

MISURE E STATISTICA

Lezione 4

Accuracy, precision, errors

- Precision with which a quantity is measured is fundamental: detection of sources, variability of sources, detection of features
- Background/instrumental noise
- Statistical fluctuations
- How to define error or accuracy in measurements based on counting statistics
- General principle
- Standard deviation limit $\pm\sigma$: where 2/3 of measurements fall
- Review of basic concepts with application to astrophysical measurements (no mathematical demonstrations)
- How to assign error bars to measurements or values derived from them
- How to fit data to theoretical models by least squares method and χ^2 fit evaluation

- Instrumental characteristic background noise
- Measurement of background (off-source pointing) and subtraction from on-source data
- Residual statistical and systematic noise

Statistical fluctuations

- Probability of detecting an integer number x of events when the average (mean) number of events over a large number of trials is m (not derived here)
- *Poisson distribution*

$$P_x = \frac{m^x e^{-m}}{x!}$$

- Normalization

$$\sum_{x=0}^{x=\infty} P_x = 1$$

- Valid for discrete, statistically independent events

Table 6.1. Sample values of Poisson function P_x

x :	0	1	2	3	4	5	6	7^a	8	9
$m = 1$	0.368	0.368	0.184	0.061	0.015	0.003	0.001	7E-5	9E-6	1E-6
$m = 2$	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.001	2E-4
$m = 3$	0.050	0.149	0.224	0.224	0.168	0.101	0.050	0.022	0.008	0.003
$m = 4^b$	0.018	0.073	0.147	0.195	0.195	0.156	0.104	0.060	0.030	0.013
$m = 6^c$	0.002	0.015	0.045	0.089	0.134	0.161	0.161	0.138	0.103	0.069
$m = 10^d$	5E-5	5E-4	0.002	0.008	0.019	0.038	0.063	0.090	0.113	0.125

^a The notation 7E-5 indicates 7×10^{-5} .

^b The values of P_x for $m = 4$ at $x = 10$ and 11 are 0.005 and 0.002 respectively.

^c The values of P_x for $m = 6$ at $x = 10-14$ are 0.041, 0.023, 0.011, 0.005, 0.002.

^d The values of P_x for $m = 10$ at $x = 10-18$ are: 0.125, 0.114, 0.095, 0.073, 0.052, 0.035, 0.022, 0.013, 0.007.

