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Bayesian model comparison techniques and prior-independent results

Focusing on neutrino physics

Torino, Informal Seminar, 04/11/2019

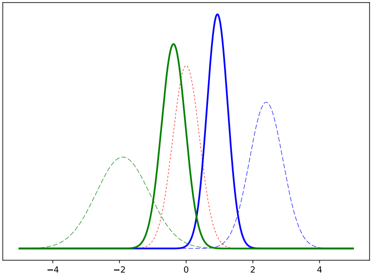
1 ***Basics of Bayesian probability***
- Probability and Bayes
- Parameter inference
- Bayesian model comparison

2 ***Neutrino mass ordering***
- How to constrain the mass ordering
- Subtleties in the Bayesian analysis
- Constraints on the mass ordering

3 ***Neutrino masses from cosmology***
- The current status
- Non-probabilistic limits

4 ***What about model extensions?***
- Model marginalization
- Non-probabilistic limits

5 ***Conclusions***



Probability

What is probability?

a frequency

"the number of times the event occurs over the total number of trials, in the limit of an infinite series of equiprobable repetitions"

another subtle point:
"randomness" of the trial series

what is really "random"?

do we properly know the initial state (and do not cheat)?

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Advantages:

- recovers frequentist on the long run;
- can be applied when frequentist cannot;
- no need to assume a distribution of possible data;
- deals effortlessly with nuisance parameters (*marginalization*);
- relies on *prior information*.

Bayes' theorem

how to deal with **Bayesian probability**?

given hypothesis H, data d, some information I (true):

$p(\theta)$

Posterior probability:

what we know after

Bayes theorem:

$$p(H|d,I) = \frac{p(d|H,I)\,p(H|I)}{p(d|I)}$$

$\pi(\theta)$

Prior probability:

what we knew before

Likelihood: $\mathcal{L}(\theta)$

sampling distribution of data, given that H is true

Marginal likelihood:

or "Bayesian evidence",

$$p(d|I) \equiv \sum_H p(d|H,I)\,p(H|I)$$

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$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

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model comparison

(Bayesian) Parameter inference

Parameter inference = what we learn on the parameters, given:

data d → build $\mathcal{L}(\theta)$ from d & $\mathcal{M}_0$

model $\mathcal{M}_0$ → build $\mathcal{L}(\theta)$ from d & $\mathcal{M}_0$

model $\mathcal{M}_0$ → parameters θ (& priors)

(physical ϕ + nuisance ψ)

Full posterior:

$$p(\theta|d,\mathcal{M}_0) \propto \mathcal{L}(\theta) \times \pi(\theta|\mathcal{M}_0)$$

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Marginalize over nuisance to obtain posterior for physical:

$$p(\phi|d,\mathcal{M}_0) \propto \int_{\Omega_\psi} \mathcal{L}(\phi,\psi)\pi(\phi,\psi|\mathcal{M}_0) d\psi$$

marginalize over all the parameters except one (two)

→ 1D (2D) posterior

Credible intervals from the posterior

Credible interval α?

range of values within which an unobserved parameter value falls
with a particular subjective probability α

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Analogous to frequentist confidence intervals α

Bayesian credible interval:

- bounds as fixed;
- estimated parameter as a random variable.

Frequentist confidence interval:

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Credible intervals are not uniquely defined!

highest posterior density interval: narrowest interval, includes values of highest probability density

equal-tailed interval: same probability of being below or above the interval

interval for which the mean is the central point

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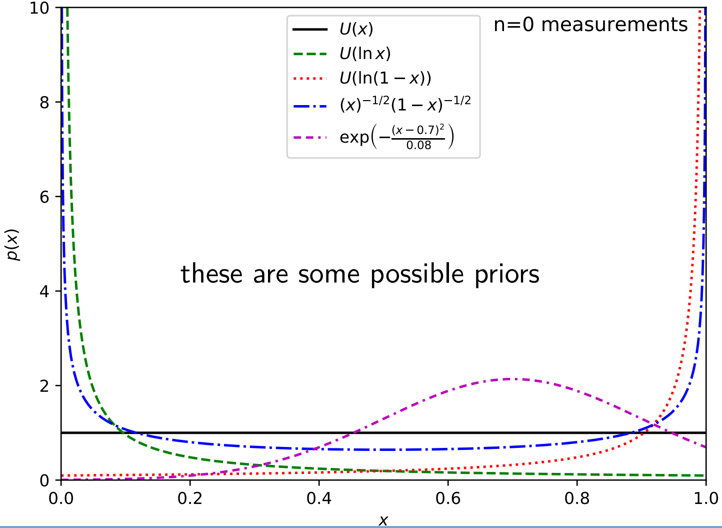
likelihood $\mathcal{L}(x) \propto \exp[-(x-0.2)^2/(2\cdot 0.3^2)]$



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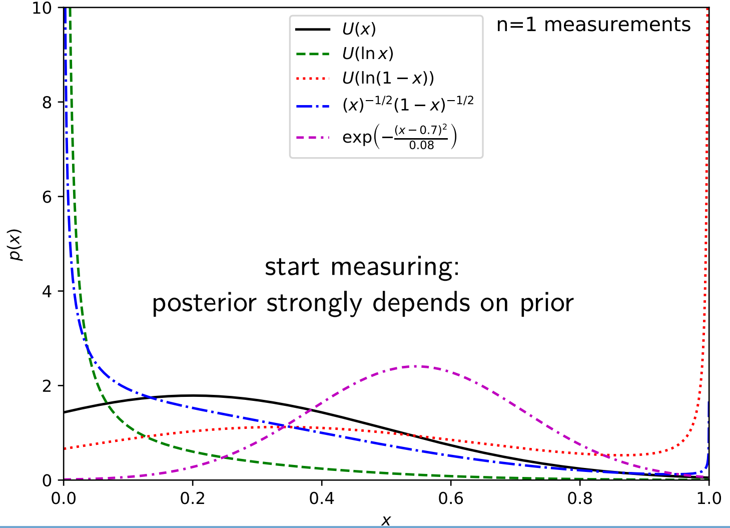

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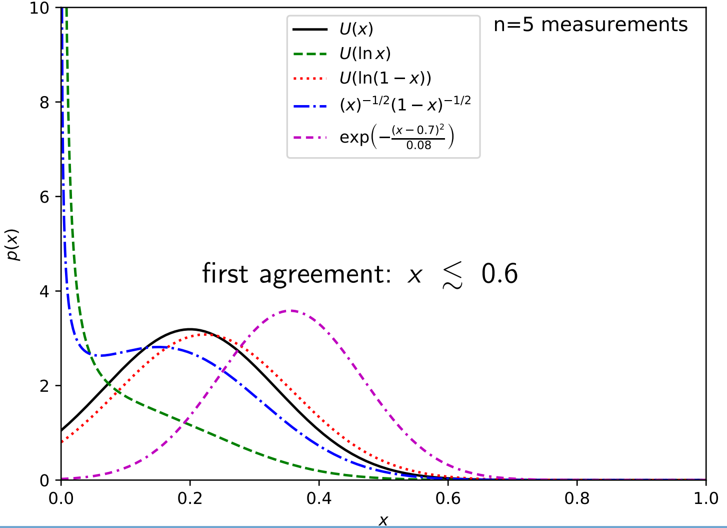

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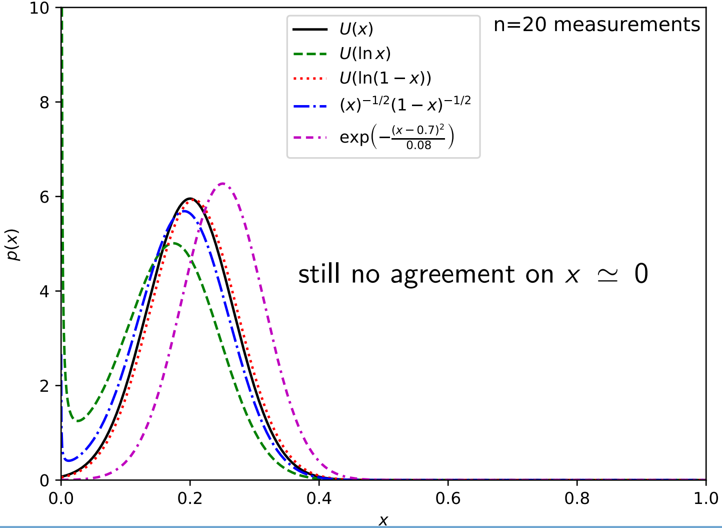

