Instantons in (deformed) gauge theories from RNS open strings

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This talk is mostly based on...


Outline

Introduction

Instantons from perturbative strings
- The set-up
- The $\mathcal{N} = 1$ gauge theory from open strings
- The ADHM moduli space of the $\mathcal{N} = 1$ theory
- The instanton profile

Deformations of gauge theories from closed strings
- The $\mathcal{N} = 1/2$ gauge theory
- The deformed ADHM moduli space
- The deformed instanton solution

Conclusions and perspectives
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Instantons in (deformed) gauge theories from strings
Field theory from strings

- **String theory** as a tool to study **field** theories.
- A single string scattering amplitude reproduces, for $\alpha' \rightarrow 0$, a sum of Feynman diagrams:

\[ \alpha' \rightarrow 0 \quad \Rightarrow \quad \overline{X} + \overline{H} + \ldots \]

- Moreover,

String theory $S'$-matrix elements $\Rightarrow$ Field theory eff. actions

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String amplitudes

- A $N$-point string amplitude $A_N$ is schematically given by

$$A_N = \int_\Sigma \langle V_{\phi_1} \cdots V_{\phi_N} \rangle_\Sigma$$

- $V_{\phi_i}$ is the vertex for the emission of the field $\phi_i$:

$$V_{\phi_i} \equiv \phi_i V_{\phi_i}$$

- $\Sigma$ is a Riemann surface of a given topology
- $\langle \ldots \rangle_\Sigma$ is the v.e.v. in C.F.T. on $\Sigma$. 
In the contemporary string perspective, we can in particular study **gauge theories** by considering the lightest d.o.f. of **open strings** suspended between **D-branes** in a well-suited limit \( \alpha' \to 0 \) with **gauge quantities fixed**.
Many useful outcomes

- **perturbative amplitudes** (many gluons, ...) via string techniques;
- construction of “realistic” extensions of Standard model (D-brane worlds);
- **AdS/CFT** and its extensions to non-conformal cases;
- hints about non-perturbative aspects (Matrix models á la Dijkgraaf-Vafa, certain cases of gauge/gravity duality, ...);
- description of **gauge instantons** moduli space by means of D3/D(-1) systems.
We will focus mostly on the stringy description of instantons. [Witten, 1995, Douglas, 1995, Dorey et al, 1999], ...

Our goal is to show how the stringy description of instantons via $D3/D(-1)$ systems is more than a convenient book-keeping for the description of instanton moduli space à la ADHM.

The $D(-1)$’s represent indeed the sources responsible for the emission of the non-trivial gauge field profile in the instanton solution.
Open strings interact with closed strings. We can turn on a closed string background and still look at the massless open string d.o.f.

In this way, deformations of the gauge theory are naturally suggested by their string realization. Such deformations are characterized by

- new geometry in (super)space-time;
- new mathematical structures;
- new types of interactions and couplings.
Deformations by closed string backgrounds

- **Open strings** interact with **closed strings**. We can turn on a **closed string background** and still look at the **massless open string d.o.f.**.

- In this way, **deformations** of the gauge theory are naturally suggested by their string realization. Such deformations are characterized by
  - new **geometry** in (super)space-time;
  - new **mathematical structures**;
  - new types of **interactions and couplings**.
Non-(anti)commutative theories

- The most famous example is that of (gauge) field theories in the background of the $B^{\mu\nu}$ field of the NS-NS sector of closed string. One gets non-commutative field theories, i.e. theories defined on a non-commutative space-time.
- Another case, recently attracting attention, is that of gauge (and matter) fields in the background of a “graviphoton” field strength $C_{\mu\nu}$ from the Ramond-Ramond sector of closed strings. These turn out to be defined on a non-anticommutative superspace.

[Ooguri–Vafa, 2003, de Boer et al, 2003, Seiberg, 2003], ...
Instantons from perturbative strings
Usual string perturbation

- The lowest-order world-sheets $\Sigma$ in the string perturbative expansion are
  - spheres for closed strings, disks for open strings.
- Closed or open vertices have vanishing tadpoles on them:
  \[
  \langle \mathcal{V}_{\phi_{\text{closed}}} \rangle_{\text{sphere}} = 0 , \quad \langle \mathcal{V}_{\phi_{\text{open}}} \rangle_{\text{disk}} = 0 .
  \]
- No tadpoles $\leadsto$ these surfaces can describe only the trivial vacua around which ordinary perturbation theory is performed, but are inadequate to describe non-perturbative backgrounds!
Closed string tadpoles and D-brane solutions

- The microscopic realization of supergravity p-brane solutions as Dp-branes (Polchinski) changes drastically the situation!

\[ \phi_{\text{closed}} \rightarrow \langle \nu_{\phi_{\text{closed}}} \rangle_{\text{disk}_p} \neq 0 \]

(The F.T. of) this diagram gives directly the leading long-distance behaviour of the Dp-brane SUGRA solution.
The microscopic realization of supergravity $p$-brane solutions as $D_p$-branes (Polchinski) changes drastically the situation!

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(The F.T. of) this diagram gives directly the leading long-distance behaviour of the $D_p$-brane SUGRA solution.
This approach can be generalized to the non-perturbative sector of open strings, in particular to instantons of gauge theories.

The world-sheets corresponding to instantonic backgrounds are mixed disks, with boundary partly on a D-instanton.

In this case, this diagram should give the leading long-distance behaviour of the instanton solution.

\[ \langle \nu_{\phi_{\text{open}}} \rangle_{\text{mixed disk}} \neq 0 \]
Consider the $k = 1$ instanton of SU(2) theory

$$A_c^\mu(x) = 2 \frac{\eta_{\mu\nu}^{c}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2}$$

$\eta_{\mu\nu}^{c}$ are the self-dual 't Hooft symbols, and $F_{\mu\nu}$ is self-dual.
With a singular gauge transf. → so-called singular gauge: $(F_{\mu\nu}$ still self-dual despite the $\bar{\eta}_c^{\mu\nu}$)

\[
A^c_\mu(x) = 2\rho^2 \bar{\eta}_c^{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^2 \left[(x - x_0)^2 + \rho^2\right]}
\]

\[
\simeq 2\rho^2 \bar{\eta}_c^{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4} \left(1 - \frac{\rho^2}{(x - x_0)^2} + \ldots\right)
\]

winding # -1 map

$S_3^{x_0}$

$\mathbb{R}^4$

$S_3 = SU(2)$
**Parameters (moduli) of \( k = 1 \) sol. in \( SU(2) \) theory:**

<table>
<thead>
<tr>
<th>moduli</th>
<th>meaning</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0^\mu )</td>
<td>center</td>
<td>4</td>
</tr>
<tr>
<td>( \rho )</td>
<td>size</td>
<td>1</td>
</tr>
<tr>
<td>( \vec{\theta} )</td>
<td>orientation(^(*))</td>
<td>3</td>
</tr>
</tbody>
</table>

\(^(*)\) from “large” gauge transf.s \( A \rightarrow U(\theta)AU^\dagger(\theta) \)
For an $SU(N)$ theory, embed the $SU(2)$ instanton in $SU(N)$:

$$A_\mu = U \begin{pmatrix} 0_{N-2 \times N-2} & 0 \\ 0 & A_\mu^{SU(2)} \end{pmatrix} U^\dagger$$

Thus, there are $4N - 5$ moduli parametrizing $\frac{SU(N)}{SU(N-2) \times U(1)}$ → total # of parameters: $4N$.

For instanton $\# k$ in $SU(N)$: total # of moduli: $4Nk$, described by ADHM construction: moduli space as a HiperKähler quotient.

Atiyah, Drinfeld, Hitchin, Manin
Instanton charge and D-instantons

- The world-volume action of \(N\) D\(_p\)-branes with a \(U(N)\) gauge field \(F\) is

\[
\text{D.B.I. } + \int_{D_p} \left[ C_{p+1} + \frac{1}{2} C_{p-3} \text{Tr}(F \wedge F) + \ldots \right]
\]

- A gauge instanton (i.e. \(\text{Tr}(F \wedge F) \neq 0\)) \(\leadsto\) a localized charge for the RR field \(C_{p-3}\) \(\sim\) a localized D\((p-3)\)-brane inside the D\(_p\)-branes.

