Large Eddy Simulations in Plasma Astrophysics. Weekly Compressible Turbulence in Local Interstellar Medium

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Goals

• To develop Large Eddy Simulation for compressible MHD turbulence
• To propose subgrid-scale models for compressible MHD case and make their comparisons and an analysis.
• To carry out test computations of MHD flows
• To apply LES approach to MHD flows in space plasma.
Homogeneous, isotropic turbulent plasma motions are described in terms of a single electrically conducting fluid model in the following manner:

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} &= - \frac{\partial \rho u_j}{\partial x_j} \\
\frac{\partial \rho u_i}{\partial t} &= - \frac{\partial}{\partial x_j} \left( \rho u_i u_j + p \delta_{ij} - \sigma_{ij} - \frac{1}{4\pi} B_j B_i + \frac{1}{8\pi} B^2 \right) \\
\frac{\partial B_i}{\partial t} &= - \frac{\partial}{\partial x_j} (B_i u_j - B_j u_i) + \eta \nabla^2 B_i \\
\frac{\partial B_j}{\partial x_j} &= 0
\end{aligned}
\]
Filtered MHD equations using ordinary filtering

\[
\frac{\partial \bar{\rho} \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \bar{\rho} \bar{u}_i \bar{u}_j + \bar{p} \delta_{ij} - \bar{\sigma}_{ij} + \frac{\bar{B}^2}{8\pi} \delta_{ij} - \frac{1}{4\pi} \bar{B}_j \bar{B}_i \right) = -\frac{\partial}{\partial x_j} \left( \bar{\rho} u_j u_i - \bar{\rho} \bar{u}_i \bar{u}_j \right) + \\
+ \frac{1}{4\pi} \frac{\partial}{\partial x_j} \left( \bar{B}_j \bar{B}_i - \bar{B}_i \bar{B}_j \right) - \frac{\partial}{\partial t} \left( \bar{\rho} \bar{u}_i - \bar{\rho} \bar{u}_i \right)
\]

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho} \bar{u}_j}{\partial x_j} = -\frac{\partial}{\partial x_j} \left( \bar{\rho} u_j - \bar{\rho} \bar{u}_j \right)
\]

additional terms, which are necessary to parametrize!

It requires additional numerical resources
Filtering operation - 2

To simplify equations describing turbulent MHD flow with variable density it is convenient to use Favre filtering (known as mass-weighted filtering) so that to avoid the appearance of extra SGS terms. Therefore, Favre filtering is used in this work.

\[ \tilde{f} = \frac{\rho f}{\bar{\rho}} \]

\[ f = \tilde{f} + f'' \]

The Favre filtered velocity:

\[ \tilde{u}_j = \frac{\rho u_j}{\bar{\rho}} = \frac{\int_a^b \rho u_j G(x_j - x'_j, \Delta_j) \, dx'_j}{\int_a^b \rho(x_j) G(x_j - x'_j, \Delta_j) \, dx_j}. \]

Properties:

- \( \rho u'' \neq 0 \)
- \( \tilde{u} \neq \tilde{u} \)
- \( \tilde{u} \neq 0 \)
Filtered MHD equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho \tilde{u}_i}{\partial x_j} &= 0 \\
\frac{\partial \rho \tilde{u}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho \tilde{u}_i \tilde{u}_j + p \delta_{ij} - \frac{1}{\text{Re}} \tilde{\sigma}_{ij} + \frac{\overline{B}^2}{2M_A^2} - \frac{1}{M_A^2} \overline{B_iB_j} \right) &= -\frac{\partial \tau^u_{ji}}{\partial x_j} \\
\frac{\partial \overline{B}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \tilde{u}_j \overline{B}_i - \overline{B}_j \tilde{u}_i \right) - \frac{1}{\text{Re}_m} \frac{\partial^2 \overline{B}_i}{\partial x_j^2} &= -\frac{\partial \tau^b_{ji}}{\partial x_j}
\end{align*}
\]

On the right-hand sides of equations the terms designate influence of subgrid terms on the filtered part. To determine these terms, special turbulent parametrizations based on large-scale values describing turbulent MHD flow must be used.

