Neutrino Physics

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Neutrino Unbound: http://www.nu.to.infn.it

Torino, May 2011

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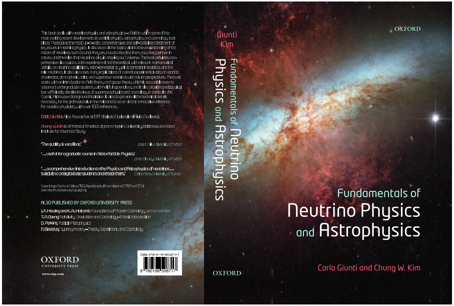

C. Giunti and C.W. Kim
Fundamentals of Neutrino Physics
and Astrophysics
Oxford University Press
15 March 2007 – 728 pages

Part I: Theory of Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
- Number of Flavor and Massive Neutrinos?
- Sterile Neutrinos

Part II: Neutrino Oscillations

- Neutrino Oscillations in Vacuum
- CPT, CP and T Symmetries
- Two-Neutrino Oscillations
- Neutrino Oscillations in Matter

Part III: Phenomenology

- Solar Neutrinos and KamLAND
- Atmospheric and LBL Oscillation Experiments
- Phenomenology of Three-Neutrino Mixing
- Absolute Scale of Neutrino Masses
- Anomalies Beyond Three-Neutrino Mixing
- Conclusions

Part I

Theory of Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
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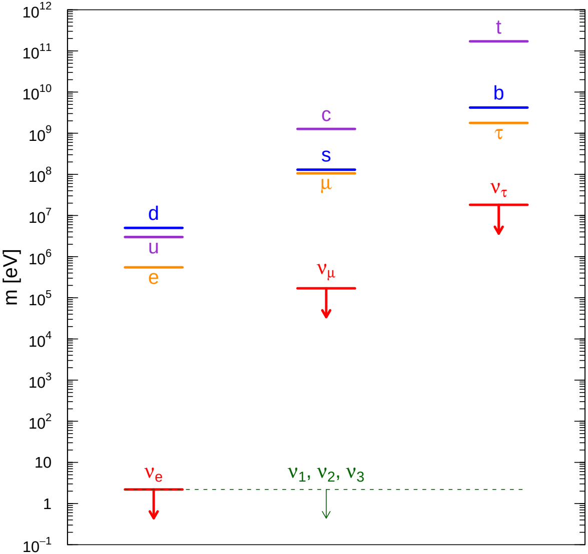
Fermion Mass Spectrum



Dirac Neutrino Masses and Mixing

- Dirac Neutrino Masses and Mixing
  - Dirac Mass
  - Higgs Mechanism in SM
  - Dirac Lepton Masses
  - Three-Generations Dirac Neutrino Masses
  - Massive Chiral Lepton Fields
  - Massive Dirac Lepton Fields
  - Quantization
  - Mixing
  - Flavor Lepton Numbers
  - Total Lepton Number
  - Mixing Matrix
  - Standard Parameterization of Mixing Matrix
  - CP Violation
    - Example: $\vartheta_{12} = 0$
    - Example: $\vartheta_{13} = \pi/2$
    - Example: $m_{\nu_2} = m_{\nu_3}$
  - Jarlskog Rephasing Invariant

Dirac Mass

- Dirac Equation: $(i\partial\!\!\!/ - m)\,\nu(x) = 0 \qquad (\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu)$
- Dirac Lagrangian: $\mathscr{L}(x) = \overline{\nu}(x)\,(i\partial\!\!\!/ - m)\,\nu(x)$
- Chiral decomposition: $\nu_L \equiv P_L \nu\,, \qquad \nu_R \equiv P_R \nu\,, \qquad \nu = \nu_L + \nu_R$

$$P_L \equiv \frac{1-\gamma^5}{2}\,, \quad P_R \equiv \frac{1+\gamma^5}{2}$$

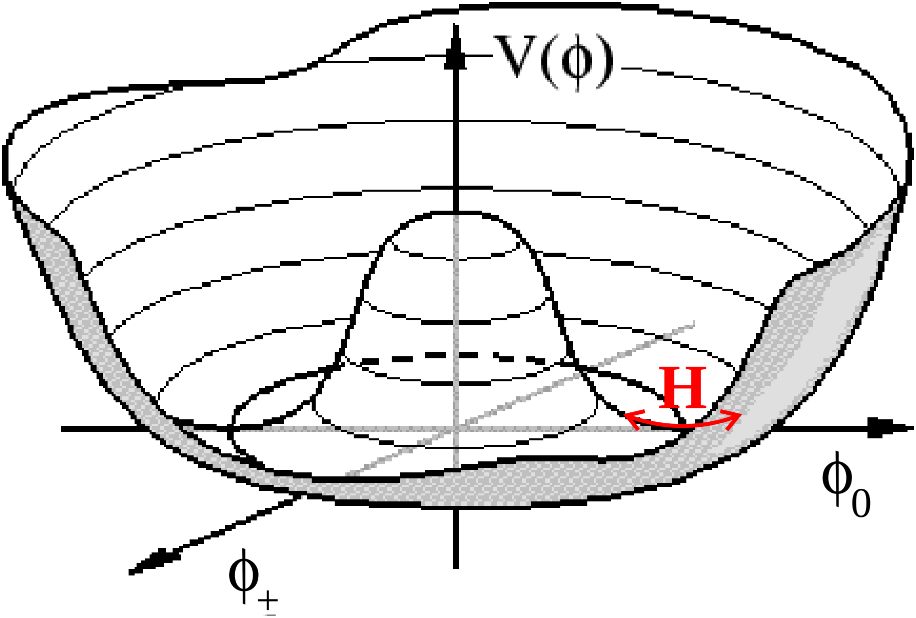
$$P_L^2 = P_L\,, \quad P_R^2 = P_R\,, \quad P_L + P_R = 1\,, \quad P_L P_R = P_R P_L = 0$$

$$\mathscr{L} = \overline{\nu_L} i\partial\!\!\!/ \nu_L + \overline{\nu_R} i\partial\!\!\!/ \nu_R - m\,(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L)$$

- In SM only $\nu_L \Longrightarrow$ no Dirac mass
- Oscillation experiments have shown that neutrinos are massive
- Simplest extension of the SM: add ν_R

Higgs Mechanism in SM

- Higgs Doublet: $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$ $\qquad |\Phi|^2 = \Phi^\dagger \Phi = \phi_+^\dagger \phi_+ + \phi_0^\dagger \phi_0$
- Higgs Lagrangian: $\mathscr{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(|\Phi|^2)$
- Higgs Potential: $V(|\Phi|^2) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4$
- $\mu^2 < 0$ and $\lambda > 0 \quad \Longrightarrow \quad V(|\Phi|^2) = \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2$, with $\quad v \equiv \sqrt{-\frac{\mu^2}{\lambda}}$
- Vacuum: V_{min} for $|\Phi|^2 = \frac{v^2}{2} \Longrightarrow \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$
- Spontaneous Symmetry Breaking: $\text{SU}(2)_L \times \text{U}(1)_Y \to \text{U}(1)_Q$
- Unitary Gauge: $\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$


V(ϕ)
H
ϕ0
ϕ±

Dirac Lepton Masses

$$L_L \equiv \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \qquad \ell_R \qquad \nu_R$$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}_{H,\mathrm{L}} = -y^{\ell}\,\overline{L_L}\,\Phi\,\ell_R - y^{\nu}\,\overline{L_L}\,\widetilde{\Phi}\,\nu_R + \text{H.c.}$$

Unitary Gauge

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2\,\Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathscr{L}_{H,\mathrm{L}} = & - \frac{y^{\ell}}{\sqrt{2}} \begin{pmatrix} \overline{\nu_L} & \overline{\ell_L} \end{pmatrix} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \ell_R \\ & - \frac{y^{\nu}}{\sqrt{2}} \begin{pmatrix} \overline{\nu_L} & \overline{\ell_L} \end{pmatrix} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix} \nu_R + \text{H.c.} \end{aligned}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\, y^{\ell}\, \frac{v}{\sqrt{2}}\, \overline{\ell_L}\, \ell_R - y^{\nu}\, \frac{v}{\sqrt{2}}\, \overline{\nu_L}\, \nu_R$$

$$-\, \frac{y^{\ell}}{\sqrt{2}}\, \overline{\ell_L}\, \ell_R\, H - \frac{y^{\nu}}{\sqrt{2}}\, \overline{\nu_L}\, \nu_R\, H + \mathrm{H.c.}$$

$$m_{\ell} = y^{\ell}\, \frac{v}{\sqrt{2}} \qquad\qquad m_{\nu} = y^{\nu}\, \frac{v}{\sqrt{2}}$$

$$g_{\ell H} = \frac{y^{\ell}}{\sqrt{2}} = \frac{m_{\ell}}{v} \qquad\qquad g_{\nu H} = \frac{y^{\nu}}{\sqrt{2}} = \frac{m_{\nu}}{v}$$

$$v = \left(\sqrt{2}\, G_{\mathrm{F}}\right)^{1/2} = 246\, \mathrm{GeV}$$

Three-Generations Dirac Neutrino Masses

$L'_{eL} \equiv \begin{pmatrix} \nu'_{eL} \\ \ell'_{eL} \equiv e'_L \end{pmatrix}$	$L'_{\mu L} \equiv \begin{pmatrix} \nu'_{\mu L} \\ \ell'_{\mu L} \equiv \mu'_L \end{pmatrix}$	$L'_{\tau L} \equiv \begin{pmatrix} \nu'_{\tau L} \\ \ell'_{\tau L} \equiv \tau'_L \end{pmatrix}$
$\ell'_{eR} \equiv e'_R$	$\ell'_{\mu R} \equiv \mu'_R$	$\ell'_{\tau R} \equiv \tau'_R$
ν'_{eR}	$\nu'_{\mu R}$	$\nu'_{\tau R}$