- Instanton-charge \(k\) sol.s of 3+1 dims. \(SU(N)\) gauge theories
  \(k\) D-instantons inside \(N\) D3-branes

[Witten, 1995, Douglas, 1995, Dorey et al, 1999], ...
Stringy description of gauge instantons

\[
\begin{array}{c|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\text{D3} & - & - & - & - & * & * & * & * & * & * \\
\text{D(-1)} & * & * & * & * & * & * & * & * & * & * \\
\end{array}
\]

\(N\) D3 branes \(u = 1, \ldots, N\)

\(k\) D(-1) branes \(i = 1, \ldots, k\)

D3/D3, C-P: \(uv\)

D(-1)/D(-1), C-P: \(ij\)

D(-1)/D3, C-P: \(iu\)
Disk amplitudes and effective actions

Usual disks:
- $D_3$
- $D(-1)$

Mixed disks:
- $D_3$
- $D(-1)$

-effective actions

$\mathcal{N} = 4$ SYM action

$\alpha' \to 0$ field theory limit

ADHM measure

D(-1)/D(-1) and mixed

Typeset with \texttt{LaTeX} using the \texttt{beamer} class
We will now discuss a bit more in detail the stringy description of instantons, focusing on the case of pure SU($N$), $\mathcal{N} = 1$ SYM.

Though for simplicity we well discuss mostly its “bosonic” part, this is the supersymmetric theory we will later deform to $\mathcal{N} = 1/2$. 
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The $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold

- Type IIB string theory on target space

$$\mathbb{R}^4 \times \frac{\mathbb{R}^6}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

Decompose $x^M \rightarrow (x^\mu, x^a), (\mu = 1, \ldots 4, \ a = 5, \ldots, 10)$.

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset SO(6)$ is generated by
  - $g_1$: a rotation by $\pi$ in the 7-8 and by $-\pi$ in the 9-10 plane;
  - $g_2$: a rotation by $\pi$ in the 5-6 and by $-\pi$ in the 9-10 plane.

- The origin is a fixed point $\Rightarrow$ the orbifold is a singular, non-compact, Calabi-Yau space.
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- The origin is a fixed point $\Rightarrow$ the orbifold is a singular, non-compact, Calabi-Yau space.
Residual supersymmetry

- Of the 8 spinor weights of $\text{SO}(6)$, $\vec{\lambda} = (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$, only
  
  $\vec{\lambda}^{(+)} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, \hspace{1cm} $\vec{\lambda}^{(-)} = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

  are invariant ones w.r.t. the generators $g_{1,2}$. They are the orbifold realization of the $2(=8/4)$ Killing spinors of the CY.

- We remain with $8(=32/4)$ real susies in the bulk.

- Only two spin fields survive the orbifold projection:
  
  $s(\pm) = e^{\pm \frac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3)}$

  where $\varphi_i$ bosonize the $\text{SO}(6)$ current algebra.
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$$S(\pm) = e^{\pm \frac{i}{2}(\varphi_1 + \varphi_2 + \varphi_3)}$$

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Place $\mathcal{N}$ fractional D3 branes, localized at the orbifold fixed point. The branes preserve $4 = 8/2$ real supercharges.

- The Chan-Patons of open strings attached to fractional branes transform in an irrep of $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- The fractional branes must sit at the orbifold fixed point.
Fractional D3-branes

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Fractional D3 branes and pure $\mathcal{N} = 1$ gauge theory

- Spectrum of massless open strings attached to $N$ fractional D3's of a given type corresponds to $\mathcal{N} = 1$ pure $U(N)$ gauge theory. Schematically,

\[
\begin{align*}
\text{NS:} & \quad \begin{cases}
\psi_\mu^\alpha & \rightarrow A_\mu \\
\bar{\psi}_\alpha & \rightarrow \text{no scalars!}
\end{cases} \\
\text{R:} & \quad \begin{cases}
S^\alpha S^{(+)} & \rightarrow \Lambda_\alpha \\
S^{\dot{\alpha}} S^{(-)} & \rightarrow \Lambda_{\dot{\alpha}}
\end{cases}
\end{align*}
\]

- The standard action is retrieved from disk amplitudes in the $\alpha' \rightarrow 0$ limit:

\[
S = \frac{1}{g_{\text{YM}}^2} \int d^4 x \ Tr \left( \frac{1}{2} F_{\mu \nu}^2 - 2 \bar{\Lambda}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha} \beta} \Lambda_{\beta} \right).
\]
Fractional D3 branes and pure $\mathcal{N} = 1$ gauge theory

- Spectrum of **massless open strings** attached to $N$ fractional D3's of a given type corresponds to $\mathcal{N} = 1$ pure $U(N)$ gauge theory. Schematically,

\[ \text{NS: } \begin{cases} \psi^\mu \\ \phi^a \end{cases} \rightarrow A_\mu \quad \text{no scalars!} \quad \text{R: } \begin{cases} S^\alpha S^{(+)} \\ S^{\dot{\alpha}} S^{(-)} \end{cases} \rightarrow \Lambda_\alpha \quad \Lambda_{\dot{\alpha}} \]

- The standard **action** is retrieved from disk amplitudes in the $\alpha' \rightarrow 0$ limit:

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The action can be obtained from cubic diagram only introducing the (anti-selfdual) auxiliary field $H_{\mu\nu} \equiv H_c \bar{\eta}_{\mu\nu}^c$:

$$S' = \frac{1}{g_{YM}^2} \int d^4x \ Tr \left\{ \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \partial^\mu A^\nu + 2i \partial_\mu A_\nu \left[ A^\mu, A^\nu \right] - 2 \bar{\Lambda}_\alpha \bar{\not{D}} \bar{\alpha}^\beta \Lambda_\beta + H_c H^c + H_c \bar{\eta}_{\mu\nu}^c \left[ A^\mu, A^\nu \right] \right\} ,$$

Integrating out $H_c$ gives $H_{\mu\nu} \propto \left[ A_\mu, A_\nu \right]$ and the usual action.
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$$- 2 \bar{\Lambda}_\dot{\alpha} \bar{D}^{\dot{\alpha}\beta} \Lambda_\beta + H_c H_c^c + H_c \bar{\eta}^{c}_{\mu\nu} [A^\mu, A^\nu] \Big\} ,$$

Integrating out $H_c$ gives $H_{\mu\nu} \propto [A_\mu, A_\nu]$ and the usual action.
The auxiliary field $H_{\mu\nu}$ is associated to the (non-BRST invariant) vertex

$$V_H(y; p) = (2\pi \alpha') \frac{H_{\mu\nu}(p)}{2} \psi^\nu \psi^\mu(y) e^{i\sqrt{2\pi\alpha'} p \cdot X(y)}.$$ 

We have then, for instance,

$$\frac{1}{2} \langle \langle V_H V_A V_A \rangle \rangle = -\frac{1}{g_{YM}^2} \text{Tr} \left( H_{\mu\nu}(p_1) A^\mu(p_2) A^\nu(p_3) \right) + \text{other ordering} \sim \text{last term in the previous action.}$$

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The auxiliary field $H_{\mu\nu}$ is associated to the (non-BRST invariant) vertex

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Moduli spectrum in the $\mathcal{N} = 1$ case

**D(-1)/D(-1) strings**

With $k$ D(-1)’s, all vertices have Chan-Paton factors in the adjoint of $U(k)$.

**Neveu-Schwarz sector**

The vertices surviving the orbifold projection are

$$V_a(y) = \left(2\pi\alpha'\right)^{\frac{1}{2}} g_0 \, a_\mu \, \psi^\mu(y) \, e^{-\phi(y)}.$$
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Here $g_0$ is the coupling on the D(-1) theory:

$$C_0 = \frac{1}{2\pi^2 \alpha'^2} \frac{1}{g_0^2} = \frac{8\pi^2}{g_{YM}^2}.$$
Moduli spectrum in the $\mathcal{N} = 1$ case

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$C_0 = \text{normaliz. of disks with (partly) D(-1) boundary. Since } g_{\text{YM}} \text{ is fixed as } \alpha' \to 0, g_0 \text{ blows up.}$
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- The moduli $a_\mu$ are rescaled with powers of $g_0$ so that their interactions survive when $\alpha' \to 0$ with $g_{YM}^2$ fixed.
Moduli spectrum in the $\mathcal{N} = 1$ case

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- The moduli $a_\mu$ have dimension (length) $\sim$ positions of the (multi)center of the instanton
Moduli spectrum in the $\mathcal{N} = 1$ case

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With $k$ D(-1)’s, all vertices have Chan-Paton factors in the adjoint of $U(k)$.

