tr} \Big\{ \frac{1}{2} F_{\mu\nu}^2 + 2\, D_\mu \bar{\phi}\, D^\mu \phi - 2\, \bar{\Lambda}_{\dot{\alpha}A} \bar{D\!\!\!/}^{\dot{\alpha}\beta} \Lambda_\beta^A$$
$$+ \mathrm{i}\sqrt{2}\, g\, \bar{\Lambda}_{\dot{\alpha}A} \epsilon^{AB} \big[\phi, \bar{\Lambda}_B^{\dot{\alpha}} \big] + \mathrm{i}\sqrt{2}\, g\, \Lambda^{\alpha A} \epsilon_{AB} \big[\bar{\phi}, \Lambda_\alpha^B \big] + g^2 \big[\phi, \bar{\phi} \big]^2 \Big\}$$

Scalar v.e.v's and low energy effective action

- We are interested in the l.e.e.a. on the Coulomb branch parametrized by the v.e.v.'s of the adjoint chiral superfields:

$$\langle \Phi_{uv} \rangle \equiv \langle \phi_{uv} \rangle = a_{uv} = a_u \, \delta_{uv} \ , \quad u, v = 1, \ldots, N \ , \quad \sum_u a_u = 0$$

  breaking $SU(N) \to U(1)^{N-1}$ [we focus for simplicity on SU(2)]

- Up to two-derivatives, $\mathcal{N} = 2$ susy forces the effective action for the chiral multiplet Φ in the Cartan direction to be of the form

$$S_{\text{eff}}[\Phi] = \int d^4x \, d^4\theta \, \mathcal{F}(\Phi) + \mathrm{c.c}$$

- We want to discuss the instanton corrections to the prepotential $\mathcal{F}$ ▸ Recall in our string set-up

Instanton calculus by mixed string diagrams

Instantons and D-instantons

- Consider the Wess-Zumino term of the effective action for a stack of D3 branes:

$$\text{D. B. I.} + \int_{\mathrm{D}_3} \left[C_3 + \frac{1}{2} C_0 \,\mathrm{Tr}\big(F \wedge F \big) \right]$$

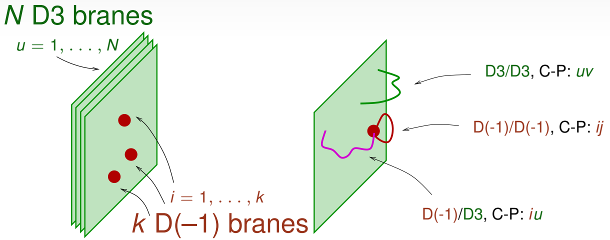
The topological density of an instantonic configuration corresponds to a localized source for the RR scalar C_0, i.e., to a distribution of D-instantons on the D3's.

- Instanton-charge k solutions of 3+1 dims. SU(N) gauge theories correspond to k D-instantons inside N D3-branes.

[Witten 1995,Douglas 1995, Dorey 1999, ...]

Stringy description of gauge instantons

	1	2	3	4	5	6	7	8	9	10
D3	—	—	—	—	$*$	$*$	$*$	$*$	$*$	$*$
D(−1)	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$



Moduli vertices and instanton parameters

Open strings with at least one end on a D(-1) carry no momentum: they are moduli, rather than fields $\leftrightarrow$ parameters of the instanton.

	ADHM	Meaning	Vertex	Chan-Paton
-1/-1 (NS)	a'_μ	*centers*	$\psi^\mu(z)\mathrm{e}^{-\varphi(z)}$	adj. $\mathrm{U}(k)$
	χ	*aux.*	$\overline{\Psi}(z)\mathrm{e}^{-\varphi(z)}$	⋮
(aux. vert.)	D_c	*Lagrange mult.*	$\bar\eta^c_{\mu\nu}\psi^\nu(z)\psi^\mu(z)$	⋮
(R)	$M^{\alpha A}$	*partners*	$S_\alpha(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	⋮
	$\lambda_{\dot\alpha A}$	*Lagrange mult.*	$S^{\dot\alpha}(z)S^A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	⋮
-1/3 (NS)	$w_{\dot\alpha}$	*sizes*	$\Delta(z)S^{\dot\alpha}(z)\,\mathrm{e}^{-\varphi(z)}$	$k \times \overline{N}$
	$\bar w_{\dot\alpha}$	⋮	$\overline{\Delta}(z)S^{\dot\alpha}(z)\mathrm{e}^{-\varphi(z)}$	⋮
(R)	μ^A	*partners*	$\Delta(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	⋮
	$\bar\mu^A$	⋮	$\overline{\Delta}(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	⋮

Super-coordinates and centred moduli

- Among the -1/-1 moduli we we can single out the center x_0^μ and its super-partners $\theta^{\alpha A}$: Back

$$
\begin{aligned}
a'^\mu &= x_0^\mu \, \mathbb{1}_{k\times k} + y_c^\mu \, T^c \quad (T^c = \text{gen.s of SU}(k)) \\
M^{\alpha A} &= \theta^{\alpha A} \, \mathbb{1}_{k\times k} + \zeta_c^{\alpha A} \, T^c
\end{aligned}
$$

  The moduli x_0^μ and $\theta^{\alpha A}$ decouple from many interactions and play the rôle of superspace coords

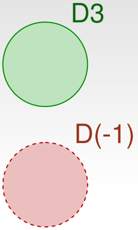
- We will distinguish the moduli $\mathcal{M}_{(k)}$ into

$$
\mathcal{M}_{(k)} \to \Big(x_0, \theta \, ; \, \widehat{\mathcal{M}}_{(k)} \Big)
$$

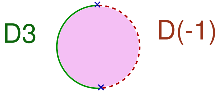
  $\widehat{\mathcal{M}}_{(k)}$ are the centred moduli