Lepton-Higgs Yukawa Lagrangian

$$\mathscr{L}_{H,\mathrm{L}} = - \sum_{\alpha,\beta=e,\mu,\tau} \left[Y'^{\ell}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \Phi \, \ell'_{\beta R} + Y'^{\nu}_{\alpha\beta} \, \overline{L'_{\alpha L}} \, \widetilde{\Phi} \, \nu'_{\beta R} \right] + \mathrm{H.c.}$$

Unitary Gauge

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \qquad \widetilde{\Phi} = i\sigma_2 \, \Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right) \sum_{\alpha,\beta=e,\mu,\tau} \left[Y'^{\ell}_{\alpha\beta}\, \overline{\ell'_{\alpha L}}\, \ell'_{\beta R} + Y'^{\nu}_{\alpha\beta}\, \overline{\nu'_{\alpha L}}\, \nu'_{\beta R} \right] + \mathrm{H.c.}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right) \left[\overline{\boldsymbol{\ell}'_L}\, Y'^{\ell}\, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L}\, Y'^{\nu}\, \boldsymbol{\nu}'_R \right] + \mathrm{H.c.}$$

$$\boldsymbol{\ell}'_L \equiv \begin{pmatrix} e'_L \\ \mu'_L \\ \tau'_L \end{pmatrix} \qquad \boldsymbol{\ell}'_R \equiv \begin{pmatrix} e'_R \\ \mu'_R \\ \tau'_R \end{pmatrix} \qquad \boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \qquad \boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu'_{eR} \\ \nu'_{\mu R} \\ \nu'_{\tau R} \end{pmatrix}$$

$$Y'^{\ell} \equiv \begin{pmatrix} Y'^{\ell}_{ee} & Y'^{\ell}_{e\mu} & Y'^{\ell}_{e\tau} \\ Y'^{\ell}_{\mu e} & Y'^{\ell}_{\mu\mu} & Y'^{\ell}_{\mu\tau} \\ Y'^{\ell}_{\tau e} & Y'^{\ell}_{\tau\mu} & Y'^{\ell}_{\tau\tau} \end{pmatrix} \qquad Y'^{\nu} \equiv \begin{pmatrix} Y'^{\nu}_{ee} & Y'^{\nu}_{e\mu} & Y'^{\nu}_{e\tau} \\ Y'^{\nu}_{\mu e} & Y'^{\nu}_{\mu\mu} & Y'^{\nu}_{\mu\tau} \\ Y'^{\nu}_{\tau e} & Y'^{\nu}_{\tau\mu} & Y'^{\nu}_{\tau\tau} \end{pmatrix}$$

$$M'^{\ell} = \frac{v}{\sqrt{2}}\, Y'^{\ell} \qquad M'^{\nu} = \frac{v}{\sqrt{2}}\, Y'^{\nu}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}'_L}\, Y'^{\ell}\, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L}\, Y'^{\nu}\, \boldsymbol{\nu}'_R\right] + \mathrm{H.c.}$$

Diagonalization of Y'^{ℓ} and Y'^{ν} with unitary V^{ℓ}_L, V^{ℓ}_R, V^{ν}_L, V^{ν}_R

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \mathbf{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \mathbf{n}_R$$

Unitary transformations are allowed
because they leave invariant the kinetic terms in the Lagrangian

$$\begin{aligned}
\mathscr{L}_{\text{kin}} &= \overline{\boldsymbol{\ell}'_L} i\partial\!\!\!/ \boldsymbol{\ell}'_L + \overline{\boldsymbol{\ell}'_R} i\partial\!\!\!/ \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L} i\partial\!\!\!/ \boldsymbol{\nu}'_L + \overline{\boldsymbol{\nu}'_R} i\partial\!\!\!/ \boldsymbol{\nu}'_R \\
&= \overline{\boldsymbol{\ell}_L} V^{\ell\dagger}_L i\partial\!\!\!/ V^{\ell}_L \boldsymbol{\ell}_L + \ldots \\
&= \overline{\boldsymbol{\ell}_L} i\partial\!\!\!/ \boldsymbol{\ell}_L + \overline{\boldsymbol{\ell}_R} i\partial\!\!\!/ \boldsymbol{\ell}_R + \overline{\boldsymbol{\nu}_L} i\partial\!\!\!/ \boldsymbol{\nu}_L + \overline{\boldsymbol{\nu}_R} i\partial\!\!\!/ \boldsymbol{\nu}_R
\end{aligned}$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}'_L}\, Y'^{\ell}\, \boldsymbol{\ell}'_R + \overline{\boldsymbol{\nu}'_L}\, Y'^{\nu}\, \boldsymbol{\nu}'_R\right] + \mathrm{H.c.}$$

$$\boldsymbol{\ell}'_L = V^{\ell}_L\, \boldsymbol{\ell}_L \qquad \boldsymbol{\ell}'_R = V^{\ell}_R\, \boldsymbol{\ell}_R \qquad \boldsymbol{\nu}'_L = V^{\nu}_L\, \mathbf{n}_L \qquad \boldsymbol{\nu}'_R = V^{\nu}_R\, \mathbf{n}_R$$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\boldsymbol{\ell}_L} V^{\ell\dagger}_L Y'^{\ell} V^{\ell}_R \boldsymbol{\ell}_R + \overline{\boldsymbol{\nu}_L} V^{\nu\dagger}_L Y'^{\nu} V^{\nu}_R \boldsymbol{\nu}_R\right] + \mathrm{H.c.}$$

$$V^{\ell\dagger}_L \; Y'^{\ell} \; V^{\ell}_R \; = \; Y^{\ell} \qquad Y^{\ell}_{\alpha\beta} = y^{\ell}_{\alpha}\, \delta_{\alpha\beta} \qquad (\alpha, \beta = e, \mu, \tau)$$

$$V^{\nu\dagger}_L \; Y'^{\nu} \; V^{\nu}_R \; = \; Y^{\nu} \qquad Y^{\nu}_{kj} = y^{\nu}_{k}\, \delta_{kj} \qquad (k, j = 1, 2, 3)$$

Real and Positive y^{ℓ}_{α}, y^{ν}_{k}

$$V^{\dagger}_L\, Y'\, V_R \; = \; Y \qquad \Longleftrightarrow \qquad \underset{18}{Y'} \; = \; \underset{9}{V_L} \; \underset{3}{Y} \; \underset{9}{V^{\dagger}_R}$$

Massive Chiral Lepton Fields

$\ell_L = V_L^{\ell\dagger}\,\ell'_L \equiv \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$	$\ell_R = V_R^{\ell\dagger}\,\ell'_R \equiv \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$
$\mathbf{n}_L = V_L^{\nu\dagger}\,\boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$	$\mathbf{n}_R = V_R^{\nu\dagger}\,\boldsymbol{\nu}'_R \equiv \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix}$

$$\begin{aligned}\mathscr{L}_{H,\mathrm{L}} &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\overline{\ell_L}\,Y^\ell\,\ell_R + \overline{\mathbf{n}_L}\,Y^\nu\,n_R\right] + \mathrm{H.c.} \\ &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\sum_{\alpha=e,\mu,\tau} y_\alpha^\ell\,\overline{\ell_{\alpha L}}\,\ell_{\alpha R} + \sum_{k=1}^{3} y_k^\nu\,\overline{\nu_{kL}}\,\nu_{kR}\right] + \mathrm{H.c.}\end{aligned}$$

Massive Dirac Lepton Fields

$$\ell_\alpha \equiv \ell_{\alpha L} + \ell_{\alpha R} \qquad (\alpha = e, \mu, \tau)$$

$$\nu_k = \nu_{kL} + \nu_{kR} \qquad (k = 1, 2, 3)$$

$$\mathscr{L}_{H,\mathrm{L}} = - \sum_{\alpha=e,\mu,\tau} \frac{y^\ell_\alpha \, v}{\sqrt{2}} \, \overline{\ell_\alpha} \, \ell_\alpha - \sum_{k=1}^{3} \frac{y^\nu_k \, v}{\sqrt{2}} \, \overline{\nu_k} \, \nu_k \qquad \text{Mass Terms}$$

$$- \sum_{\alpha=e,\mu,\tau} \frac{y^\ell_\alpha}{\sqrt{2}} \, \overline{\ell_\alpha} \, \ell_\alpha \, H - \sum_{k=1}^{3} \frac{y^\nu_k}{\sqrt{2}} \, \overline{\nu_k} \, \nu_k \, H \qquad \text{Lepton-Higgs Couplings}$$

