Theory of Neutrino Oscillations

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- ⇝ Critical Discussion of Standard Plane Wave Theory of Neutrino Oscillations
- ⇝ Covariant Plane Wave Derivation of Neutrino Oscillations
- ⇝ Necessity of a Wave Packet Treatment
- ⇝ Quantum Mechanical Wave-Packet Approach
- ⇝ Neutrino Wave Packets in Quantum Field Theory

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Standard Theory of Neutrino Oscillations in Vacuum

[Eliezer, Swift, NPB 105 (1976) 45] , [Fritzsch, Minkowski, PLB 62 (1976) 72] , [Bilenky, Pontecorvo, Nuovo Cim. Lett. 17 (1976) 569]

[Bilenky, Pontecorvo, Phys. Rep. 41 (1978) 225]

Neutrino Production: $j_\rho^{\mathrm{CC}} = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} \gamma_\rho \ell_{\alpha L}$ $\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL}$ Fields

$$\langle 0|\nu_{\alpha L}|\nu_\beta\rangle = \sum_{k,j} U_{\alpha k} U_{\beta j}^* \underbrace{\langle 0|\nu_{kL}|\nu_j\rangle}_{\propto \delta_{kj}} \propto \sum_k U_{\alpha k} U_{\beta k}^* = \delta_{\alpha\beta}$$

$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$ States

$$\mathcal{H}|\nu_k\rangle = E_k|\nu_k\rangle \quad \Rightarrow \quad |\nu_k(t)\rangle = e^{-iE_k t}|\nu_k\rangle \quad \Rightarrow \quad |\nu_\alpha(t)\rangle = \sum_k U_{\alpha k}^* e^{-iE_k t}|\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k}|\nu_\beta\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k}\right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t)} |\nu_\beta\rangle$$

Transition Probability: $P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta|\nu_\alpha(t)\rangle|^2 = \left|\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t)\right|^2 = \left|\sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k}\right|^2$

$$P_{\nu_\alpha \to \nu_\beta}(t) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-i\left(E_k - E_j\right) t\right]$$

Relativistic Approximation + Assumption $p_k = p = E$ [neutrinos with the same momentum propagate in the same direction]

$$E_k = \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p} = E + \frac{m_k^2}{2E} \implies E_k - E_j \simeq \frac{\Delta m_{kj}^2}{2E}$$

$$\boxed{\Delta m_{kj}^2 \equiv m_k^2 - m_j^2}$$

Approximation $t \simeq L$ $\implies$

$$\boxed{P_{\nu_\alpha \to \nu_\beta}(L) \simeq \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-i \frac{\Delta m_{kj}^2 L}{2E}\right]}$$

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_{k} |U_{\alpha k}|^2 |U_{\beta k}|^2$$ $\Leftarrow$ constant term

$$+ 2\text{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$ $\Leftarrow$ oscillating term

$\Updownarrow$

COHERENCE

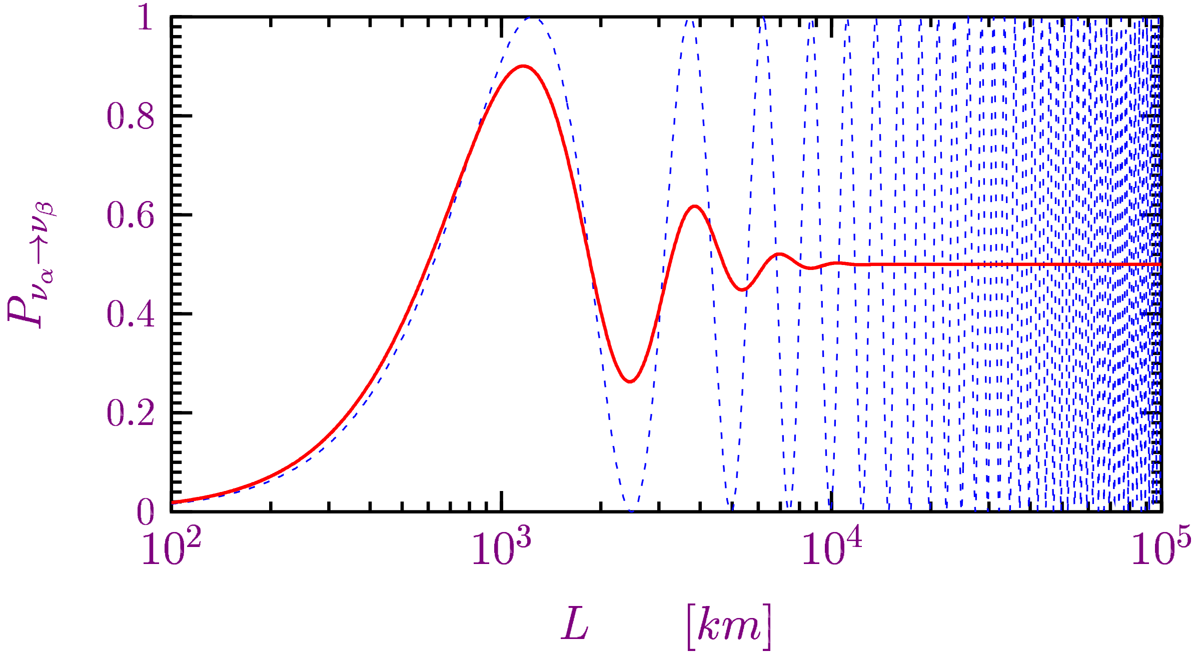
Two-Neutrino Mixing ($k = 1, 2$)

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \qquad \Delta m^2 \equiv \Delta m^2_{21} \equiv m_2^2 - m_1^2$$

Transition Probability ($\alpha \neq \beta$): $$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

Survival Probability ($\alpha = \beta$): $$P_{\nu_\alpha \to \nu_\alpha}(L, E) = 1 - P_{\nu_\alpha \to \nu_\beta}(L, E)$$

Averaged Transition Probability: $$\langle P_{\nu_\alpha \to \nu_\beta} \rangle = \frac{1}{2} \sin^2 2\vartheta$$



$\Delta m^2 = 10^{-3}\,\mathrm{eV}^2$

$\sin^2 2\vartheta = 1$

$\langle E \rangle = 1\,\mathrm{GeV}$

$\Delta E = 0.2\,\mathrm{GeV}$

Main Assumptions of Standard Theory

(A1) Neutrinos are extremely relativistic particles OK!

