Complex Networks
Third Lecture

Plan of the lectures

I. Introduction

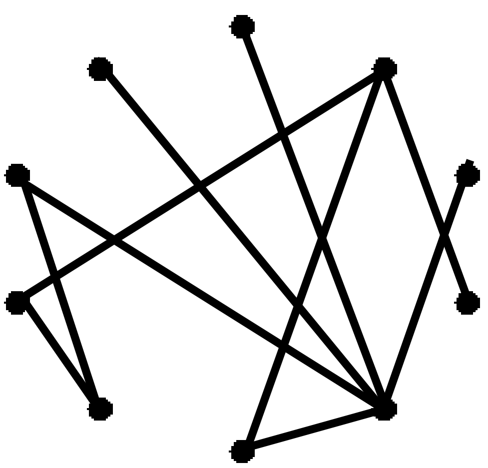
II. Networks: basic definitions

III. Models

IV. Community Detection

Usual random graphs: Erdös-Renyi model (1960)

N points, links with probability p:
static random graphs



Average number of edges:
$<E> = pN(N-1)/2$

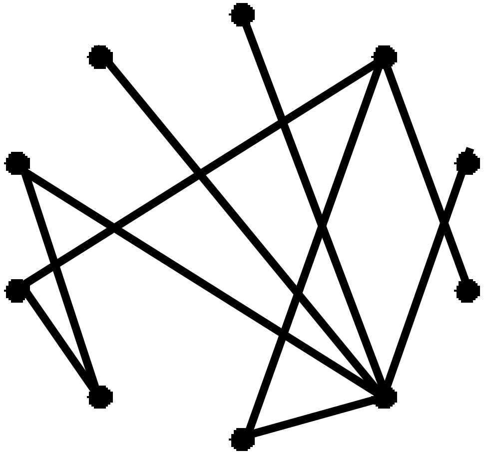
Average degree:
$<k> = p(N-1)$

$p=c/N$ to have
finite average degree

Erdös-Renyi model (1960)

<k> < 1: many small subgraphs

< k > > 1: giant component + small subgraphs



Erdös-Renyi model (1960)

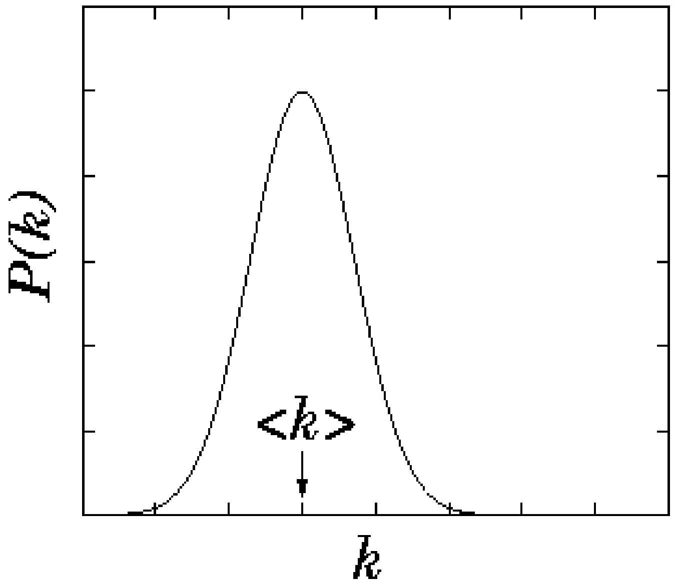
Probability to have a node of degree k

- connected to k vertices,
- not connected to the other N-k-1

$$P(k) = C^{k}_{N-1}\, p^{k}\, (1-p)^{N-k-1}$$

Large N, fixed pN=< k > : Poisson distribution

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^{k}}{k!}$$



Exponential decay at large k

Erdös-Renyi model (1960)

Poisson degree distribution

Small clustering: $\langle C \rangle = p = \langle k \rangle / N$

Short distances $l = \log(N)/\log(\langle k \rangle)$
(number of neighbors at distance d: $\langle k \rangle^d$)

“Real” Networks are different!

(1) The number of nodes (N) is NOT fixed.

Networks continuously expand by the addition of new nodes:

Examples:
WWW: addition of new documents
Citation: publication of new papers

(2) Attachment is NOT uniform.

A node is linked with higher probability to a node that already has a large number of links.