Charged Lepton and Neutrino Masses

$$m_\alpha = \frac{y^\ell_\alpha \, v}{\sqrt{2}} \quad (\alpha = e, \mu, \tau) \qquad\qquad m_k = \frac{y^\nu_k \, v}{\sqrt{2}} \qquad (k = 1, 2, 3)$$

Lepton-Higgs coupling $\propto$ Lepton Mass

Quantization

$$\nu_k(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\, 2E_k} \sum_{h=\pm1} \left[a_k^{(h)}(p)\, u_k^{(h)}(p)\, e^{-ip\cdot x} + b_k^{(h)\dagger}(p)\, v_k^{(h)}(p)\, e^{ip\cdot x} \right]$$

$$p^0 = E_k = \sqrt{\vec{p}^2 + m_k^2}$$

$$(\not{p} - m_k)\, u_k^{(h)}(p) = 0$$
$$(\not{p} + m_k)\, v_k^{(h)}(p) = 0$$

$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, u_k^{(h)}(p) = h\, u_k^{(h)}(p)$$
$$\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\, v_k^{(h)}(p) = -h\, v_k^{(h)}(p)$$

$$\{a_k^{(h)}(p), a_k^{(h')\dagger}(p')\} = \{b_k^{(h)}(p), b_k^{(h')\dagger}(p')\} = (2\pi)^3\, 2E_k\, \delta^3(\vec{p} - \vec{p}')\, \delta_{hh'}$$
$$\{a_k^{(h)}(p), a_k^{(h')}(p')\} = \{a_k^{(h)\dagger}(p), a_k^{(h')\dagger}(p')\} = 0$$
$$\{b_k^{(h)}(p), b_k^{(h')}(p')\} = \{b_k^{(h)\dagger}(p), b_k^{(h')\dagger}(p')\} = 0$$
$$\{a_k^{(h)}(p), b_k^{(h')}(p')\} = \{a_k^{(h)\dagger}(p), b_k^{(h')\dagger}(p')\} = 0$$
$$\{a_k^{(h)}(p), b_k^{(h')\dagger}(p')\} = \{a_k^{(h)\dagger}(p), b_k^{(h')}(p')\} = 0$$

Mixing

Charged-Current Weak Interaction Lagrangian

$$\mathscr{L}_{\mathrm{I}}^{(\mathrm{CC})} = -\frac{g}{2\sqrt{2}} j_W^\rho W_\rho + \mathrm{H.c.}$$

Weak Charged Current: $j_W^\rho = j_{W,\mathrm{L}}^\rho + j_{W,\mathrm{Q}}^\rho$

Leptonic Weak Charged Current

$$j_{W,\mathrm{L}}^\rho = \sum_{\alpha=e,\mu,\tau} \overline{\nu'_\alpha} \gamma^\rho \left(1 - \gamma^5\right) \ell'_\alpha = 2 \sum_{\alpha=e,\mu,\tau} \overline{\nu'_{\alpha L}} \gamma^\rho \ell'_{\alpha L} = 2 \overline{\boldsymbol{\nu}'_L} \gamma^\rho \boldsymbol{\ell}'_L$$

$$\underline{\boldsymbol{\ell}'_L = V_L^\ell \boldsymbol{\ell}_L} \qquad \underline{\boldsymbol{\nu}'_L = V_L^\nu \mathbf{n}_L}$$

$$j_{W,\mathrm{L}}^\rho = 2 \overline{\mathbf{n}_L} V_L^{\nu\dagger} \gamma^\rho V_L^\ell \boldsymbol{\ell}_L = 2 \overline{\mathbf{n}_L} V_L^{\nu\dagger} V_L^\ell \gamma^\rho \boldsymbol{\ell}_L = 2 \overline{\mathbf{n}_L} U^\dagger \gamma^\rho \boldsymbol{\ell}_L$$

Mixing Matrix

$$U^\dagger = V_L^{\nu\dagger} V_L^\ell \qquad \boxed{U = V_L^{\ell\dagger} V_L^\nu}$$

- Definition: Left-Handed Flavor Neutrino Fields

$$\nu_L = U\,\mathbf{n}_L = V_L^{\ell\dagger}\,\nu'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- They allow us to write the Leptonic Weak Charged Current as in the SM:

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\nu_L}\,\gamma^{\rho}\,\ell_L = 2\sum_{\alpha=e,\mu,\tau}\overline{\nu_{\alpha L}}\,\gamma^{\rho}\,\ell_{\alpha L}$$

- Each left-handed flavor neutrino field is associated with the corresponding charged lepton field which describes a massive charged lepton:

$$j^{\rho}_{W,\mathrm{L}} = 2\left(\overline{\nu_{eL}}\,\gamma^{\rho}\,e_L + \overline{\nu_{\mu L}}\,\gamma^{\rho}\,\mu_L + \overline{\nu_{\tau L}}\,\gamma^{\rho}\,\tau_L\right)$$

- In practice left-handed flavor neutrino fields are useful for calculations in the SM approximation of massless neutrinos (interactions).

- If neutrino masses must be taken into account, it is necessary to use

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\ell_L = 2\sum_{k=1}^{3}\sum_{\alpha=e,\mu,\tau}U^{*}_{\alpha k}\,\overline{\nu_{kL}}\,\gamma^{\rho}\,\ell_{\alpha L}$$

Flavor Lepton Numbers

Flavor Neutrino Fields are useful for defining
Flavor Lepton Numbers
as in the SM

	L_e	L_μ	L_τ
$(\nu_e\,,\,e^-)$	$+1$	0	0
$(\nu_\mu\,,\,\mu^-)$	0	$+1$	0
$(\nu_\tau\,,\,\tau^-)$	0	0	$+1$

	L_e	L_μ	L_τ
$(\nu_e^c\,,\,e^+)$	-1	0	0
$(\nu_\mu^c\,,\,\mu^+)$	0	-1	0
$(\nu_\tau^c\,,\,\tau^+)$	0	0	-1

$$\boxed{L = L_e + L_\mu + L_\tau}$$

Standard Model: Lepton numbers are conserved

$$\mathscr{L}^{\mathrm{D}}_{\text{mass}} = - \begin{pmatrix} \overline{\nu_{eL}} & \overline{\nu_{\mu L}} & \overline{\nu_{\tau L}} \end{pmatrix} \begin{pmatrix} m^{\mathrm{D}}_{ee} & m^{\mathrm{D}}_{e\mu} & m^{\mathrm{D}}_{e\tau} \\ m^{\mathrm{D}}_{\mu e} & m^{\mathrm{D}}_{\mu\mu} & m^{\mathrm{D}}_{\mu\tau} \\ m^{\mathrm{D}}_{\tau e} & m^{\mathrm{D}}_{\tau\mu} & m^{\mathrm{D}}_{\tau\tau} \end{pmatrix} \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} + \text{H.c.}$$

L_e, L_μ, L_τ are not conserved

L is conserved: $\quad L(\nu_{\alpha R}) = L(\nu_{\beta L}) \Rightarrow |\Delta L| = 0$

- Leptonic Weak Charged Current is invariant under the global U(1) gauge transformations

$$\ell_{\alpha L} \to e^{i\varphi_\alpha} \ell_{\alpha L} \qquad \nu_{\alpha L} \to e^{i\varphi_\alpha} \nu_{\alpha L} \qquad (\alpha = e, \mu, \tau)$$

- If neutrinos are massless (SM), Noether's theorem implies that there is, for each flavor, a conserved current:

$$j_\alpha^\rho = \overline{\nu_{\alpha L}} \gamma^\rho \nu_{\alpha L} + \overline{\ell_\alpha} \gamma^\rho \ell_\alpha \qquad\qquad \partial_\rho j_\alpha^\rho = 0$$

and a conserved charge:

$$\mathrm{L}_\alpha = \int \mathrm{d}^3 x \, j_\alpha^0(x) \qquad\qquad \partial_0 \mathrm{L}_\alpha = 0$$

$$
\begin{aligned}
:\mathrm{L}_\alpha: \; = & \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \, 2E} \left[a_{\nu_\alpha}^{(-)\dagger}(p) \, a_{\nu_\alpha}^{(-)}(p) - b_{\nu_\alpha}^{(+)\dagger}(p) \, b_{\nu_\alpha}^{(+)}(p) \right] \\
& + \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \, 2E} \sum_{h=\pm 1} \left[a_{\ell_\alpha}^{(h)\dagger}(p) \, a_{\ell_\alpha}^{(h)}(p) - b_{\ell_\alpha}^{(h)\dagger}(p) \, b_{\ell_\alpha}^{(h)}(p) \right]
\end{aligned}
$$

- Lepton-Higgs Yukawa Lagrangian:

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\sum_{\alpha=e,\mu,\tau} y_\alpha^\ell\, \overline{\ell_{\alpha L}}\, \ell_{\alpha R} + \sum_{k=1}^{3} y_k^\nu\, \overline{\nu_{kL}}\, \nu_{kR}\right] + \mathrm{H.c.}$$

- Mixing: $\nu_{\alpha L} = \sum_{k=1}^{3} U_{\alpha k}\, \nu_{kL} \quad \Longleftrightarrow \quad \nu_{kL} = \sum_{\alpha=e,\mu,\tau} U_{\alpha k}^*\, \nu_{\alpha L}$

$$\mathscr{L}_{H,\mathrm{L}} = -\left(\frac{v+H}{\sqrt{2}}\right)\sum_{\alpha=e,\mu,\tau}\left[y_\alpha^\ell\, \overline{\ell_{\alpha L}}\, \ell_{\alpha R} + \overline{\nu_{\alpha L}} \sum_{k=1}^{3} U_{\alpha k}\, y_k^\nu\, \nu_{kR}\right] + \mathrm{H.c.}$$

- Invariant for

$$\ell_{\alpha L} \to e^{i\varphi_\alpha}\, \ell_{\alpha L}\,, \quad \nu_{\alpha L} \to e^{i\varphi_\alpha}\, \nu_{\alpha L}$$

$$\ell_{\alpha R} \to e^{i\varphi_\alpha}\, \ell_{\alpha R}\,, \quad \sum_{k=1}^{3} U_{\alpha k}\, y_k^\nu\, \nu_{kR} \to e^{i\varphi_\alpha} \sum_{k=1}^{3} U_{\alpha k}\, y_k^\nu\, \nu_{kR}$$

- But kinetic part of neutrino Lagrangian is not invariant

$$\mathscr{L}_{\text{kinetic}}^{(\nu)} = \sum_{\alpha=e,\mu,\tau} \overline{\nu_{\alpha L}} i \partial\!\!\!/ \nu_{\alpha L} + \sum_{k=1}^{3} \overline{\nu_{kR}} i \partial\!\!\!/ \nu_{kR}$$

because $\sum_{k=1}^{3} U_{\alpha k}\, y_k^\nu\, \nu_{kR}$ is not a unitary combination of the ν_{kR}'s

Total Lepton Number

- Dirac neutrino masses violate conservation of Flavor Lepton Numbers
- Total Lepton Number is conserved, because Lagrangian is invariant under the global $\mathrm{U}(1)$ gauge transformations

$$\nu_{kL} \to e^{i\varphi}\,\nu_{kL}\,, \qquad \nu_{kR} \to e^{i\varphi}\,\nu_{kR} \qquad (k=1,2,3)$$
$$\ell_{\alpha L} \to e^{i\varphi}\,\ell_{\alpha L}\,, \qquad \ell_{\alpha R} \to e^{i\varphi}\,\ell_{\alpha R} \qquad (\alpha=e,\mu,\tau)$$

- From Noether's theorem:

$$j^\rho = \sum_{k=1}^{3} \overline{\nu_k}\,\gamma^\rho\,\nu_k + \sum_{\alpha=e,\mu,\tau} \overline{\ell_\alpha}\,\gamma^\rho\,\ell_\alpha \qquad \partial_\rho j^\rho = 0$$

Conserved charge: $\mathrm{L}_\alpha = \int \mathrm{d}^3x\, j^0_\alpha(x) \qquad \partial_0 \mathrm{L}_\alpha = 0$

$$:\!\mathrm{L}\!: \;= \sum_{k=1}^{3} \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E} \sum_{h=\pm1} \left[a^{(h)\dagger}_{\nu_k}(p)\, a^{(h)}_{\nu_k}(p) - b^{(h)\dagger}_{\nu_k}(p)\, b^{(h)}_{\nu_k}(p) \right]$$
$$+ \sum_{\alpha=e,\mu,\tau} \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2E} \sum_{h=\pm1} \left[a^{(h)\dagger}_{\ell_\alpha}(p)\, a^{(h)}_{\ell_\alpha}(p) - b^{(h)\dagger}_{\ell_\alpha}(p)\, b^{(h)}_{\ell_\alpha}(p) \right]$$

Mixing Matrix

- Leptonic Weak Charged Current: $j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\boldsymbol{\ell}_L$

- $U = V_L^{\ell\dagger}\,V_L^{\nu} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \equiv \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$

- Unitary $N{\times}N$ matrix depends on N^2 independent real parameters

$$N = 3 \qquad \Longrightarrow \qquad \begin{array}{ll} \dfrac{N\,(N-1)}{2} = 3 & \qquad \text{Mixing Angles} \\ \dfrac{N\,(N+1)}{2} = 6 & \qquad \text{Phases} \end{array}$$

- Not all phases are physical observables

- Only physical effect of mixing matrix occurs through its presence in the Leptonic Weak Charged Current

- Weak Charged Current: $j^{\rho}_{W,\mathrm{L}} = 2 \sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}} \, U^{*}_{\alpha k} \, \gamma^{\rho} \, \ell_{\alpha L}$
- Apart from the Weak Charged Current, the Lagrangian is invariant under the global phase transformations (6 arbitrary phases)

$$\nu_k \to e^{i\varphi_k} \, \nu_k \quad (k = 1,2,3) \, , \qquad \ell_\alpha \to e^{i\varphi_\alpha} \, \ell_\alpha \quad (\alpha = e, \mu, \tau)$$

- Performing this transformation, the Charged Current becomes

$$j^{\rho}_{W,\mathrm{L}} = 2 \sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}} \, e^{-i\varphi_k} \, U^{*}_{\alpha k} \, e^{i\varphi_\alpha} \, \gamma^{\rho} \, \ell_{\alpha L}$$

$$j^{\rho}_{W,\mathrm{L}} = 2 \, \underbrace{e^{-i(\varphi_1-\varphi_e)}}_{1} \sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}} \, \underbrace{e^{-i(\varphi_k-\varphi_1)}}_{2} \, U^{*}_{\alpha k} \, \underbrace{e^{i(\varphi_\alpha-\varphi_e)}}_{2} \, \gamma^{\rho} \, \ell_{\alpha L}$$

- There are 5 arbitrary phases of the fields that can be chosen to eliminate 5 of the 6 phases of the mixing matrix
- 5 and not 6 phases of the mixing matrix can be eliminated because a common rephasing of all the fields leaves the Charged Current invariant $\iff$ conservation of Total Lepton Number.

- The mixing matrix contains 1 Physical Phase.

- It is convenient to express the 3×3 unitary mixing matrix only in terms of the four physical parameters:

3 Mixing Angles and 1 Phase

Standard Parameterization of Mixing Matrix

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}$$

$c_{ab} \equiv \cos \vartheta_{ab}$ $\quad$ $s_{ab} \equiv \sin \vartheta_{ab}$ $\quad$ $0 \leq \vartheta_{ab} \leq \dfrac{\pi}{2}$ $\quad$ $0 \leq \delta_{13} \leq 2\pi$

3 Mixing Angles ϑ_{12}, ϑ_{23}, ϑ_{13} and 1 Phase δ_{13}

Standard Parameterization

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Phase Convention

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta_{23}} \\ 0 & -s_{23} e^{-i\delta_{13}} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of Different Parameterization

$$U = \begin{pmatrix} c'_{12} & s'_{12} e^{-i\delta'_{12}} & 0 \\ -s'_{12} e^{i\delta'_{12}} & c'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{pmatrix} \begin{pmatrix} c'_{13} & 0 & s'_{13} \\ 0 & 1 & 0 \\ -s'_{13} & 0 & c'_{13} \end{pmatrix}$$

CP Violation

- $U \neq U^* \quad \Longrightarrow \quad$ CP Violation
- General conditions for CP violation (14 conditions):
  1. No two charged leptons or two neutrinos are degenerate in mass (6 conditions)
  2. No mixing angle is equal to 0 or $\pi/2$ (6 conditions)
  3. The physical phase is different from 0 or π (2 conditions)
- These 14 conditions are combined into the single condition $\det C \neq 0$

$$C = -i\,[M'^{\nu}\,M'^{\nu\dagger}, M'^{\ell}\,M'^{\ell\dagger}]$$

$$\det C = -2\,J\left(m_{\nu_2}^2 - m_{\nu_1}^2\right)\left(m_{\nu_3}^2 - m_{\nu_1}^2\right)\left(m_{\nu_3}^2 - m_{\nu_2}^2\right)\left(m_{\mu}^2 - m_{e}^2\right)\left(m_{\tau}^2 - m_{e}^2\right)\left(m_{\tau}^2 - m_{\mu}^2\right)$$

- Jarlskog rephasing invariant: $J = c_{12}s_{12}c_{23}s_{23}c_{13}^2s_{13}\sin\delta_{13}$ (stand. par.)