**Neveu-Schwarz sector**
The vertices surviving the orbifold projection are

$$V_a(y) = \left(2\pi\alpha'\right)^{\frac{1}{2}} g_0 a_\mu \psi^\mu(y) e^{-\phi(y)}.$$  

Moreover, we have the auxiliary vertex decoupling the quartic interactions

$$V_D(y) = \left(2\pi\alpha'\right) \frac{Dc \bar{\eta}^c}{2} \psi^\nu \psi^\mu(y).$$
Moduli spectrum in the $\mathcal{N} = 1$ case

**D(-1)/D(-1) strings**
With $k$ D(-1)’s, all vertices have Chan-Paton factors in the adjoint of $U(k)$.

**Ramond sector**
The vertices surviving the orbifold projection are

\[
V_M(y) = (2\pi \alpha')^{\frac{3}{4}} \frac{g_0}{\sqrt{2}} M'^\alpha S_\alpha(y) S^{(-)}(y) e^{-\frac{1}{2} \phi(y)},
\]

\[
V_\lambda(y) = (2\pi \alpha')^{\frac{3}{4}} \lambda_\dot{\alpha} S^{\dot{\alpha}}(y) S^{(+)}(y) e^{-\frac{1}{2} \phi(y)}.
\]

- $M'^\alpha$ has dimensions of (length)$^{\frac{1}{2}}$, $\lambda_\dot{\alpha}$ of (length)$^{-\frac{3}{2}}$. 

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D(-1)/D3 strings
All vertices have Chan-Patons in the bifundamental of $U(k) \times U(N)$.

Neveu-Schwarz sector
The vertices surviving the orbifold projection are

$$V_w(y) = (2\pi \alpha')^{1/2} \frac{g_0}{\sqrt{2}} w_{\dot{\alpha}} \Delta(y) S^{\dot{\alpha}}(y) e^{-\phi(y)},$$

$$V_{\bar{w}}(y) = (2\pi \alpha')^{1/2} \frac{g_0}{\sqrt{2}} \bar{w}_{\dot{\alpha}} \bar{\Delta}(y) S^{\dot{\alpha}}(y) e^{-\phi(y)},$$

- The (anti-)twist fields $\Delta, \bar{\Delta}$ switch the b.c.'s on the $X^\mu$ string fields.
Moduli spectrum in the $\mathcal{N} = 1$ case

**D(-1)/D3 strings**

All vertices have Chan-Patons in the bifundamental of $U(k) \times U(N)$.

**Neveu-Schwarz sector**

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- $w$ and $\bar{w}$ have dimensions of (length) and are related to the size of the instanton solution.
Moduli spectrum in the $\mathcal{N} = 1$ case

**D(-1)/D3 strings**

All vertices have Chan-Patons in the bifundamental of $U(k) \times U(N)$.

**Ramond sector**

The vertices surviving the orbifold projection are

\[
V_\mu(y) = (2\pi \alpha')^{3/4} \frac{g_0}{\sqrt{2}} \mu \Delta(y) S^{(-)}(y) e^{-\frac{1}{2} \phi(y)},
\]

\[
V_{\bar{\mu}}(y) = (2\pi \alpha')^{3/4} \frac{g_0}{\sqrt{2}} \bar{\mu} \bar{\Delta}(y) S^{(-)}(y) e^{-\frac{1}{2} \phi(y)}.
\]

- The fermionic moduli $\mu, \bar{\mu}$ have dimensions of $(\text{length})^{1/2}$. 

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The $\mathcal{N} = 1$ moduli action

- (Mixed) disk diagrams with the above moduli, for $\alpha' \to 0$ yield

$$S_{\text{mod}} = \text{tr} \left\{ -iD_c \left( W^c + i\tilde{\eta}^c_{\mu\nu} [a'{}^{\mu}, a'{}^{\nu}] \right) -i\lambda^{\dot{\alpha}} \left( w^{u}{}_{\dot{\alpha}} \bar{\mu}_u + \mu^u \bar{w}_{\dot{\alpha}u} + [a'_{\alpha\dot{\alpha}}, M'{}^{\alpha}] \right) \right\}$$

where $\left( W^c \right)^i_j = w^{iu}{}_{\dot{\alpha}} (\tau^c)^{\dot{\alpha}}_{\dot{\beta}} \bar{w}^{\dot{\beta}u} j$

- $D_c$ and $\lambda^{\dot{\alpha}} \sim$ Lagrange multipliers for the (super)ADHM constraints.
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\]

where

\[
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\]

- $D_c$ and $\lambda^{\dot{\alpha}} \sim$ Lagrange multipliers for the (super)ADHM constraints.
The $\mathcal{N} = 1$ ADHM constraints

- The ADHM constraints are three $k \times k$ matrix eq.s

\[ W^c + i \tilde{\eta}^c_{\mu\nu} [a'_{\mu}, a'_{\nu}] = 0. \]

- and their fermionic counterparts

\[ w^u_{\dot{\alpha}} \bar{\mu}_u + \mu^u \bar{w}_{\dot{\alpha}u} + [a'_{\dot{\alpha} \dot{\alpha}'}, M'^{\dot{\alpha}}] = 0. \]

- Once these constraints are satisfied, the moduli action vanishes.
The $\mathcal{N} = 1$ ADHM constraints

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- Once these constraints are satisfied, the moduli action vanishes.
Parameter counting

- E.g., for the bosonic parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a''^{\mu}$</td>
<td>$4k^2$</td>
</tr>
<tr>
<td>$w_\alpha, \bar{w}_\alpha$</td>
<td>$4kN$</td>
</tr>
<tr>
<td>ADHM constraints</td>
<td>$-3k^2$</td>
</tr>
<tr>
<td>Global $U(k)$ inv.</td>
<td>$-k^2$</td>
</tr>
<tr>
<td>True moduli</td>
<td>$4kN$</td>
</tr>
</tbody>
</table>

- After imposing the constraints, more or less

  $a''^{\mu} \leadsto$ multi-center positions, ...

  $w_\alpha, \bar{w}_\alpha \leadsto$ size, orientation inside $SU(N)$,
Mixed disks = sources for gauge theory fields. The amplitude for emitting a gauge field is

\[ A^I_\mu(p) = \langle \mathcal{V} A^I_\mu (-p) \rangle_{\text{m.d}} = \langle \langle V \bar{w} \mathcal{V} A^I_\mu (-p) V_w \rangle \rangle = i (T^I)^u_v p^\nu \bar{\eta}^{c\mu}_{\nu\mu} (w^u_{\dot{\alpha}} (\tau^c_{\dot{\alpha}} \dot{\beta} (w^\dot{\beta} v) e^{-ip \cdot x_0}.}
The instanton solution from mixed disks

- **Mixed disks** = sources for gauge theory fields. The amplitude for emitting a gauge field is

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= i (T^I)_{\nu}^u p^v \bar{\eta}_{\nu\mu}^c (w^u_{\dot{\alpha}} (\tau^c_{\dot{\alpha}} \dot{\beta} \bar{w}^\beta_v) e^{-ip \cdot x_0}.
\]

- \( \mathcal{V}^I_{A_{\mu}}(-p) \): no polariz., outgoing \( p \), 0-picture

\[
\mathcal{V}^I_{A_{\mu}}(z; -p) = 2iT^I (\partial X_{\mu} - ip \cdot \psi \psi_{\mu}) e^{-ip \cdot X(z)}
\]
The instanton solution from mixed disks

- **Mixed disks = sources** for gauge theory fields. The amplitude for emitting a gauge field is

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\]