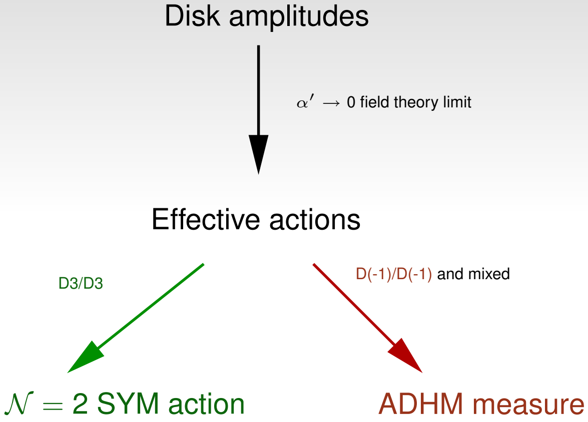
Disk amplitudes and effective actions

Usual disks:



Mixed disks:





The action for the moduli

From disk diagrams with insertion of moduli vertices, in the field theory limit we extract the ADHM moduli action (at fixed k)

$$\mathcal{S}_{\text{mod}} = \mathcal{S}_{\text{bos}}^{(k)} + \mathcal{S}_{\text{fer}}^{(k)} + \mathcal{S}_{\text{c}}^{(k)}$$

with

$$\begin{aligned}
\mathcal{S}_{\text{bos}}^{(k)} &= \text{tr}_k\Big\{ -2\,[\chi^\dagger, a'_\mu][\chi, a'^\mu] + \chi^\dagger \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi + \chi \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi^\dagger \Big\} \\
\mathcal{S}_{\text{fer}}^{(k)} &= \text{tr}_k\Big\{ \mathrm{i}\,\frac{\sqrt{2}}{2}\,\bar\mu^A \epsilon_{AB}\mu^B\chi^\dagger - \mathrm{i}\,\frac{\sqrt{2}}{4}\,M^{\alpha A}\epsilon_{AB}[\chi^\dagger, M^B_\alpha]\Big\} \\
\mathcal{S}_{\text{c}}^{(k)} &= \text{tr}_k\Big\{ -\mathrm{i}D_c\big(W^c + \mathrm{i}\bar\eta^c_{\mu\nu}\,[a'^\mu, a'^\nu]\big) \\
&\qquad -\mathrm{i}\lambda^{\dot\alpha}_A\big(\bar\mu^A w_{\dot\alpha} + \bar{w}_{\dot\alpha}\mu^A + [a'_{\alpha\dot\alpha}, M'^{\alpha A}]\big)\Big\}
\end{aligned}$$

- In $\mathcal{S}_{\text{c}}^{(k)}$ the bosonic and fermionic ADHM constraints appear (string moduli span the unconstrained parameter space)

Auxiliary moduli

The quartic interactions in $\mathcal{S}^{(k)}_{\text{bos}}$ can be disentangled using auxiliary moduli Y_μ, $X_{\dot\alpha}$ and $\bar X_{\dot\alpha}$:

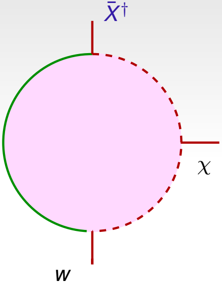
$$\begin{aligned}\mathcal{S}'^{(k)}_{\text{bos}} &= \text{tr}_k\Big\{2\,Y^\dagger_\mu Y^\mu + 2\,Y^\dagger_\mu\,[a'^\mu,\chi] + 2\,Y_\mu\,[a'^\mu,\chi^\dagger] \\ &+ \bar X^\dagger_{\dot\alpha} X^{\dot\alpha} + \bar X_{\dot\alpha} X^{\dagger\,\dot\alpha} + \bar X^\dagger_{\dot\alpha} w^{\dot\alpha}\chi + \bar X_{\dot\alpha} w^{\dot\alpha}\chi^\dagger - \chi\bar w_{\dot\alpha} X^{\dagger\,\dot\alpha} - \chi^\dagger \bar w_{\dot\alpha} X^{\dot\alpha}\Big\}\end{aligned}$$

- The corresponding auxiliary vertices are

$$\begin{aligned}V_Y(z) &= \sqrt{2}g_0\,Y_\mu\,\overline{\Psi}(z)\,\psi^\mu(z)\\ V_X(z) &= g_0\,X_{\dot\alpha}\,\Delta(z)S^{\dot\alpha}(z)\,\overline{\Psi}(z)\\ V_{\bar X}(z) &= g_0\,\bar X_{\dot\alpha}\,\overline{\Delta}(z)S^{\dot\alpha}(z)\,\overline{\Psi}(z)\end{aligned}$$

An example

One of the terms in $\mathcal{S}'^{(k)}_{\rm bos}$ (involving the auxiliary moduli):

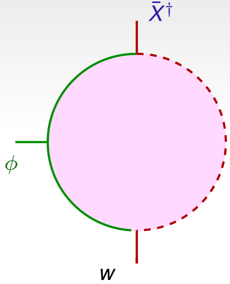


$$\left\langle\!\left\langle V_{\bar{X}^\dagger} V_w V_\chi \right\rangle\!\right\rangle$$
$$\equiv C_0 \int \frac{\prod_i dz_i}{dV_{\rm CKG}} \times \langle V_{\bar{X}^\dagger}(z_1) V_w(z_2) V_\chi(z_3)\rangle$$
$$= \ldots = \mathrm{tr}_k\left\{ \bar{X}^\dagger_{\dot\alpha}\, w^{\dot\alpha}\, \chi \right\}$$

- Here $C_0 = 8\pi^2/g^2$ is the normalization of $D(-1)$ disks.