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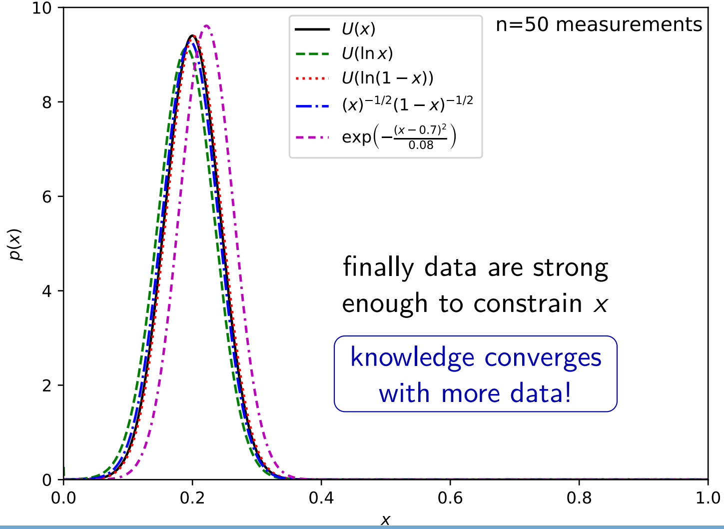
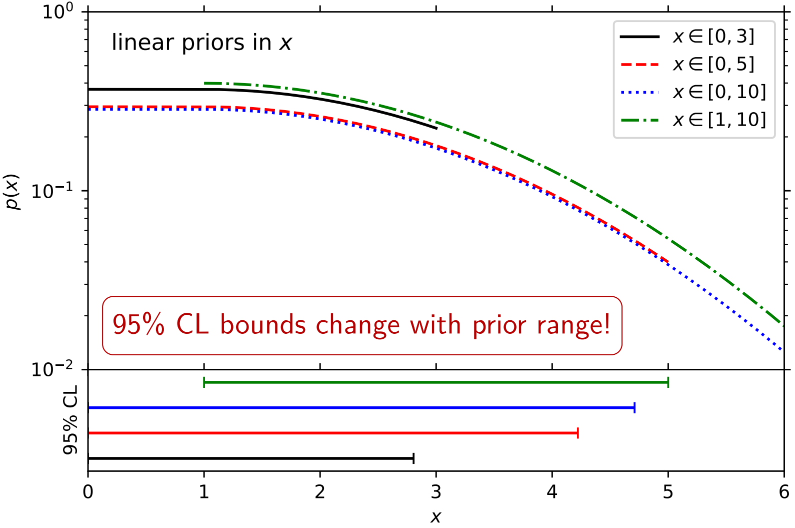


Prior dependence in parameter estimation - II

other example: need to measure $x > 0$ (Σm_ν?)

likelihood $\mathcal{L}(x) \propto \exp[-(x-1)^2/(2\cdot 1^2)]$ for $x > 1$, constant otherwise



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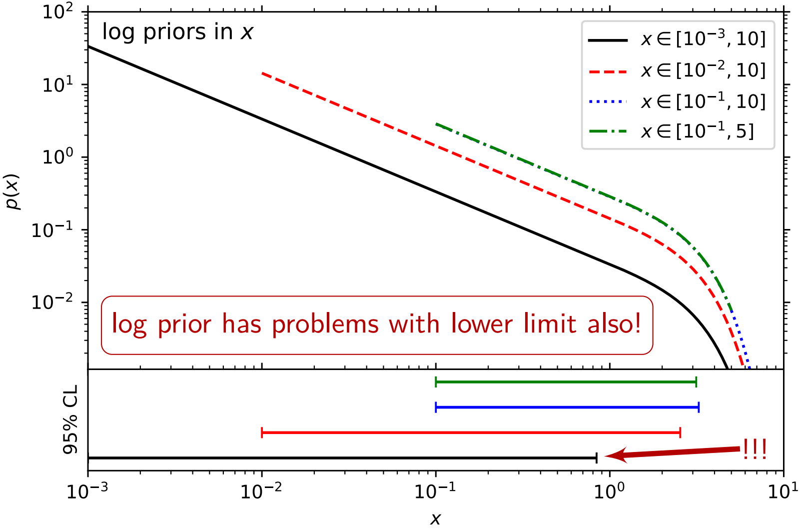

Bayesian evidence

"Bayesian evidence" or "Marginal likelihood"

$$p(d|\mathcal{M}) = Z = \int_{\Omega_{\mathcal{M}}} \mathcal{L}(d|\theta,\mathcal{M})\,\pi(\theta|\mathcal{M})\,d\theta$$

integrate over all possible (continuous) parameters of model $\mathcal{M}$
(given that $\mathcal{M}$ is true)

What if there are several possible models $\mathcal{M}_i$?

use Z_i to perform bayesian model comparison

Warning: compare models given the same data!

Model posterior:

$$p(\mathcal{M}_i|d) \propto \pi(\mathcal{M}_i)\,Z_i$$

← proportional to constant that depends only on data

given model prior $\pi(\mathcal{M}_i)$

Bayes factor

Posterior odds of $\mathcal{M}_1$ versus $\mathcal{M}_2$:

$$\frac{p(\mathcal{M}_1|d)}{p(\mathcal{M}_2|d)} = B_{1,2}\,\frac{\pi(\mathcal{M}_1)}{\pi(\mathcal{M}_2)}$$

Bayes factor:

$$B_{1,2} = \frac{Z_1}{Z_2} \quad \Rightarrow \quad \ln B_{1,2} = \ln Z_1 - \ln Z_2$$

if priors are the same $[\pi(\mathcal{M}_1) = \pi(\mathcal{M}_2)]$,
$B_{1,2}$ tells which model is preferred:

$B_{1,2} > 1$ ($\ln B_{1,2} > 0$)

$\mathcal{M}_1$ preferred

$B_{1,2} < 1$ ($\ln B_{1,2} < 0$)

$\mathcal{M}_2$ preferred

$\exp(|\ln B_{1,2}|)$ tells the odds in favor of preferred model

Occam's razor

what the Bayesian model comparison tells us?

Best model strikes optimum balance between

Quality of fit → Occam's razor ← Predictivity

the simplest theory that fits data is preferred

model with more parameters → better fit (usually)

→ are all the parameters needed?

Bayes factor penalizes unnecessarily complex models!

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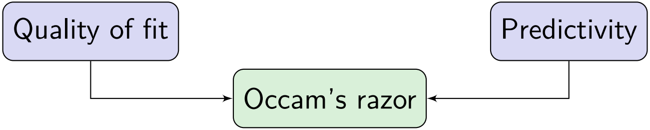

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what if we compare same model and different priors?

Bayesian evidence depends on priors!

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Bayes factor DOES NOT penalize models with parameters that are unconstrained by the data

Prior dependence in the Bayesian evidence

Bayesian evidences depend on priors!

$$\text{likelihood: } \mathcal{L}(x) \propto \begin{cases} 1 & \text{for } x \leq 1 \\ \exp[-(x-1)^2/(2\cdot 1^2)] & \text{for } x > 1 \end{cases}$$

linear prior		log prior	
range	Z	range	Z
$0 \leq x \leq 3$	0.180	$10^{-3} \leq x \leq 10$	0.192
$0 \leq x \leq 5$	0.135	$10^{-2} \leq x \leq 10$	0.172
$0 \leq x \leq 10$	0.070	$10^{-1} \leq x \leq 10$	0.151
$1 \leq x \leq 10$	0.056	$10^{-1} \leq x \leq 5$	0.177

linear prior $x \in [a, b]$ is $\propto 1/(b-a)$

irrelevant for Bayes factor
if the compared models
have the parameter x in common

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towards Lindley's paradox:

use $\mathcal{L}(x) \propto \exp[-x^2/(2\Sigma^2)]$,
$\pi(x) \propto \exp[-(x - N\sigma_t)^2/(2\sigma^2)]$,
with $\sigma_t = \sqrt{\sigma^2 + \Sigma^2}$

$$Z = \exp(-N^2/2) \Big/ \left(\sqrt{2\pi}\,\sigma_t\right)$$

Prior dependence in the Bayesian evidence

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max evidence for a given likelihood $\mathcal{L}(x)$?

Select a Dirac delta centered on the $\hat{x}$
that gives the maximum of the likelihood

useful estimate of the **max Bayes factor**, in particular for nested models

$\mathcal{M}_1$: free x
$\mathcal{M}_0$: $\mathcal{M}_1|\ x = x_0$

$$B_{01} = \frac{\mathcal{L}(x_0)}{\int \mathrm{d}x\, \mathcal{L}(x)\, \pi(x)} \geq \frac{\mathcal{L}(x_0)}{\mathcal{L}(\hat{x})} = \frac{\mathcal{L}(x_0)}{\int \mathrm{d}x\, \mathcal{L}(x)\, \delta(x-\hat{x})}$$

maximum likelihood ratio

you will never find a prior that gives a better B_{01} than this!

useful for prior-independent estimates of B_{01}

Jeffreys' scale

[Trotta, arxiv:0803.4089]

odds in favor of the preferred model:

$$\exp(|\ln B_{1,2}|) : 1$$

strength of preference according to Jeffreys' scale:

$\|\ln B_{1,2}\|$	Odds	$N\sigma$	strength of evidence
< 1.0	$\lesssim 3:1$	< 1.1	inconclusive
$\in [1.0, 2.5]$	$(3-12):1$	$1.1-1.7$	weak
$\in [2.5, 5.0]$	$(12-150):1$	$1.7-2.7$	moderate
$\in [5.0, 10]$	$(150-2.2\times10^4):1$	$2.7-4.1$	strong
$\in [10, 15]$	$(2.2\times10^4-3.3\times10^6):1$	$4.1-5.1$	very strong
> 15	$> 3.3\times10^6:1$	> 5.1	decisive

odds & strength always valid

$N\sigma$ correspondence is valid only given equal model priors
and that only two models are possible
(see e.g. neutrino mass ordering: normal OR inverted)

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$\in [3,6]$	$(20-400):1$	$2.0-3.0$	moderate
$\in [6,10]$	$(400-2.2\times10^4):1$	$3.0-4.1$	strong
$\in [10,15]$	$(2.2\times10^4-3.3\times10^6):1$	$4.1-5.1$	very strong
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Can we extend to more than two (mutually exclusive) models?