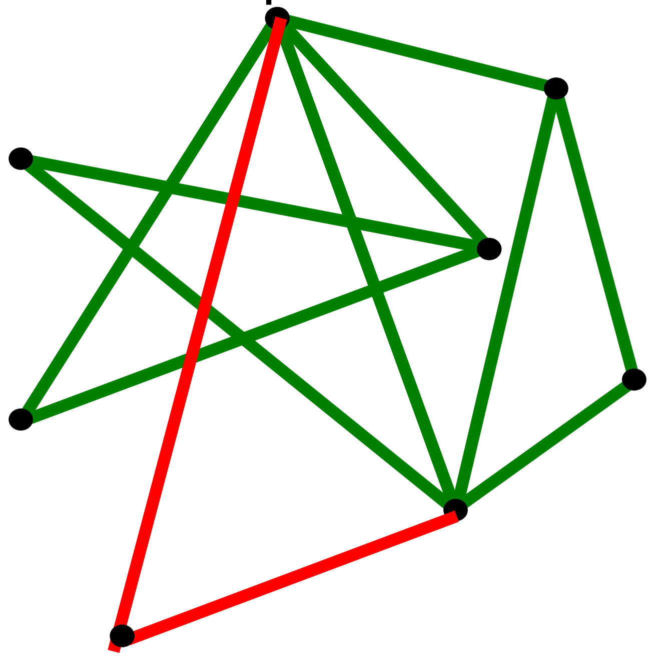
Examples :
WWW : new documents link to well known sites
Citation : well cited papers are more likely to be cited again

Microscopic mechanism:
An example

(1) *GROWTH* **:** At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

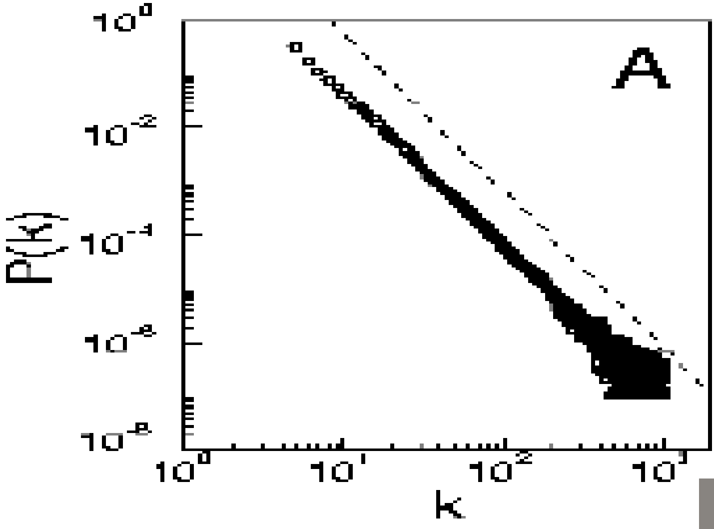
(2) *PREFERENTIAL ATTACHMENT* **:**
The probability **Π** that a new node will be connected to node *i* depends on the connectivity k_i of that node