Introducing scalar v.e.v.'s

To evaluate the effect of a v.e.v. $\langle \Phi_{uv} \rangle = a_{uv} = a_u \, \delta_{uv}$ compute systematically mixed disk diagrams with a constant scalar ϕ emitted from the D3 boundary. Example: Back



$$\left\langle\!\!\left\langle V_{\bar{X}^\dagger} \, V_{\phi=a} V_w \right\rangle\!\!\right\rangle = \ldots = \mathrm{tr}_k \left\{ \bar{X}^\dagger_{\dot\alpha} \, a \, w^{\dot\alpha} \right\}$$

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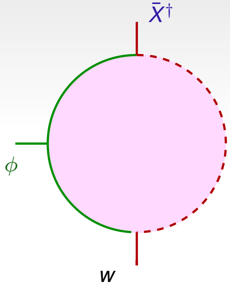

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- The resulting moduli action is obtained with the shifts Back

$$\chi_{ij} \, \delta_{uv} \;\to\; \chi_{ij} \, \delta_{uv} - \delta_{ij} \, a_{uv} \; , \quad \chi^\dagger_{ij} \, \delta_{uv} \;\to\; \chi^\dagger_{ij} \, \delta_{uv} - \delta_{ij} \, \bar{a}_{uv}$$

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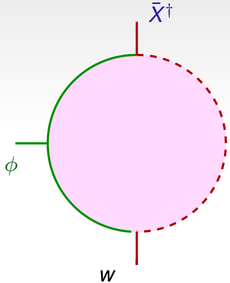

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- The action does not depend on the center super-coordinates x_0^{μ} and $\theta^{\alpha A}$ Recall

Holomorphicity, Q-exactness

In the action $\mathcal{S}_{\text{mod}}(a, \bar{a}; \mathcal{M}_{(k)})$ the v.e.v.'s a and $\bar{a}$ are not on the same footing: a does not appear in the fermionic action.

- The moduli action has the form

$$\mathcal{S}_{\text{mod}}(a, \bar{a}) = Q\,\Xi$$

where Q is the scalar twisted supercharge:

$$Q^{\dot{\alpha}B} \xrightarrow{\text{top. twist}} Q^{\dot{\alpha}\dot{\beta}} \,, \quad Q \equiv \frac{1}{2}\,\epsilon_{\dot{\alpha}\dot{\beta}}\, Q^{\dot{\alpha}\dot{\beta}}$$

- The parameter $\bar{a}$ appears only in the gauge fermion Ξ
- The instanton partition function

$$Z^{(k)}(a) \equiv \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-\mathcal{S}_{\text{mod}}(a,\bar{a})}$$

is independent of $\bar{a}$: variation w.r.t this parameter is Q-exact.

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Instanton calculus from the string standpoint

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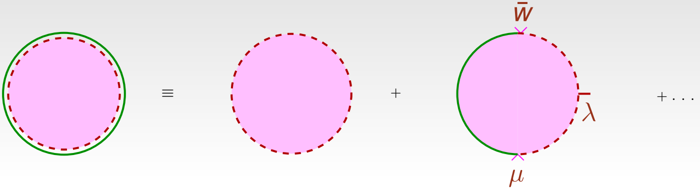

$$\stackrel{\alpha'\to 0}{\simeq} \; -\frac{8\pi^2\, k}{g^2} \; - \; \mathcal{S}_{\rm mod}$$

(the “pure” D(-1) disks yields kC_0 [Polchinski, 1994])

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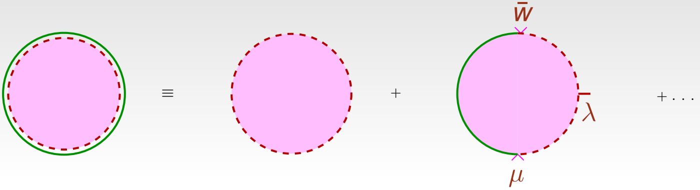

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- The combinatorics of boundaries [Polchinski, 1994] is such that these D-instanton diagrams exponentiate

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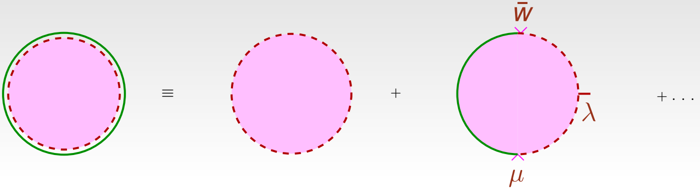


$$\stackrel{\alpha'\to 0}{\simeq} \; -\frac{8\pi^2 k}{g^2} \quad - \quad \mathcal{S}_{\rm mod}$$

(the “pure” D(-1) disks yields kC_0 [Polchinski, 1994])

- The moduli must be integrated over (path integral → ordinary integral):

$$Z^{(k)} = \int d\mathcal{M}_{(k)} \mathrm{e}^{-\frac{8\pi^2 k}{g^2} - \mathcal{S}_{\rm mod}}$$

Field-dependent moduli action

Consider correlators of D3/D3 fields, e.g of the scalar ϕ in the Cartan direction, in presence of k D-instantons. It turns out that

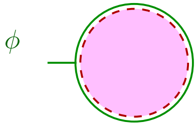
[Green-Gutperle 2000, Billò et al 2002]

- the dominant contribution to $\langle \phi_1 \dots \phi_n \rangle$ is from n one-point amplitudes on disks with moduli insertions. The result can be encoded in extra moduli-dependent vertices for ϕ's, i.e. in extra terms in the moduli action containing such one-point functions

$$\mathcal{S}_{\text{mod}}(\varphi; \mathcal{M}) = \phi(x_0) J_\phi(\widehat{\mathcal{M}}) + \mathcal{S}_{\text{mod}}(\widehat{\mathcal{M}})$$