- **N.B.** From now on we set \( k = 1 \), i.e., we consider instanton number 1.
Mixed disks = sources for gauge theory fields. The amplitude for emitting a gauge field is

\[ A^I_\mu(p) = \left\langle V^I_{A\mu}(-p) \right\rangle_{\text{m.d}} = \langle V_{\bar{w}} V^I_{A\mu}(-p) V_w \rangle = i \left( T^I \right)_u^v p^\nu \bar{\eta}^{c\nu\mu} \left( w^{u}_{\dot{\alpha}} (\tau^c)_{\dot{\alpha} \dot{\beta}} \bar{w}^{\dot{\beta}}_{v} \right) e^{-ip \cdot x_0}. \]

\[ x_0 = \text{pos. of the D(-1). Broken transl. invariance in the D3 world-volume } \sim \text{ “tadpole”} \]

\[ \left\langle e^{-ip \cdot X} \right\rangle_{\text{m.d}} \propto e^{ip \cdot x_0}. \]
The classical profile

- The classical profile of the gauge field emitted by the mixed disk is obtained by attaching a free propagator and Fourier transforming:

\[
A^I_\mu(x) = \int \frac{d^4 p}{(2\pi)^2} A^I_\mu(p) \frac{1}{p^2} e^{ip\cdot x}
\]

\[
= 2 (T^I)^{\nu}_u \left( \bar{T}^c \right)^{u}_v \eta^{c}_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4},
\]

where \((T^I)^{\nu}_u\) are the \(U(N)\) generators and

\[
(T^c)^{u}_v = w^{u}_{\dot{\alpha}} (\tau^c_{\dot{\alpha}\dot{\beta}}) \bar{w}^{\dot{\beta}}_{v}.
\]
In the above solution we still have the \textit{unconstrained} moduli $\bar{w}, w$. 
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We must still impose the bosonic ADHM constraints

$$W^c \equiv w^u_\dot{\alpha}(\tau^c)^\dot{\alpha}_\dot{\beta} \bar{w}^{\dot{\beta}}_\nu = 0.$$
In the above solution we still have the unconstrained moduli $\bar{w}, w$. 

Iff $W^c = 0$, the $N \times N$ matrices

\[ (t_c)^u_v \equiv \frac{1}{2\rho^2} \left( w_{\dot{\alpha}}^u (\tau_c)^{\dot{\beta}}_{\dot{\gamma}} \bar{w}_{\dot{\gamma}}^{\dot{\beta}} v \right), \]

where

\[ 2\rho^2 = w_{\dot{\alpha}}^u \bar{w}_{\dot{\alpha}}^u , \]

satisfy an $su(2)$ subalgebra: $[t_c, t_d] = i\epsilon_{cde} t_e$. 

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The classical instanton profile

The gauge vector profile can be written as

$$A^I_{\mu}(x) = 4\rho^2 \text{Tr} \left( T^I t_c \right) \tilde{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4}$$
The classical instanton profile

- The gauge vector profile can be written as

\[ A^I_\mu(x) = 4\rho^2 \text{Tr} (T^I t_c) \tilde{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4} \]

- This is a moduli-dependent (through \( t_c \)) embedding in \( \text{su}(N) \) of the \( \text{su}(2) \) instanton connection in
  - large-distance leading approx. (\(|x - x_0| \gg \rho|\)
  - singular gauge

Recall the singular gauge
The mixed disks emit also a gaugino $\Lambda^\alpha, I \sim$ account for its leading profile in the super-instanton solution.

Subleading terms in the long-distance expansion of the solution arise from emission diagrams with more moduli insertions.

At the field theory level, they correspond to having more source terms.
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Subleading terms in the long-distance expansion of the solution arise from emission diagrams with more moduli insertions.

At the field theory level, they correspond to having more source terms.
**Question**: Why **singular** gauge?

- **Instanton** produced by a **point-like source**, the D(-1), inside the D3 → **singular** at the location of the source.

- In the singular gauge, rapid fall-off of the fields → eq.s of motion reduce to **free eq.s at large distance** → "perturbative" solution in terms of the **source term**.

- **non-trivial properties** of the instanton profile from the region near the singularity through the embedding

\[
S^3 \ni SU(2) \subset SU(N)
\]
Deformations of gauge theories from closed strings
Non-anticommutative theories and RR backgrounds

\( C_{\mu \nu} \) **RR background: new geometry**

- A class of “deformed” field theories, recently attracting attention, is that of gauge (and matter) fields in the background of a “graviphoton” field strength \( C_{\mu \nu} \) from the Ramond-Ramond sector of closed strings.

- These turn out to be defined on a non-anticommutative superspace, where the, say, anti-chiral fermionic coordinates satisfy

\[
\{ \theta^{\dot{\alpha}}, \theta^{\dot{\beta}} \} \propto C^{\dot{\alpha} \dot{\beta}} \propto (\sigma^{\mu \nu})_{\dot{\alpha} \dot{\beta}} C_{\mu \nu}.
\]
$C_{\mu\nu}$ **RR background: new geometry**

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\]
Non-anticommutative theories and RR backgrounds

\( C_{\mu \nu} \) RR background: new structure

- The superspace deformation can be rephrased as a modification of the product among functions, which now becomes

\[
    f(\theta) \ast g(\theta) = f(\theta) \exp \left( -\frac{1}{2} \left[ \frac{\partial}{\partial \theta^{\dot{\alpha}}} \ C^{\dot{\alpha} \dot{\beta}} \frac{\partial}{\partial \theta^{\dot{\beta}}} \right] \right) g(\theta) .
\]

- There are also new interactions between the gauge and matter fields: see later in the talk.
Non-anticommutative theories and RR backgrounds
Non-anticommutative theories and RR backgrounds

Plan

- We shall analyze the deformation of $\mathcal{N} = 1$ pure gauge theory induced by a RR “graviphoton” $C_{\mu\nu}$, the so-called $\mathcal{N} = 1/2$ gauge theory.

- We shall discuss how to derive explicitly the $\mathcal{N} = 1/2$ theory from string diagrams (in the traditional RNS formulation).

- Moreover we will derive from string diagrams the instantonic solutions of this theory and their ADHM moduli space.
The graviphoton background

- RR vertex in 10D, in the symmetric superghost picture:
  \[ \mathcal{F}_{\dot{A}\dot{B}} S^{\dot{A}} e^{-\phi/2}(z) \tilde{S}^{\dot{B}} e^{-\tilde{\phi}/2}(\tilde{z}) \. \]

  Bispinor \( \mathcal{F}_{\dot{A}\dot{B}} \mapsto 1\text{-}, 3\text{-} \) and a.s.d. 5-form field strengths.

- On \( \mathbb{R}^4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), a surviving 4D bispinor vertex is
  \[ \mathcal{F}_{\dot{\alpha}\dot{\beta}} S^{\dot{\alpha}} S^{(+)} e^{-\phi/2}(z) \tilde{S}^{\dot{\beta}} \tilde{S}^{(+)} e^{-\tilde{\phi}/2}(\tilde{z}) \. \]

  with \( \mathcal{F}_{\dot{\alpha}\dot{\beta}} = \mathcal{F}_{\dot{\beta}\dot{\alpha}} \).

- Decomposing the 5-form along the holom. 3-form of the CY
  \( \mapsto \) an a.s.d. 2-form in 4D
  \[ C_{\mu\nu} \propto \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} \, . \]

Typeset with \LaTeX\ using the \texttt{beamer} class.
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Bispinor \( \mathcal{F}_{\dot{A}\dot{B}} \sim 1-, 3- \) and a.s.d. 5-form field strengths.

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  the graviphoton f.s. of \( \mathcal{N} = 1/2 \) theories.
Inserting graviphotons in disk amplitudes

- Conformally mapping the disk to the upper half $z$-plane, the D3 boundary conditions on spin fields read

$$S^\dot{\alpha} S^{(+)}(z) \equiv \tilde{S}^\dot{\alpha} \tilde{S}^{(+)}(\bar{z}) \bigg|_{z=\bar{z}}.$$

(opposite sign for $\tilde{S}^\alpha \tilde{S}^{(-)}(\bar{z})$).