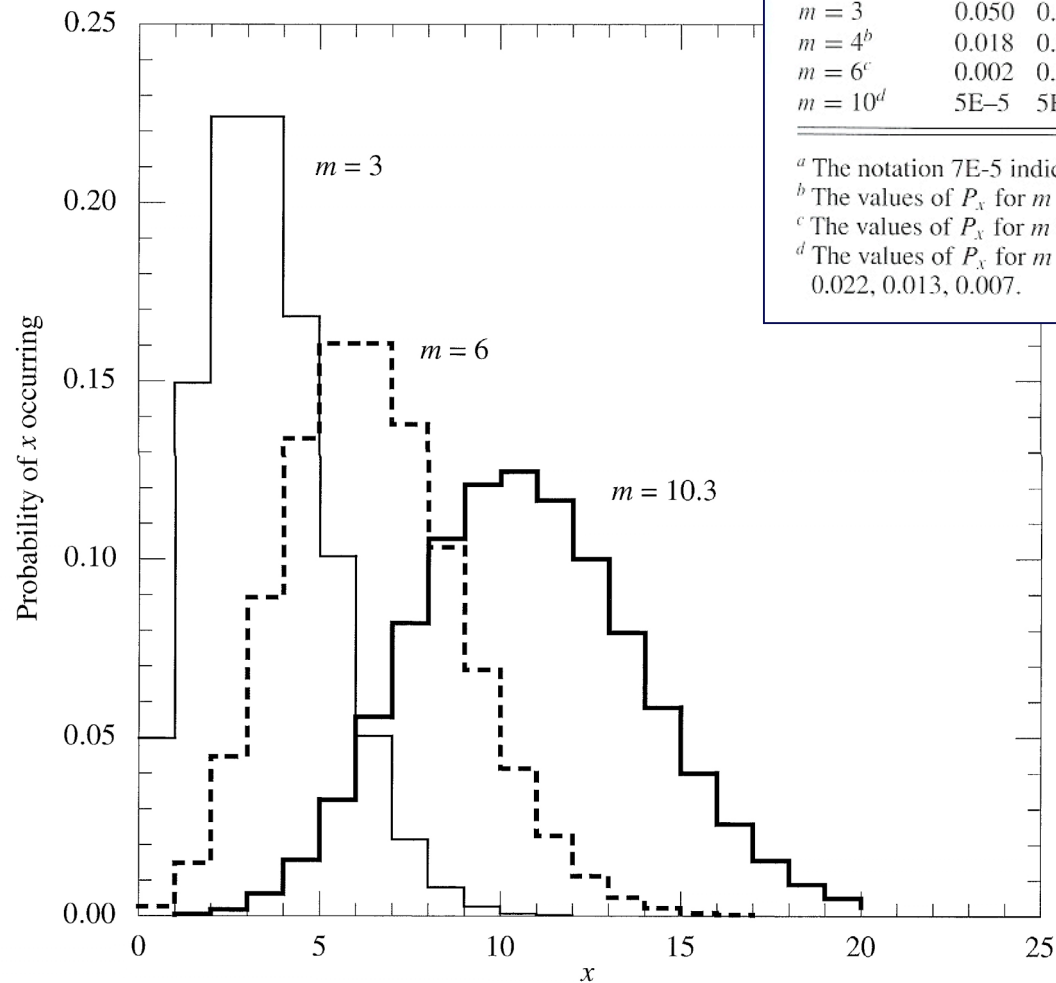


Figure 6.7. The Poisson distribution for small mean numbers, $m = 3.0, 6.0$ and 10.3 . The ordinate gives the probability of the value x occurring, for the given mean value. Note the asymmetry of the histograms.

$$P_x = \frac{m^x e^{-m}}{x!}$$

Normal Gaussian Distribution

- Continuous and symmetrical normal distribution (De Moivre 1735)
- Differential probability of finding the value x within a differential interval dx

$$dP_x = \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left[-\frac{(x-m)^2}{2\sigma_w^2}\right] dx$$

- Two parameters:
 - m is the true value of the quantity to measure
 - σ_w a width parameter
- Normalized to unity by integration over all x
- For $x=m\pm\sigma_w$ probability falls to $e^{-0.5}=0.601$ of its maximum; for $x=m\pm 2^{1/2}\sigma_w$ probability falls to $e^{-1}=0.37$
- The FWHM is for $2.36\sigma_w$
- For large values of m the normal distribution with $\sigma_w=m^{1/2}$ approaches the Poisson distribution (not derived)