(A2) Neutrinos produced in CC weak interaction processes together with charged leptons α^+ are described by the flavor state $|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$

Correct approximation for ultrarelativistic ν's [Giunti, Kim, Lee, PRD 45 (1992) 2414]

(A3) Massive neutrino states $|\nu_k\rangle$ have the same momentum $p_k = p$ ("Equal Momentum Assumption") and different energies: $E_k \simeq E + \frac{m_k^2}{2E}$

Unrealistic assumption, forbidden by energy-momentum conservation and Lorentz invariance, but gives correct result (as well as the "Equal Energy Assumption")

[Winter, LNC 30 (1981) 101], [Giunti, Kim, FPL 14 (2001) 213], [Giunti, MPLA 16 (2001) 2363], [Giunti, hep-ph/0302026]

(A4) Propagation Time $T \simeq L$ Source-Detector Distance OK!

$\Updownarrow$

WAVE PACKETS

Simplest Example of Neutrino Production: $\pi^+ \to \mu^+ + \nu_\mu \quad \pi^- \to \mu^- + \bar{\nu}_\mu$

two-body decay $\Longrightarrow$ fixed kinematics $\quad \boxed{E_k^2 = p_k^2 + m_k^2}$

π at rest:
$$\begin{cases} p_k^2 = \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2 - \dfrac{m_k^2}{2}\left(1 + \dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \\[2ex] E_k^2 = \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2 + \dfrac{m_k^2}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) + \dfrac{m_k^4}{4\,m_\pi^2} \end{cases}$$

0^{th} order: $m_k = 0 \Rightarrow p_k = E_k = E = \dfrac{m_\pi}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) \simeq 30\,\text{MeV}$

1^{st} order: $\boxed{E_k \simeq E + \xi \dfrac{m_k^2}{2E}} \qquad \boxed{p_k \simeq E - (1 - \xi)\dfrac{m_k^2}{2E}} \qquad \xi = \dfrac{1}{2}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right) \simeq 0.2$

$\nwarrow \quad \nearrow$

general!

Equal Momentum of Massive Neutrinos?

(standard assumption: neutrinos with same momentum propagate in same direction)

$$p_k \simeq E - (1 - \xi)\,\frac{m_k^2}{2E}$$

the special case $\xi = 1 \Rightarrow p_k = p_j = E$ in general does not correspond to reality

BUT

for Extremely Relativistic Neutrinos the phase of $P_{\nu_\alpha \to \nu_\beta}$ is independent from ξ

$\Downarrow$

Equal Momentum Assumption gives correct Oscillation Phase

Different Momentum Contributions $\Longleftrightarrow$ Lorentz–Invariant Oscillations

$$|\nu_k(x,t)\rangle = e^{-iE_k t + ip_k x}|\nu_k\rangle \quad \Longrightarrow \quad |\nu_\alpha(x,t)\rangle = \sum_k U_{\alpha k}^{*}\, e^{-iE_k t + ip_k x}\, |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_{\beta=e,\mu,\tau} U_{\beta k}\, |\nu_\beta\rangle$$

$$|\nu_\alpha(x,t)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U_{\alpha k}^{*} e^{-iE_k t + ip_k x} U_{\beta k} \right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(x,t)} |\nu_\beta\rangle$$

Transition Probability: $P_{\nu_\alpha \to \nu_\beta}(x,t) = |\langle \nu_\beta | \nu_\alpha(x,t) \rangle|^2 = \left|\mathcal{A}_{\nu_\alpha \to \nu_\beta}(x,t)\right|^2$

LORENTZ INVARIANT

OSCILLATION PROBABILITY

$$P_{\nu_\alpha \to \nu_\beta}(x,t) = \left| \sum_k U_{\alpha k}^{*}\, e^{-iE_k t + ip_k x}\, U_{\beta k} \right|^2$$

[Dolgov, Morozov, Okun, Shchepkin, NPB 502 (1997) 3], [Dolgov, hep-ph/0004032], [Dolgov, Phys. Rept. 370 (2002) 333],

[Giunti, Kim, FPL 14 (2001) 213], [Bilenky, Giunti, IJMPA 16 (2001) 3931], [Beuthe, Phys. Rept. 375 (2003) 105]

important: Flavor is Lorentz Invariant $\Leftrightarrow$ different observers measure same $P_{\nu_\alpha \to \nu_\beta}$

ultrarelativistic neutrinos $\Longrightarrow$ $t \simeq x = L$ source-detector distance

$$E_k t - p_k x \simeq (E_k - p_k)\, L = \frac{E_k^2 - p_k^2}{E_k + p_k}\, L = \frac{m_k^2}{E_k + p_k}\, L \simeq \frac{m_k^2}{2E}\, L$$

$$P_{\nu_\alpha \to \nu_\beta}(L) = \left| \sum_k U_{\alpha k}^* \, e^{-i m_k^2 L / 2E} \, U_{\beta k} \right|^2$$

$$= \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2 \mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E} \right)$$

STANDARD OSCILLATION PROBABILITY!

$$\frac{\Delta m_{kj}^2 L}{2E} = 2\pi \quad \Longrightarrow \quad L_{kj}^{\mathrm{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}$$

Oscillation Length

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2 \mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-2\pi i \frac{L}{L_{kj}^{\mathrm{osc}}} \right)$$