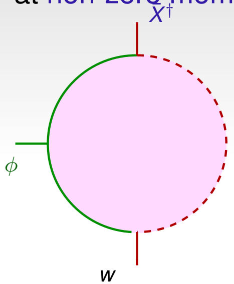
with

$$\phi(x_0) J_\phi(\widehat{\mathcal{M}}) \quad =$$



Moduli action with the unbroken scalar ϕ

The relevant one-point diagrams are those already computed to describe the dependence on the v.e.v. a. We just insert the vertex V_ϕ at non-zero momentum. ▸ Recall



$$\left\langle\!\left\langle V_{\bar{X}^\dagger} V_\phi V_w \right\rangle\!\right\rangle = \ldots = \mathrm{tr}_k \left\{ \bar{X}^\dagger_{\dot\alpha}\, \phi(x_0)\, w^{\dot\alpha} \right\}$$

We get a dependence on the instanton location x_0 (x from now on)

- The field-dependent action $\mathcal{S}_{\rm mod}(\phi; \mathcal{M})$ is thus simply obtained by

$$a \longrightarrow \phi(x)$$

Effective action for the unbroken multiplet

Other non-zero diagrams couple the components of the gauge supermultiplet to the moduli, related by the Ward identities of the susies broken by the D(-1).

- Example:

$$\left\langle\!\left\langle V_{\bar{X}^\dagger} V_{\delta\phi} V_w \right\rangle\!\right\rangle = \left\langle\!\left\langle V_{\bar{X}^\dagger} \left[\theta^{\alpha A} Q_{\alpha A}, V_\Lambda\right] V_w \right\rangle\!\right\rangle = -\left\langle\!\left\langle V_{\bar{X}^\dagger} V_\Lambda V_w \int V_\theta \right\rangle\!\right\rangle$$

  The contribution of the last diagram can be obtained simply by letting $\phi \to \theta\Lambda$

- This iterates: further couplings with higher components of Φ and more θ-insertions

Effective action for the unbroken multiplet

Other non-zero diagrams couple the components of the gauge supermultiplet to the moduli, related by the Ward identities of the susies broken by the D(-1).

- The superfield-dependent moduli action $\mathcal{S}_{\text{mod}}(\Phi; \mathcal{M})$ is obtained by simply letting

$$a \to \Phi(x, \theta)$$

Contributions to the prepotential

Integrating over the moduli the interactions described by $\mathcal{S}_{\rm mod}(\Phi; \mathcal{M}(k))$ one gets the effective action for the long-range multiplet Φ induced by the k-th instanton sector:

$$S_{\rm eff}^{(k)}[\Phi] = \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\rm mod}(\Phi;\mathcal{M}(k))}$$

Correspondingly, the prepotential for the low energy $\mathcal{N} = 2$ theory is given by the centred instanton partition function

$$\mathcal{F}^{(k)}(\Phi) = \int d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\rm mod}(\Phi;\mathcal{M}(k))}$$

- The superfield $\Phi(x,\theta)$ is a constant w.r.t. $\widehat{\mathcal{M}}_{(k)}$. We can compute $\mathcal{F}^{(k)}$ fixing $\Phi(x,\theta) \to a$ and using the results of the literature

[see e.g. Dorey et al, 2002]

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Contributions to the prepotential (2)

One finds (Λ is the dynamical scale)

$$\mathcal{F}^{(k)}(\Phi) = c_k \, \Phi^2 \left(\frac{\Lambda}{\Phi} \right)^{4k} \quad .$$

- Λ^{4k} stems from the term $\exp(-8\pi k/g^2)$, using the β-function of the $\mathcal{N} = 2$, SU(2) theory
- The coefficients c_k (the hard part!) were finally determined using a deformation of the moduli action which localizes the integration

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

We will now embed this localization deformation in our string set-up, recognizing it as the effect of a graviphoton background and deriving gravitational corrections to the non-perturbative prepotential

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Deformation from a graviphoton background

The Weyl multiplet

- The field content of $\mathcal{N} = 2$ sugra:

$$h_{\mu\nu} \text{ (metric)} , \quad \psi_\mu^{\alpha A} \text{ (gravitini)} , \quad C_\mu \text{ (graviphoton)}$$

  can be organized in a chiral Weyl multiplet:

$$W^+_{\mu\nu}(x,\theta) = \mathcal{F}^+_{\mu\nu}(x) + \theta\chi^+_{\mu\nu}(x) + \frac{1}{2}\,\theta\sigma^{\lambda\rho}\theta\, R^+_{\mu\nu\lambda\rho}(x) + \cdots$$

  (${\chi_{\mu\nu}}^{\alpha A}$ is the gravitino field strength)
- These fields arise from massless vertices of type IIB strings on $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$

Graviphoton vertex

The graviphoton vertex, connected by the supercharges to the other fields in the Weyl multiplet, is given by

$$\begin{aligned} V_{\mathcal{F}}(z,\bar{z}) &= \frac{1}{4\pi}\mathcal{F}^{\alpha\beta AB}(p) \\ &\times \left[S_\alpha(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}S_\beta(\bar{z})S_B(\bar{z})\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\right]\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})} \end{aligned}$$

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- We will insert the closed string vertices in disk diagrams bounded by the branes → suitable identifications between left- and right-movers taken into account

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- The bi-spinor graviphoton polarization is given by

$$\mathcal{F}^{(\alpha\beta)[AB]} = \frac{\sqrt{2}}{4}\,\mathcal{F}^{+}_{\mu\nu}\,(\sigma^{\mu\nu})^{\alpha\beta}\,\epsilon^{AB}$$

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- A different RR field, with a similar structure, will be useful:

$$V_{\bar{\mathcal{F}}}(z,\bar{z}) = \frac{1}{4\pi}\bar{\mathcal{F}}^{\alpha\beta\hat{A}\hat{B}}(p) \times \left[S_\alpha(z)S_{\hat{A}}(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}S_\beta(\bar{z})S_{\hat{B}}(\bar{z})\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\right]\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})}$$

$\hat{A},\hat{B} = 3,4 \leftrightarrow$ odd "internal" spin fields ‣ Recall

Effect of the graviphoton on the instanton measure

Let us investigate the effect of a graviphoton v.e.v.

$$\langle W^{+}_{\mu\nu} \rangle \equiv \langle \mathcal{F}^{+}_{\mu\nu} \rangle \equiv f_{\mu\nu}$$

on the moduli measure.