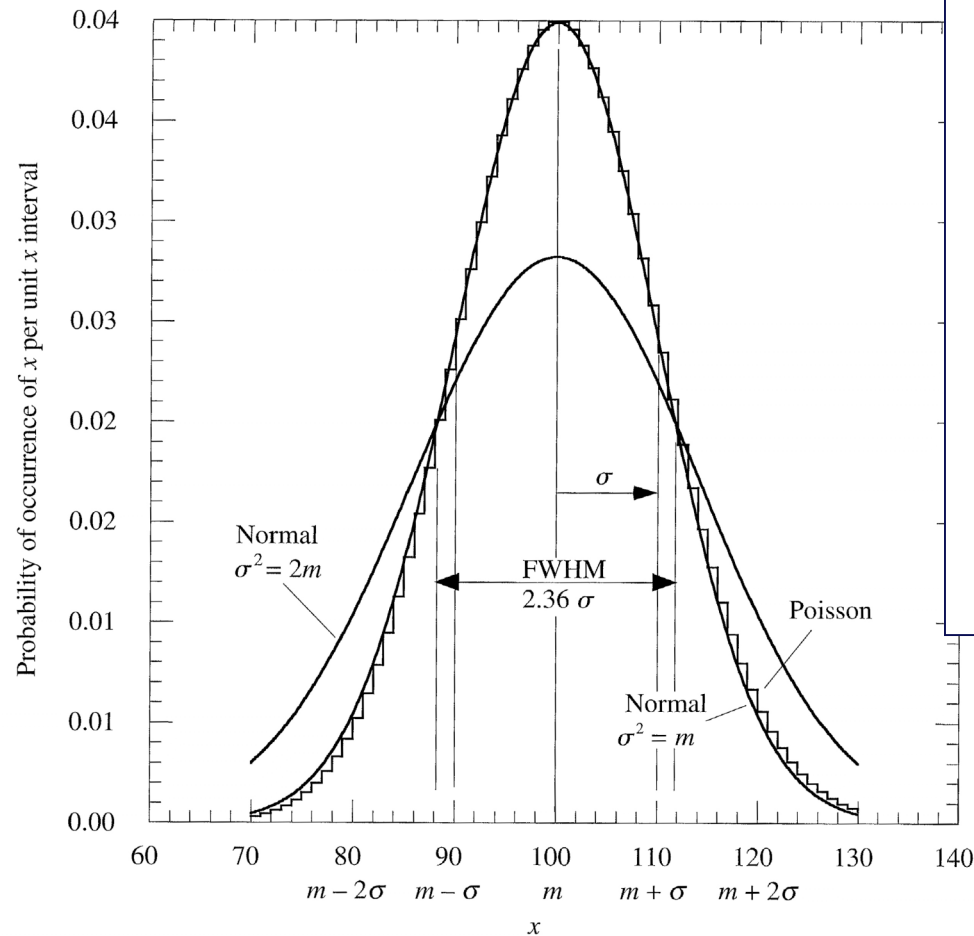
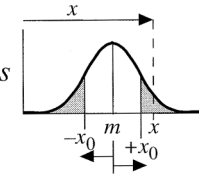


Figure 6.8. The Poisson (step curve) and normal distributions (smooth curves) for the mean value $m = 100$. The normal distribution is given for two values of the width parameter σ_w which is shown in the text to be equal to the standard deviation σ . The Poisson distribution approximates well the normal distribution if the latter has $\sigma = \sqrt{m}$. Note the slight asymmetry of the Poisson distribution relative to the normal distribution. The standard deviation and full width half maximum widths are shown for the higher normal peak; the two normal curves happen to cross at the FWHM point.

Table 6.2. Normal distribution probabilities



$\left(\frac{x_0}{\sigma}\right)^a$	Area (shaded) at $ x - m > x_0^b$	$\left(\frac{x_0}{\sigma}\right)^a$	Area (shaded) at $ x - m > x_0^b$
0	1.00	2.5	0.0124
0.5	0.617	3.0	0.00270
1.0	0.317	3.5	4.65×10^{-4}
1.2	0.230	4.0	6.34×10^{-5}
1.4	0.162	5.0	5.73×10^{-7}
1.6	0.110	6.0	2.0×10^{-9}
1.8	0.0719	7.0	2.6×10^{-12}
2.0	0.0455		

^a Ratio of deviation x_0 to standard deviation σ . The standard deviation σ is equal to σ_w , the width parameter of the distribution.

^b Probability of occurrence of deviation greater than $\pm x_0$.

$$dP_x = \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left[-\frac{(x-m)^2}{2\sigma_w^2}\right] dx$$

68% of the area under the curve falls within $m \pm \sigma_w$, 32% outside

4.5% outside $m \pm 2\sigma_w$

0.27% outside $m \pm 3\sigma_w$ (a miracle!)

Variance and standard deviation

- Width of measured distribution gives range of values obtained from set of n individual measurements of x
- *Root-mean-square deviation* or *standard deviation* σ
- *Variance*

$$\sigma^2 \equiv \frac{1}{n} \sum_{i=1}^{i=n} (x_i - m)^2$$

- m is the true mean obtainable only with $n \rightarrow \infty$
- For finite n , the average x_{av} approximates m
- *Practical variance*

$$\sigma^2 \equiv \frac{1}{n-1} \sum_{i=1}^{i=n} (x_i - x_{av})^2$$

- Express variance in terms of probability

$$\sigma^2 \equiv \sum_{x=-\infty}^{x=\infty} (x - m)^2 P_x$$

- Variance of Poisson distribution

$$\sigma^2 = \sum_{x=-\infty}^{x=\infty} (x - m)^2 \frac{m^x e^{-m}}{x!} = m$$

- Variance of normal distribution

$$\sigma^2 = \int_{x=-\infty}^{x=\infty} (x - m)^2 dP_x = \frac{1}{\sqrt{2\pi} \sigma_w} \int_{x=-\infty}^{x=\infty} (x - m)^2 \exp\left[-\frac{(x - m)^2}{2\sigma_w^2}\right] dx = \sigma_w^2$$