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$



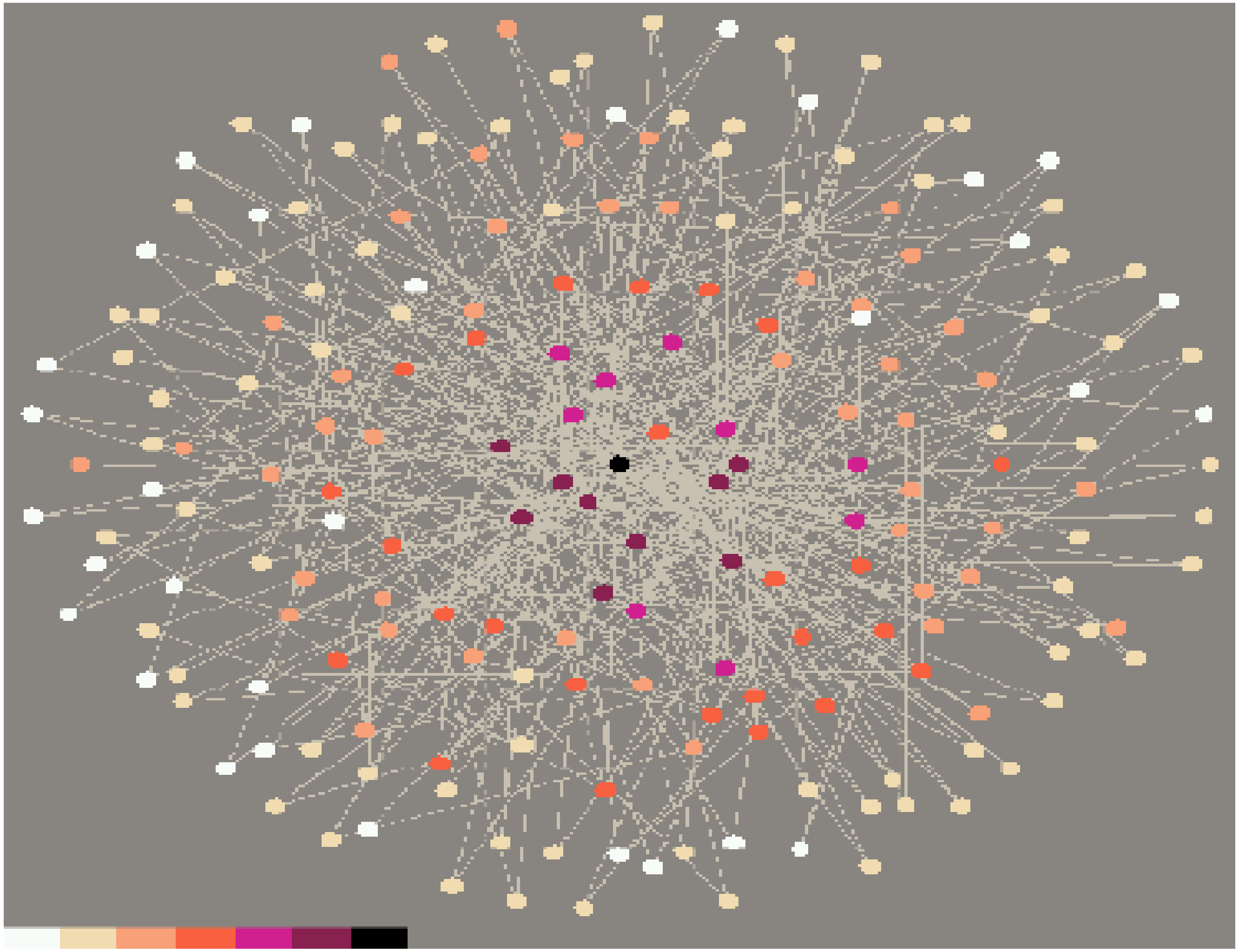
$$P(k,t) \sim \frac{2m^2}{k^3}$$

A.-L.Barabási, R. Albert, Science **286,** 509 (1999)



Connectivity distribution

BA network



Problem with directed graphs

Natural extension: $$P(k_i^{in}) = \frac{k_i^{in}}{\Sigma_j k_j^{in}}$$

What happens if $k_i^{in} = 0$? $$P(k_i^{in}) = 0!$$

Nodes with zero indegree will never receive links! Bad!

Microscopic mechanism:

Preferential attachment with attractivity

(1) ***GROWTH*** **:** At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

(2) ***PREFERENTIAL ATTACHMENT*** **:**
The probability **Π** that a new node will be connected to node *i* depends on the connectivity k_i of that node and a constant k_0 (**attractivity**), with $-m < k_0 < \infty$

$$P(k_i) = \frac{k_i + k_0}{\Sigma_j (k_j + k_0)}$$

S. N. Dorogovtev, J. F. F. Mendes, A. N. Samukhin, Phys. Rev. Lett. **85,** 4633 (2000)

Degree distribution:

$$P(k) \sim k^{-(3 + k_0/m)}$$

Extension to directed graphs:

$$P(k_i^{in}) = \frac{k_i^{in} + k_0}{\Sigma_j (k_j^{in} + k_0)} \qquad P(k^{in}) \sim (k^{in})^{-(2 + k_0/m)}$$

Problem of nodes with zero indegree solved!

Microscopic mechanism:

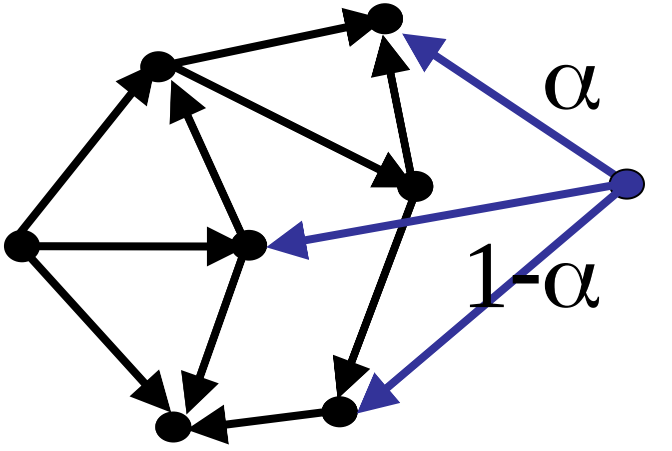
Copying model

Growing network:

a. Selection of a vertex

b. Introduction of a new vertex

c. The new vertex copies m links of the selected one



d. Each new link is kept with proba α, rewired at random with proba 1-α

J. M. Kleinberg, S. R. Kumar, P. Raghavan, S. Rajagopalan, A. Tomkins, Proc. Int. Conf. Combinatorics & Computing, LNCS **1627,** 1 (1999)

Microscopic mechanism:

Copying model

Probability for a vertex to receive a new link at time t:

- Due to random rewiring: $(1-\alpha)/t$

- Because it is neighbour of the selected vertex:

  $k_{in}/(mt)$

effective preferential attachment, without a priori knowledge of degrees!

Microscopic mechanism:

Copying model

Degree distribution:

$$P(k_{in}) \sim (k_0 + k_{in})^{-\frac{2-\alpha}{1-\alpha}}$$

=> Heavy-tails

=> model for the evolution of genetic networks