- We have to consider disk amplitudes with open string moduli vertices on the boundary and closed string graviphoton vertices in the interior which survive in the field theory limit $\alpha' \to 0$.
- We will consider also insertions of vertices of type $V_{\bar{\mathcal{F}}}$, with constant polarization $\bar{\mathcal{F}}^{+}_{\mu\nu} = \bar{f}_{\mu\nu}$

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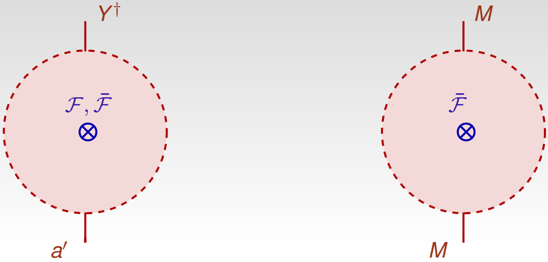
Non-zero diagrams

Very few diagrams contribute.



- The only one involving the true graviphoton is

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle \equiv C_0 \int \frac{dz_1\, dz_2\, dw d\bar{w}}{dV_{\mathrm{CKG}}} \langle V_{Y^\dagger}(z_1)\, V_{a'}(z_2)\, V_{\mathcal{F}}(w,\bar{w})\rangle$$

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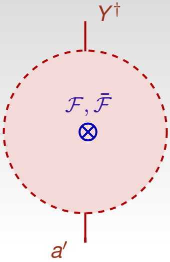

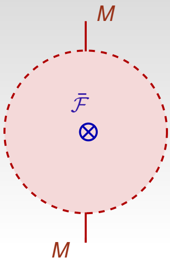

- More explicitly,

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle = \frac{1}{4\pi}\mathrm{tr}_k\left\{ Y^\dagger_\mu\, a'_\nu\, f_{\lambda\rho} \right\}\left(\sigma^{\lambda\rho}\right)^{\alpha\beta}\epsilon^{AB}\int \frac{dz_1\, dz_2 dw d\bar{w}}{dV_{\mathrm{CKG}}} \times$$

$$\langle \mathrm{e}^{-\varphi(z_2)}\mathrm{e}^{-\frac{1}{2}\varphi(w)}\mathrm{e}^{-\frac{1}{2}\varphi(\bar{w})}\rangle\langle \Psi(z_1)S_A(w)S_B(\bar{w})\rangle$$

$$\langle \psi^\mu(z_1)\psi^\nu(z_2)S_\alpha(w)S_\beta(\bar{w})\rangle$$

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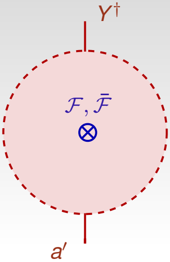

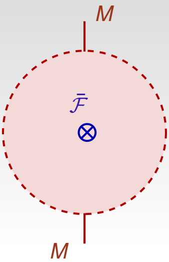

- Result: (same also with $\bar{f}^{\mu\nu}$)

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle = -4\mathrm{i}\,\mathrm{tr}_k \left\{ Y^\dagger_\mu\, a'_\nu\, f^{\mu\nu} \right\}$$

- Moreover, term with fermionic moduli and a $V_{\bar{\mathcal{F}}}$:

$$\left\langle\!\!\left\langle V_M\, V_M\, V_{\bar{\mathcal{F}}} \right\rangle\!\!\right\rangle = \frac{1}{4\sqrt{2}}\mathrm{tr}_k \left\{ M^{\alpha A} M^{\beta B} \bar{f}_{\mu\nu} \right\} (\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{AB}$$

The deformed moduli action

Including the backgrounds $f, \bar{f}$ besides the chiral v.e.v.'s $a, \bar{a}$: ▸ Back

$$
\begin{aligned}
&\mathcal{S}_{\text{mod}}(a, \bar{a}; f, \bar{f}) = \\
&-\mathrm{tr}_k\Big\{\big([\chi^\dagger, a'_{\alpha\dot{\beta}}] + 2\bar{f}_c(\tau^c a')_{\alpha\dot{\beta}}\big)\big([\chi, a'^{\dot{\beta}\alpha}] + 2f_c(a'\tau^c)^{\dot{\beta}\alpha}\big) \\
&-\big(\chi^\dagger \bar{w}_{\dot{\alpha}} - \bar{w}_{\dot{\alpha}}\,\bar{a}\big)\big(w^{\dot{\alpha}}\chi - a\,w^{\dot{\alpha}}\big) - \big(\chi\bar{w}_{\dot{\alpha}} - \bar{w}_{\dot{\alpha}}\,a\big)\big(w^{\dot{\alpha}}\chi^\dagger - \bar{a}\,w^{\dot{\alpha}}\big)\Big\} \\
&+\mathrm{i}\,\frac{\sqrt{2}}{2}\,\mathrm{tr}_k\Big\{\bar{\mu}^A\epsilon_{AB}\big(\mu^B\chi^\dagger - \bar{a}\,\mu^B\big) \\
&-\frac{1}{2}M^{\alpha A}\epsilon_{AB}\big([\chi^\dagger, M^B_\alpha] + 2\,\bar{f}_c\,(\tau^c)_{\alpha\beta}M^{\beta B}\big)\Big\} + S_{\text{c}}^{(k)}
\end{aligned}
$$