- *Significance of measurements*
 - If a source has an average luminosity of 100 photons in a given time interval, the standard deviation is $\sigma = 10$
 - The probability of measuring 130 photons is 3σ away $P_x=0.0027$, i.e. one chance over 370
 - Measuring >130 photons is outside the limit of statistical fluctuations, i.e. the measure is most likely a true variation in luminosity
 - Repeating measurement is however required to improve significance
 - Statistical traps (ex.: measurements with CCD: the statistical error must be multiplied by the number of pixels)

Background

- Data are contaminated by background events
- Sets of two measurements are made, one with the source in the field of view, one offset from the source
- Manipulation of data to calculate other physical quantities
- Propagation of errors
- Simple argument to evaluate maximum errors on calculated quantities
 - Error of summed measurements of variables

$$z = x \pm y \quad dz = dx + dy$$

$$\left| dz \right|_{\max} = \left| dx \right|_{\max} + \left| dy \right|_{\max}$$

- Fractional error of multiplied quantities (quotient of variables)

$$z = xy \quad dz = xdy + ydx$$

$$\left| \frac{dz}{z} \right|_{\max} = \left| \frac{dx}{x} \right|_{\max} + \left| \frac{dy}{y} \right|_{\max}$$

- Realistic evaluation of errors on quantities from independent measurements
- Use variance to take into account distribution function
 - Error in sum or difference

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$$

for $\sigma_x = \sigma_y$ one obtains $\sigma_z = \sqrt{2}\sigma_x$ instead of $2\sigma_x$

for $\sigma_x \gg \sigma_y$ one obtains $\sigma_z \approx \sigma_x$

- Error in a product or quotient

$$\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

Background subtraction

- S = number of counts expected from a source in a time Δt
- B = number of counts expected from background in a time Δt
- Measure of sky with source = $S + B$, measure of sky without source = B , signal = $(S + B) - B$
- Fluctuations on uncorrelated measurements

$$\sigma_S^2 = \sigma_{S+B}^2 + \sigma_B^2$$

- For Poisson distribution

$$\sigma_{S+B} = \sqrt{(S + B)} \quad \sigma_B = \sqrt{B}$$

$$\sigma_S^2 = S + 2B$$

- If $\sigma_S \ll S$ the measurement is of high quality; typically one assign good *significance* to 3σ results, i.e. $S \geq 3\sigma$
- *Signal-to-noise ratio* (S/N ratio), number of standard deviations

$$\frac{S}{\sigma_S} = \frac{S}{\sqrt{S + 2B}}$$

- Intensity of a source (for Δt precisely known and same for S, B)

$$r_S = \frac{S}{\Delta t} \pm \frac{\sigma_S}{\Delta t} \quad \text{counts/sec}$$

Low and high background limits

- $B \ll S$ low background

$$\frac{S}{\sigma_S} \approx \frac{S}{\sqrt{S}} = \sqrt{S} = \sqrt{r_S \Delta t}$$

- Significance increases for a given source with $(\Delta t)^{1/2}$; to increase significance one must increase integration time
- $B \gg S$ high background

$$\frac{S}{\sigma_S} \approx \frac{S}{\sqrt{2B}} = \frac{r_S \Delta t}{\sqrt{2r_B \Delta t}} = \frac{r_S}{\sqrt{2r_B}} \sqrt{\Delta t}$$

- Again significance increases with $(\Delta t)^{1/2}$, but the coefficient is small and large integration times are required to have good S/N
- Compare sensitivities of two detectors (one high B , one low B) on same source with same integration time Δt

$$\left(\frac{S}{\sigma_S} \right)_{B \gg S} = \sqrt{\frac{r_S}{2r_B}} \left(\frac{S}{\sigma_S} \right)_{B \ll S}$$