How to compute the model posterior

[SG+, PRD 99 (2019) 021301]

Assume N models, equal model prior probabilities:

$$\pi_i \equiv \pi(\mathcal{M}_i) \qquad \pi_i = \pi_j \quad \forall i,j \qquad \sum_i \pi_i = 1 \to \pi_i = 1/N$$

Compute model posterior probabilities:

$$p_i \equiv p(\mathcal{M}_i|d) \qquad p_i = A\pi_i Z_i \quad \text{with } A \text{ constant} \qquad \sum_i p_i = 1$$

$$\sum_i^N p_i = A\sum_i^N \pi_i Z_i = 1 \quad \Longrightarrow \quad p_i = \pi_i Z_i \Big/ \sum_j^N \pi_j Z_j = \pi_i \Big/ \sum_j^N \pi_j B_{ji}$$

Selecting a generic $\mathcal{M}_0$ as a reference, we have:

$$\boxed{p_0 = \left(\sum_i^N B_{i0}\right)^{-1}}$$

the sum includes $B_{00} = 1$

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example 1: $N = 2$

$$p_0 = 1/(1 + B_{10})$$
$$p_1 = B_{10}/(1 + B_{10})$$

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example 1: $N = 2$

$$p_0 = 1/(1 + B_{10})$$
$$p_1 = B_{10}/(1 + B_{10})$$

example 2: $N = 8$

assume $B_{i0} \simeq e^{-5}$ ($i \neq 0$) to get

$$p_0 = 1/(1 + \sum_{i\neq 0} B_{i0}) \simeq 0.955$$

strong? no, only 2σ!

Model posterior with many models

$$p_i = Z_i \bigg/ \sum_j^N Z_j = B_{i0} \bigg/ \sum_j^N B_{j0}$$

Do the result depend on N?

Does $p_0 \to 0$ when $N \to \infty$?

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ΛCDM ⟵ this will probably be the favorite one

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ΛCDM

+1 parameter

$+r$ $+\Sigma m_\nu$ $+N_{\rm eff}$ $+w$ $+\Omega_k$ $+Y_p$ $+A_{\rm lens}$ $+\ldots$

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+2 parameters

$+\Sigma m_\nu + N_{\rm eff}$ $+N_{\rm eff} + m_s^{\rm eff}$ $+w_0 + w_a$ $+\alpha_s + \beta_s$ $+Y_p + N_{\rm eff}$

$+r + \alpha_s$ $+A_{\rm lens} + \Sigma m_\nu$ $+\alpha_s + N_{\rm eff}$ $+\ldots$

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ΛCDM

+1 parameter

$+r \quad +\Sigma m_\nu \quad +N_{\rm eff} \quad +w \quad +\Omega_k \quad +Y_p \quad +A_{\rm lens} \quad +\ldots$

+2 parameters

$+\Sigma m_\nu + N_{\rm eff} \quad +N_{\rm eff} + m_s^{\rm eff} \quad +w_0 + w_a \quad +\alpha_s + \beta_s \quad +Y_p + N_{\rm eff}$

$+r + \alpha_s \quad +A_{\rm lens} + \Sigma m_\nu \quad +\alpha_s + N_{\rm eff} \quad +\ldots$

+3 parameters (and so on...)

Model posterior with many models

$$p_i = Z_i \Big/ \sum_j^N Z_j = B_{i0} \Big/ \sum_j^N B_{j0}$$

Do the result depend on N?

Does $p_0 \to 0$ when $N \to \infty$?

in principle one should consider all the possible models,
starting from the simplest one under consideration
(e.g. ΛCDM in cosmology) and then extending it

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$+1$ parameter

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$+r + \alpha_s$ $\quad +A_{\rm lens} + \Sigma m_\nu$ $\quad +\alpha_s + N_{\rm eff}$ $\quad + \ldots$

Complexity increases: more and more penalized by Occam's razor

$+3$ parameters (and so on...)

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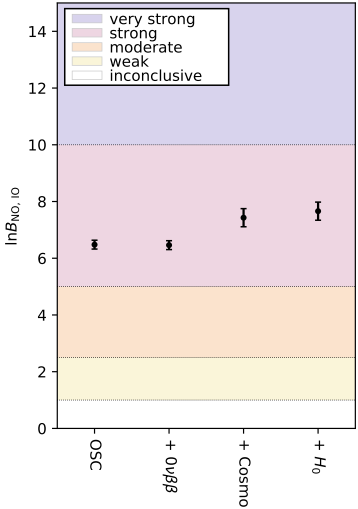
$+\sum m_\nu + N_{\rm eff}$ $+N_{\rm eff} + m_s$ $+w_0 + w_a$ $+\alpha_s + \beta_s$ $+r_p + N_{\rm eff}$

but beware: unconstrained parameters...

$+3$ parameters (and so on...)


very strong
strong
moderate
weak
inconclusive
$\ln B_{\rm NO,\, IO}$
0
2
4
6
8
10
12
14
OSC
+ $0\nu\beta\beta$
+ Cosmo
+ H_0

Neutrinos and their masses

Normal ordering (NO)

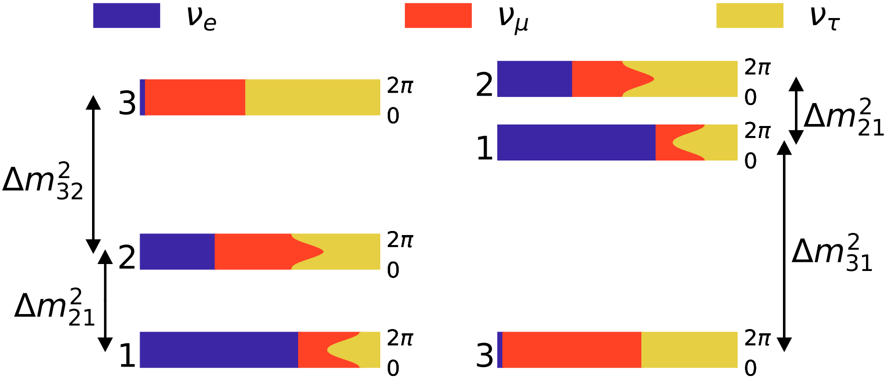
$m_1 < m_2 < m_3$

$\sum m_k \gtrsim 0.06$ eV

Inverted ordering (IO)

$m_3 < m_1 < m_2$

$\sum m_k \gtrsim 0.1$ eV



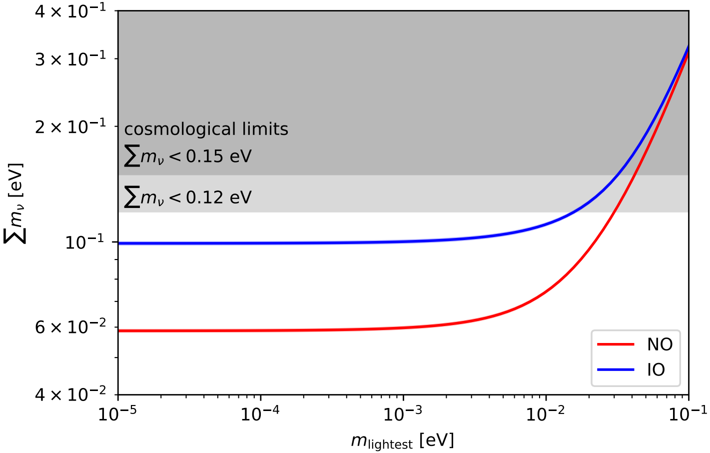
Absolute scale unknown!

Can we constrain the mass ordering using bounds on $\sum m_\nu$?

From cosmology...

Warning: model dependent content!

How the limit change when considering extensions of the ΛCDM model?



Warning: $\sum m_\nu \lesssim 0.1$ eV at 95% CL
does not mean IO disfavored at 95% CL!

Can current data tell us the neutrino mass ordering?

1. [Hannestad, Schwetz, 2016]: extremely weak (2:1, 3:2) preference for NO (cosmology + [Bergstrom et al., 2015] neutrino oscillation fit) Bayesian approach;
2. [Gerbino et al, 2016]: extremely weak (up to 3:2) preference for NO (cosmology only), Bayesian approach;
3. [Simpson et al., 2017]: strong preference for NO (cosmological limits on $\sum m_\nu$ + constraints on Δm^2_{21} and $|\Delta m^2_{31}|$) Bayesian approach;
4. [Schwetz et al., 2017], "Comment on ..."[Simpson et al., 2017]: effect of prior?
5. [Capozzi et al., 2017]: 2σ preference for NO (cosmology + [Capozzi et al., 2016, updated 2017] neutrino oscillation fit) frequentist approach;
6. [Caldwell et al., 2017] very mild indication for NO (cosmology + neutrinoless double-beta decay + [Esteban et al., 2016] readapted oscillation results) Bayesian approach;
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Parameterizing neutrino masses

[SG+, JCAP 03 (2018) 11]

[Simpson et al, 2017]

using m_1, m_2, m_3 (A)

[Caldwell et al, 2017]

using $m_{\rm lightest}, \Delta m^2_{21}, |\Delta m^2_{31}|$ (B)

intuition says: (B) is closer to observable quantities! Better than (A)?