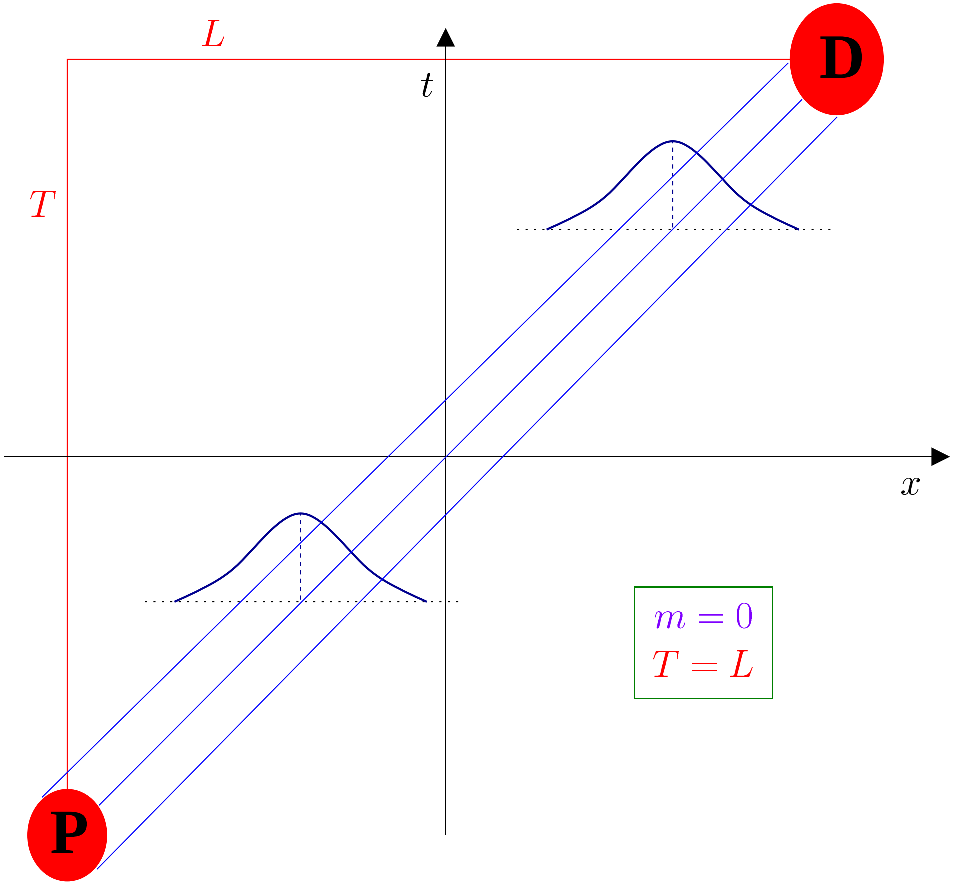
$t \simeq x = L \quad \Longleftrightarrow \quad$ Wave Packets

Other Motivations:

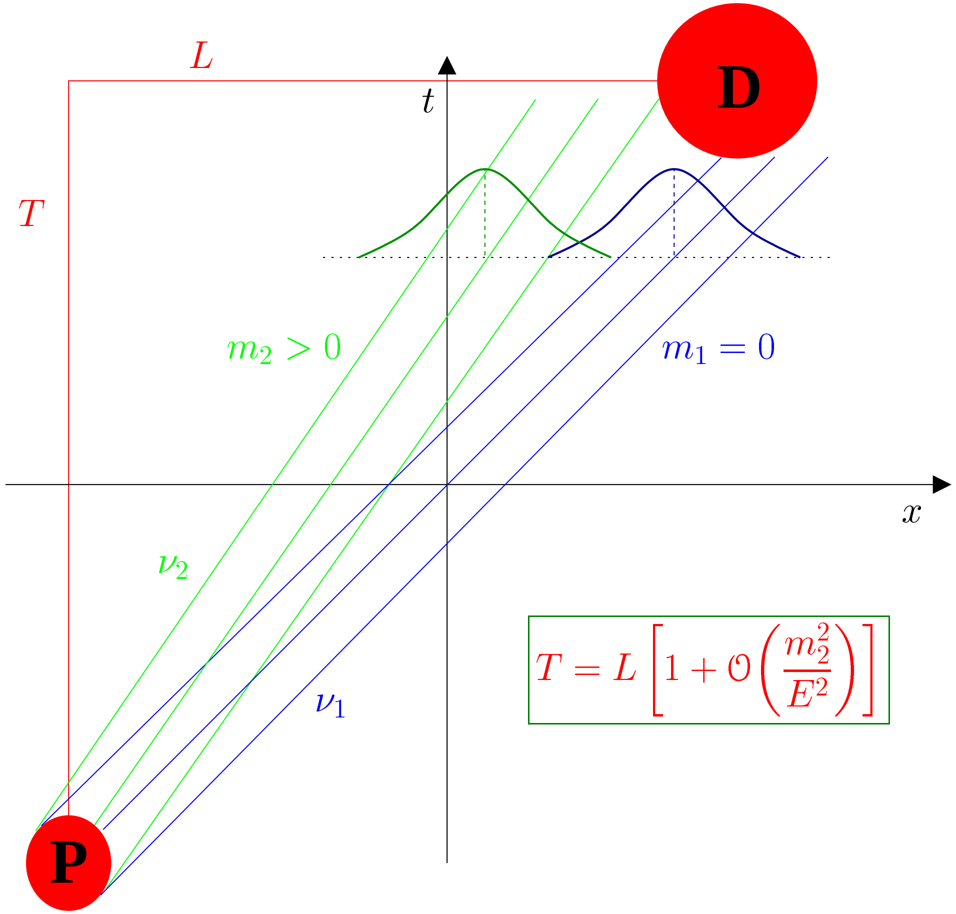
[Kayser, PRD 24 (1981) 110]

[Giunti, hep-ph/0302026]

- ⇝ Localization of Production and Detection Processes
- ⇝ Exact Energy-Momentum conservation would imply creation and detection of only one massive neutrino (neutrino mass measurement)



$m = 0$
$T = L$



$$T = L \left[1 + \mathcal{O}\left(\frac{m_2^2}{E^2} \right) \right]$$

Corrections to $T = L$

Size of wave packets is determined by coherence size of Production Process δt_{P}