Subgrid scale (SGS) or Subfilter-scale (SFS) terms
Smagorinsky model for MHD

\[ \tau_{ij}^u - \frac{1}{3} \tau_{kk}^u \delta_{ij} = -2 \nu_t (\tilde{S}_{ij} - \frac{1}{3} \tilde{S}_{kk} \delta_{ij}) \quad \tau_{kk}^u = 2Y_1 \bar{\rho} \bar{\Delta}^2 |\tilde{S}^u|^2 \quad |\tilde{S}^u| = (2S_{ij}S_{ij})^{1/2} \]

**Turbulent viscosity:**  \( \nu_t = C_1 \bar{\rho} \bar{\Delta}^2 |\tilde{S}^u| \)

\[ \tilde{S}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) \quad \text{- large-scale strain rate tensor} \]

\[ \tilde{J}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{B}_i}{\partial x_j} - \frac{\partial \tilde{B}_j}{\partial x_i} \right) \quad \text{- large-scale magnetic rotation tensor} \]

\[ \tau_{ij}^b - \frac{1}{3} \tau_{kk}^b \delta_{ij} = -2 \eta_t \tilde{J}_{ij} \]

**Turbulent diffusivity:**  \( \eta_t = D_1 \bar{\Delta}^2 |j| \)
Kolmogorov model for MHD

If the filter length is in the inertial subrange of the completely developed turbulence, the kinetic subgrid energy and the magnetic subgrid-energy dissipation can be assumed to be only time dependent and constant in space. These parametrizations are based on the Kolmogorov scaling model:

\[ \nu_t = C_2 \overline{\rho} \Delta^{4/3} \quad - \text{turbulent viscosity} \]

\[ \tau_{kk}^u = 2Y_2 \overline{\rho} \Delta^{4/3} |\tilde{S}^u| \quad - \text{isotropic term} \]

\[ \eta_t = D_2 \Delta^{4/3} \quad - \text{turbulent magnetic diffusivity} \]
Cross-helicity model

The cross helicity is

\[ H^c = \int_V (u \cdot B) dV \]

The cross-helicity is related to the transfer between kinetic and magnetic energies caused by the Lorentz force. Therefore, the cross helicity allows one to estimate the energy exchange between large and small scales in the LES method:

\[ \nu_t = C_3 \rho \Delta^2 | \tilde{S}^u_{ij} \tilde{S}^b_{ij} |^{1/2} \quad - \text{turbulent viscosity} \]

\[ \tau_{kk}^u = 2Y_3 \rho \Delta^2 | \tilde{f} || \tilde{S}^u_t | \quad - \text{isotropic term} \]

\[ \eta_t = D_3 \Delta^2 \text{sgn}( \tilde{j} \tilde{\omega} ) | \tilde{j} \tilde{\omega} |^{1/2} \quad - \text{turbulent magnetic diffusivity} \]
The scale-similarity model is not of the eddy-viscosity-type. It is based on the assumption that the component of the SGS most active in the energy transfer from large to small scales can be estimated with sufficient accuracy from the smallest resolved scale, which can be obtained by filtering the SGS quantities

\[
\tau_{ij}^u = \bar{\rho}((\tilde{u}_j \tilde{u}_i) - \tilde{u}_j \tilde{u}_i) - \frac{1}{M_A^2} (\overline{B_i B_j} - \overline{B_j B_i})
\]

\[
\tau_{ij}^b = (\tilde{u}_i \overline{B_j} - \overline{B_j} \tilde{u}_i) - (\overline{B_i} \tilde{u}_j - \tilde{u}_j \overline{B_i})
\]
Mixed model for MHD

The mixed model is a combination of two models: the scale similarity and the Smagorinsky models.

\[
\tau_{ij}^{u} - \frac{1}{3} \tau_{kk}^{u} \delta_{ij} = -2C_5 \bar{\rho} \bar{\Delta}^2 \left| \tilde{S}^{u} \right| (\tilde{S}_{ij} - \frac{1}{3} \tilde{S}_{kk} \delta_{ij}) + \bar{\rho}((\tilde{u}_j \tilde{u}_i) - \tilde{u}_j \tilde{u}_i) - \frac{1}{M_A^2} (\overline{B_i B_j} - \overline{B_j B_i})
\]

\[
\tau_{kk}^{u} = 2Y_5 \bar{\rho} \bar{\Delta}^2 \left| \tilde{S}^{u} \right|^2
\]

\[
\tau_{ij}^{b} = -2D_5 \bar{\Delta}^2 \left| \tilde{j} \right| \tilde{J}_{ij} + (\tilde{u}_i \overline{B_j} - \overline{B_j} \tilde{u}_i) - (\overline{B_i \tilde{u}_j} - \tilde{u}_j \overline{B_i})
\]
Numerical realization - 1

• MHD equations written in the conservative form.
• Finite difference numerical code of the fourth order accuracy.
• Third order low-storage Runge-Kutta method for time integration.
• The Courant-Friedrichs-Levy stability condition (CFL condition) on the time step.
• Gaussian filter of the fourth order accuracy.
• The skew-symmetric form for the nonlinear terms.
The left boundary for Re is chosen so as to provide a regime of developed turbulence, and the right boundary is the compromise between obtaining adequate DNS results and the necessity of carrying out the comparative analysis with subgrid-scale LES models. The value of the right boundary of the considered interval for magnetic Reynolds number is defined so that we investigate decaying compressible MHD turbulence, and the probability of occurrence of dynamo-processes in three-dimensional charged fluid flow increases with Re_m. The left boundary for Re_m is determined to express the role of magnetic effects in MHD flow. Mach number is limited by one because in this work approximation of polytropy gas is accepted. The flow with the value Mach number less than 0.2 is not interesting from the point of view of studying compressible flow.