Should we use linear or logarithmic priors on m_k ($m_{\rm lightest}$)?

Can data help to select (A) or (B), linear or log?

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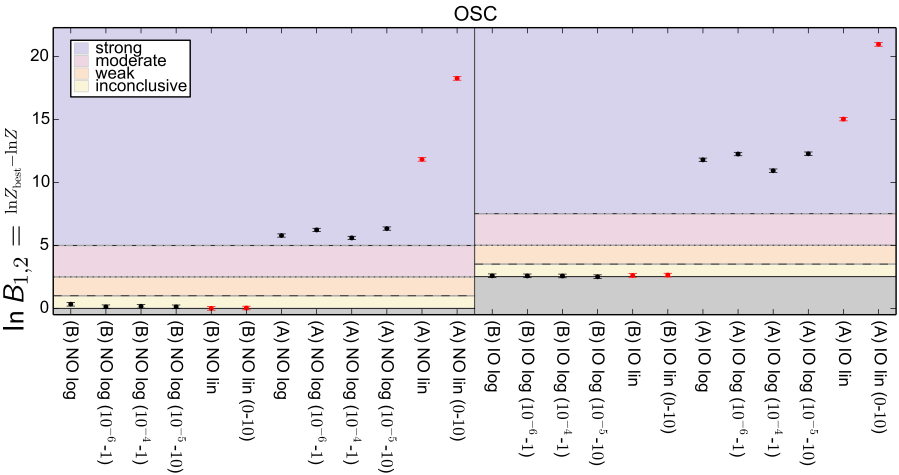
Case A			Case B		
Parameter	Prior	Range	Parameter	Prior	Range
m_1/eV	linear log	$0-1$ $10^{-5}-1$	$m_{\rm lightest}/\rm eV$	linear log	$0-1$ $10^{-5}-1$
m_2/eV	linear log	$0-1$ $10^{-5}-1$	$\Delta m^2_{21}/\rm eV^2$	linear	$5\times10^{-5}-10^{-4}$
m_3/eV	linear log	$0-1$ $10^{-5}-1$	$\lvert\Delta m^2_{31}\rvert/\rm eV^2$	linear	$1.5\times10^{-3}-3.5\times10^{-3}$

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]



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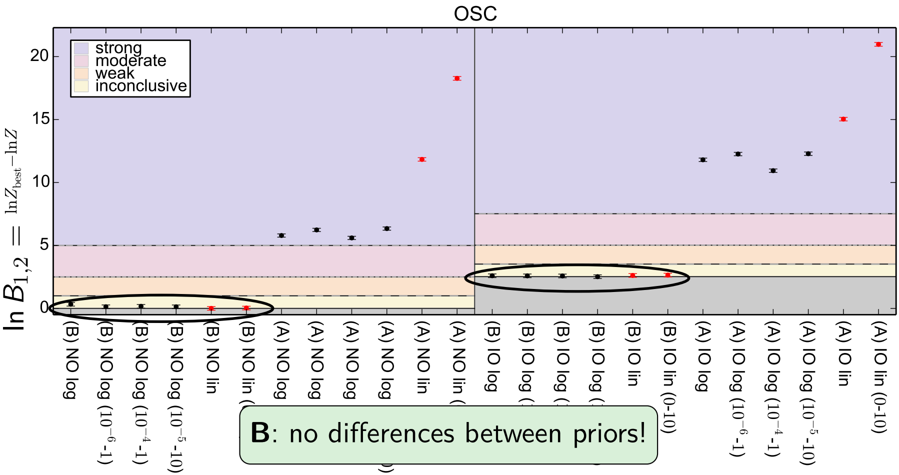
[SG+, JCAP 03 (2018) 11]

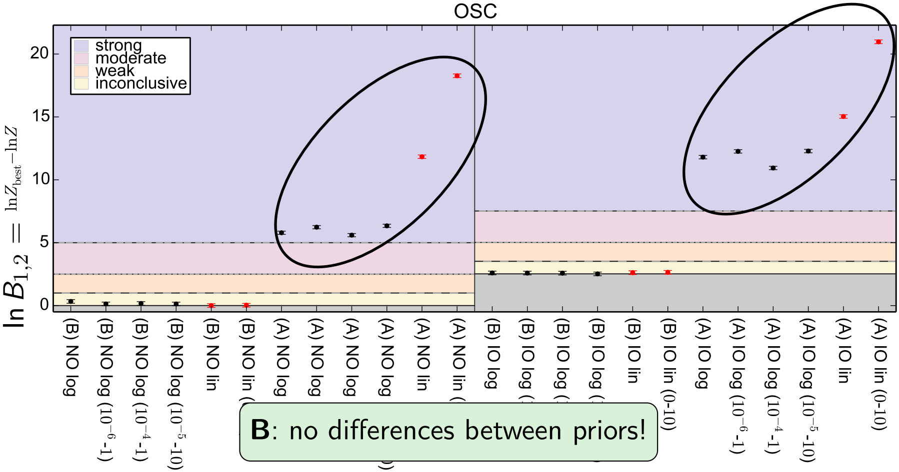


B: no differences between priors!

(no penalty for unconstrained parameter m_{lightest}!)

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]



B: no differences between priors!

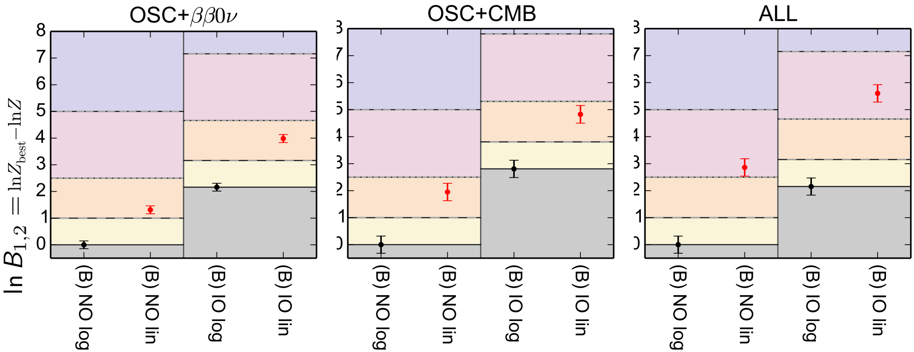
(no penalty for unconstrained parameter $m_{\rm lightest}$!)

A: always strongly disfavored!

(waste of parameter space, no unconstrained parameters due to Δm^2_{i1}!)

Comparing parameterizations/priors

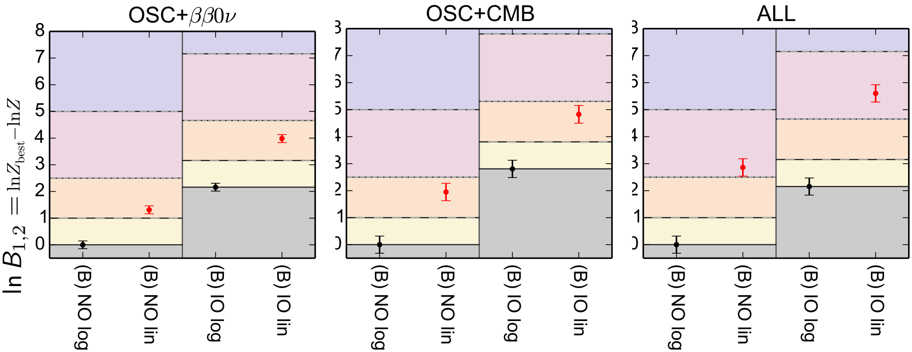
[SG+, JCAP 03 (2018) 11]



compare linear versus **logarithmic**

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]

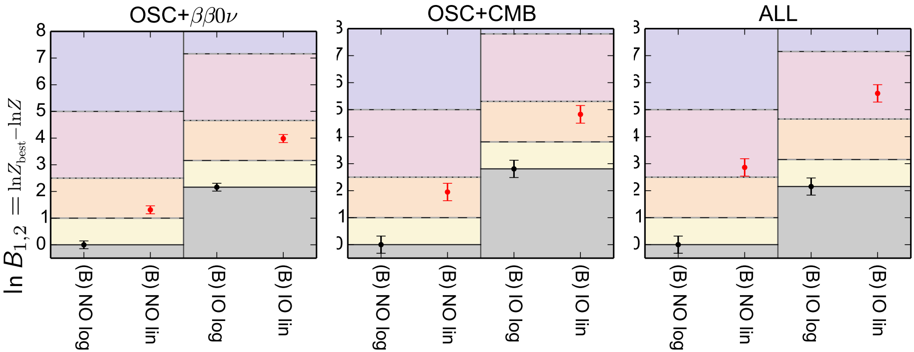


compare linear versus **logarithmic**

log priors are
weakly-to-moderately more efficient

Comparing parameterizations/priors

[SG+, JCAP 03 (2018) 11]



compare linear versus **logarithmic**

log priors are
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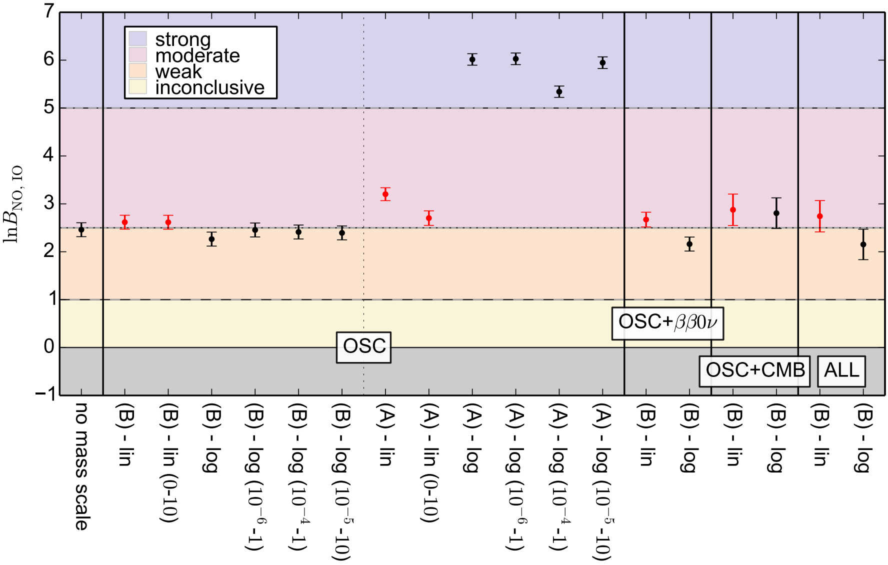
summary: case B, log prior is better!