- When closed string vertices are inserted in a D3 disk,

$$\tilde{S}^\dot{\alpha} \tilde{S}^{(+)}(\bar{z}) \to S^\dot{\alpha} S^{(+)}(\bar{z}).$$
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Disk amplitudes with a graviphoton

Start inserting a graviphoton vertex:

\[ \langle \langle V_\lambda V_\lambda V_\lambda V_F \rangle \rangle \]

where

\[ V_F(z, \bar{z}) = F_{\dot{\alpha} \dot{\beta}} S^{\dot{\alpha}} S^{(+)\dot{\beta}} e^{-\phi/2}(z) S^{\dot{\beta}+} e^{-\phi/2}(\bar{z}) \]

\[ \implies \text{we need two } S^{(-)} \text{ operators to "saturate the charge"} \]
Disk amplitudes with a graviphoton

Start inserting a graviphoton vertex:

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where

\[ V_F(z, \bar{z}) = \mathcal{F}_{\dot{\alpha} \dot{\beta}} S^{\dot{\alpha}+}(z) S^{\dot{\beta}+}(\bar{z}) e^{-\phi/2} \]

\[ \implies \text{we need two } S^{(-)} \text{ operators to “saturate the charge”} \]
Disk amplitudes with a graviphoton

We insert therefore two chiral gauginos:

\[
\langle V_\Lambda V_\Lambda V_A V_F \rangle
\]

with vertices

\[
V_\Lambda(y; p) = (2\pi \alpha')^{\frac{3}{4}} \Lambda^\alpha(p) S_\alpha S'(-) e^{-\frac{1}{2} \phi(y)} e^{i \sqrt{2\pi \alpha'} p \cdot X(y)}.
\]

Without other insertions, however,

\[
\langle S^{\dot{\alpha}} S^{\dot{\beta}} S_\alpha S_\beta \rangle \propto \epsilon^{\dot{\alpha} \dot{\beta}} \epsilon_{\alpha \beta}
\]

vanishes when contracted with \( F_{\dot{\alpha} \dot{\beta}} \).
Disk amplitudes with a graviphoton

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\[ \left\langle V_\Lambda V_\Lambda V_A V_\mathcal{F} \right\rangle \]

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\[ V_\Lambda (y; p) = (2\pi \alpha')^{\frac{3}{4}} \Lambda^\alpha (p) S_\alpha S(-) e^{-\frac{1}{2}\phi(y)} \]
\[ \times e^{i\sqrt{2\pi\alpha'} p \cdot X(y)} . \]

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To cure this problem, insert a **gauge field** vertex:

\[
\langle \langle V_\Lambda V_\Lambda V_A V_\mathcal{F} \rangle \rangle
\]

that must be in the 0 picture:

\[
V_A(y; p) = 2i \left(2\pi\alpha'\right)^{\frac{1}{2}} A_\mu(p) \\
\left(\partial X^\mu(y) + i \left(2\pi\alpha'\right)^{\frac{1}{2}} p \cdot \psi \psi^\mu(y)\right) \\
e^{i\sqrt{2\pi\alpha'} p \cdot X(y)}
\]

\[\leadsto\] finally, we may get a **non-zero result**!
To cure this problem, insert a **gauge field** vertex:

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\[
V_A(y; p) = 2i (2\pi \alpha')^{1/2} A_\mu(p) \\
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e^{i\sqrt{2\pi \alpha'}p \cdot X(y)}
\]

\[\sim\sim\text{finally, we may get a non-zero result!}\]
Evaluation of the amplitude

We have

\[ \langle V_\Lambda V_\Lambda V_A V_\mathcal{F} \rangle \equiv C_4 \int \frac{\prod_i dy_i dz d\bar{z}}{dV_{\text{CGK}}} \langle V_\Lambda(y_1; p_1) V_\Lambda(y_2; p_2) V_A(y_3; p_3) V_\mathcal{F}(z, \bar{z}) \rangle \]

where the normalization for a D3 disk is

\[ C_4 = \frac{1}{\pi^2 \alpha'^2} \frac{1}{g_{\text{YM}}^2} \]

and the SL(2, \mathbb{R})-invariant volume is

\[ dV_{\text{CGK}} = \frac{dy_a dy_b dy_c}{(y_a - y_b)(y_b - y_c)(y_c - y_a)} \]

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Explicit expression of the amplitude

- Altogether, the explicit expression is

\[
\langle \langle V_\Lambda V_\Lambda V_A V_F \rangle \rangle = \frac{8}{g_{YM}^2} (2\pi\alpha')^{\frac{1}{2}} \text{Tr} \left( \Lambda^\alpha(p_1) \Lambda^\beta(p_2) p_3^\nu A^\mu(p_3) \right) \mathcal{F}_{\dot{\alpha}\dot{\beta}} \\
\times \int \frac{\prod_i dy_i dz d\bar{z}}{dV_{\text{CKG}}} \left\{ \langle S_\alpha(y_1) S_\beta(y_2) : \psi^\nu \psi^\mu : (y_3) S^{\dot{\alpha}}(z) S^{\dot{\beta}}(\bar{z}) \rangle \\
\times \langle S^{(-)}(y_1) S^{(-)}(y_2) S^{(+)}(z) S^{(+)}(\bar{z}) \rangle \\
\times \langle e^{-\frac{1}{2}\phi(y_1)} e^{-\frac{1}{2}\phi(y_2)} e^{-\frac{1}{2}\phi(z)} e^{-\frac{1}{2}\phi(\bar{z})} \rangle \\
\times \langle e^{i\sqrt{2\pi\alpha'}p_1 \cdot X(y_1)} e^{i\sqrt{2\pi\alpha'}p_2 \cdot X(y_2)} e^{i\sqrt{2\pi\alpha'}p_3 \cdot X(y_3)} \rangle \right\}. 
\]
Evaluation of the amplitude: correlators

- The relevant correlators are:
Evaluation of the amplitude: correlators

- The relevant correlators are:

  1. **Superghosts**

    \[
    \langle e^{-\frac{1}{2}\phi(y_1)} e^{-\frac{1}{2}\phi(y_2)} e^{-\frac{1}{2}\phi(z)} e^{-\frac{1}{2}\phi(\bar{z})} \rangle = \left[ (y_1 - y_2) (y_1 - z) (y_1 - \bar{z}) (y_2 - z) (y_2 - \bar{z}) (z - \bar{z}) \right]^{-\frac{1}{4}}.
    \]
Evaluation of the amplitude: correlators

- The relevant correlators are:

  2. **Internal spin fields**

     \[ \langle S(-)(y_1)S(-)(y_2)S(+)zS(+)\bar{z} \rangle \]

     \[ = (y_1 - y_2)^{\frac{3}{4}} (y_1 - z)^{-\frac{3}{4}} (y_1 - \bar{z})^{-\frac{3}{4}} (y_2 - z)^{-\frac{3}{4}} (y_2 - \bar{z})^{-\frac{3}{4}} \]

     \[ \times (z - \bar{z})^{\frac{3}{4}} \]
Evaluation of the amplitude: correlators

The relevant correlators are:

3. **4D spin fields**

\[ \langle S_\gamma (y_1) S_\delta (y_2) : \psi^\mu \psi^\nu : (y_3) S^\dot{\alpha} (z) S^\dot{\beta} (\bar{z}) \rangle \]

\[ = \frac{1}{2} (y_1 - y_2)^{-\frac{1}{2}} (z - \bar{z})^{-\frac{1}{2}} \]

\[ \times \left( (\sigma^{\mu\nu})_\gamma^\delta \varepsilon^{\dot{\alpha} \dot{\beta}} \frac{(y_1 - y_2)}{(y_1 - y_3)(y_2 - y_3)} \right. \]

\[ + \varepsilon^{\gamma \delta} (\bar{\sigma}^{\mu\nu})^{\dot{\alpha} \dot{\beta}} \frac{(z - \bar{z})}{(y_3 - z)(y_3 - \bar{z})} \]

\[ \quad \times \left( (\sigma^{\mu\nu})_\gamma^\delta \varepsilon^{\dot{\alpha} \dot{\beta}} \frac{(y_1 - y_2)}{(y_1 - y_3)(y_2 - y_3)} \right). \]
Evaluation of the amplitude: correlators

- The relevant correlators are:

4. Momentum factors

\[ \langle e^{i\sqrt{2\pi\alpha'}p_1 \cdot X(y_1)} e^{i\sqrt{2\pi\alpha'}p_2 \cdot X(y_2)} e^{i\sqrt{2\pi\alpha'}p_3 \cdot X(y_3)} \rangle \text{ on shell} \rightarrow 1. \]
Evaluation of the amplitude: $SL(2, \mathbb{R})$ fixing

- We may, for instance, choose

$$y_1 \to \infty, \quad z \to i, \quad \bar{z} \to -i.$$  

- The remaining integrations turn out to be

$$\int_{-\infty}^{+\infty} dy_2 \int_{-\infty}^{y_2} dy_3 \frac{1}{(y_2^2 + 1)(y_3^2 + 1)} = \frac{\pi^2}{2}.$$  

Symmetry factor $1/2$ and other ordering compensate each other.
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Symmetry factor $1/2$ and other ordering compensate each other.
We finally obtain for $\langle \langle V_\Lambda V_\Lambda V_A V_F \rangle \rangle$ the result

$$\frac{8\pi^2}{g_{\text{YM}}^2} (2\pi \alpha')^{1/2} \text{Tr} \left( \Lambda(p_1) \cdot \Lambda(p_2) p_3^\nu A^\mu(p_3) \right) F_{\dot{\alpha} \dot{\beta}} (\bar{\sigma}_{\nu\mu})^{\dot{\alpha} \dot{\beta}}.$$

This result is finite for $\alpha' \to 0$ if we keep constant

$$C_{\mu\nu} \equiv 4\pi^2 (2\pi \alpha')^{1/2} F_{\dot{\alpha} \dot{\beta}} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha} \dot{\beta}}$$

$C_{\mu\nu}$, of dimension (length) will be exactly the one of $\mathcal{N} = 1/2$ theory.