- The constraint part of the action, $S_{\text{c}}^{(k)}$, is not modified

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&+\mathrm{i}\,\frac{\sqrt{2}}{2}\,\mathrm{tr}_k\Big\{\bar\mu^A\epsilon_{AB}\big(\mu^B\chi^\dagger - \bar{a}\,\mu^B\big) \\
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\end{aligned}
$$

- The effect of the $f, \bar{f}$ background amounts to the shift Recall

$$
[\chi,(\bullet)_\alpha] \to [\chi,(\bullet)_\alpha] + 2f_c(\tau^c\,\bullet)_\alpha\;,\quad [\chi^\dagger,(\bullet)_\alpha] \to [\chi^\dagger,(\bullet)_\alpha] + 2\bar{f}_c(\tau^c\,\bullet)_\alpha
$$

Holomorphicity, Q-exactness

Also the deformed moduli action has the form

$$\mathcal{S}_{\text{mod}}(a, \bar{a}; f, \bar{f}) = Q\,\Xi$$

where Q is the scalar twisted supercharge.

- The parameters $\bar{a}$,$\bar{f}_c$ appear only in the gauge fermion Ξ
- The instanton partition function

$$Z^{(k)} \equiv \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-\mathcal{S}_{\text{mod}}(a,\bar{a};f,\bar{f})}$$

  is independent of $\bar{a}$,$\bar{f}_c$: variation w.r.t these parameters is Q-exact.

Graviphoton and localization

The moduli action obtained inserting the graviphoton background coincides exactly with the "deformed" action considered in the literature to localize the moduli space integration if we set

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

$$f_c = \frac{\varepsilon}{2}\,\delta_{3c}\ ,\quad \bar{f}_c = \frac{\bar{\varepsilon}}{2}\,\delta_{3c}\ ,$$

and moreover (referring to the notations in the above ref.s)

$$\varepsilon = \bar{\varepsilon}\ ,\quad \varepsilon = \epsilon_1 = -\epsilon_2$$

- The "shift" rule which yields the deformed action was interpreted as "gauging" the chiral rotations in 4d Euclidean space which are symmetries of the ADHM constraints

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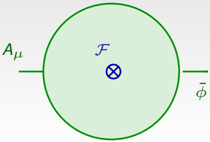
The localization deformation of the $\mathcal{N} = 2$ ADHM construction is produced, in the type IIB string realization, by a graviphoton background

Deformation of the gauge action



- The graviphoton background can be inserted also in D3 disks, producing extra terms in the gauge theory action on the D3 branes
- Collecting all diagrams, one gets

$$S_{\text{SYM}} + \int d^4x \text{Tr}\Big\{ -2\,\text{i}\,g\,F_{\mu\nu}\bar\phi f^{\mu\nu} - g^2\big(\bar\phi f^{\mu\nu}\big)^2 \Big\}$$

(in agreement with couplings between gauge and Weyl multiplets in $\mathcal{N}=2$ sugra)

- At linear order in g, field eq.s for ϕ

$$D^2\phi = -\text{i}\sqrt{2}g\epsilon_{AB}\Lambda^{\alpha A}\Lambda_\alpha^B - 2\text{i}\,g\,f_{\mu\nu}F^{\mu\nu}$$

agree with the one implied by the deformed ADHM construction

Weyl multiplet dependence of the effective prepotential

- Just as for the case of the scalar v.e.v's only, we can compute

$$\mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))$$

containing the one-point couplings of the fields in the gauge and Weyl multiplets to the moduli by simply promoting

$$a \to \Phi(x;\theta)\ , \quad f_{\mu\nu} \to W_{\mu\nu}(x,\theta)$$

- The l.e.e.a. for Φ and W^+ in the instanton # k sector is

$$S_{\text{eff}}^{(k)}[\Phi, W^+] = \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))}$$

and the prepotential reads thus

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Expansion of the prepotential

$\Phi(x,\theta)$ and $W^{+}_{\mu\nu}(x,\theta)$ are constant w.r.t. the integration variables $\widehat{\mathcal{M}}_{(k)}$.
We can compute $\mathcal{F}^{(k)}(a;f)$ and reinstate the full multiplets in the result.

- From the explicit form of $\mathcal{S}_{\rm mod}(a,0;f,0)$ Recall it follows that partition function $\mathcal{F}^{(k)}(a;f)$ is invariant under

$$a, f_{\mu\nu} \to -a, -f_{\mu\nu}$$

- We need a regular expansion for $f \to 0$, and no odd powers of $af_{\mu\nu}$, Altogether, reinstating the superfields,

$$\mathcal{F}^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} c_{k,h}\, \Phi^2 \left(\frac{\Lambda}{\Phi}\right)^{4k} \left(\frac{W^+}{\Phi}\right)^{2h}$$

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The non-perturbative prepotential

Sum over the instanton sectors:

$$\mathcal{F}_{\text{n.p.}}(\Phi, W^+) = \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} C_h(\Lambda, \Phi)(W^+)^{2h}$$

with

$$C_h(\Lambda, \Phi) = \sum_{k=1}^{\infty} c_{k,h} \, \frac{\Lambda^{4k}}{\Phi^{4k+2h-2}}$$

- Many different terms in the eff. action connected by susy. Saturating the θ integration with four θ's all from W^+

$$\int d^4x \; C_h(\Lambda, \phi) \, (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

Freezing $\phi \to a$, this is a purely gravitational F-term

The non-perturbative prepotential

Sum over the instanton sectors:

$$\mathcal{F}_{\text{n.p.}}(\Phi, W^{+}) = \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(\Phi, W^{+}) = \sum_{h=0}^{\infty} C_h(\Lambda, \Phi)(W^{+})^{2h}$$

with

$$C_h(\Lambda, \Phi) = \sum_{k=1}^{\infty} c_{k,h} \frac{\Lambda^{4k}}{\Phi^{4k+2h-2}}$$