<table>
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<tr>
<th>Case</th>
<th>Re_I</th>
<th>Re</th>
<th>Re_m</th>
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## CASE #1 - 1

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<td>Kolmogorov model</td>
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<td>Cross-helicity model</td>
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<td>Scale-similarity model</td>
<td>Marker +</td>
</tr>
<tr>
<td>Mixed model</td>
<td>Dashed-dotted</td>
</tr>
</tbody>
</table>
CASE #1 - 2

\[ \varepsilon_\mu = 2\mu \tilde{S}_{ij} \tilde{S}_{ij} \]

molecular dissipation

\[ \chi_\mu = -\tau_{ij}^u \tilde{S}_{ij} \]

kinetic energy subgrid-scale dissipation

\[ \varepsilon_\eta = \eta |\tilde{j}|^2 \]

magnetic molecular dissipation

\[ \chi_b = -\tau_{ij}^b \tilde{J}_{ij} \]

magnetic energy subgrid-scale dissipation
CASE #1 - 3

Fluctuating parts:

\[ \nu_i = \bar{u} - \langle \rho \bar{u}_i \rangle / \langle \rho \rangle \]

\[ b_i = \bar{B}_i - \langle \bar{B}_i \rangle \]
CASE #1 - 4

Kurtosis (or flatness) of a velocity component:

\[ Ku_j = \frac{\langle u_j^4 \rangle}{\langle (u_j^2) \rangle^2} \]

Kurtosis (or flatness) of a magnetic field:

\[ Kb_j = \frac{\langle B_j^4 \rangle}{\langle (B_j^2) \rangle^2} \]
CASE #1 - 5

Skewness of a velocity component:

$$S_{u_j} = \frac{\langle u_j^3 \rangle}{\langle u_j^2 \rangle^{3/2}}$$

Skewness of a magnetic field:

$$S_{b_j} = \frac{\langle B_j^3 \rangle}{\langle B_j^2 \rangle^{3/2}}$$
Kinetic energy

Ms = 1
Re_m = 2

Ms = 0.2
Re_m = 20
• For kinetic energy, larger divergence of LES results was observed with a decrease in magnetic Reynolds number using various SGS closures. The scale-similarity model shows the worst results, however, the other SGS closures increase calculation accuracy.

• The changing of Reynolds number produces qualitatively similar results, as the initial conditions of velocity and magnetic fields are the same, and therefore Taylor Reynolds number does not have a significant impact on the choice of subgrid parameterizations in our computations.

• Mach number $M_s$ exerts essential influence on results of modeling. The divergence between DNS and LES results for kinetic energy increases with $M_s$.

• Generally, the Smagorinsky model and the cross-helicity model yield the best accordance with DNS under various Mach number.
Magnetic energy - 1

- $M_s = 1$
- $M_s = 0.2$
- $Re_m = 2$
- $Re_m = 20$
• The differences between SGS models for magnetic energy are shown to decrease with reducing magnetic Reynolds number and all models above demonstrate good agreement with DNS results at small value of number Re$_m$.

• The effect of subgrid-scale closures increases with magnetic Reynolds number for modeling of compressible MHD turbulence, but the rate of dissipation of the magnetic energy decreases with increasing Re$_m$.

• Generally, the best results are shown for the Smagorinsky, the Kolmogorov, and the cross-helicity models for evolution of the magnetic energy.

• The deviations in results for magnetic energy decrease with increasing Ms. It is necessary to notice, that magnetic energy reaches a stationary level more rapidly with reducing Mach number.
Cross-helicity - 1

Ms = 1
Re_m = 2

Ms = 0.2
Re_m = 20
For the cross-helicity, the influence of subgrid-scale parametrizations increases with magnetic Reynolds number.

The scale-similarity model demonstrates the worst results. In the presence of adequate SGS parametrization improves calculation accuracy.