We get an extra term in the gauge theory action:

$$\frac{1}{2} \int \frac{d^4 x}{16\pi^2} \text{Tr} \left( \Lambda \cdot \Lambda \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \right) C_{\mu\nu}.$$
Final result for the amplitude

- We finally obtain for $\langle \langle V_{\Lambda} V_{\Lambda} V_{A} V_{\mathcal{F}} \rangle \rangle$ the result

$$\frac{8\pi^2}{g_{\mathrm{YM}}^2} (2\pi\alpha')^{\frac{1}{2}} \text{Tr} \left( \Lambda(p_1) \cdot \Lambda(p_2) p_3^\nu A^\mu(p_3) \right) \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{\nu\mu})^{\dot{\alpha}\dot{\beta}} .$$

- This result is **finite** for $\alpha' \to 0$ if we keep constant

$$C_{\mu\nu} \equiv 4\pi^2 (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} .$$

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Final result for the amplitude

- We finally obtain for \( \langle \langle V_\Lambda V_\Lambda V_A V_F \rangle \rangle \) the result
  \[
  \frac{8\pi^2}{g_{YM}^2} (2\pi\alpha')^{1/2} \text{Tr} \left( \Lambda(p_1) \cdot \Lambda(p_2) p_3^\nu A^\mu(p_3) \right) \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{\nu\mu})^{\dot{\alpha}\dot{\beta}}.
  \]
- This result is finite for \( \alpha' \to 0 \) if we keep constant
  \[
  C_{\mu\nu} \equiv 4\pi^2 (2\pi\alpha')^{1/2} \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}
  \]
- \( C_{\mu\nu} \), of dimension (length) will be exactly the one of \( \mathcal{N} = 1/2 \) theory.
- We get an extra term in the gauge theory action:
  \[
  \frac{i}{g_{YM}^2} \int d^4x \ \text{Tr} \left( \Lambda \cdot \Lambda \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \right) C_{\mu\nu}.
  \]
Another contribute

- Another possible diagram with a graviphoton insertion is

\[ \langle \langle V_\Lambda V_\Lambda V_H V_F \rangle \rangle. \]
Another possible diagram with a graviphoton insertion is

\[ \langle \langle V_{\Lambda} V_{\Lambda} V_{H} V_{\mathcal{F}} \rangle \rangle. \]

Recall that the auxiliary field vertex in the 0 picture is

\[ V_{H}(y; p) = \left(2\pi\alpha'\right) \frac{H_{\mu\nu}(p)}{2} \psi^\nu \psi^\mu(y) e^{i\sqrt{2\pi\alpha'} p \cdot X(y)}. \]
Another possible diagram with a graviphoton insertion is

\[ \left\langle V_\Lambda V_\Lambda V_H V_\mathcal{F} \right\rangle. \]

The evaluation of this amplitude parallels exactly the previous one and contributes to the field theory action the term:

\[
\frac{1}{2g_{\text{YM}}^2} \int d^4x \ Tr \left( \Lambda \cdot \Lambda H^{\mu \nu} \right) C_{\mu \nu},
\]

having introduced \( C_{\mu \nu} \) as above.
Another possible diagram with a graviphoton insertion is
\[ \langle V_{\Lambda} V_{\Lambda} V_{H} V_{\mathcal{F}} \rangle. \]

All other amplitudes involving $\mathcal{F}$ vertices either
- vanish because of their tensor structure;
- vanish in the $\alpha' \to 0$ limit, with $C_{\mu\nu}$ fixed.
The deformed gauge theory action

- From **disk diagrams** with **RR insertions** we obtain, in the field theory limit

\[ \alpha' \to 0 \quad \text{with} \quad C_{\mu\nu} \equiv 4\pi^2 (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} \quad \text{fixed} \]

the action

\[
\tilde{S}' = \frac{1}{g_{YM}^2} \int d^4x \text{ Tr} \left\{ \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \partial^\mu A^\nu + 2i \partial_\mu A_\nu [A^\mu, A^\nu] \\
- 2\bar{\Lambda}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}\dot{\beta}} \Lambda_\beta + i \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \Lambda \cdot \Lambda C_{\mu\nu} \\
+ H_c H^c + H_c \bar{\eta}^c_{\mu\nu} \left( [A^\mu, A^\nu] + \frac{1}{2} \Lambda \cdot \Lambda C^{\mu\nu} \right) \right\}.
\]
The deformed gauge theory action

Integrating on the auxiliary field $H_c$, we get

$$
\tilde{S} = \frac{1}{g_{YM}^2} \int d^4 x \ Tr \left\{ \frac{1}{2} F_{\mu\nu}^2 - 2\bar{\Lambda} \hat{\partial} \hat{\alpha} \beta \Lambda_{\beta} \\
+ i F^{\mu\nu} \Lambda \cdot \Lambda C_{\mu\nu} - \frac{1}{4} \left( \Lambda \cdot \Lambda C_{\mu\nu} \right)^2 \right\} \\
= \frac{1}{g_{YM}^2} \int d^4 x \ Tr \left\{ \left( F_{\mu\nu}^{(-)} + \frac{i}{2} \Lambda \cdot \Lambda C_{\mu\nu} \right)^2 + \frac{1}{2} F_{\mu\nu} \tilde{F}^{\mu\nu} \\
- 2\bar{\Lambda} \hat{\partial} \hat{\alpha} \beta \Lambda_{\beta} \right\},
$$

i.e., exactly the action of Seiberg’s $\mathcal{N} = 1/2$ gauge theory.
The deformed gauge theory action

- Integrating on the auxiliary field $H_c$, we get

$$
\tilde{S} = \frac{1}{g_{YM}^2} \int d^4x \; \text{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 - 2 \bar{\Lambda} \dot{\Lambda} \bar{D} \dot{\Lambda} \beta \Lambda \beta \\
+ i F_{\mu\nu} \Lambda \cdot \Lambda C_{\mu\nu} - \frac{1}{4} \left( \Lambda \cdot \Lambda C_{\mu\nu} \right)^2 \right\} \\
= \frac{1}{g_{YM}^2} \int d^4x \; \text{Tr} \left\{ \left( F_{\mu\nu}^{(-)} + \frac{i}{2} \Lambda \cdot \Lambda C_{\mu\nu} \right)^2 \right\} + \frac{1}{2} F_{\mu\nu} \tilde{F}^{\mu\nu} - 2 \bar{\Lambda} \dot{\Lambda} \bar{D} \dot{\Lambda} \beta \Lambda \beta \right\}.
$$

How is the instantonic sector affected?
The graviphoton in D(-1) disks

- Inserting $V_F$ in a disk with all boundary on D(-1)’s is perfectly analogous to the D3 case (but we have non momenta).