[C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039, Z. Phys. C 29 (1985) 491]

[O. W. Greenberg, Phys. Rev. D 32 (1985) 1841]

[I. Dunietz, O. W. Greenberg, Dan-di Wu, Phys. Rev. Lett. 55 (1985) 2935]

Example: $\vartheta_{12} = 0$

$$U = R_{23} R_{13} W_{12}$$

$$W_{12} = \begin{pmatrix} \cos\vartheta_{12} & \sin\vartheta_{12} e^{-i\delta_{12}} & 0 \\ -\sin\vartheta_{12} e^{-i\delta_{12}} & \cos\vartheta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vartheta_{12} = 0 \qquad \Longrightarrow \qquad W_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}$$

real mixing matrix $\qquad U = R_{23} R_{13}$

Example: $\vartheta_{13} = \pi/2$

$$U = R_{23} W_{13} R_{12}$$

$$W_{13} = \begin{pmatrix} \cos\vartheta_{13} & 0 & \sin\vartheta_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -\sin\vartheta_{13} e^{i\delta_{13}} & 0 & \cos\vartheta_{13} \end{pmatrix}$$

$$\vartheta_{13} = \pi/2 \qquad \Longrightarrow \qquad W_{13} = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -e^{i\delta_{13}} & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}e^{i\delta_{13}} & 0 \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}e^{i\delta_{13}} & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ |U_{\mu 1}| e^{i\lambda_{\mu 1}} & |U_{\mu 2}| e^{i\lambda_{\mu 2}} & 0 \\ |U_{\tau 1}| e^{i\lambda_{\tau 1}} & |U_{\tau 2}| e^{i\lambda_{\tau 2}} & 0 \end{pmatrix}$$

$$\lambda_{\mu 1} - \lambda_{\mu 2} = \lambda_{\tau 1} - \lambda_{\tau 2} \pm \pi \qquad \lambda_{\tau 1} - \lambda_{\mu 1} = \lambda_{\tau 2} - \lambda_{\mu 2} \pm \pi$$

$$\nu_k \to e^{i\varphi_k} \nu_k \quad (k = 1, 2, 3), \qquad \ell_\alpha \to e^{i\varphi_\alpha} \ell_\alpha \quad (\alpha = e, \mu, \tau)$$

$$U \to \begin{pmatrix} e^{-i\varphi_e} & 0 & 0 \\ 0 & e^{-i\varphi_\mu} & 0 \\ 0 & 0 & e^{-i\varphi_\tau} \end{pmatrix} \begin{pmatrix} 0 & 0 & e^{-i\delta_{13}} \\ |U_{\mu 1}| e^{i\lambda_{\mu 1}} & |U_{\mu 2}| e^{i\lambda_{\mu 2}} & 0 \\ |U_{\tau 1}| e^{i\lambda_{\tau 1}} & |U_{\tau 2}| e^{i\lambda_{\tau 2}} & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & e^{i(-\delta_{13} - \varphi_e + \varphi_3)} \\ |U_{\mu 1}| e^{i(\lambda_{\mu 1} - \varphi_\mu + \varphi_1)} & |U_{\mu 2}| e^{i(\lambda_{\mu 2} - \varphi_\mu + \varphi_2)} & 0 \\ |U_{\tau 1}| e^{i(\lambda_{\tau 1} - \varphi_\tau + \varphi_1)} & |U_{\tau 2}| e^{i(\lambda_{\tau 2} - \varphi_\tau + \varphi_2)} & 0 \end{pmatrix}$$

$$\varphi_1 = 0 \qquad \varphi_\mu = \lambda_{\mu 1} \qquad \varphi_\tau = \lambda_{\tau 1} \qquad \varphi_2 = \varphi_\mu - \lambda_{\mu 2} = \lambda_{\mu 1} - \lambda_{\mu 2}$$

$$\varphi_2 = \varphi_\tau - \lambda_{\tau 2} \pm \pi = \lambda_{\tau 1} - \lambda_{\tau 2} \pm \pi = \lambda_{\mu 1} - \lambda_{\mu 2}$$ OK!

$$U = \begin{pmatrix} 0 & 0 & \pm 1 \\ |U_{\mu 1}| & |U_{\mu 2}| & 0 \\ |U_{\tau 1}| & -|U_{\tau 2}| & 0 \end{pmatrix}$$

Example: $m_{\nu_2} = m_{\nu_3}$

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\ell_L$$

$$U = R_{12}R_{13}W_{23} \qquad \Longrightarrow \qquad j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,W_{23}^{\dagger}R_{13}^{\dagger}R_{12}^{\dagger}\,\gamma^{\rho}\,\ell_L$$

$$W_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_{23} & \sin\vartheta_{23}e^{-i\delta_{23}} \\ 0 & -\sin\vartheta_{23}e^{-i\delta_{23}} & \cos\vartheta_{23} \end{pmatrix}$$

$$W_{23}\mathbf{n}_L = \mathbf{n}'_L \qquad R_{12}R_{13} = U' \qquad \Longrightarrow \qquad j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}'_L}\,U'^{\dagger}\,\gamma^{\rho}\,\ell_L$$

ν_2 and ν_3 are indistinguishable

drop the prime $\Longrightarrow$ $j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\ell_L$

real mixing matrix $U = R_{12}R_{13}$

Jarlskog Rephasing Invariant

- Simplest rephasing invariants: $|U_{\alpha k}| = U_{\alpha k} U_{\alpha k}^{*}$, $U_{\alpha k} U_{\alpha j}^{*} U_{\beta k}^{*} U_{\beta j}$

$$\mathfrak{Im}\left[U_{\alpha k} U_{\alpha j}^{*} U_{\beta k}^{*} U_{\beta j}\right] = \pm J$$

$$J = \mathfrak{Im}\left[U_{e2} U_{e3}^{*} U_{\mu 2}^{*} U_{\mu 3}\right] = \mathfrak{Im} \begin{pmatrix} \cdot & \circ & \times \\ \cdot & \times & \circ \\ \cdot & \cdot & \cdot \end{pmatrix}$$

- In standard parameterization:

$$\begin{aligned} J &= c_{12} s_{12} c_{23} s_{23} c_{13}^{2} s_{13} \sin\delta_{13} \\ &= \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{23} \cos\vartheta_{13} \sin 2\vartheta_{13} \sin\delta_{13} \end{aligned}$$

- Jarlskog invariant is useful for quantifying CP violation in a parameterization-independent way
- All measurable CP-violation effects depend on J.

Maximal CP Violation

- Maximal CP violation is defined as the case in which $|J|$ has its maximum possible value

$$|J|_{\max} = \frac{1}{6\sqrt{3}}$$

- In the standard parameterization it is obtained for

$$\vartheta_{12} = \vartheta_{23} = \pi/4 \,, \quad s_{13} = 1/\sqrt{3} \,, \quad \sin\delta_{13} = \pm 1$$

- This case is called Trimaximal Mixing. All the absolute values of the elements of the mixing matrix are equal to $1/\sqrt{3}$:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \mp\frac{i}{\sqrt{3}} \\ -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} \mp \frac{i}{2\sqrt{3}} & -\frac{1}{2} \mp \frac{i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \mp i \\ -e^{\pm i\pi/6} & e^{\mp i\pi/6} & 1 \\ e^{\mp i\pi/6} & -e^{\pm i\pi/6} & 1 \end{pmatrix}$$

GIM Mechanism

[S.L. Glashow, J. Iliopoulos, L. Maiani, Phys. Rev. D 2 (1970) 1285]

- The unitarity of V_L^ℓ, V_R^ℓ and V_L^ν implies that the expression of the neutral weak current in terms of the lepton fields with definite masses is the same as that in terms of the primed lepton fields:

$$\begin{aligned} j^\rho_{Z,\mathrm{L}} =& 2\, g_L^\nu\, \overline{\boldsymbol{\nu}_L'}\, \gamma^\rho\, \boldsymbol{\nu}_L' + 2\, g_L^l\, \overline{\boldsymbol{\ell}_L'}\, \gamma^\rho \boldsymbol{\ell}_L' + 2\, g_R^l\, \overline{\boldsymbol{\ell}_R'}\, \gamma^\rho \boldsymbol{\ell}_R' \\ =& 2\, g_L^\nu\, \overline{\mathbf{n}_L}\, V_L^{\nu\dagger}\, \gamma^\rho\, V_L^\nu\, \mathbf{n}_L + 2\, g_L^l\, \overline{\boldsymbol{\ell}_L}\, V_L^{\ell\dagger}\, \gamma^\rho\, V_L^\ell\, \boldsymbol{\ell}_L + 2\, g_R^l\, \overline{\boldsymbol{\ell}_R}\, V_R^{\ell\dagger}\, \gamma^\rho\, V_R^\ell\, \boldsymbol{\ell}_R \\ =& 2\, g_L^\nu\, \overline{\mathbf{n}_L}\, \gamma^\rho\, \mathbf{n}_L + 2\, g_L^l\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho \boldsymbol{\ell}_L + 2\, g_R^l\, \overline{\boldsymbol{\ell}_R}\, \gamma^\rho \boldsymbol{\ell}_R \end{aligned}$$