The Smagorinsky model shows the best results for the cross-helicity both for high and for low Mach numbers.
MHD EQUATIONS FOR HEAT CONDUCTING FLUID -1

The governing system of compressible electrically conducting fluid is written in the following form:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0; \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho u_i u_j + p \delta_{ij} - \sigma_{ij} \right) - \frac{1}{c} \epsilon_{ijk} j_j B_k = 0; \\
\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j} \left( (\rho E + p) u_j + q_j - \sigma_{ij} u_i \right) - \Xi j = 0; \\
\frac{\partial B_i}{\partial t} = \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} (B_l u_m) + \eta \frac{\partial^2 B_i}{\partial x_j^2}; \\
\frac{\partial B_i}{\partial x_i} = 0.
\]
The MHD approximation implies that the energy of the electric field is much less than that of the magnetic field. In this case, the electric field is eliminated from the governing system of equations, and flow characteristics are expressed in terms of magnetic field. Using Maxwell equations for electrodynamic field, we transform MHD equations and reduce to the following dimensionless form:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0; \]

\[ \frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho u_i u_j + p \delta_{ij} - \frac{1}{Re} \sigma_{ij} + \frac{B^2}{2M_a^2} \delta_{ij} - \frac{1}{M_a^2} B_j B_i \right) = 0; \]

\[ \frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} \left( u_j B_i - u_i B_j \right) - \frac{1}{Re_m} \frac{\partial^2 B_i}{\partial x_j^2} = 0; \]

\[ \frac{\partial}{\partial t} \left( \rho E + \frac{B^2}{2M_a^2} \right) + \frac{\partial}{\partial x_j} \left( (\rho E + P) u_j + \frac{1}{PrReM_s^2(\gamma - 1)} q_j - \frac{1}{Re} \sigma_{ij} u_i - \frac{1}{M_a^2} B_j B_i u_i \right) - \]

\[ - \frac{\partial}{\partial x_j} \left( \frac{\eta}{Re_m M_a^2} B_i \left( \frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i} \right) \right) = 0. \]
Filtered MHD equations for heat-conducting plasma

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho \tilde{u}_j}{\partial x_j} &= 0 \\
\frac{\partial \rho \tilde{u}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho \tilde{u}_i \tilde{u}_j + \tilde{p} \delta_{ij} - \frac{1}{\text{Re}} \tilde{\sigma}_{ij} + \frac{B_i B_j}{2M_A^2} - \frac{1}{M_A^2} B_i B_j \right) &= -\frac{\partial \tau_{ji}^u}{\partial x_j} \\
\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \tilde{u}_j B_i - B_j \tilde{u}_i \right) - \frac{1}{\text{Re}_m} \frac{\partial^2 B_i}{\partial x_j^2} &= -\frac{\partial \tau_{ji}^b}{\partial x_j} \\
\frac{\partial \rho \tilde{E}}{\partial t} + \frac{\partial}{\partial x_j} \left[ (\tilde{E} + \tilde{P}) \tilde{u}_j - \frac{1}{M_A^2} \tilde{B}_i \tilde{B}_j \tilde{u}_i \right] + \frac{\partial}{\partial x_j} \left[ \tilde{q}_j \frac{\eta}{\text{Pr} \text{Re} M_s^2 (\gamma - 1)} \right] - \frac{1}{\text{Re}} \tilde{\sigma}_{ij} \tilde{u}_i \right] &= -\frac{\partial}{\partial x_j} \left( \frac{1}{\gamma M_s^2} Q_j + \frac{1}{2} J_j + \frac{1}{2M_a^2} V_j - \frac{1}{M_s^2} G_j \right)
\end{align*}
\]

Equation of state: \( \bar{p} = \frac{\tilde{E}}{\gamma M_s^2} \quad E = \rho e + \frac{1}{2} \rho u_i u_i + \frac{1}{2M_a^2} B_i B_i \) \( \epsilon = \frac{T \rho}{\gamma (\gamma - 1) M_s^2} \)

total energy \hspace{1cm} internal energy
Since compressibility effects and temporal dynamics of temperature defined from the total energy equation depend nontrivially on the Mach number, in this work we consider three cases:
the Mach number $M_s = 0.38$, that is, the flow is moderately compressible;
the Mach number $M_s = 0.70$, when compressibility plays an important role in turbulent fluid flow;
the Mach number $M_s = 1.11$ corresponding to appearance of strong discontinuity in essentially compressible flow.

In all three numerical experiments, the following dimensionless parameters for computations are used: the hydrodynamic Reynolds number $Re = 281$, the microscale (Taylor) Reynolds number $Re_l = 43$, the magnetic Reynolds number $Re_m = 10$, the magnetic Mach number $Ma = 1.2$, the Prandtl number $Pr = 1.0$ and the ratio of the specific heats $1.5$. 