Comparing the mass orderings

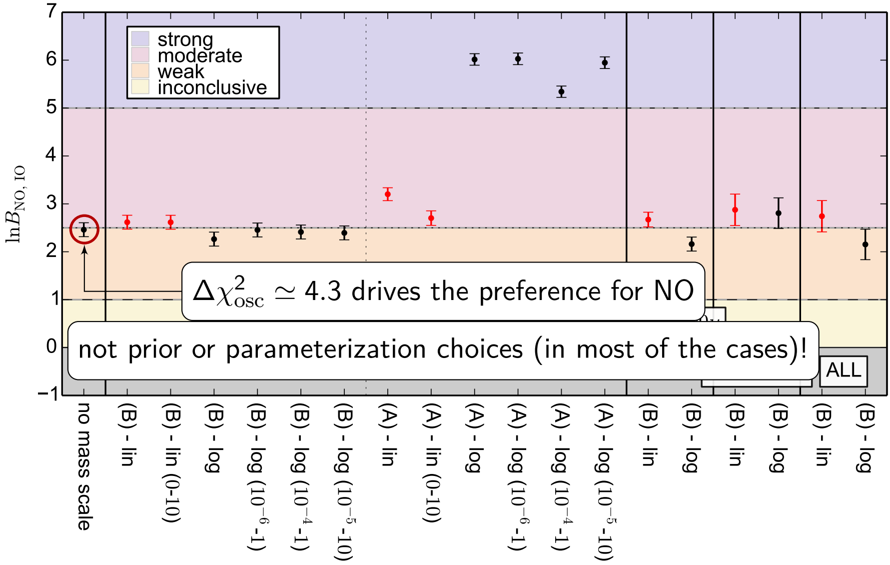
[SG+, JCAP 03 (2018) 11]



Note: only oscillation data until the end of 2017 are included!

Comparing the mass orderings

[SG+, JCAP 03 (2018) 11]



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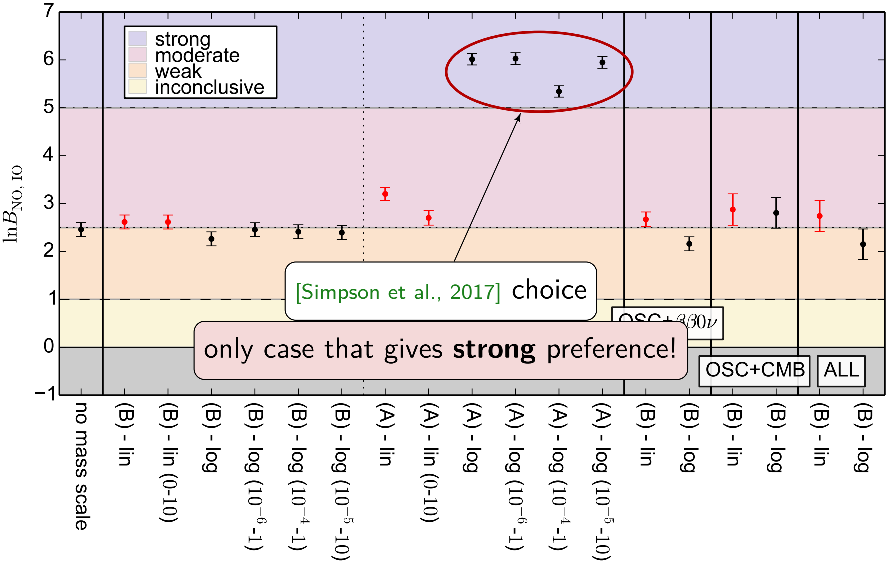

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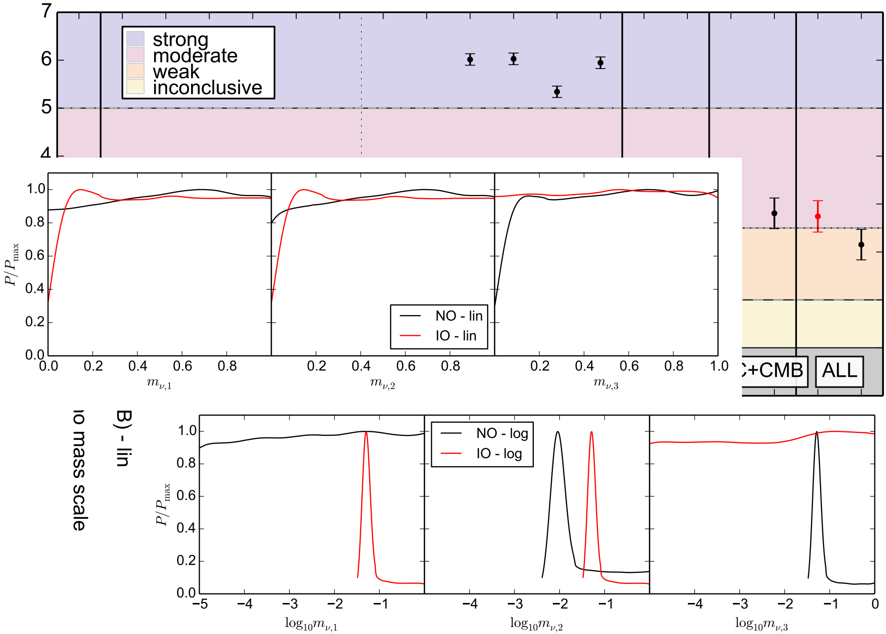

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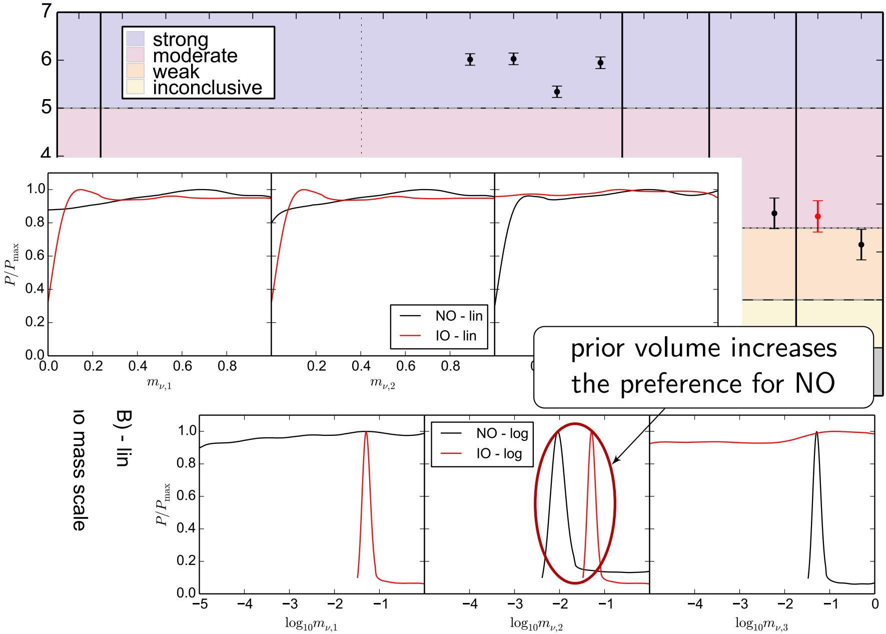


Results in 2018

[de Salas+, Frontiers 5 (2018) 36]
http://globalfit.astroparticles.es/

Bayes theorem for models:

$$p(\mathcal{M}|d) \propto Z_\mathcal{M}\pi(\mathcal{M})$$

Bayesian evidence:

$$Z_\mathcal{M} = \int_{\Omega_\mathcal{M}} \mathcal{L}(\theta)\,\pi(\theta)\,d\theta$$