- The unitarity of U implies the same expression for the neutral weak current in terms of the flavor neutrino fields $\boldsymbol{\nu}_L = U\, \mathbf{n}_L$:

$$\begin{aligned} j^\rho_{Z,\mathrm{L}} &= 2\, g_L^\nu\, \overline{\boldsymbol{\nu}_L}\, U\, \gamma^\rho\, U^\dagger\, \boldsymbol{\nu}_L + 2\, g_L^l\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho \boldsymbol{\ell}_L + 2\, g_R^l\, \overline{\boldsymbol{\ell}_R}\, \gamma^\rho \boldsymbol{\ell}_R \\ &= 2\, g_L^\nu\, \overline{\boldsymbol{\nu}_L}\, \gamma^\rho\, \boldsymbol{\nu}_L + 2\, g_L^l\, \overline{\boldsymbol{\ell}_L}\, \gamma^\rho \boldsymbol{\ell}_L + 2\, g_R^l\, \overline{\boldsymbol{\ell}_R}\, \gamma^\rho \boldsymbol{\ell}_R \end{aligned}$$

Lepton Numbers Violating Processes

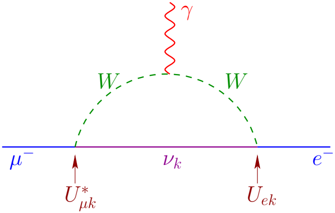
Dirac mass term allows L_e, L_μ, L_τ violating processes

Example: $\mu^\pm \to e^\pm + \gamma\,, \qquad \mu^\pm \to e^\pm + e^+ + e^-$

$$\boxed{\mu^- \to e^- + \gamma}$$

$\sum_k U_{\mu k}^* U_{ek} = 0 \Longrightarrow$ only part of ν_k propagator $\propto m_k$ contributes

$$\Gamma = \frac{G_F m_\mu^5}{192\pi^3} \underbrace{\frac{3\alpha}{32\pi} \left| \sum_k U_{\mu k}^* U_{ek} \frac{m_k^2}{m_W^2} \right|^2}_{\text{BR}}$$



Suppression factor: $\frac{m_k}{m_W} \lesssim 10^{-11}$ for $m_k \lesssim 1\,\text{eV}$

$(\text{BR})_{\text{the}} \lesssim 10^{-47} \qquad (\text{BR})_{\text{exp}} \lesssim 10^{-11}$

Majorana Neutrino Masses and Mixing

Two-Component Theory of a Massless Neutrino

[L. Landau, Nucl. Phys. 3 (1957) 127], [T.D. Lee, C.N. Yang, Phys. Rev. 105 (1957) 1671], [A. Salam, Nuovo Cim. 5 (1957) 299]

- Dirac Equation: $(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$
- Chiral decomposition of a Fermion Field: $\psi = \psi_L + \psi_R$
- Equations for the Chiral components are coupled by mass:

$$i\gamma^{\mu}\partial_{\mu}\psi_L = m\,\psi_R$$
$$i\gamma^{\mu}\partial_{\mu}\psi_R = m\,\psi_L$$

- They are decoupled for a massless fermion: Weyl Equations (1929)

$$i\gamma^{\mu}\partial_{\mu}\psi_L = 0$$
$$i\gamma^{\mu}\partial_{\mu}\psi_R = 0$$

- A massless fermion can be described by a single chiral field ψ_L or ψ_R (Weyl Spinor).

- ψ_L and ψ_R have only two independent components: in the chiral representation

$$\psi_L = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix} \qquad \psi_R = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ 0 \\ 0 \end{pmatrix}$$

- The possibility to describe a physical particle with a Weyl spinor was rejected by Pauli in 1933 because it leads to parity violation ($\psi_L \overset{P}{\rightleftharpoons} \psi_R$)

- The discovery of parity violation in 1956-57 invalidated Pauli's reasoning, opening the possibility to describe massless particles with Weyl spinor fields $\Longrightarrow$ Two-component Theory of a Massless Neutrino (1957)

- $V - A$ Charged-Current Weak Interactions $\Longrightarrow$ ν_L

- In the 1960s, the Two-component Theory of a Massless Neutrino was incorporated in the SM through the assumption of the absence of ν_R

Majorana Equation

- Can a two-component spinor describe a massive fermion? Yes! (E. Majorana, 1937)
- Trick: ψ_R and ψ_L are not independent: $\boxed{\psi_R = \mathcal{C}\,\overline{\psi_L}^T}$
- $\mathcal{C}\,\overline{\psi_L}^T$ is right-handed: $P_R \mathcal{C}\,\overline{\psi_L}^T = \mathcal{C}\,\overline{\psi_L}^T$ $\quad (\mathcal{C}\,\gamma_\mu^T\,\mathcal{C}^{-1} = -\gamma_\mu)$
- Majorana Equation: $\boxed{i\gamma^\mu \partial_\mu \psi_L = m\,\mathcal{C}\,\overline{\psi_L}^T}$
- Majorana Field: $\psi = \psi_L + \psi_R = \psi_L + \mathcal{C}\,\overline{\psi_L}^T$
- Majorana Condition: $\boxed{\psi = \mathcal{C}\,\overline{\psi}^T = \psi^C}$
- Only two independent components: $\psi = \begin{pmatrix} i\sigma^2 \chi_L^* \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{L2}^* \\ -\chi_{L1}^* \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$

- $\psi = \psi^C$ implies the equality of particle and antiparticle

- Only neutral fermions can be Majorana particles

- For a Majorana field, the electromagnetic current vanishes identically:

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi^C}\gamma^\mu\psi^C = -\psi^T\mathcal{C}^\dagger\gamma^\mu\mathcal{C}\overline{\psi}^T = \overline{\psi}\mathcal{C}\gamma^{\mu\,T}\mathcal{C}^\dagger\psi = -\overline{\psi}\gamma^\mu\psi = 0$$

Majorana Lagrangian

Dirac Lagrangian

$$
\begin{aligned}
\mathscr{L}^{\mathrm{D}} &= \overline{\nu}\,(i\partial\!\!\!/ - m)\,\nu \\
&= \overline{\nu_L} i\partial\!\!\!/ \nu_L + \overline{\nu_R} i\partial\!\!\!/ \nu_R - m\,(\overline{\nu_R}\,\nu_L + \overline{\nu_L}\,\nu_R)
\end{aligned}
$$

$$\nu_R \quad \to \quad \nu_L^C = \mathcal{C}\,\overline{\nu_L}^T$$

$$\frac{1}{2}\mathscr{L}^{\mathrm{D}} \quad \to \quad \overline{\nu_L}\, i\partial\!\!\!/\, \nu_L - \frac{m}{2}\left(-\nu_L^T\,\mathcal{C}^\dagger\,\nu_L + \overline{\nu_L}\,\mathcal{C}\,\overline{\nu_L}^T\right)$$

Majorana Lagrangian

$$
\begin{aligned}
\mathscr{L}^{\mathrm{M}} &= \overline{\nu_L}\, i\partial\!\!\!/\, \nu_L - \frac{m}{2}\left(-\nu_L^T\,\mathcal{C}^\dagger\,\nu_L + \overline{\nu_L}\,\mathcal{C}\,\overline{\nu_L}^T\right) \\
&= \overline{\nu_L}\, i\partial\!\!\!/\, \nu_L - \frac{m}{2}\left(\overline{\nu_L^C}\,\nu_L + \overline{\nu_L}\,\nu_L^C\right)
\end{aligned}
$$

- Majorana Field: $\nu = \nu_L + \nu_L^C$

- Majorana Condition: $\nu^C = \nu$

- Majorana Lagrangian: $\mathscr{L}^{\mathrm{M}} = \dfrac{1}{2}\,\overline{\nu}\,(i\partial\!\!\!/ - m)\,\nu$

- The factor $1/2$ distinguishes the Majorana Lagrangian from the Dirac Lagrangian

- Quantized Dirac Neutrino Field:

$$\nu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\, 2E} \sum_{h=\pm1} \left[a^{(h)}(p)\, u^{(h)}(p)\, e^{-ip\cdot x} + b^{(h)\,\dagger}(p)\, v^{(h)}(p)\, e^{ip\cdot x} \right]$$

- Quantized Majorana Neutrino Field $[b^{(h)}(p) = a^{(h)}(p)]$

$$\nu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\, 2E} \sum_{h=\pm1} \left[a^{(h)}(p)\, u^{(h)}(p)\, e^{-ip\cdot x} + a^{(h)\dagger}(p)\, v^{(h)}(p)\, e^{ip\cdot x} \right]$$

- A Majorana field has half the degrees of freedom of a Dirac field

Majorana Antineutrino?