CASE STUDIES
Time dynamics of kinetic and magnetic energy
$M=0.38$

Time evolution of cross-helicity and temperature
$M = 0.38$

Time dynamics the skewness of the temperature and the parameter $Ft$

$$St = \frac{\langle T^3 \rangle}{(\langle T^2 \rangle)^{3/2}}$$  - skewness of the temperature

$$Ft = (\langle (T - \langle T \rangle)^2 \rangle)^{1/2}$$  - parameter, describing temperature fluctuations
Kinetic and magnetic energy spectra.
\[ M = 0.70 \]

Time dynamics of kinetic and magnetic energy
M = 0.70

Time evolution of cross-helicity and temperature
M = 0.70

Kinetic and magnetic energy spectra.
M = 1.11

Time dynamics of kinetic and magnetic energy
Time evolution of cross-helicity and temperature
LES for Heat-Conductive plasma

• The system of the filtered MHD equations with the total energy equation using the mass-weighted filtering procedure has been obtained. Novel subgrid-scale terms arise in total energy equation due to the presence of energy equation.

• New subgrid-scale models for the SGS terms, appearing after filtering procedure in the total energy equation in the presence of magnetic field, are suggested.

• Consideration of the SGS terms in the energy equation scarcely affects the kinetic and the magnetic energy even at high Mach numbers, while for the temperature (same as for the internal energy) the presence of SGS models in the energy equation is an important condition for improvement of calculation accuracy of thermodynamic quantities.

• Generally, LES method using explicit mass-weighted filtering demonstrates good results for modeling of electrically and heat conducting fluid in MHD turbulence when the medium is weakly or moderately compressible.
Compressible MHD equations

The system of equations of compressible magnetohydrodynamic turbulence in the presence of external force is written in the following form:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} & = - \frac{\partial \rho u_j}{\partial x_j} \\
\frac{\partial \rho u_i}{\partial t} & = - \frac{\partial}{\partial x_j} \left( \rho u_i u_j + p \delta_{ij} - \sigma_{ij} - \frac{1}{4\pi} B_j B_i + \frac{1}{8\pi} B^2 \right) + F_i^u \\
\frac{\partial B_i}{\partial t} & = - \frac{\partial}{\partial x_j} \left( B_i u_j - B_j u_i \right) + \eta \nabla^2 B_i + F_i^b \\
\frac{\partial B_j}{\partial x_j} & = 0
\end{align*}
\]

Driving forces

Polytropic relation: \( p = \rho^\gamma \)
Linear forcing

Idea essentially consists in adding a force proportional to the fluctuating velocity. Linear forcing resembles a turbulence when forced with a mean velocity gradient, that is, a shear. This force appears as a term in the equation for fluctuating velocity that corresponds to a production term in the equation of turbulent kinetic energy.

The equation for the fluctuating part of the velocity in a compressible MHD turbulent flow are written as

\[
\rho \left[ \frac{\partial \dot{u}_i}{\partial t} + U_j \frac{\partial \dot{u}_i}{\partial x_j} \right] = - \frac{\partial \dot{p}}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_j} - \left[ \rho \dot{u}_j \frac{\partial U_i}{\partial x_j} - \rho \langle \dot{u}_j \frac{\partial \dot{u}_i}{\partial x_j} \rangle \right] - \frac{\partial}{\partial x_j} \frac{\dot{B}^2}{8\pi} + \frac{1}{4\pi} \left[ \beta_j \frac{\partial \dot{B}_i}{\partial x_j} + \dot{B}_j \frac{\partial \beta_i}{\partial x_j} \right] - \frac{1}{4\pi} \left[ \dot{B}_j \frac{\partial \dot{B}_i}{\partial x_j} - \langle \dot{B}_j \frac{\partial \dot{B}_i}{\partial x_j} \rangle \right]
\]

Here following decomposition referred to as the Reynolds decomposition is used:

\[
u_i = U_i + \dot{u}_i, \quad B_i = \beta_i + \dot{B}_i, \quad \dot{B}_i = \beta_i + \dot{B}_i, \quad p = P + \dot{p}, \quad \sigma_{ij} = \Sigma_{ij} + \dot{\sigma}_{ij}
\]
Linear forcing

In symbolic terms, derivation of turbulent kinetic energy equation can be written as
$$\langle u \cdot NS \text{ eq.} \rangle - U \langle NS \text{ eq.} \rangle$$
which yields:

$$\frac{\partial}{\partial t} \left\langle \frac{1}{2} \rho \dot{u}^2 \right\rangle + \frac{\partial}{\partial x_j} \left( \left( \frac{1}{2} \rho \dot{u}^2 \right) U_j + \left( \frac{1}{2} \rho \dot{u}^2 \dot{u}_j \right) - \langle \beta_{ij} \dot{u}_i \rangle \right) = -\langle \dot{u}_i \frac{\partial \rho \dot{p}}{\partial x_i} \rangle + \langle \dot{u}_i \frac{\partial \sigma_{ij}}{\partial x_j} \rangle - \langle \rho \dot{u}_i \dot{u}_j \frac{\partial U_i}{\partial x_j} \rangle - \langle \beta_{ij} \frac{\partial \dot{u}_i}{\partial x_j} \rangle$$