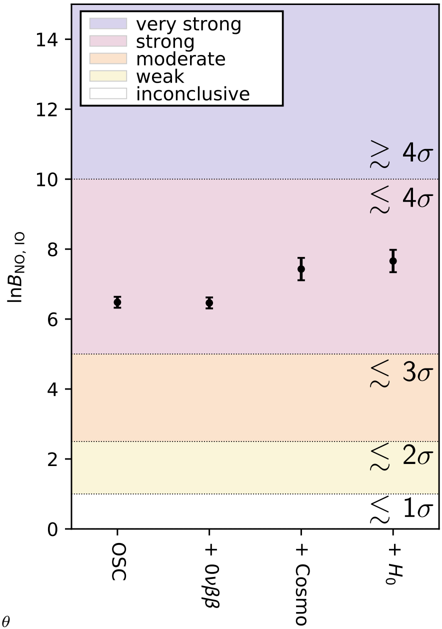
Bayes factor NO vs IO:

$$B_{\mathrm{NO,IO}} = Z_{\mathrm{NO}}/Z_{\mathrm{IO}}$$

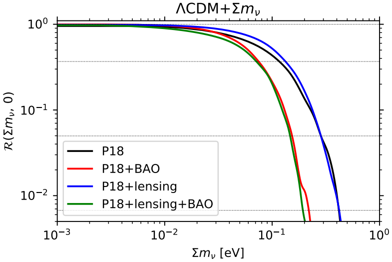
Posterior probability:

$$\begin{aligned} P_{\mathrm{NO}} &= B_{\mathrm{NO,IO}}/(B_{\mathrm{NO,IO}}+1) \\ P_{\mathrm{IO}} &= 1/(B_{\mathrm{NO,IO}}+1) \end{aligned}$$

$N\sigma$ from $P_{\mathrm{NO}} = \mathrm{erf}(N/\sqrt{2})$



$\pi(\mathcal{M})$ model prior
$p(\mathcal{M}|d)$ model posterior
$\mathcal{L}(\theta)$ likelihood
$\Omega_\mathcal{M}$ parameter space, for parameters θ



ΛCDM+Σm_ν
$\mathcal{R}(\Sigma m_\nu, 0)$
Σm_ν [eV]
10^0
10^{-1}
10^{-2}
10^{-3}
P18
P18+BAO
P18+lensing
P18+lensing+BAO

Playing with priors

Bayes theorem:

$$p(\theta|d,\mathcal{M}) = \mathcal{L}(\theta)\frac{\pi(\theta|\mathcal{M})}{Z_{\mathcal{M}}}$$

posterior depends on prior!

parameters θ, model $\mathcal{M}$, data d $\quad$ $\pi(\theta|\mathcal{M})$ prior $\quad$ $p(\theta|d,\mathcal{M})$ posterior $\quad$ $\mathcal{L}(\theta)$ likelihood $\quad$ $Z_{\mathcal{M}}$ Bayesian evidence

Playing with priors

Bayes theorem:

$$p(\theta|d,\mathcal{M}) = \mathcal{L}(\theta)\frac{\pi(\theta|\mathcal{M})}{Z_{\mathcal{M}}}$$

posterior depends on prior!

[Planck 2018]: prior
$0 < \Sigma m_\nu < \mathcal{O}(1)$ eV

strongest upper limit (95%):
$\Sigma m_\nu < 113$ meV
(CMB+lens+BAO+SN)

corresponding to
$\Sigma m_\nu < 53.6$ meV (68%)

below minimum for NO!
does it make sense?

parameters θ, model $\mathcal{M}$, data d $\pi(\theta|\mathcal{M})$ prior $p(\theta|d,\mathcal{M})$ posterior $\mathcal{L}(\theta)$ likelihood $Z_{\mathcal{M}}$ Bayesian evidence

Playing with priors

Bayes theorem:

$$p(\theta|d,\mathcal{M}) = \mathcal{L}(\theta)\frac{\pi(\theta|\mathcal{M})}{Z_{\mathcal{M}}}$$

posterior depends on prior!

Different limits if you consider simply $\Sigma m_\nu > 0$ or you take into account oscillation results. . .

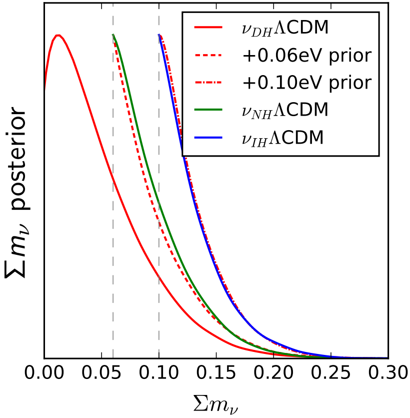
[Wang+, 2017]

degenerate (DH)

vs normal (NH)

vs inverted (IH) hierarchy

(i.e. change the prior lower bound)



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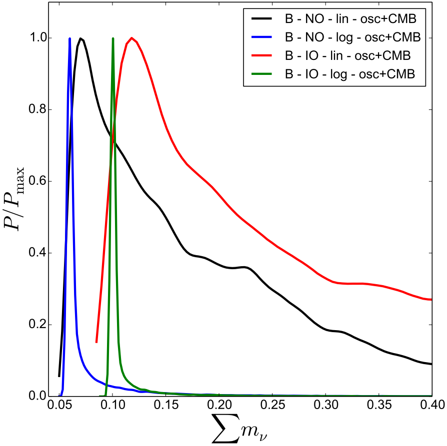
[SG+, 2018]

logarithmic

vs **lin**ear prior

on $m_{\rm lightest}$

You can artificially tighten the bounds on Σm_ν with different priors. . .



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Prior-independent parameter constraints

[SG, arxiv:1910.06646]

Bayes theorem (again!): $p(\theta|d, \mathcal{M}_i) = \pi(\theta|\mathcal{M}_i)\, \mathcal{L}_{\mathcal{M}_i}(\theta)/\, Z_i$

We usually present 1-dim marginalized posterior distributions:

- 1-dim ⟶ function of x
- marginalized ⟶ over params ψ

$$p(x|d) = \int_{\Omega_\psi} d\psi\, p(x, \psi|\mathcal{M}_i, d)$$

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Assume that prior is separable: $\pi(\theta|\mathcal{M}_i) = \pi(x)\cdot\pi(\psi|\mathcal{M}_i)$

$$p(x|d) = \frac{\pi(x)}{Z_i}\int_{\Omega_\psi} d\psi\,\pi(\psi|\mathcal{M}_i)\mathcal{L}_{\mathcal{M}_i}(x,\psi)$$

$\equiv Z_i^x$ Bayesian evidence of model $\mathcal{M}_i|_{\text{fixed}\,x}$
independent of $\pi(x)$ but not of x

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Rewrite a bit:

[Astone, 1999]
[D'Agostini, 2000]
relative belief updating ratio

$$\mathcal{R}(x_1,x_2|d) \equiv \frac{Z_i^{x_1}}{Z_i^{x_2}} = \frac{p(x_1|d)/\pi(x_1)}{p(x_2|d)/\pi(x_2)}$$

→ independent of $\pi(x)$!

Interpreting $\mathcal{R}(x, x_0|d)$

[SG, arxiv:1910.06646]

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Rewrite in a more familiar form:

$$\frac{p(x|d)}{p(x_0|d)} = \mathcal{R}(x, x_0|d)\frac{\pi(x)}{\pi(x_0)}$$

see $\frac{p(\mathcal{M}_i|d)}{p(\mathcal{M}_j|d)} = B_{ij}\frac{\pi(\mathcal{M}_i)}{\pi(\mathcal{M}_j)}$

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not a probability distribution!!

DON'T USE FOR PROBABILISTIC LIMITS

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x_0 is limit to which data are insensitive to x, e.g. $x_0 = 0$ (if x is Σm_ν)

$\mathcal{R}(x, x_0|d)$ describes how **data** update our initial beliefs on x

→ $\mathcal{R} \simeq 1$ $(x \to x_0)$: data are insensitive to x

→ $\mathcal{R} \to 0$ $(x \gg x_0)$: data disfavor x, regardless of prior

Interpreting $\mathcal{R}(x, x_0|d)$

[SG, arxiv:1910.06646]

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$\mathcal{R}(x, x_0|d)$ describes how **data** update our initial beliefs on x

- → $\mathcal{R} \simeq 1$ $(x \to x_0)$: data are insensitive to x
- → $\mathcal{R} \to 0$ $(x \gg x_0)$: data disfavor x, regardless of prior

we can use $\mathcal{R}$ to derive a (non-probabilistic) "sensitivity bound x_s"

$x > x_s$ disfavored because $\ln \mathcal{R}(x, x_0|d) < -s$, with $s = 3$ or 5

levels s as from Jeffreys scale for Bayes factors

Interpreting $\mathcal{R}(x, x_0|d)$

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Rewrite in a more familiar form:

see $\frac{p(\mathcal{M}_i|d)}{p(\mathcal{M}_j|d)} = B_{ij} \frac{\pi(\mathcal{M}_i)}{\pi(\mathcal{M}_j)}$

$$\frac{p(x|d)}{p(x_0|d)} = \mathcal{R}(x, x_0|d) \frac{\pi(x)}{\pi(x_0)}$$

→ it's the same as a Bayes factor!
not a probability distribution!!