- A Majorana neutrino is the same as a Majorana antineutrino
- Neutrino interactions are described by the CC and NC Lagrangians

$$\mathscr{L}_{\text{I,L}}^{\text{CC}} = -\frac{g}{\sqrt{2}} \left(\overline{\nu_L} \, \gamma^\mu \, \ell_L \, W_\mu + \overline{\ell_L} \, \gamma^\mu \, \nu_L \, W_\mu^\dagger \right)$$

$$\mathscr{L}_{\text{I},\nu}^{\text{NC}} = -\frac{g}{2 \cos \vartheta_{\text{W}}} \, \overline{\nu_L} \, \gamma^\mu \, \nu_L \, Z_\mu$$

- In practice, since detectable neutrinos are always ultrarelativistic, the neutrino mass can be neglected in interactions

- In interaction amplitudes we neglect corrections of order m/E

- Dirac: $\begin{cases} \nu_L \begin{cases} \text{destroys left-handed neutrinos} \\ \text{creates right-handed antineutrinos} \end{cases} \\ \overline{\nu_L} \begin{cases} \text{destroys right-handed antineutrinos} \\ \text{creates left-handed neutrinos} \end{cases} \end{cases}$

- Majorana: $\begin{cases} \nu_L \begin{cases} \text{destroys left-handed neutrinos} \\ \text{creates right-handed neutrinos} \end{cases} \\ \overline{\nu_L} \begin{cases} \text{destroys right-handed neutrinos} \\ \text{creates left-handed neutrinos} \end{cases} \end{cases}$

- Common definitions:

  Majorana neutrino with negative helicity $\equiv$ neutrino

  Majorana neutrino with positive helicity $\equiv$ antineutrino

Lepton Number

$$\cancel{L = +1} \quad \longleftarrow \quad \boxed{\nu = \nu^C} \quad \longrightarrow \quad \cancel{L = -1}$$

$$\nu_L \quad \Longrightarrow \quad L = +1 \qquad\qquad \nu_L^C \quad \Longrightarrow \quad L = -1$$

$$\mathscr{L}^{\mathrm{M}} = \overline{\nu_L}\, i\partial\!\!\!/\, \nu_L - \frac{m}{2}\left(\overline{\nu_L^C}\,\nu_L + \overline{\nu_L}\,\nu_L^C\right)$$

Total Lepton Number is not conserved: $\boxed{\Delta L = \pm 2}$

Best process to find violation of Total Lepton Number:

Neutrinoless Double-β Decay

$$\mathcal{N}(A,Z) \to \mathcal{N}(A,Z+2) + 2e^- + \cancel{2\bar{\nu}_e} \qquad (\beta\beta_{0\nu}^-)$$

$$\mathcal{N}(A,Z) \to \mathcal{N}(A,Z-2) + 2e^+ + \cancel{2\nu_e} \qquad (\beta\beta_{0\nu}^+)$$

CP Symmetry

- Under a CP transformation

$$\begin{aligned}
\mathrm{U}_{\mathrm{CP}}\nu_L(x)\mathrm{U}_{\mathrm{CP}}^{-1} &= \xi_\nu^{\mathrm{CP}}\,\gamma^0\,\nu_L^C(x_{\mathrm{P}})\\
\mathrm{U}_{\mathrm{CP}}\nu_L^C(x)\mathrm{U}_{\mathrm{CP}}^{-1} &= -\xi_\nu^{\mathrm{CP}*}\,\gamma^0\,\nu_L(x_{\mathrm{P}})\\
\mathrm{U}_{\mathrm{CP}}\overline{\nu_L}(x)\mathrm{U}_{\mathrm{CP}}^{-1} &= \xi_\nu^{\mathrm{CP}*}\,\overline{\nu_L^C}(x_{\mathrm{P}})\,\gamma^0\\
\mathrm{U}_{\mathrm{CP}}\overline{\nu_L^C}(x)\mathrm{U}_{\mathrm{CP}}^{-1} &= -\xi_\nu^{\mathrm{CP}}\,\overline{\nu_L}(x_{\mathrm{P}})\,\gamma^0
\end{aligned}$$

with $|\xi_\nu^{\mathrm{CP}}|^2 = 1$, $x^\mu = (x^0, \vec{x})$, and $x_{\mathrm{P}}^\mu = (x^0, -\vec{x})$

- The theory is CP-symmetric if there are values of the phase ξ_ν^{CP} such that the Lagrangian transforms as

$$\mathrm{U}_{\mathrm{CP}}\mathscr{L}(x)\mathrm{U}_{\mathrm{CP}}^{-1} = \mathscr{L}(x_{\mathrm{P}})$$

in order to keep invariant the action $I = \int \mathrm{d}^4x\,\mathscr{L}(x)$

- The Majorana Mass Term

$$\mathscr{L}^{\mathrm{M}}_{\mathrm{mass}}(x) = -\frac{1}{2}\, m \left[\overline{\nu_L^C}(x)\, \nu_L(x) + \overline{\nu_L}(x)\, \nu_L^C(x) \right]$$

transforms as

$$\mathrm{U}_{\mathrm{CP}} \mathscr{L}^{\mathrm{M}}_{\mathrm{mass}}(x) \mathrm{U}_{\mathrm{CP}}^{-1} = -\frac{1}{2}\, m \Big[-(\xi_\nu^{\mathrm{CP}})^2\, \overline{\nu_L}(x_{\mathrm{P}})\, \nu_L^C(x_{\mathrm{P}}) - (\xi_\nu^{\mathrm{CP}*})^2\, \overline{\nu_L^C}(x_{\mathrm{P}})\, \nu_L(x_{\mathrm{P}}) \Big]$$

- $\mathrm{U}_{\mathrm{CP}} \mathscr{L}^{\mathrm{M}}_{\mathrm{mass}}(x) \mathrm{U}_{\mathrm{CP}}^{-1} = \mathscr{L}^{\mathrm{M}}_{\mathrm{mass}}(x_{\mathrm{P}})$ for $\boxed{\xi_\nu^{\mathrm{CP}} = \pm i}$

- The one-generation Majorana theory is CP-symmetric

- The Majorana case is different from the Dirac case, in which the CP phase ξ_ν^{CP} is arbitrary

No Majorana Neutrino Mass in the SM

- Majorana Mass Term $\propto \left[\nu_L^T \, \mathcal{C}^\dagger \, \nu_L - \overline{\nu_L} \, \mathcal{C} \, \overline{\nu_L}^T\right]$ involves only the neutrino left-handed chiral field ν_L, which is present in the SM (one for each lepton generation)
- Eigenvalues of the weak isospin I, of its third component I_3, of the hypercharge Y and of the charge Q of the lepton and Higgs multiplets:

	I	I_3	Y	$Q = I_3 + \frac{Y}{2}$
lepton doublet $L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	-1	0 -1
lepton singlet ℓ_R	0	0	-2	-1
Higgs doublet $\Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_0(x) \end{pmatrix}$	$1/2$	$1/2$ $-1/2$	$+1$	1 0

- $\nu_L^T \, \mathcal{C}^\dagger \, \nu_L$ has $I_3 = 1$ and $Y = -2 \Longrightarrow$ needed Higgs triplet with $Y = 2$

Effective Majorana Mass

- Dimensional analysis: Fermion Field $\sim [E]^{3/2}$ Boson Field $\sim [E]$
- Dimensionless action: $I = \int d^4x \, \mathscr{L}(x) \Longrightarrow \mathscr{L}(x) \sim [E]^4$
- Kinetic terms: $\overline{\psi} i \partial\!\!\!/ \psi \sim [E]^4$, $(\partial_\mu \phi)^\dagger \partial^\mu \phi \sim [E]^4$
- Mass terms: $m \overline{\psi} \psi \sim [E]^4$, $m^2 \phi^\dagger \phi \sim [E]^4$
- CC weak interaction: $g \, \overline{\nu_L} \gamma^\rho \ell_L W_\rho \sim [E]^4$
- Yukawa couplings: $y \, \overline{L_L} \Phi \ell_R \sim [E]^4$
- Product of fields $\mathscr{O}_d$ with energy dimension $d \equiv$ dim-d operator
- $\mathscr{L}_{(\mathscr{O}_d)} = C_{(\mathscr{O}_d)} \mathscr{O}_d \quad \Longrightarrow \quad C_{(\mathscr{O}_d)} \sim [E]^{4-d}$
- $\mathscr{O}_{d>4}$ are not renormalizable

- SM Lagrangian includes all $\mathscr{O}_{d \leq 4}$ invariant under $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$
- SM cannot be considered as the final theory of everything
- SM is an effective low-energy theory
- It is likely that SM is the low-energy product of the symmetry breaking of a high-energy unified theory
- It is plausible that at low-energy there are effective non-renormalizable $\mathscr{O}_{d>4}$ [S. Weinberg, Phys. Rev. Lett. 43 (1979) 1566]
- All $\mathscr{O}_d$ must respect $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, because they are generated by the high-energy theory which must include the gauge symmetries of the SM in order to be effectively reduced to the SM at low energies

- $\mathscr{O}_{d>4}$ is suppressed by a coefficient $\mathcal{M}^{4-d}$, where $\mathcal{M}$ is a heavy mass characteristic of the symmetry breaking scale of the high-energy unified theory:

$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{g_5}{\mathcal{M}} \mathscr{O}_5 + \frac{g_6}{\mathcal{M}^2} \mathscr{O}_6 + \ldots$$