where $\beta_{ij} = \frac{\dot{B}_i \dot{B}_j}{4\pi} - \frac{\dot{B}^2}{8\pi} \delta_{ij}$ - turbulent magnetic tensor

production of turbulent energy per unit volume per unit time resulting from the interaction between the Reynolds stress and the mean shear.
Linear forcing

\[ F_i^u = \Theta \rho u_i \quad \text{- driving term proportional to the velocity} \]

coefficient which is determined from a balance of kinetic energy for a statistically stationary state:

\[ \Theta = \frac{1}{3 \langle \rho \rangle u_{rms}^2} \left[ \langle u_j \frac{\partial}{\partial x_j} \rho \delta_{ij} \rangle + \varepsilon + \frac{1}{8\pi} \langle u_j \frac{\partial}{\partial x_j} B^2 \delta_{ij} \rangle \right] \]

\[ \varepsilon = -\langle u_j \frac{\partial \sigma}{\partial x_j} \rangle \quad \text{- mean dissipation rate of turbulent energy into heat} \]

\[ 1/(\langle \rho u^2 \rangle) = 1/(3 \langle \rho \rangle u_{rms}^2) \]
Linear forcing

The equation for the fluctuating part of the magnetic field in a compressible MHD turbulence is written as:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\langle \mathbf{u} \times \mathbf{B} \rangle) + \nabla \times (\mathbf{U} \times \dot{\mathbf{B}}) + \eta \nabla^2 \mathbf{B}$$

where

$$\nabla \times (\mathbf{U} \times \dot{\mathbf{B}}) = \left(-\mathbf{B}_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial \mathbf{B}_i}{\partial x_j} - \dot{\mathbf{B}}_i \frac{\partial U_j}{\partial x_j}\right)$$

The equation for the turbulent magnetic energy follows from averaging and the results is:

$$\frac{\partial}{\partial t} \langle \frac{\dot{B}^2}{8\pi} \rangle + \frac{\partial}{\partial x_j} \left[ \left( \frac{\dot{B}^2}{8\pi} \right) U_j + \left( \frac{\dot{B}^2}{8\pi} \right) \dot{u}_j \right] = \left( \frac{\dot{B}_i \dot{B}_j}{4\pi} \right) \frac{\partial U_i}{\partial x_j} + \left( \frac{\dot{B}_i \dot{B}_j}{4\pi} \right) \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\eta}{4\pi} \langle \dot{B}_i \frac{\partial^2 \dot{B}_i}{\partial x_j^2} \rangle$$

source term for the production of the magnetic turbulent energy as a consequence of the interaction between the magnetic field and the mean fluid shear.
Linear forcing

\[ F_i^b = \Psi B_i \] - the driving force

\[ \downarrow \]

coefficient which is determined from a balance of magnetic energy for a statistically stationary state:

\[ \Psi = \frac{\chi}{3B_{\text{rms}}^2} \]

\[ \chi = \langle \eta B_i (\partial^2 B_i / \partial x_j^2) \rangle \] - resistive dissipation of turbulent magnetic energy

\[ B_{\text{rms}}^2 = \langle B^2 \rangle / 3 \] - root-mean-square magnetic field
Filtered MHD equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho \tilde{u}_j}{\partial x_j} &= 0 \\
\frac{\partial \rho \tilde{u}_i}{\partial t} + \frac{\partial \rho \tilde{u}_i}{\partial x_j} \left( \rho \tilde{u}_i \tilde{u}_j + \tilde{p} \delta_{ij} - \frac{1}{\text{Re}} \tilde{\sigma}_{ij} + \frac{\tilde{B}^2}{2M_A^2} - \frac{1}{M_A^2} \tilde{B}_i \tilde{B}_j \right) &= - \frac{\partial \tau_{ji}^u}{\partial x_j} + F_i^u \\
\frac{\partial \tilde{B}_i}{\partial t} + \frac{\partial}{\partial x_j} (\tilde{u}_j \tilde{B}_i - \tilde{B}_j \tilde{u}_i) - \frac{1}{\text{Re}_m} \frac{\partial^2 \tilde{B}_i}{\partial x_j^2} &= - \frac{\partial \tau_{ji}^b}{\partial x_j} + F_i^b
\end{align*}
\]

On the right-hand sides of equations the terms designate influence of subgrid terms on the filtered part. To determine these terms, special turbulent parametrizations based on large-scale values describing turbulent MHD flow must be used.