- Many different terms in the eff. action connected by susy. Saturating the θ integration with four θ's all from W^{+}

$$\int d^4x \; C_h(\Lambda, \phi)\,(R^{+})^2(\mathcal{F}^{+})^{2h-2}$$

Freezing $\phi \to a$, this is a purely gravitational F-term

Evaluation via localization

- To determine the coefficients $c_{k,h}$, constant bkg values $\Phi \to a$ and $W^{+}_{\mu\nu} \to f_{\mu\nu}$ are enough.
- The localization deformation of the multi-instanton partition function $Z^{(k)}(a,\varepsilon)$ is obtained for

$$f_{\mu\nu} = \frac{1}{2}\,\varepsilon\,\eta^{3}_{\mu\nu}\,, \quad \bar{f}_{\mu\nu} = \frac{1}{2}\,\bar{\varepsilon}\,\eta^{3}_{\mu\nu}$$

  - (Holomorphicity:) $Z^{(k)}(a,\varepsilon)$ does not smoothly depend on $\bar{\varepsilon}$
  - However, $\bar{\varepsilon} = 0$ is a limiting case: some care is needed
- $\mathcal{F}^{(k)}(a;\varepsilon)$ is well-defined. $S^{(k)}[a;\varepsilon]$ diverges because of the (super)volume integral $\int d^4x\, d^4\theta$
  - $\bar{\varepsilon}$ regularizes the superspace integration by a Gaussian term. One can then work with the effective action, i.e., the *full* instanton partition function

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Example: the case $k = 1$

The moduli action in the $k = 1$ sector (at $\bar{a} = 0$) is

$$\mathcal{S}_{\text{mod}}^{(k=1)} = -2\bar{\varepsilon}\varepsilon\, x^2 - \frac{\bar{\varepsilon}}{2}\,\theta^{\alpha A}\epsilon_{AB}(\tau_3)_{\alpha\beta}\theta^{\beta B} + \widehat{\mathcal{S}}_{\text{mod}}^{(k=1)}(a)$$

The last term does not depend on $\varepsilon, \bar{\varepsilon}$

- Using the $\varepsilon, \bar{\varepsilon}$-independence of the centred partition function

$$\mathcal{F}^{(k=1)}(a) = \int d\widehat{\mathcal{M}}^{(k=1)} \mathrm{e}^{-\frac{8\pi^2}{g^2} - \widehat{\mathcal{S}}_{\text{mod}}^{(k=1)}(a)}$$

we have for the $k = 1$ instanton partition function

$$Z^{(k=1)}(a, \varepsilon) = \int d^4x\, d^4\theta \mathrm{e}^{-2\bar{\varepsilon}\varepsilon x^2 - \frac{1}{2}\bar{\varepsilon}\,\theta\cdot\theta}\, \mathcal{F}^{(k=1)}(a) = \frac{1}{\varepsilon^2}\mathcal{F}^{(k=1)}(a)$$

- Effectively, with the full deformation, we have the rule

$$\int d^4x\, d^4\theta \to \frac{1}{\varepsilon^2}$$

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The deformed partition function vs the prepotential

- With $\varepsilon, \bar{\varepsilon}$, the partition function $Z(a; \varepsilon)$ can be computed:
  - (super)volume divergences $\rightarrow \varepsilon$ singularities
  - a and $\varepsilon, \bar{\varepsilon}$ deformations localize completely the integration over moduli space which can be carried out

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

- With $\bar{\varepsilon} \neq 0$ (complete localization) a trivial superposition of instantons of charges k_i contributes to the sector $k = \sum k_i$
- Such disconnected configurations do *not* contribute when $\bar{\varepsilon} = 0$
- The partition function computed by localization corresponds to the exponential of the non-perturbative prepotential:

$$
\begin{aligned}
Z(a; \varepsilon) &= \exp\left(\frac{\mathcal{F}_{\text{n.p.}}(a, \varepsilon)}{\varepsilon^2}\right) = \exp\left(\sum_{k=1}^{\infty} \frac{\mathcal{F}^{(k)}(a, \varepsilon)}{\varepsilon^2}\right) \\
&= \exp\left(\sum_{h=0}^{\infty} \sum_{k=1}^{\infty} c_{k,h} \frac{\varepsilon^{2h-2}}{a^{2h}} \left(\frac{\Lambda}{a}\right)^{4k}\right)
\end{aligned}
$$

Summarizing

- The computation via localization techniques of the multi-instanton partition function $Z(a; \varepsilon)$ determines the coefficients $c_{k,h}$ which appear in the gravitational F-terms of the $\mathcal{N} = 2$ effective action

$$\int d^4x \; C_h(\Lambda, \phi) \, (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

via the relation

$$C_h(\Lambda, \phi) = \sum_{k=1}^{\infty} c_{k,h} \, \frac{\Lambda^{4k}}{\phi^{4k+2h-2}}$$

- The very same gravitational F-terms can been extracted in a completely different way: topological string amplitudes on suitable Calabi-Yau manifolds

Summarizing

- The computation via localization techniques of the multi-instanton partition function $Z(a;\varepsilon)$ determines the coefficients $c_{k,h}$ which appear in the gravitational F-terms of the $\mathcal{N}=2$ effective action

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Relation to topological strings on CY

Geometrical engineering and topological strings

- SW: low energy $\mathcal{N} = 2 \leftrightarrow$ (auxiliary) Riemann surface
- Geometrical engineering: embed directly the low energy theory into string theory as type IIB on a suitable local CY manifold $\mathfrak{M}$

  [Kachru et al 1995, Klemm et al 1996-97]

  - geometric moduli of $\mathfrak{M} \leftrightarrow$ gauge theory data (Λ, a);
  - The coefficients C_h in the l.e.e.a. gravitational F-terms

$$C_h \, (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

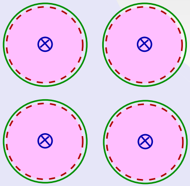
  are given by topological string amplitudes at genus h