- Since $r_S \ll r_B$ significance is much less in high background observations for same integration time

Bright and faint source observations

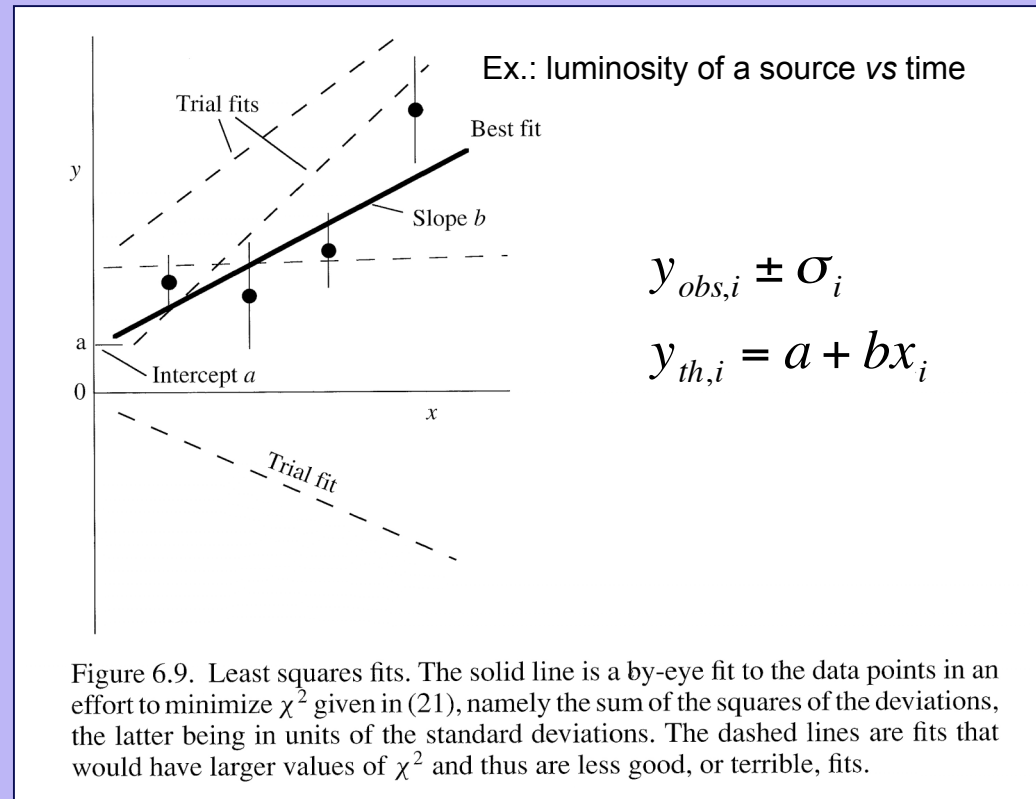
- When source dominates
 - Statistical noise comes from the source itself
 - r_S and r_B increase together
 - $S/N \propto (r_S)^{1/2}$
- When background dominates
 - Noise does not depend on intensity of source
 - $S/N \propto r_S$
- Detectors in the two cases have different efficiencies
 - Proportional counters have high background, but can have large effective areas
 - Focusing systems have low background, but small effective areas
 - For bright sources a high-background large area systems can have much larger S/N in a given time and allow variability measures
 - Focusing instruments are low background systems and are required to detect faint sources and structures

Finding parameters and checking hypotheses

- Data are used to derive parameters and check consistency with theory
- Least squares fit
- Chi square definition