($\delta t_{\mathrm{P}} \gtrsim \delta x_{\mathrm{P}}$ because coherence region must be causally connected)

velocity of neutrino wave packets: $$v_k = \frac{p_k}{E_k} \simeq 1 - \frac{m_k^2}{2E^2}$$

Wave packets arrive at Detection Process at different times: $$t_k = \frac{L}{v_k} \simeq L \left(1 + \frac{m_k^2}{2E^2} \right)$$

$$m_k > m_j \quad \Longrightarrow \quad v_k < v_j \quad \Longrightarrow \quad t_k > t_j$$

average time: $$\bar{t} = \frac{t_k + t_j}{2} \simeq L \left(1 + \frac{\overline{m_{kj}^2}}{2E^2} \right) \qquad \overline{m_{kj}^2} = \frac{m_k^2 + m_j^2}{2}$$

wave packets overlap with detection process $\Longrightarrow$ range of $T \simeq [\bar{t} - \delta t, \bar{t} + \delta t]$

$$\delta t \simeq \sqrt{\delta t_{\mathrm{P}}^2 + \delta t_{\mathrm{D}}^2}$$

phase of oscillations: $\Phi_{kj} = (p_k - p_j)\,L - (E_k - E_j)\,T$

$$T = L \quad \Longrightarrow \quad \Phi_{kj} = -\frac{\Delta m^2_{kj} L}{2E} \qquad \text{correction:} \quad \Delta\Phi_{kj} \simeq -(E_k - E_j)\left(\frac{\overline{m^2_{kj}}}{2E^2}\,L \pm \delta t\right)$$

$$E_k \simeq E + \xi\,\frac{m^2_k}{2E} \quad \Longrightarrow \quad E_k - E_j \simeq \xi\,\frac{\Delta m^2_{kj}}{2E} \quad \Longrightarrow \quad \Delta\Phi_{kj} \simeq -\xi\,\frac{\Delta m^2_{kj} L}{2E}\left(\frac{\overline{m^2_{kj}}}{2E^2} \pm \frac{\delta t}{L}\right)$$

$\overline{m^2_{kj}} \ll E^2$ (ultrarelativistic neutrinos) $\qquad \delta t \ll L^{\text{osc}}_{kj} \lesssim L$

flux energy spectrum
+
detector energy resolution
+
distance uncertainty

$\Longrightarrow$

oscillations observable if $\Phi_{kj} \sim 1 \Rightarrow \dfrac{\Delta m^2_{kj} L}{2E} \sim 1$

$$\Delta\Phi_{kj} \simeq -\xi\,\frac{\Delta m^2_{kj} L}{2E}\Bigg(\underbrace{\frac{\overline{m^2_{kj}}}{2E^2}}_{\text{negligible}} \pm \underbrace{\frac{\delta t}{L}}_{\text{negligible}}\Bigg)$$

phase practically constant during wave packets overlap with detection process

Very Important: ξ is irrelevant $\Rightarrow$ oscillations of ultrarelativistic neutrinos are independent from the kinematics of the production process

Coherence Length

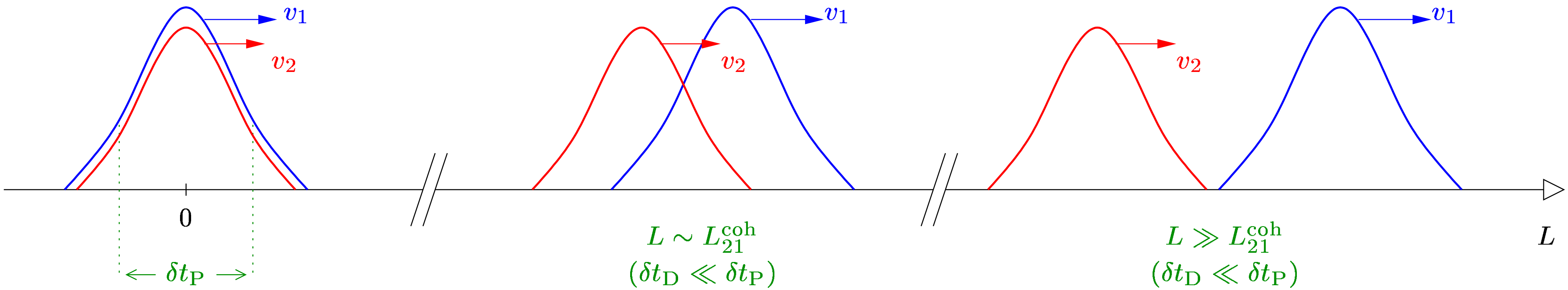
[Nussinov, PLB 63 (1976) 201], [Kiers, Nussinov, Weiss, PRD 53 (1996) 537]

Wave Packets have different velocities and separate

different massive neutrinos can interfere
if and only if
wave packets arrive with $\delta t_{kj} < \delta t_{\mathrm{D}}$

$\Longrightarrow \quad L \lesssim L^{\mathrm{coh}}_{kj}$

$$|\delta t_{kj}| \simeq |v_k - v_j|\, T \simeq \frac{|\Delta m^2_{kj}|}{2E^2} L \qquad \Longrightarrow \qquad L^{\mathrm{coh}}_{kj} \sim \frac{2E^2}{|\Delta m^2_{kj}|}\sqrt{\delta t_{\mathrm{P}}^2 + \delta t_{\mathrm{D}}^2}$$



Quantum Mechanical Wave Packet Model

[Giunti, Kim, Lee, PRD 44 (1991) 3635], [Giunti, Kim, PRD 58 (1998) 017301]

also called "Intermediate Wave Packet Model" [Beuthe, Phys. Rept. 375 (2003) 105]

neglecting mass effects in amplitudes of production and detection processes

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* \int \mathrm{d}p \, \psi_k^P(p) \, |\nu_k(p)\rangle \qquad |\nu_\beta\rangle = \sum_k U_{\beta k}^* \int \mathrm{d}p \, \psi_k^D(p) \, |\nu_k(p)\rangle$$

$$\begin{aligned} \mathcal{A}_{\alpha\beta}(x,t) &= \langle \nu_\beta | \, e^{-i\widehat{E}t + i\widehat{P}x} \, |\nu_\alpha\rangle \\ &= \sum_k U_{\alpha k}^* \, U_{\beta k} \int \mathrm{d}p \, \psi_k^P(p) \, \psi_k^{D*}(p) \, e^{-iE_k(p)t + ipx} \end{aligned}$$