DON'T USE FOR PROBABILISTIC LIMITS

x_0 is limit to which data are insensitive to x, e.g. $x_0 = 0$ (if x is Σm_ν)

$\mathcal{R}(x, x_0|d)$ describes how **data** update our initial beliefs on x

→ $\mathcal{R} \simeq 1$ $(x \to x_0)$: data are insensitive to x

→ $\mathcal{R} \to 0$ $(x \gg x_0)$: data disfavor x, regardless of prior

we can use $\mathcal{R}$ to derive a (non-probabilistic) "sensitivity bound x_s"

$x > x_s$ disfavored because $\ln \mathcal{R}(x, x_0|d) < -s$, with $s = 3$ or 5

x_s is a hedge "which separates the region in which we are, and where we see nothing, from the the region we cannot see" [D'Agostini, 2000]

An example with Planck 2018

[SG, arxiv:1910.06646]

relative belief updating ratio

$$\mathcal{R}(x, x_0|d) \equiv \frac{p(x|d)/\pi(x)}{p(x_0|d)/\pi(x_0)}$$

Numerically easy to compute: fix $\pi(x)$, get $p(x|d)$ normally and divide

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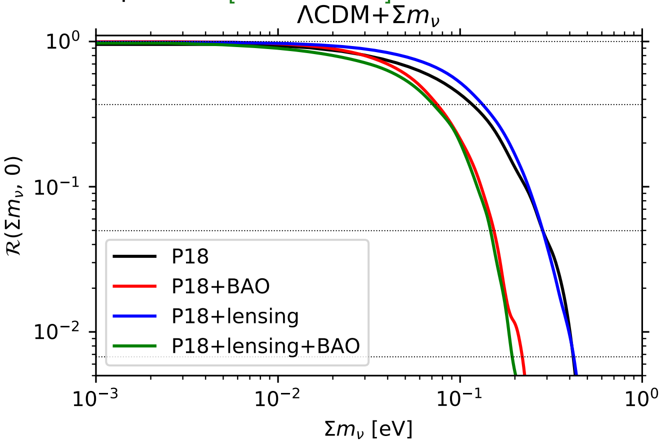

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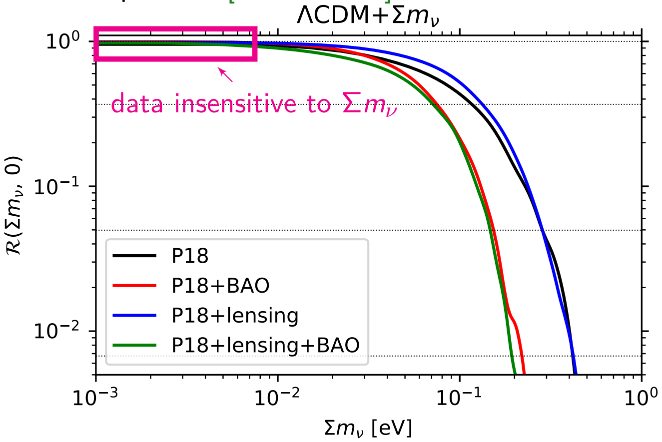

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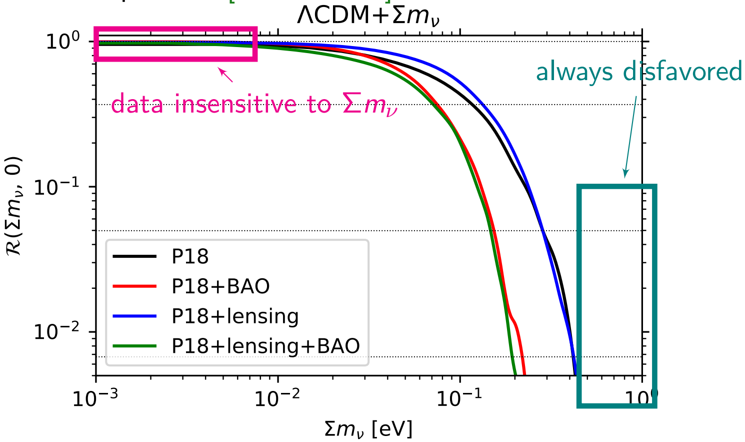

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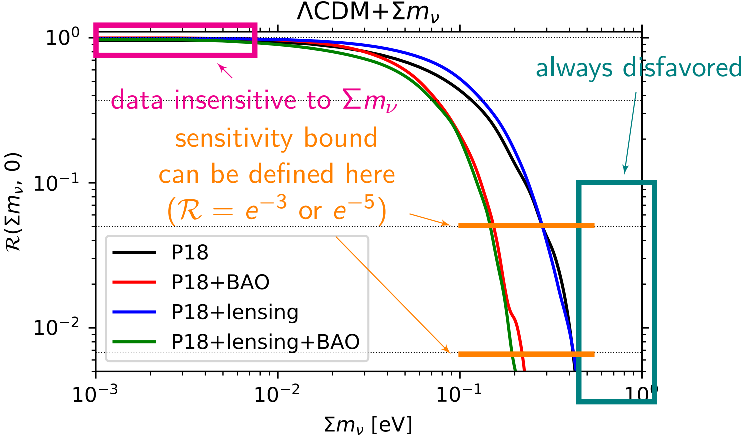

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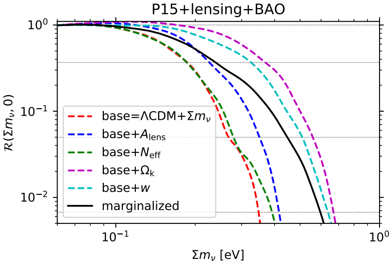

Playing with the baseline model

what if we release the assumption of the ΛCDM model?

CMB TT + lens **CMB TT,TE,EE**		CMB TT + lens + BAO **CMB TT,TE,EE + BAO**
$\Sigma m_\nu < 0.68$ eV $\Sigma m_\nu < 0.49$ eV	[Planck 2015] ΛCDM	$\Sigma m_\nu < 0.25$ eV $\Sigma m_\nu < 0.17$ eV

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free dark energy equation of state $w \neq -1$

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$\Sigma m_\nu < 0.96$ eV	[Di Valentino+, 2015] eCDM 12-parameters cosmological model, ΛCDM based	$\Sigma m_\nu < 0.53$ eV

Marginalize over models?

[SG+, PRD 99 (2019) 021301]

We usually marginalize over parameters:

$$p(\theta|d,\mathcal{M}_0) \propto \int \mathcal{L}(\theta,\psi)\pi(\theta,\psi|\mathcal{M}_0)d\psi$$

Can we marginalize over models?

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$$p(\theta|d,\mathcal{M}_0) \propto \int \mathcal{L}(\theta,\psi)\pi(\theta,\psi|\mathcal{M}_0)d\psi$$

Can we marginalize over models?

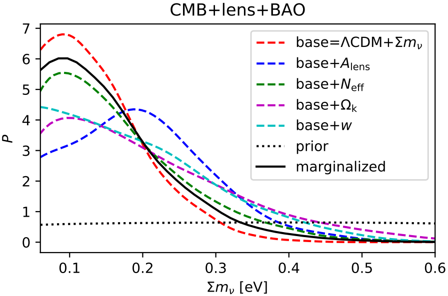
Yes, if we know the model posteriors:

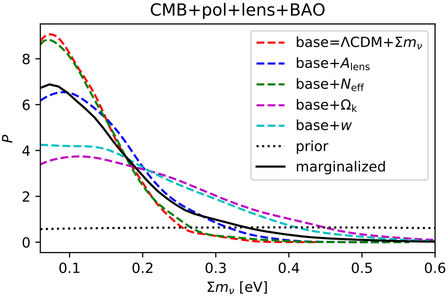
$$p(\theta|d) = \sum_i^N p(\theta|d,\mathcal{M}_i)\, p_i$$

Select a model $\mathcal{M}_0$ and use $p_i = \pi_i Z_i/(\sum \pi_j Z_j)$:

$$p(\theta|d) = \sum_i^N p(\theta|d,\mathcal{M}_i)\pi_i Z_i \Bigg/ \sum_j^N \pi_j Z_j$$

$p(\theta|d)$ is a model-marginalized posterior for θ, given the data d





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model	$\ln B_{i0}$	Σm_ν [eV]	$\ln B_{i0}$	Σm_ν [eV]
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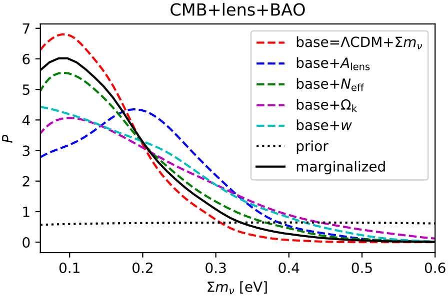
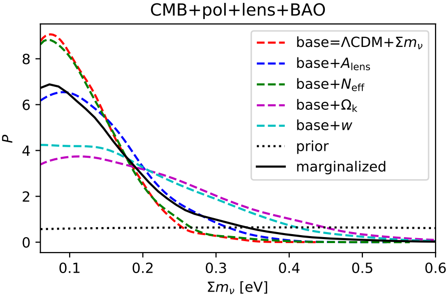
base+$A_{\rm lens}$	-2.6	< 0.38	-2.4	< 0.29
base+$N_{\rm eff}$	-1.5	< 0.37	-2.3	< 0.25
base+$\Omega_{\rm k}$	-10.3	< 0.47	-7.3	< 0.45
base+w	-1.4	< 0.42	-0.1	< 0.42
marginalized	–	< 0.33	–	< 0.35
p_0	0.65		0.48	

Model-marginalization applied to Σm_ν

[SG+, PRD 99 (2019) 021301]