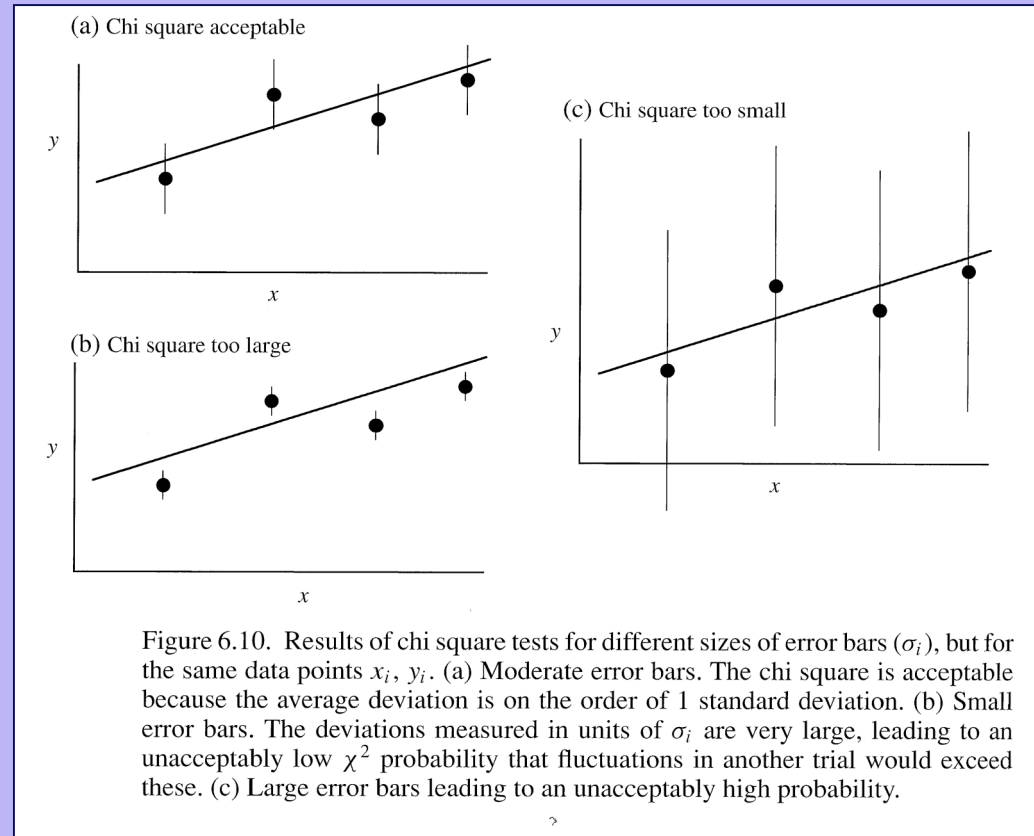
$$\chi^2 = \sum_i \left[\frac{(y_{obs,i} - y_{th,i})^2}{\sigma_i} \right]$$

- Minimizing χ^2 gives more weight to points with small error bars
- Vary a and b to get best fit
- Best fit is not necessarily consistent with data; χ^2 may be too bad or too good



Chi square test

- Obtain best fit and value of χ^2
- If the trial function were the true function and one made a similar set of measurements, what is the probability to find a larger χ^2 ?
- Chi square statistics:
 - (a) probability 0.1-0.9, fluctuations in data are compatible with any other set of measurements
 - (b) probability < 0.02 , χ^2 large, data have improbably large deviations
 - (c) probability > 0.98 , χ^2 small, data are clustered around curve, do not fluctuate following statistics, wrong error bars



- Statistics of χ^2 (not derived) is plotted in a plane (χ^2, f)
- f are defined *degrees of freedom*: number of data points n less the number of variable parameters p in the theoretical function