Gaussian Approximation of Wave Packets

$$\psi_k^P(p) = \left(2\pi\sigma_{pP}^2\right)^{-1/4} \exp\left[-\frac{(p-p_k)^2}{4\sigma_{pP}^2}\right] \qquad \psi_k^D(p) = \left(2\pi\sigma_{pD}^2\right)^{-1/4} \exp\left[-\frac{(p-p_k)^2}{4\sigma_{pD}^2}\right]$$

the value of p_k is determined by the production process (causality)

$$\mathcal{A}_{\alpha\beta}(x,t) \propto \sum_k U_{\alpha k}^* U_{\beta k} \int \mathrm{d}p \, \exp\left[-iE_k(p)t + ipx - \frac{(p-p_k)^2}{4\sigma_p^2}\right]$$

global energy-momentum uncertainty: $\dfrac{1}{\sigma_p^2} = \dfrac{1}{\sigma_{pP}^2} + \dfrac{1}{\sigma_{pD}^2}$

sharply peaked wave packets

$$\sigma_p \ll E_k^2(p_k)/m_k \implies E_k(p) = \sqrt{p^2 + m_k^2} \simeq E_k + v_k\,(p - p_k)$$

$$E_k = E_k(p_k) = \sqrt{p_k^2 + m_k^2} \qquad v_k = \left.\frac{\partial E_k(p)}{\partial p}\right|_{p=p_k} = \frac{p_k}{E_k}$$

group velocity

$$\mathcal{A}_{\alpha\beta}(x,t) \propto \sum_k U_{\alpha k}^* U_{\beta k} \, \exp\left[-iE_k t + ip_k x - \underbrace{\frac{(x - v_k t)^2}{4\sigma_x^2}}\right]$$

suppression factor for $|x - v_k t| \gtrsim \sigma_x$
due to size of wave packets

$$\boxed{\sigma_x \, \sigma_p = \frac{1}{2}}$$

global space-time uncertainty: $\sigma_x^2 = \sigma_{xP}^2 + \sigma_{xD}^2$

$$
\begin{aligned}
-E_k t + p_k x &= -\left(E_k - p_k\right) x + E_k \left(x - t\right) = -\frac{E_k^2 - p_k^2}{E_k + p_k}\, x + E_k \left(x - t\right) \\
&= -\frac{m_k^2}{E_k + p_k}\, x + E_k \left(x - t\right) \simeq -\frac{m_k^2}{2E}\, x + E_k \left(x - t\right)
\end{aligned}
$$

$$
\mathcal{A}_{\alpha\beta}(x,t) \propto \sum_k U_{\alpha k}^{*}\, U_{\beta k}\, \exp\left[\underbrace{- i\, \frac{m_k^2}{2E}\, x}_{\text{standard phase for } t = x} \underbrace{+ i E_k \left(x - t\right)}_{\text{additional phase for } t \neq x} - \frac{\left(x - v_k t\right)^2}{4\sigma_x^2} \right]
$$

Space-Time Flavor Transition Probability

$$P_{\alpha\beta}(x,t) \propto \sum_{k,j} U_{\alpha k}^{*}\, U_{\beta k}\, U_{\alpha j}\, U_{\beta j}^{*} \exp\Bigg[\underbrace{-\,i\,\frac{\Delta m_{kj}^{2} x}{2E}}_{\text{standard phase for } t=x} \underbrace{+\,i\,(E_k - E_j)\,(x-t)}_{\text{additional phase for } t \neq x}\Bigg]$$

$$\times \exp\Bigg[\underbrace{-\,\frac{\left(x-\overline{v}_{kj}t\right)^2}{4\sigma_x^2}}_{\substack{\text{suppression factor for}\\ |x-\overline{v}_{kj}t| \gtrsim \sigma_x \\ \text{due to size of wave packets}}} \underbrace{-\,\frac{\left(v_k - v_j\right)^2 t^2}{8\sigma_x^2}}_{\substack{\text{suppression factor due to}\\ \text{separation of wave packets}}}\Bigg]$$

$$v_k = \frac{p_k}{E_k} \simeq 1 - \frac{m_k^2}{2E^2} \qquad \overline{v}_{kj} = \frac{v_k + v_j}{2} \simeq 1 - \frac{m_k^2 + m_j^2}{4E^2}$$

Oscillations in Space: $P_{\alpha\beta}(L) \propto \int \mathrm{d}t \, P_{\alpha\beta}(L,t)$

Gaussian integration over $\mathrm{d}t$

$$P_{\alpha\beta}(L) \propto \sum_{k,j} U_{\alpha k}^{*} \, U_{\beta k} \, U_{\alpha j} \, U_{\beta j}^{*} \, \exp\left[-i \, \frac{\Delta m_{kj}^{2} L}{2E}\right]$$

$$\times \underbrace{\sqrt{\frac{2}{v_k^2 + v_j^2}}}_{\simeq 1} \exp\left[- \underbrace{\frac{(v_k - v_j)^2}{v_k^2 + v_j^2}}_{\simeq (\Delta m_{kj}^2)^2 / 8E^4} \frac{L^2}{4\sigma_x^2} - \underbrace{\frac{(E_k - E_j)^2}{v_k^2 + v_j^2}}_{\simeq \xi^2 (\Delta m_{kj}^2)^2 / 8E^2} \sigma_x^2 \right]$$

$$\times \exp \underbrace{\left[i \, (E_k - E_j) \left(1 - \frac{2 \overline{v}_{kj}^2}{v_k^2 + v_j^2} \right) L \right]}_{\text{negligible}}$$