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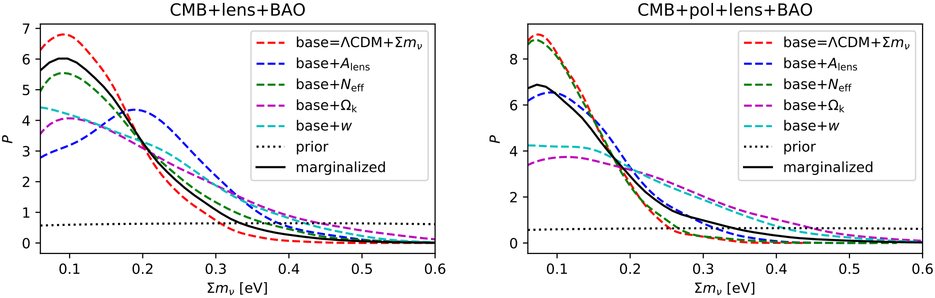

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Prior-free + model-marginalized bounds

[SG, arxiv:1910.06646]

Model marginalization formula: $p(x|d) = \sum_i p(x|d, \mathcal{M}_i)\, p(\mathcal{M}_i|d)$

parameter posterior, same as before: $p(x|d, \mathcal{M}_i) = \pi(x|\mathcal{M}_i)\, Z_i^x / Z_i$

prior independent

Model posterior: $p(\mathcal{M}_i|d) = \dfrac{Z_i\, \pi(\mathcal{M}_i)}{\sum_j Z_j\, \pi(\mathcal{M}_j)}$

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assume same prior $\pi(x) = \pi(x|\mathcal{M}_i)$ for each $\mathcal{M}_i$

$$\sum_j Z_j\,\pi(\mathcal{M}_j) = \frac{\sum_i Z_i^x\,\pi(\mathcal{M}_i)}{p(x|d)/\pi(x)} = \frac{\sum_i Z_i^{x_0}\,\pi(\mathcal{M}_i)}{p(x_0|d)/\pi(x_0)}$$

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finally: $\mathcal{R}(x, x_0|d) \equiv \dfrac{\sum_i Z_i^x\, \pi(\mathcal{M}_i)}{\sum_j Z_j^{x_0}\, \pi(\mathcal{M}_j)} = \dfrac{p(x|d)/\pi(x)}{p(x_0|d)/\pi(x_0)}$ model marginalized!

same meaning as before

A model-marginalized example

[SG, arxiv:1910.06646]

relative belief updating ratio

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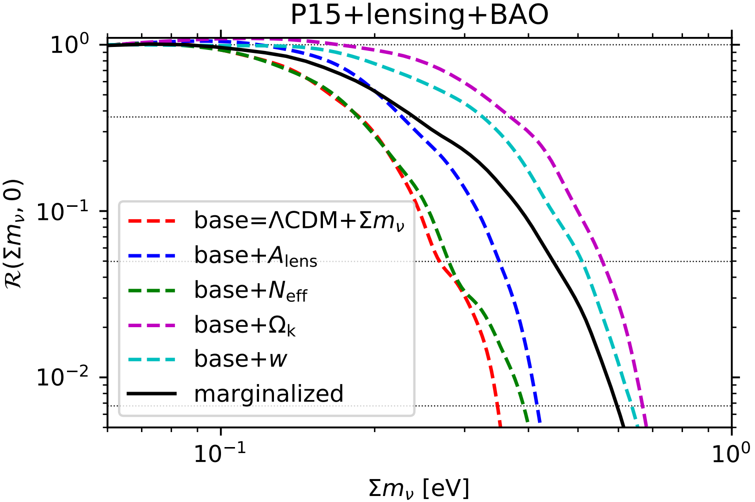

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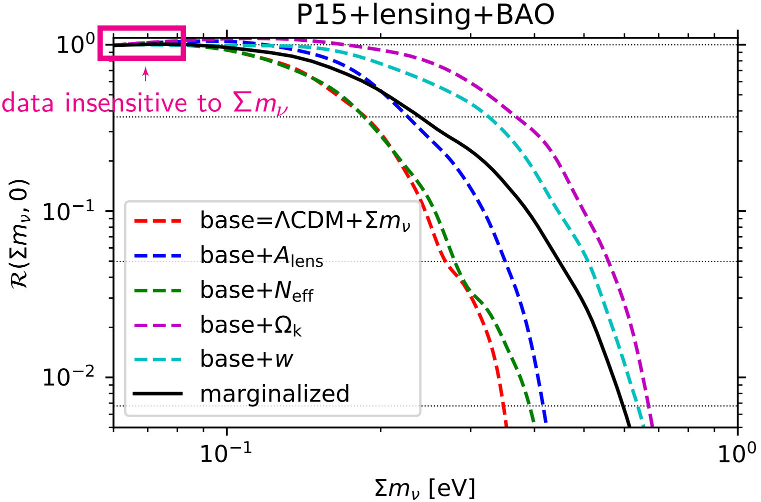

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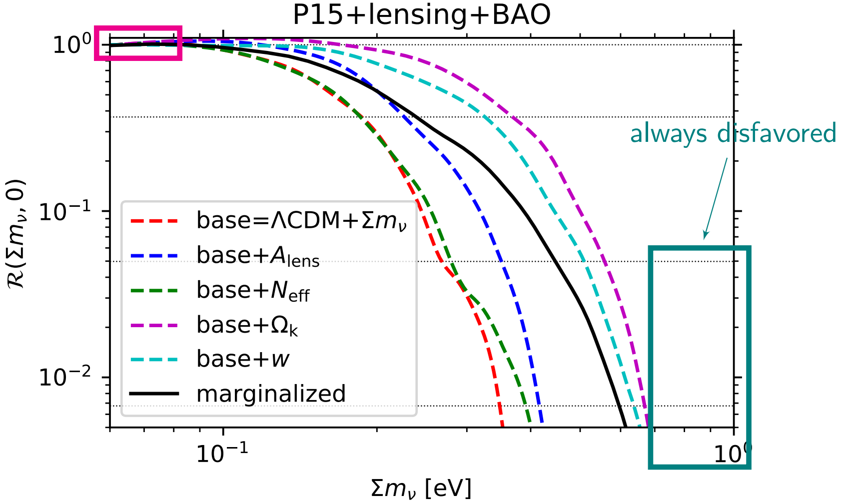

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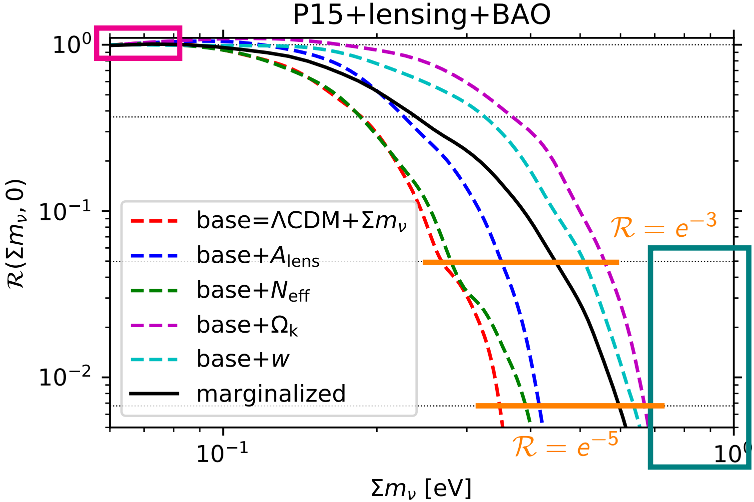

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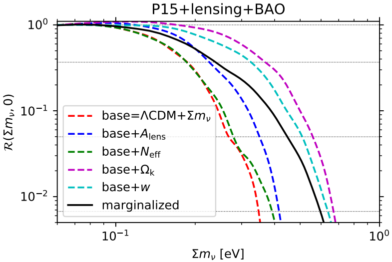

Conclusions

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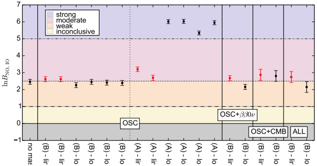
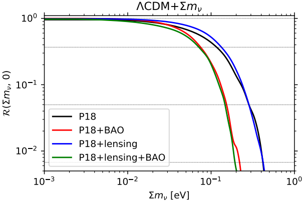
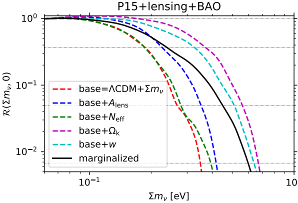


2 Model comparison techniques to present **prior-independent** results!



3 **marginalization** **over different models/priors** is also possible



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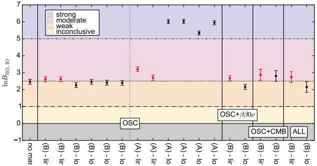

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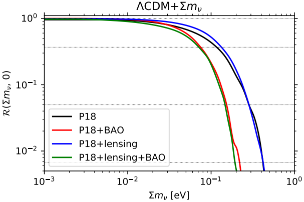

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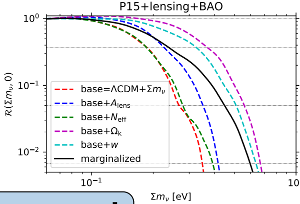

Thank you for the attention!