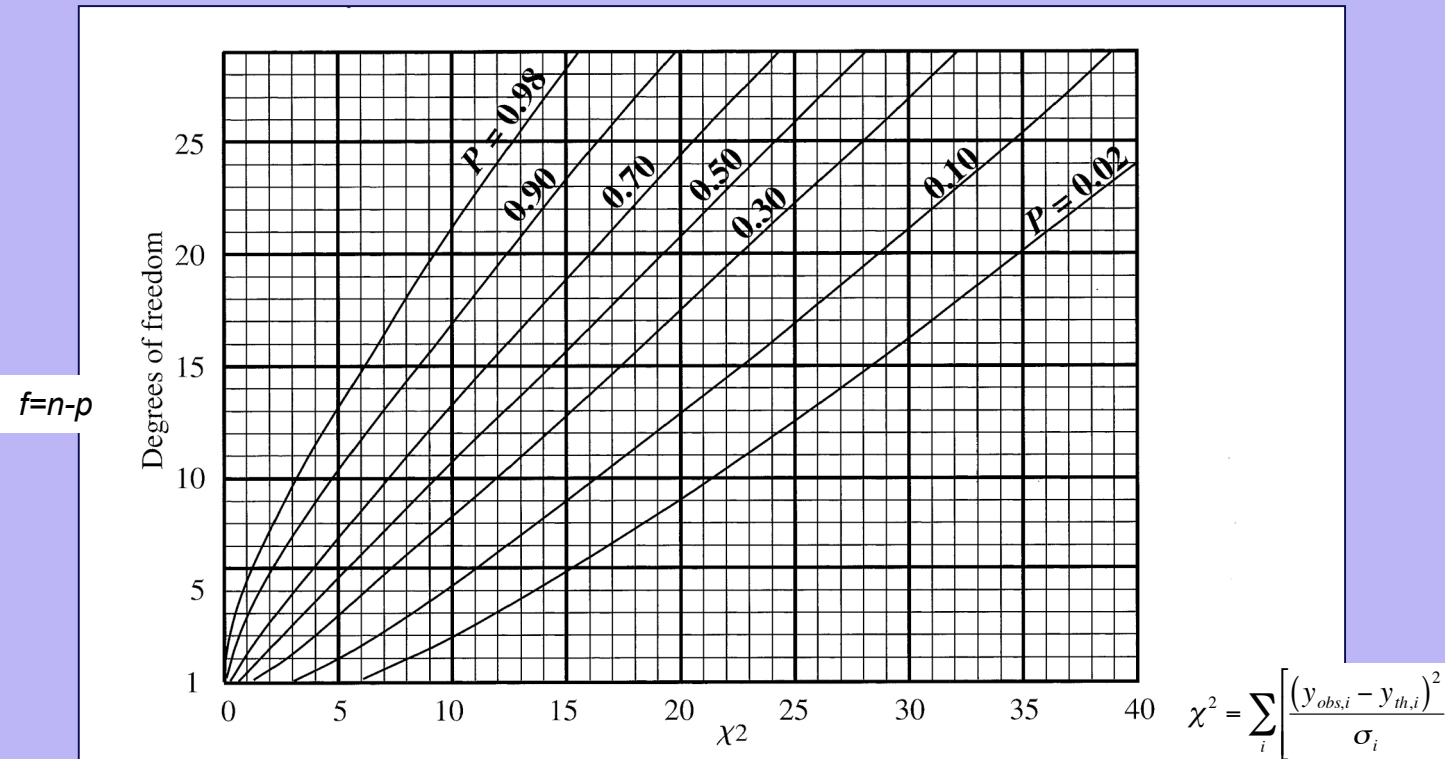


Figure 6.11. The probabilities P for the chi square test. The ordinate is the number of degrees of freedom (number of data points less number of variable parameters in trial function), and the abscissa is the value of χ^2 . The curves give the probability that χ^2 would have a greater value in another set of measurements. [Adapted from Evans, *The Atomic Nucleus*, McGraw-Hill, 1955, p. 776, with permission.]

- Why degrees of freedom ?
 - Straight line fit has 2 parameters, a and b
 - $n = 20$ data points, $f = 18$
 - Fitting a straight line on 2 points with 2 parameters is doable, $\chi^2 = 0$
 - Only point 3 makes $\chi^2 \neq 0$, 18 points make $\chi^2 \rightarrow 18$
- The typical value of each term of χ^2 for normal fluctuations is $\sim(1\sigma/\sigma)$, which means that $\chi^2 \sim f$
- Values $\chi^2 \gg f$ or $\chi^2 \ll f$ do not correspond to normal statistical fluctuations
- In astronomy *reduced chi square* is used

$$\chi_v^2 = \frac{\chi^2}{f}$$

that must be around unity:

- $f = 10 \rightarrow 0.49 < \chi_v^2 < 1.6$
- $f = 200 \rightarrow 0.87 < \chi_v^2 < 1.13$