Ultrarelativistic Neutrinos: $p_k \simeq E - (1 - \xi) \, \frac{m_k^2}{2E}$ $\qquad E_k \simeq E + \xi \, \frac{m_k^2}{2E}$

$$P_{\alpha\beta}(L) = \sum_{k,j} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left[-i \frac{\Delta m_{kj}^2 L}{2E}\right]$$
$$\times \exp\left[-\left(\frac{\Delta m_{kj}^2 L}{4\sqrt{2} E^2 \sigma_x}\right)^2 - 2\xi^2 \left(\frac{\Delta m_{kj}^2 \sigma_x}{4E}\right)^2\right]$$

Oscillation Lengths: $L_{kj}^{\text{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}$ Coherence Lengths: $L_{kj}^{\text{coh}} = \frac{4\sqrt{2} E^2}{|\Delta m_{kj}^2|} \sigma_x$

$$P_{\alpha\beta}(L) = \sum_{k,j} U_{\alpha k}^{*} U_{\beta k} U_{\alpha j} U_{\beta j}^{*} \exp\left[-2\pi i \frac{L}{L_{kj}^{\text{osc}}}\right]$$
$$\times \exp\left[-\left(\frac{L}{L_{kj}^{\text{coh}}}\right)^2 - 2\pi^2 \xi^2 \left(\frac{\sigma_x}{L_{kj}^{\text{osc}}}\right)^2\right]$$

new localization term: $\exp\left[-2\pi^2\xi^2\left(\frac{\sigma_x}{L^{\text{osc}}_{kj}}\right)^2\right]$

interference is suppressed for $\sigma_x \gtrsim L^{\text{osc}}_{kj}$ $(\xi \sim 1)$

equivalent to neutrino mass measurement

uncertainty of neutrino mass measurement: $m_k^2 = E_k^2 - p_k^2$

$$\delta m_k^2 \simeq \sqrt{(2\,E_k\,\delta E_k)^2 + (2\,p_k\,\delta p_k)^2} \sim 4\,E\,\sigma_p$$

$$\sigma_p = \frac{1}{2\,\sigma_x}\,, \qquad E = \frac{|\Delta m^2_{kj}|L^{\text{osc}}_{kj}}{4\pi} \quad \Longrightarrow \quad \delta m_k^2 \sim \frac{|\Delta m^2_{kj}|L^{\text{osc}}_{kj}}{\sigma_x}$$

$$\boxed{\sigma_x \gtrsim L^{\text{osc}}_{kj} \quad \Longleftrightarrow \quad \delta m_k^2 \lesssim |\Delta m^2_{kj}|}$$

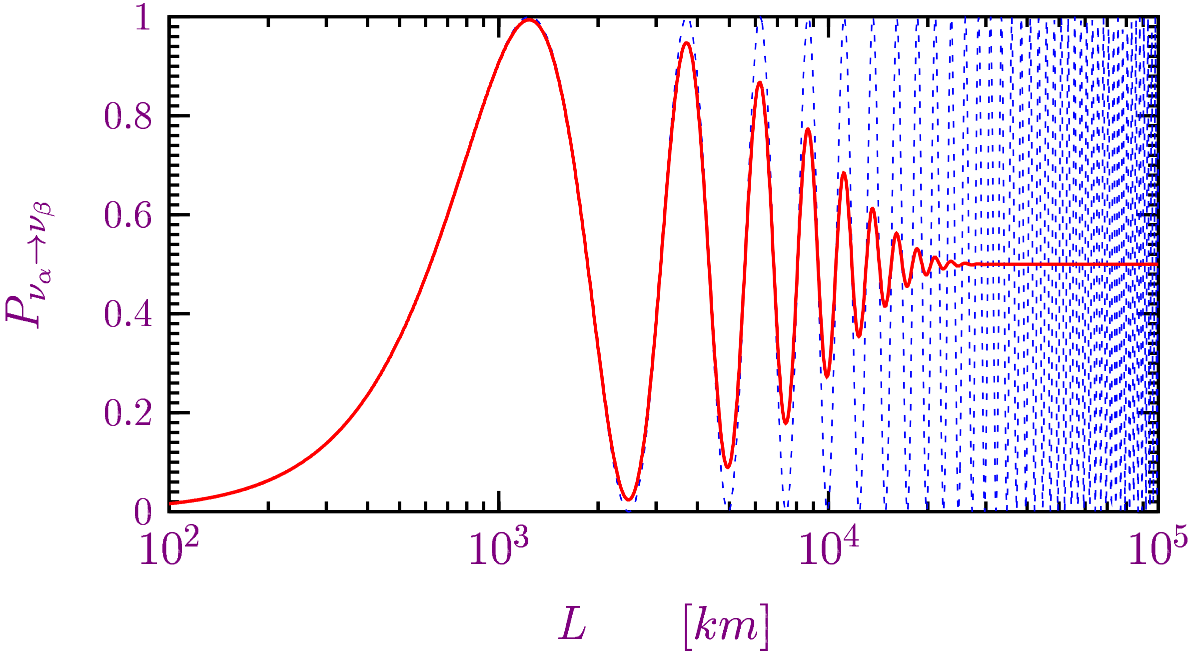
$\Downarrow$

! only one massive neutrino !

Two-Neutrino Mixing: Decoherence for $L \gtrsim L^{\text{coh}}$

$\Delta m^2 = 10^{-3}\,\text{eV}^2 \qquad \sin^2 2\vartheta = 1 \qquad E = 1\,\text{GeV} \qquad \sigma_p = 50\,\text{MeV}$

$$L^{\text{osc}} = \frac{4\pi E}{\Delta m^2} = 2480\,\text{km} \qquad\qquad L^{\text{coh}} = \frac{4\sqrt{2}E^2}{|\Delta m^2|}\,\sigma_x = 11163\,\text{km}$$



Achievements of the Quantum Mechanical Wave Packet Model

Confirmed Standard Oscillation Length:

$$L^{\text{osc}}_{kj} = \frac{4\pi E}{\Delta m^2_{kj}}$$

Derived Coherence Length:

$$L^{\text{coh}}_{kj} = \frac{4\sqrt{2}E^2}{|\Delta m^2_{kj}|}\,\sigma_x$$

problem: flavor states in production and detection processes have to be assumed

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} \int \mathrm{d}p\, \psi^P_k(p)\, |\nu_k(p)\rangle \qquad |\nu_\beta\rangle = \sum_k U^*_{\beta k} \int \mathrm{d}p\, \psi^D_k(p)\, |\nu_k(p)\rangle$$

calculation of neutrino production and detection?