- The only possible diagram is

$$\langle V_M V_M V_D V_F \rangle = \frac{\pi^2}{2} 2\pi\alpha' \frac{1}{2} \text{tr} \left( M' \cdot M' D_c \right) \tilde{\eta}^c_{\mu\nu} F_{\dot{\alpha} \dot{\beta}} (\tilde{\sigma}_\nu \mu)^{\dot{\alpha} \dot{\beta}}$$

$$= -\frac{1}{2} \text{tr} \left( M' \cdot M' D_c \right) C^c,$$

where

$$C^c = \frac{1}{4} \tilde{\eta}^c_{\mu\nu} C^{\mu\nu}.$$
Inserting $V_F$ in a disk with all boundary on $D(-1)$'s is perfectly analogous to the $D3$ case (but we have non momenta).

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\left\langle V_M V_M V_D V_F \right\rangle = \frac{\pi^2}{2} 2\pi\alpha' \left( M' \cdot M' D_c \right) \bar{\eta}_{\mu\nu} F_{\dot{\alpha}\dot{\beta}} \left( \bar{\sigma}_{\nu\mu} \right)^{\dot{\alpha}\dot{\beta}}
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= -\frac{1}{2} \text{tr} \left( M' \cdot M' D_c \right) C^c,
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where

\[
C^c = \frac{1}{4} \bar{\eta}_{\mu\nu} C^{\mu\nu}.
\]
The graviphoton in mixed disks

- We can also insert $V_{\mathcal{F}}$ in a disk with mixed b.c.’s.
- There is a possible diagram

\[ \langle \langle V_{\bar{\mu}} V_{\mu} V_D V_{\mathcal{F}} \rangle \rangle \]
The graviphoton in mixed disks

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- There is a possible diagram:

$$\langle \langle V_{\bar{\mu}} V_\mu V_D V_{\mathcal{F}} \rangle \rangle$$

- We have different b.c.s on the two parts of the boundary, but the spin fields in the RR vertex $V_{\mathcal{F}}$ have the same identification on both:

$$S^{\dot{\alpha}} S^{(+)}(z) = S^{\dot{\alpha}} S^{(+)}(\bar{z}) \bigg|_{z=\bar{z}}.$$

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Marco Billò  Instantons in (deformed) gauge theories from strings
The graviphoton in mixed disks

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- There is a possible diagram

$$\langle \langle V_{\bar{\mu}} V_{\mu} V_D V_{\mathcal{F}} \rangle \rangle$$

- This is because we chose $D(-1)$’s to represent instantons with self-dual f.s. and $\mathcal{F}_{\mu\nu}$ to be anti-self-dual.
The graviphoton in mixed disks

- We can also insert $V_F$ in a disk with mixed b.c.’s.
- There is a possible diagram
  \[ \langle \langle V_{\bar{\mu}} V_\mu V_D V_F \rangle \rangle \]
- The $\mu, \bar{\mu}$ vertices contain bosonic twist fields with correlator
  \[ \Delta(y_1) \bar{\Delta}(y_2) \sim (y_1 - y_2)^{-\frac{1}{2}}. \]
The graviphoton in mixed disks

- We can also insert $V_{\mathcal{F}}$ in a disk with mixed b.c.’s.
- There is a possible diagram

$$\langle \langle V_{\bar{\mu}} V_{\mu} V_D V_{\mathcal{F}} \rangle \rangle$$

- Taking into account all correlators, the SL(2, $\mathbb{R}$) gauge fixing, the integrations and the normalizations, we find the result

$$- \frac{\pi^2}{2} (2\pi \alpha')^{1/2} \text{tr} \left( \bar{\mu} u \mu^u D_c \right) \bar{\eta}_{\mu\nu} \mathcal{F}_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}^\nu \mu) \dot{\alpha} \dot{\beta}$$

$$= \frac{1}{2} \text{tr} \left( \bar{\mu} u \mu^u D_c \right) C^c .$$
Effects of the graviphoton on the moduli measure

- No other disk diagrams contribute in our $\alpha' \to 0$ limit.
- The two terms above are linear in the auxiliary field $D_c$ to deform the bosonic ADHM constraints to

$$W^c + i\bar{\eta}^c_{\mu\nu} \left[ a'^\mu, a'^\nu \right] + \frac{i}{2} \left( M' \cdot M' + \mu^u \bar{\mu}_u \right) C^c = 0 .$$

- This is the only effect of the chosen anti-self-dual graviphoton background.
- Had we chosen a self-dual graviphoton, we would have no effect.
The $\mathcal{N} = 1/2$ gauge theory

The deformed ADHM moduli space

The deformed instanton solution

**Effects of the graviphoton on the moduli measure**

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\Rightarrow$ deform the bosonic ADHM constraints to

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Effects of the graviphoton on the moduli measure

- No other disk diagrams contribute in our $\alpha' \to 0$ limit.
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  - deform the bosonic ADHM constraints to

$$W^c + i\tilde{\eta}^c_{\mu\nu} [a'^{\mu}, a'^{\nu}] + \frac{1}{2} \left( M' \cdot M' + \mu^u \bar{\mu}_u \right) C^c = 0 .$$

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The emitted gauge field in presence of $C_{\mu\nu}$

- In the graviphoton background, we have the extra emission diagram

$$
\langle \langle V_{\bar{\mu}} \mathcal{V}_{A^I_{\mu}}(-p) V_{\mu} V_{\mathcal{F}} \rangle \rangle = 2\pi^2 (2\pi \alpha')^{\frac{1}{2}} (T^I)_{uv} p^\nu (\bar{\sigma}_{\nu\mu})^{\dot{\alpha}\dot{\beta}} \mathcal{F}_{\dot{\alpha}\dot{\beta}} \mu^u \bar{\mu}_v e^{-ip\cdot x_0}
$$

$$
= \frac{1}{2} (T^I)_{uv} p^\nu \bar{\eta}^c_{\nu\mu} \mu^u \bar{\mu}_v C^c e^{-ip\cdot x_0},
$$

- No other diagrams with only two moduli contribute to the emission of a gauge field.
The emitted gauge field in presence of $C_{\mu \nu}$

- In the graviphoton background, we have the extra emission diagram

$$ \langle \langle V_{\tilde{\mu}} V_{A_{\mu}^I} (-p) V_\mu V_F \rangle \rangle $$

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$$ = \frac{1}{2} (T^I)_u p^\nu \bar{\eta}_{\nu \mu}^c \mu^u \bar{\mu}_v C^c e^{-ip \cdot x_0} $$

- No other diagrams with only two moduli contribute to the emission of a gauge field.

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The classical solution

- Taking into account also “undeformed” emission diagram discussed before, the emission amplitude is

\[ A^I_{\mu}(p) = i (T^I)^u_v p^\nu \tilde{\eta}^c_{\nu\mu} \left[ (T^c)^u_v + (S^c)^u_v \right] e^{-ip\cdot x_0} , \]

where \((T^I)^u_v\) are the \(U(N)\) generators and

\[ (T^c)^u_v = \omega^u_\dot{\alpha} (\tau^c)^{\dot{\alpha}}_{\dot{\beta}} \bar{w}^{\dot{\beta}}_v , \quad (S^c)^u_v = -\frac{i}{2} \mu^u \bar{\mu}^v C^c . \]

- From this we obtain the profile of the classical solution

\[ A^I_{\mu}(x) = \int \frac{d^4p}{(2\pi)^2} A^I_{\mu}(p) \frac{1}{p^2} e^{ip\cdot x} \]

\[ = 2 (T^I)^v_u \left[ (T^c)^u_v + (S^c)^u_v \right] \tilde{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^2} . \]
The classical solution

- Taking into account also “undeformed” emission diagram discussed before, the emission amplitude is

\[ A^I_{\mu}(p) = i (T^I)_{\nu}^{\mu} p^\nu \eta_c^{\nu\mu} \left[ (T^c)^u_v + (S^c)^u_v \right] e^{-ip \cdot x_0} , \]

where \((T^I)_{\nu}^{\mu}\) are the \(U(N)\) generators and

\[ (T^c)^u_v = w^u_{\dot{\alpha}} (\tau^c)^{\dot{\alpha}}_{\dot{\beta}} \bar{w}^\beta_v , \quad (S^c)^u_v = -\frac{i}{2} \mu^u \bar{\mu}_v C^c . \]

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The classical solution

- The above solution represents the leading term at long distance of the deformed instanton solution in the singular gauge.
- However, above appeared the unconstrained moduli $\mu, \bar{\mu}, w, \bar{w}$.
  - We need to enforce the deformed ADHM constraints, for $k = 1$:

\[
W^c + \frac{1}{2} \left( M' \cdot M' + \mu^u \bar{\mu}_u \right) C^c = 0 ,
\]

\[
w^u_\alpha \cdot \bar{\mu}_u + \mu^u \bar{w}_\alpha u = 0 .
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The classical solution