\[
\tilde{r}_{ij} = \tilde{u}_j \tilde{B}_i - \tilde{B}_j \tilde{u}_i
\]

\[
\begin{align*}
\tilde{F}_i^u &= \frac{1}{3(\tilde{\rho}) \tilde{u}_{rms}^2} \left[ \tilde{\varepsilon} + \frac{\langle \tilde{u}_j \frac{\partial}{\partial x_j} \rho' \delta_{ij} \rangle}{\gamma M_s^2} + \frac{\langle \tilde{u}_j \frac{\partial}{\partial x_j} \tilde{B}^2 \delta_{ij} \rangle}{2M_a^2} \right] \rho \tilde{u}_i \\
F_i^b &= \frac{1}{3B_{rms}^2} \left[ \frac{1}{\text{Re}_m} \tilde{B}_i \frac{\partial^2 \tilde{B}_i}{\partial x_j^2} \right] \tilde{B}_i
\end{align*}
\]

Subgrid-scale (SGS) terms
Filtered MHD equations for heat-conducting plasma

\[ \begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho \tilde{u}_j}{\partial x_j} &= 0 \\
\frac{\partial \rho \tilde{u}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho \tilde{u}_i \tilde{u}_j + p \delta_{ij} - \frac{1}{\text{Re}} \tilde{\sigma}_{ij} + \frac{B^2}{2M_A^2} - \frac{1}{M_A^2} \bar{B}_i \bar{B}_j \right) &= -\frac{\partial \tau^u_{ji}}{\partial x_j} + F_i^u \\
\frac{\partial \bar{B}_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \tilde{u}_j \bar{B}_i - \bar{B}_j \tilde{u}_i \right) - \frac{1}{\text{Re}_m} \frac{\partial^2 \bar{B}_i}{\partial x_j^2} &= -\frac{\partial \tau^b_{ji}}{\partial x_j} + F_i^b \\
\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x_j} \left[ (\tilde{E} + \bar{P}) \tilde{u}_j - \frac{1}{M_A^2} \bar{B}_i \bar{B}_j \tilde{u}_i \right] + \frac{\partial}{\partial x_j} \left[ \frac{\bar{q}_j}{\text{Pr} \text{Re} M_s^2 (\gamma - 1)} - \frac{1}{\text{Re}} \tilde{\sigma}_{ij} \tilde{u}_i \right] &= 0 \\
\frac{\partial}{\partial x_j} \left[ \frac{\eta}{\text{Re}_m M_a^2} \bar{B}_i \left( \frac{\partial \bar{B}_i}{\partial x_j} - \frac{\partial \bar{B}_j}{\partial x_i} \right) \right] &= -\frac{\partial}{\partial x_j} \left( \frac{1}{\gamma M_s^2} Q_j + \frac{1}{2} J_j + \frac{1}{2M_a^2} V_j - \frac{1}{M_s^2} G_j \right)
\end{align*} \]

Equation of state: \[ \bar{p} = \frac{\tilde{T} \rho}{\gamma M_s^2} \]

\[ E = \rho e + \frac{1}{2} \rho u_i u_i + \frac{1}{2M_a^2} B_i B_i \]

\[ e = \frac{T \rho}{\gamma (\gamma - 1) M_s^2} \]

total energy internal energy
Heat-conducting plasma

\[
\tau_{ij}^u = -2C \bar{\rho} \bar{\Delta}^2 \left| \mathbf{\tilde{S}}^u \right| (\mathbf{\tilde{S}}_{ij} - \frac{1}{3} \mathbf{\tilde{S}}_{kk} \delta_{ij}) + \frac{2}{3} Y \bar{\rho} \bar{\Delta}^2 \left| \mathbf{\tilde{S}}^u \right|^2 \delta_{ij}
\]

\[
\tau_{ij}^b = -2D \bar{\Delta}^2 \left| \mathbf{\bar{J}} \right| \mathbf{\bar{J}}_{ij}
\]

\[
Q_j = -C_s \frac{\bar{\Delta}^2 \bar{\rho} \left| \mathbf{\tilde{S}}^u \right| \partial \tilde{T}}{Pr_T \partial x_j}
\]

\[
J_j = \bar{u}_k \tau_{jk}^u
\]

\[
\frac{1}{2} \mathbf{V}_j - \mathbf{G}_j \approx \bar{B}_k \tau_{jk}^b.
\]

\[
\mathbf{\tilde{F}}_{ij}^u = \frac{1}{3 \langle \bar{\rho} \rangle \bar{u}^2_{rms}} \left[ -\overline{\mathbf{u}_j \partial \bar{T}_i} \right] + \frac{1}{\gamma M_s^2} \overline{\mathbf{u}_j \partial \bar{\rho} \bar{T} \delta_{ij}} + \frac{1}{2 M_a^2} \overline{\mathbf{u}_j \partial \bar{B}^2 \delta_{ij}} \right] \bar{\rho} \bar{u}_i
\]

\[
\mathbf{\tilde{F}}_{ij}^b = \frac{1}{3 \bar{B}^2_{rms}} \left[ \frac{1}{Re} \overline{\bar{B}_i \partial^2 \bar{B}_i} \right] \bar{B}_i
\]
Polytropic plasma - 1