$\Uparrow$

Quantum Field Theoretical Wave Packet Model

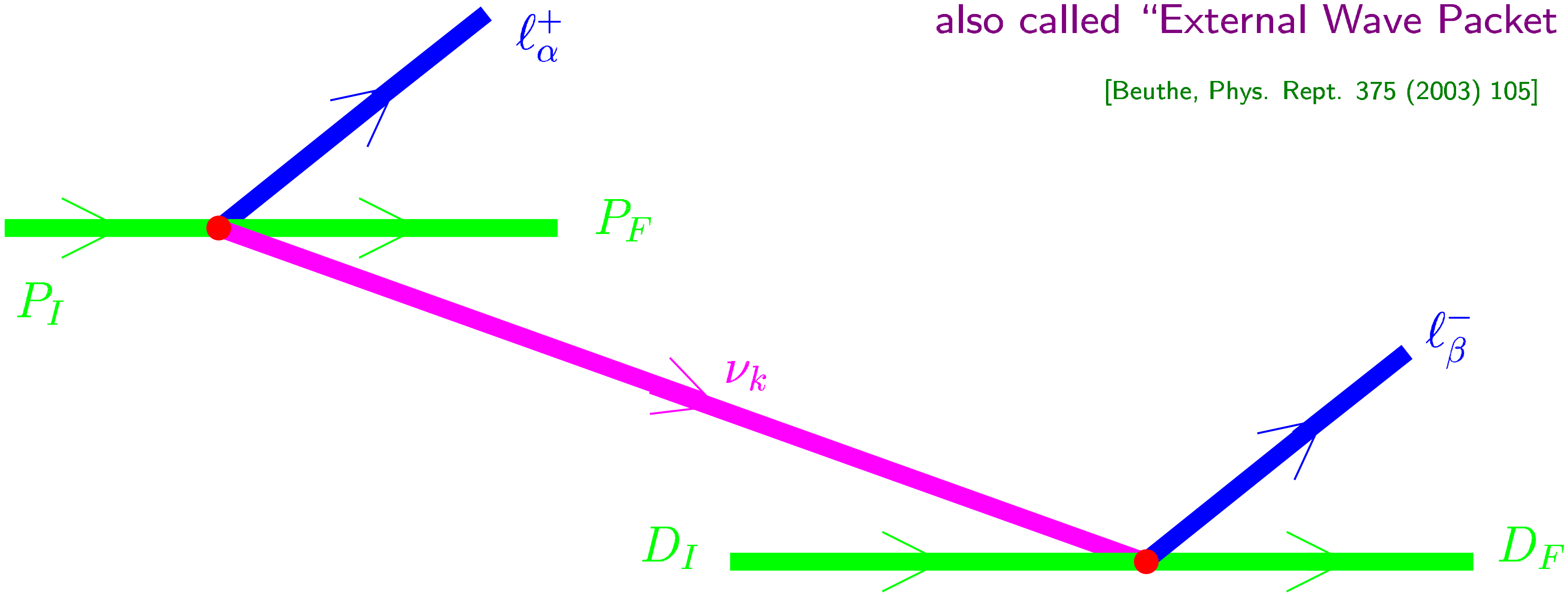
Quantum Field Model of Neutrino Oscillations with external particles in Production and Detection processes described by wave packets and intermediate virtual neutrino

[Giunti, Kim, Lee, Lee, PRD 48 (1993) 4310]
[Giunti, Kim, Lee, PLB 421 (1998) 237]
[Cardall, PRD 61 (2000) 073006]
[Beuthe, PRD 66 (2002) 013003]

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha \xrightarrow{\nu_\alpha \to \nu_\beta} \nu_\beta + D_I \to D_F + \ell_\beta^-$$

also called "External Wave Packet Model"

[Beuthe, Phys. Rept. 375 (2003) 105]



$$P_{\nu_\alpha \to \nu_\beta} \propto \left| \sum_k \left\{ P_I \to P_F + \ell_\alpha^+ + \nu_k \xrightarrow{\text{propagator}} \nu_k + D_I \to D_F + \ell_\beta^- \right\} \right|^2$$

Confirmed Standard Oscillation Length

Derived Coherence Length

"philosophical" problem of External Wave Packet Model: neutrino has no properties!

in oscillation experiments neutrinos propagate as free particles over macroscopically large distance, sometimes astronomical distances (solar, atmospheric neutrinos)

it must be possible to describe neutrinos in oscillation experiments with appropriate state, as in the quantum-mechanical approach

Neutrino Wave Packets in Quantum Field Theory

[Giunti, JHEP 11 (2002) 017]

also called "Interacting Wave Packet Model" [Beuthe, Phys. Rept. 375 (2003) 105]

In Quantum Field Theory $$|f\rangle \propto (\mathcal{S} - \mathbf{1})|i\rangle \simeq -i \int d^4x \, \mathcal{H}_I(x) \, |i\rangle$$

Entangled Final State in Production Process:

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha$$

$$|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto -i \int d^4x \, \mathcal{H}_I(x) \, |P_I\rangle$$

Disentangled by Interaction with Surrounding Medium (Measurement):

$$|\nu_\alpha\rangle \propto \langle P_F, \ell_\alpha^+ | \widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto \langle P_F, \ell_\alpha^+ | - i \int d^4x \, \mathcal{H}_I(x) \, |P_I\rangle$$