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\]

\[
w^u_{\bar{\alpha}} \bar{\mu}_u + \mu^u \bar{w}_{\bar{\alpha} u} = 0 .
\]
Using the ADHM constraints, the solution can be written as

\[ A^I_\mu(x) = 2 \left( \mathcal{M}^{cb} \text{Tr}(T^I t^b) + W^c \text{Tr}(T^I t^0) + \text{Tr}(T^I S^c) \right) \times \bar{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4}. \]
The classical solution in the true moduli space

- Using the ADHM constraints, the solution can be written as

\[
A^I_\mu(x) = 2\left(\mathcal{M}^{cb} \text{Tr}(T^I t^b) + W^c \text{Tr}(T^I t^0) + \text{Tr}(T^I S^c)\right)
\times \bar{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4}.
\]

- On the bosonic ADHM constraints,

\[
W^c = -\frac{i}{2}\left(M' \cdot M' + \mu^u \bar{\mu}_u\right) C^c \equiv \hat{W}^c.
\]

Without the RR deformation, \(W^c\) would vanish.
The classical solution in the true moduli space

Using the ADHM constraints, the solution can be written as

\[ A_I^\mu(x) = 2 \left( M^{cb} \text{Tr}(T^I t^b) + W^c \text{Tr}(T^I t^0) + \text{Tr}(T^I S^c) \right) \]
\[ \times \frac{\bar{\eta}^c_{\mu\nu} (x - x_0)^\nu (x - x_0)^4}{\eta_{\mu\nu}}. \]

The matrix \( M \) is \( M^{ab} = W^0 \sqrt{W_0^2 - |\vec{W}|^2} \left( R^{-\frac{1}{2}} \right)^{ab} \), with

\[ (R)^{ab} = W_0^2 \delta^{ab} - W^a W^b, \]

where

\[ W^0 = w_u^\alpha \bar{w}^{\dot{\alpha}}_u. \]

At \( C_c = 0 \), \( W_0^0 = 2\rho^2 \), where \( \rho = \text{size of the instanton} \).
The classical solution in the true moduli space

- Using the ADHM constraints, the solution can be written as

\[ A^I_\mu(x) = 2 \left( \mathcal{M}^{cb} \text{Tr}(T^I t^b) + W^c \text{Tr}(T^I t^0) + \text{Tr}(T^I S^c) \right) \]
\[ \times \bar{\eta}^c_{\mu \nu} \frac{(x - x_0) \nu}{(x - x_0)^4}. \]

- The \( N \times N \) matrices \( t^a \) and \( t^0 \), depending on the moduli \( w, \bar{w} \), generate a \( u(2) \) subalgebra
- The instanton field contains an abelian factor, beside \( su(2) \).

Moreover, the matrix \( (S^c)^u_v = -\frac{i}{2} \mu^u \bar{\mu}_v C^c \) commutes with this \( u(2) \), and another abelian factor.
The classical solution in the true moduli space

- Using the ADHM constraints, the solution can be written as

\[
A^I_\mu(x) = 2 \left( \mathcal{M}^{cb} \text{Tr}(T^I t^b) + W^c \text{Tr}(T^I t^0) + \text{Tr}(T^I S^c) \right) \times \bar{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4}. 
\]

- The $N \times N$ matrices $t^a$ and $t^0$, depending on the moduli $w, \bar{w}$, generate a $u(2)$ subalgebra.
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- Moreover, the matrix \((S^c)^u_v = -\frac{1}{2} \mu^u \bar{\mu}_v C^c\) commutes with this $u(2)$ another abelian factor.
An explicit case of the solution

- We can write the above general expression choosing a particular solution to the ADHM constraints, to make contact with the literature [Grassi et al, 2003, Britto et al, 2003].

- Decomposing \( u = (\dot{\alpha}, i) \) with \( \dot{\alpha} = 1, 2 \) and \( i = 3, \ldots, N \), the bosonic ADHM constraints are solved by

\[
\begin{align*}
  w^{\dot{\beta}}_{\dot{\alpha}} &= \rho \delta^{\dot{\beta}}_{\dot{\alpha}} + \frac{1}{4\rho} \hat{W}_c (\tau^c)^{\dot{\beta}}_{\dot{\alpha}} , \\
  w^i_{\dot{\alpha}} &= 0 .
\end{align*}
\]

- Having fixed \( w, \bar{w} \), the fermionic constraints are solved by

\[
\mu^{\dot{\alpha}} = \bar{\mu}_{\dot{\alpha}} = 0 .
\]

Moreover, up to a \( U(N - 2) \) rotation, we can choose a single \( \mu^i, \bar{\mu}^{\dot{\alpha}} \) being \( \neq 0 \).
An explicit case of the solution

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- Decomposing $u = (\hat{\alpha}, i)$ with $\hat{\alpha} = 1, 2$ and $i = 3, \ldots, N$, the bosonic ADHM constraints are solved by

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    w^i_{\hat{\alpha}} &= 0.
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\]

- Having fixed $w, \bar{w}$, the fermionic constraints are solved by

\[
\mu^{\hat{\alpha}} = \bar{\mu}_{\hat{\alpha}} = 0.
\]

Moreover, up to a $U(N - 2)$ rotation, we can choose a single $\mu^i$, say $\mu^3$ being $\neq 0$. 
An explicit case of the solution

- The instanton gauge field \((A_\mu)^u_v\) reduces then to

\[
(A_\mu)^{\dot{\alpha}}_{\dot{\beta}} = \left\{ \rho^2 (\tau_c)^{\dot{\alpha}}_{\dot{\beta}} - \frac{i}{4} \left( M' \cdot M' + \mu^3 \bar{\mu}_3 \right) C_c \delta^{\dot{\alpha}}_{\dot{\beta}} \right. \\
+ \frac{1}{32 \rho^2} \left( |\vec{C}|^2 (\tau_c)^{\dot{\alpha}}_{\dot{\beta}} - 2 C_c C^b (\tau_b)^{\dot{\alpha}}_{\dot{\beta}} \right) M' \cdot M' \mu^3 \bar{\mu}_3 \left( x - x_0 \right)^\nu \\
\right\} \bar{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4}
\]

and

\[
(A_\mu)^3_3 = -\frac{i}{2} \mu^3 \bar{\mu}_3 C_c \bar{\eta}^c_{\mu\nu} \frac{(x - x_0)^\nu}{(x - x_0)^4}.
\]

This agrees with [Britto et al, 2003].
Additional remarks

- The gaugino emission is *not* modified at the leading order by the RR background.

- **Subleading** terms in the long-distance expansion of the solution arise from emission diagrams with more moduli insertions.

- At the field theory level, they correspond to having more source terms.

- This, is exactly the field-theoretical procedure utilized in [Grassi et al, 2003, Britto et al, 2003] to determine the (deformed) super-instanton profile,
Conclusions and perspectives
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- Disks (partly) attached to the D(-1)'s account, in the $\alpha' \to 0$ field theory limit for
  - the ADHM construction of instanton moduli space;
  - the classical profile of the instanton solution: the mixed disks are the source for it;
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In particular, the deformation of $\mathcal{N} = 1$ gauge theory to $\mathcal{N} = 1/2$ gauge theory is exactly described in the open string set-up by the inclusion of a particular Ramond-Ramond background.

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Deformations of $\mathcal{N} = 2$ theories:
- deformations of $\mathcal{N} = 2$ superspace by RR backgrounds (work in progress);
- stringy interpretation of the deformations leading to the localization à la Nekrasov of the integrals on instanton moduli space (under investigation, in collab. also with Tor Vergata).

Derivation of the effects of constant Ramond-Ramond field strengths (gauge theory action, instantons, etc) using Berkovits’ formalism instead of RNS (work in progress).

Derivation of the instantonic sector of non-commutative gauge theory from the string realization with constant $B_{\mu\nu}$ background.
Basic references about D-instantons


Stringy realization of ADHM construction

D-brane and gauge theory solutions from string theory


A very, very partial list of references

\textbf{C-deformations}


Instantons in $C$-deformed theories