Spectra of MHD turbulence
Polytropic plasma - 2

Time dynamics of rms velocity, rms magnetic field and mean density.

Spectrum of total energy
Heat-conducting plasma -1
Heat-conducting plasma - 2
There is growing interest in observations and explanation of the spectrum of the density fluctuations in the interstellar medium. These fluctuations are responsible for radio wave scattering in the interstellar medium and cause interstellar scintillation fluctuations in the amplitude and phase of radio waves. Kolmogorov-like $k^{-5/3}$ spectrum of density fluctuations have been observed in wide range of scales in the local interstellar medium (from an outer scale of a few parsecs to scales of about 200 km).

Parameters of numerical study of local interstellar medium

For study of compressible MHD turbulence in interstellar, medium we use large eddy simulation (LES) method. Smagorinsky model for compressible MHD case for subgrid-scale parameterization is applied. The Smagorinsky model for compressible MHD turbulence showed accurate results under various range of similarity numbers.

Initial parameters: \( \text{Re} \approx 2000 \quad M_s \approx M_A \approx 2.2 \)

\( \text{Re}_m \approx 200 \) (ambipolar diffusion)

The initial isotropic turbulent spectrum was chosen for kinetic and magnetic energies in Fourier space to be close to \( \kappa^{-2} \) with random amplitudes and phases in all three directions. The choice of such spectrum as initial conditions is due to velocity perturbations with an initial power spectrum in Fourier space similar to that of developed turbulence.

The simulation domain is a cube with dimensions of \( \pi^3 \)
Decay of turbulent small-scale Mach number with time. A transition from a supersonic to a subsonic regime can be observed.

Time dynamics of the velocity divergence. The velocity divergence describing medium compressibility attenuates and the flow becomes weakly compressible with time.
Turbulent spectra in the local interstellar medium -1

The kinetic energy spectrum (left). Normalized and smoothed spectrum of kinetic energy, multiplied by \( k^{5/3} \) (right). Notice that the spectrum is close to \( k^{-3} \) in a forward cascade regime of decaying turbulence. However, there is well-defined inertial Kolmogorov-like range of \( k^{-5/3} \).
The density spectrum is the solid line and the density fluctuations spectrum is the dot line (left). Normalized and smoothed spectrum of density fluctuations, multiplied by $k^{5/3}$ (right). Both graphs (in the left figure) have spectral index close to $k^{-3}$. Moreover, there is well-defined inertial Kolmogorov-like range of $k^{-5/3}$ that confirms observation data.
Turbulent spectra in the local interstellar medium - 3

The variation of the kinetic energy spectrum with time. Dotted line displays time direction.

The variation of the kinetic energy spectrum with time. Dotted line displays time direction.
Anisotropic turbulence

Anisotropy and symmetry breakdown are caused first of all by the magnetic field at low value of the plasma beta when the role of the magnetic field is substantial. Anisotropic cascades are observed to be due to propagating compressible acoustic modes that hinder spectral transfer in the local Fourier space at high value of plasma beta when the role of the magnetic field is little. These modes in compressible MHD turbulence could be excited either by a large-scale or ambient velocity component of the background hydrodynamic turbulence.

- the Shebalin angles (or anisotropy angles)
Conclusions

- It is shown that density fluctuations are a passive scalar in a velocity field in weakly compressible magnetohydrodynamic turbulence and demonstrate Kolmogorov-like spectrum.

- The decrease of energy-containing large eddies and inertial range with time, and the increase of dissipative scale are also represented.

- It is shown, that the turbulent sonic Mach number decreases significantly from a supersonic turbulent regime, where the medium is strongly compressible, to a subsonic value of Mach number describing weakly compressible flow.

- In local interstellar medium, the transition of MHD turbulent flow from a strongly compressible to a weakly compressible state not only transforms the characteristic supersonic motion into subsonic motion, but also attenuates plasma magnetization, which is shown in this work because plasma beta increases with time, thus, role of magnetic energy decreases in comparison with plasma pressure.

- The anisotropy of turbulent flow is considered and it is demonstrated that large-scale flow shows anisotropic properties while small-scale structures are isotropic.