Localization: $$|\chi\rangle = \int d^3p \, \psi_\chi(\vec{p}; \vec{p}_\chi, \sigma_{p\chi}) \, |\chi(\vec{p}, h_\chi)\rangle \qquad (\chi = P_I, P_F, \ell_\alpha^+)$$

Neutrino State: $$|\nu_\alpha\rangle = N_\alpha \sum_k U_{\alpha k}^* \int d^3p \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p}, h) \, |\nu_k(\vec{p}, h)\rangle$$

Detection Process at $(\vec{L}, T)$: $\quad |\nu_\alpha(\vec{L}, T)\rangle = e^{-i\widehat{E}T + i\widehat{\vec{P}}\cdot\vec{L}} \, |\nu_\alpha\rangle$

$$|\nu_\alpha(\vec{L}, T)\rangle = N_\alpha \sum_k U_{\alpha k}^* \int \mathrm{d}^3p \, e^{-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}} \, e^{-S_k^P(\vec{p})} \sum_h \mathcal{A}_k^P(\vec{p}, h) \, |\nu_k(\vec{p}, h)\rangle$$

$$\nu_\beta + D_I \to D_F + \ell_\beta^-$$

Transition Amplitude: $\quad \mathcal{A}_{\alpha\beta}(\vec{L}, T) \propto \langle D_F, \ell_\beta^- | - i \int \mathrm{d}^4x \, \mathcal{H}_I(x) \, |D_I, \nu_\alpha(\vec{L}, T)\rangle$

$$\mathcal{A}_{\alpha\beta}(\vec{L}, T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \sum_h \int \mathrm{d}^3p \, \mathcal{A}_k^P(\vec{p}, h) \mathcal{A}_k^D(\vec{p}, h) \, e^{-S_k^P(\vec{p}) - S_k^D(\vec{p})} \, e^{-iE_{\nu_k}(\vec{p})T + i\vec{p}\cdot\vec{L}}$$

Ultrarelativistic Neutrinos

$$\boxed{\mathcal{A}_{\alpha\beta}(L, T) \propto \sum_k U_{\alpha k}^* U_{\beta k} \exp\left[-iE_k T + ip_k L - \frac{(L - v_k T)^2}{4\eta^2}\right]}$$

$$\eta^2 \sim \sigma_x^2 = \sigma_{xP}^2 + \sigma_{xD}^2$$

Space-Time Transition Probability: $P_{\alpha\beta}(\vec{L},T) \propto |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$

Transition Probability in Space: $P_{\alpha\beta}(\vec{L}) \propto \int \mathrm{d}T \, |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2$

Ultrarelativistic Neutrinos: $P_{\alpha\beta}(L) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2$

$$+2\mathrm{Re} \sum_{k>j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left[-2\pi i \, \frac{L}{L_{kj}^{\mathrm{osc}}} - \left(\frac{L}{L_{kj}^{\mathrm{coh}}} \right)^2 - 2\pi^2 \zeta^2 \left(\frac{\sigma_x}{L_{kj}^{\mathrm{osc}}} \right)^2 \right]$$

Oscillation Lengths: $L_{kj}^{\mathrm{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}$ Coherence Lengths: $L_{kj}^{\mathrm{coh}} = \frac{4\sqrt{2}\omega E^2}{|\Delta m_{kj}^2|} \sigma_x$

ω, ζ depend on Production and Detection processes ($\omega \sim 1$, $\zeta \sim 1$)

Estimates of Coherence Length

$$L^{\mathrm{osc}} = \frac{4\pi E}{\Delta m^2} = 2.5 \frac{(E/\mathrm{MeV})}{(\Delta m^2/\mathrm{eV}^2)} \mathrm{m} \qquad L^{\mathrm{coh}} \sim \frac{4\sqrt{2}E^2}{|\Delta m^2|}\sigma_x = 10^{12} \frac{(E^2/\mathrm{MeV}^2)}{(|\Delta m^2|/\mathrm{eV}^2)} \left(\frac{\sigma_x}{\mathrm{m}}\right) \mathrm{m}$$

Process	$\lvert\Delta m^2\rvert$	L^{osc}	σ_x	L^{coh}
$\pi \to \mu + \nu$ at rest in vacuum: $E \simeq 30\,\mathrm{MeV}$ natural linewidth	$2.5 \times 10^{-3}\,\mathrm{eV}^2$	$30\,\mathrm{km}$	$\tau_\pi \sim 10\,\mathrm{m}$	$\sim 10^{16}\,\mathrm{km}$
$\pi \to \mu + \nu$ at rest in matter: $E \simeq 30\,\mathrm{MeV}$ collision broadening	$2.5 \times 10^{-3}\,\mathrm{eV}^2$	$30\,\mathrm{km}$	$\tau_{\mathrm{col}} \sim 10^{-5}\,\mathrm{m}$	$\sim 10^{10}\,\mathrm{km}$
$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$ at rest in matter: $E \leq 50\,\mathrm{MeV}$ collision broadening	$1\,\mathrm{eV}^2$	$\leq 125\,\mathrm{m}$	$\tau_{\mathrm{col}} \sim 10^{-10}\,\mathrm{m}$	$\lesssim 10^{2}\,\mathrm{km}$
${}^7\mathrm{Be} + e^- \to {}^7\mathrm{Li} + \nu_e$ in solar core: $E \simeq 0.86\,\mathrm{MeV}$ collision broadening	$7 \times 10^{-5}\,\mathrm{eV}^2$	$31\,\mathrm{km}$	$\tau_{\mathrm{col}} \sim 10^{-9}\,\mathrm{m}$	$\sim 10^{4}\,\mathrm{km}$

Conclusions

- Standard expression for Oscillation Length of Ultrarelativistic Neutrinos is robust.

- Wave Packet Treatment is necessary for $T \simeq L \Leftrightarrow$ Oscillations in Space.

- Quantum Field Theoretical Wave Packet Models confirm Standard Oscillation Length and allows to calculate Coherence Length.

Neutrino Unbound

http://www.nu.to.infn.it

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