  [Bershadsky et al 1993-94, Antoniadis et al 1993]

- For the local CY $\mathfrak{M}_{\mathrm{SU}(2)}$ the couplings C_h were checked to coincide with those given by the deformed multi-instanton calculus as proposed by Nekrasov [Klemm et al 2002]

Microscopic vs effective string description

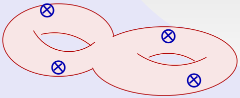
Orbifold space
with D3/D(-1) system
Moduli action depends on
gauge theory data Λ, a
open and closed strings



$2h$ disks
connected by
integration over
moduli

$\chi = 2h - 2$

Local CY manifold
Geometric moduli determined
from gauge theory data Λ, a
No branes - closed strings only



genus h Riemann surface

$\chi = 2h - 2$

Same gravitational F-term interactions

$$C_h(\Lambda, a)\,(R^+)^2(\mathcal{F}^+)^{2h-2}$$

Perspectives

Some interesting directions to go...

- Study of the instanton corrections to $\mathcal{N} = 2$ eff. theory in the gauge/gravity context: modifications of the classical solution of fD3's
- Application of similar techniques to (euclidean) D3's along a CY orbifold to derive BH partition functions in $\mathcal{N} = 2$ sugra (which OSV conjecture relates to $|Z_{\text{top}}|^2$)
- Non-perturbative corrections to $\mathcal{N} = 1$ superpotentials by Euclidean D3's along orbifold directions
- ...

Some references

Multi-instanton contributions in $\mathcal{N} = 2$

- N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Phys. Rept. **371** (2002) 231 [arXiv:hep-th/0206063].
- N. A. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831 [arXiv:hep-th/0206161].
- R. Flume and R. Poghossian, Int. J. Mod. Phys. A **18** (2003) 2541 [arXiv:hep-th/0208176].
- A. S. Losev, A. Marshakov and N. A. Nekrasov, [arXiv:hep-th/0302191]; N. Nekrasov and A. Okounkov, [arXiv:hep-th/0306238].
- N. A. Nekrasov, Class. Quant. Grav. **22** (2005) S77.

Geometrical engineering, topological amplitudes

- M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. B **405** (1993) 279 [arXiv:hep-th/9302103]; Commun. Math. Phys. **165** (1994) 311 [arXiv:hep-th/9309140].
- I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B **413** (1994) 162 [arXiv:hep-th/9307158].
- S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B **459**, 537 (1996) [arXiv:hep-th/9508155]; A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. P. Warner, Nucl. Phys. B **477**, 746 (1996) [arXiv:hep-th/9604034].
- S. Katz, A. Klemm and C. Vafa, Nucl. Phys. B **497**, 173 (1997) [arXiv:hep-th/9609239].
- A. Klemm, M. Marino and S. Theisen, JHEP **0303** (2003) 051 [arXiv:hep-th/0211216].

String perspective on instanton calculus

- E. Witten, Nucl. Phys. B **460** (1996) 335 [arXiv:hep-th/9510135].
- M. R. Douglas, J. Geom. Phys. **28**, 255 (1998) [arXiv:hep-th/9604198]; arXiv:hep-th/9512077.
- J. Polchinski, Phys. Rev. D **50** (1994) 6041 [arXiv:hep-th/9407031].
- M.B. Green and M. Gutperle, Nucl. Phys. B **498** (1997) 195, [arXiv:hep-th/9701093]; JHEP **9801** (1998) 005, [arXiv:hep-th/9711107]; Phys. Rev. D **58** (1998) 046007, [arXiv:hep-th/9804123].
- M.B. Green and M. Gutperle, JHEP **0002** (2000) 014 [arXiv:hep-th/0002011].
- M. Billo, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, JHEP **0302** (2003) 045 [arXiv:hep-th/0211250].

Some notations

String fields in the orbifold space

- In the six directions transverse to the brane,

$$Z \equiv (X^5 + \mathrm{i}X^6)/\sqrt{2}\,, \quad Z^1 \equiv (X^7 + \mathrm{i}X^8)/\sqrt{2}\,, \quad Z^2 \equiv (X^9 + \mathrm{i}X^{10})/\sqrt{2}\,,$$
$$\Psi \equiv (\psi^5 + \mathrm{i}\psi^6)/\sqrt{2}\,, \quad \Psi^1 \equiv (\psi^7 + \mathrm{i}\psi^8)/\sqrt{2}\,, \quad \Psi^2 \equiv (\psi^9 + \mathrm{i}\psi^{10})/\sqrt{2}$$

  the $\mathbb{Z}_2$ orbifold generator h acts by

$$(Z^1, Z^2) \to (-Z^1, -Z^2)\,, \quad (\Psi^1, \Psi^2) \to (-\Psi^1, -\Psi^2)$$

- Under the SO(10) → SO(4) × SO(6) induced by D3's, $S^{\dot{\mathcal{A}}} \to (S_\alpha S_{A'}, S^{\dot\alpha} S^{A'})$
- Under SO(6) → SO(2) × SO(4) induced by the orbifold, ▸ Back

$S^{A'}$	notat.	SO(2)	SO(4)	$S_{A'}$	notat.	SO(2)	SO(4)	h
S^{+++} S^{+--}	S^A $A=1,2$	$\frac{1}{2}$	$(\mathbf{2},\mathbf{1})$	S_{---} S_{-++}	S_A $A=1,2$	$-\frac{1}{2}$	$(\mathbf{2},\mathbf{1})$	$+1$
S^{-+-} S^{--+}	$S^{\hat{A}}$ $\hat{A}=3,4$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{2})$	S_{+-+} S_{++-}	$S_{\hat{A}}$ $\hat{A}=3,4$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{2})$	-1