- Analogy with $\mathscr{L}_{\text{eff}}^{(\text{CC})} \propto G_{\text{F}} \left(\overline{\nu_{eL}} \gamma^{\rho} e_L\right) \left(\overline{e_L} \gamma_{\rho} \nu_{eL}\right) + \ldots$

$$\mathscr{O}_6 \to \left(\overline{\nu_{eL}} \gamma^{\rho} e_L\right) \left(\overline{e_L} \gamma_{\rho} \nu_{eL}\right) + \ldots \qquad \frac{g_6}{\mathcal{M}^2} \to \frac{G_{\text{F}}}{\sqrt{2}} = \frac{g^2}{8 m_W^2}$$

- $\mathcal{M}^{4-d}$ is a strong suppression factor which limits the observability of the low-energy effects of the new physics beyond the SM

- The difficulty to observe the effects of the effective low-energy non-renormalizable operators increase rapidly with their dimensionality

- $\mathscr{O}_5 \quad \Longrightarrow \quad$ Majorana neutrino masses (Lepton number violation)

- $\mathscr{O}_6 \quad \Longrightarrow \quad$ Baryon number violation (proton decay)

- Only one dim-5 operator:

$$
\begin{aligned}
\mathscr{O}_5 &= (L_L^T\,\sigma_2\,\Phi)\,\mathcal{C}^\dagger\,(\Phi^T\,\sigma_2\,L_L) + \text{H.c.} \\
&= \frac{1}{2}\,(L_L^T\,\mathcal{C}^\dagger\,\sigma_2\,\vec{\sigma}\,L_L)\cdot(\Phi^T\,\sigma_2\,\vec{\sigma}\,\Phi) + \text{H.c.}
\end{aligned}
$$

$$
\mathscr{L}_5 = \frac{g_5}{2\mathcal{M}}\,(L_L^T\,\mathcal{C}^\dagger\,\sigma_2\,\vec{\sigma}\,L_L)\cdot(\Phi^T\,\sigma_2\,\vec{\sigma}\,\Phi) + \text{H.c.}
$$

- Electroweak Symmetry Breaking: $\Phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \xrightarrow[\text{Breaking}]{\text{Symmetry}} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$

- $\mathscr{L}_5 \xrightarrow[\text{Breaking}]{\text{Symmetry}} \mathscr{L}_{\text{mass}}^{\text{M}} = \frac{1}{2}\,\frac{g_5\,v^2}{\mathcal{M}}\,\nu_L^T\,\mathcal{C}^\dagger\,\nu_L + \text{H.c.} \quad \Longrightarrow \quad \boxed{m = \frac{g_5\,v^2}{\mathcal{M}}}$

- The study of Majorana neutrino masses provides the most accessible low-energy window on new physics beyond the SM

- $m \propto \frac{v^2}{\mathcal{M}} \propto \frac{m_{\text{D}}^2}{\mathcal{M}}$ natural explanation of smallness of neutrino masses (special case: See-Saw Mechanism)

- Example: $m_{\text{D}} \sim v \sim 10^2\,\text{GeV}$ and $\mathcal{M} \sim 10^{15}\,\text{GeV} \implies m \sim 10^{-2}\,\text{eV}$

Mixing of Three Majorana Neutrinos

- $\nu'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix}$

$$\mathscr{L}_{\text{mass}}^{\text{M}} = \frac{1}{2} \nu_L'^T \, \mathcal{C}^\dagger \, M^L \, \nu'_L + \text{H.c.}$$

$$= \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \nu_{\alpha L}'^T \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\beta L} + \text{H.c.}$$

- In general, the matrix M^L is a complex symmetric matrix

$$\sum_{\alpha,\beta} \nu_{\alpha L}'^T \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\beta L} = - \sum_{\alpha,\beta} \nu_{\beta L}'^T \, M^L_{\alpha\beta} \, (\mathcal{C}^\dagger)^T \, \nu'_{\alpha L}$$

$$= \sum_{\alpha,\beta} \nu_{\beta L}'^T \, \mathcal{C}^\dagger \, M^L_{\alpha\beta} \, \nu'_{\alpha L} = \sum_{\alpha,\beta} \nu_{\alpha L}'^T \, \mathcal{C}^\dagger \, M^L_{\beta\alpha} \, \nu'_{\beta L}$$

$$M^L_{\alpha\beta} = M^L_{\beta\alpha} \quad \Longleftrightarrow \quad M^L = M^{L\,T}$$

- $\mathscr{L}^{\text{M}}_{\text{mass}} = \frac{1}{2} \boldsymbol{\nu}_L'^T \, \mathcal{C}^\dagger \, M^L \, \boldsymbol{\nu}_L' + \text{H.c.}$
- $\boldsymbol{\nu}_L' = V_L^\nu \, \mathbf{n}_L \qquad \Longrightarrow \qquad \mathscr{L}^{\text{M}}_{\text{mass}} = \frac{1}{2} \boldsymbol{\nu}_L'^T \, (V_L^\nu)^T \, \mathcal{C}^\dagger \, M^L \, V_L^\nu \, \boldsymbol{\nu}_L' + \text{H.c.}$
- $(V_L^\nu)^T \, M^L \, V_L^\nu = M \,, \qquad M_{kj} = m_k \, \delta_{kj} \qquad (k, j = 1, 2, 3)$
- Left-handed chiral fields with definite mass: $\mathbf{n}_L = V_L^{\nu\dagger} \, \boldsymbol{\nu}_L' = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$

$$
\begin{aligned}
\mathscr{L}^{\text{M}}_{\text{mass}} &= \frac{1}{2} \left(\mathbf{n}_L^T \, \mathcal{C}^\dagger \, M \, \mathbf{n}_L - \overline{\mathbf{n}_L} \, M \, \mathcal{C} \, \mathbf{n}_L^T \right) \\
&= \frac{1}{2} \sum_{k=1}^{3} m_k \left(\nu_{kL}^T \, \mathcal{C}^\dagger \, \nu_{kL} - \overline{\nu_{kL}} \, \mathcal{C} \, \nu_{kL}^T \right)
\end{aligned}
$$

- Majorana fields of massive neutrinos: $\nu_k = \nu_{kL} + \nu_{kL}^C \qquad \boxed{\nu_k^C = \nu_k}$
- $\mathbf{n} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \Longrightarrow \mathscr{L}^{\text{M}} = \frac{1}{2} \sum_{k=1}^{3} \overline{\nu_k} \, (i\partial\!\!\!/ - m_k) \, \nu_k = \frac{1}{2} \, \overline{\mathbf{n}} \, (i\partial\!\!\!/ - M) \, \mathbf{n}$

Mixing Matrix

- Leptonic Weak Charged Current:

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\mathbf{n}_L}\,U^{\dagger}\,\gamma^{\rho}\,\ell_L \qquad \text{with} \qquad U = V_L^{\ell\dagger}\,V_L^{\nu}$$

- Definition of the left-handed flavor neutrino fields:

$$\boldsymbol{\nu}_L = U\,\mathbf{n}_L = V_L^{\ell\dagger}\,\boldsymbol{\nu}'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

- Leptonic Weak Charged Current has the SM form

$$j^{\rho}_{W,\mathrm{L}} = 2\,\overline{\boldsymbol{\nu}_L}\,\gamma^{\rho}\,\boldsymbol{\ell}_L = 2\sum_{\alpha=e,\mu,\tau}\overline{\nu_{\alpha L}}\,\gamma^{\rho}\,\ell_{\alpha L}$$

- Important difference with respect to Dirac case:
Two additional CP-violating phases: Majorana phases

- Majorana Mass Term $\mathscr{L}_{\text{mass}}^{\text{M}} = \frac{1}{2} \sum_{k=1}^{3} m_k \, \nu_{kL}^T \, \mathcal{C}^\dagger \, \nu_{kL} + \text{H.c.}$ is not invariant under the global U(1) gauge transformations

$$\nu_{kL} \to e^{i\varphi_k} \, \nu_{kL} \qquad (k = 1, 2, 3)$$

- Left-handed massive neutrino fields cannot be rephased in order to eliminate two Majorana phases factorized on the right of mixing matrix:

$$\boxed{U = U^{\text{D}} \, D^{\text{M}}} \qquad D^{\text{M}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

- U^{D} is analogous to a Dirac mixing matrix, with one Dirac phase
- Standard parameterization:

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_2} & 0 \\ 0 & 0 & e^{i\lambda_3} \end{pmatrix}$$

- Jarlskog rephasing invariant: $J = c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin\delta_{13}$

- $D^{\mathrm{M}} = \mathrm{diag}\Big(e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_3}\Big)$, but only two Majorana phases are physical

- All measurable quantities depend only on the differences of the Majorana phases
$$\ell_\alpha \to e^{i\varphi}\ell_\alpha \quad \Longrightarrow \quad e^{i\lambda_k} \to e^{i(\lambda_k - \varphi)}$$
$e^{i(\lambda_k - \lambda_j)}$ remains constant

- Our convention: $\lambda_1 = 0 \Longrightarrow D^{\mathrm{M}} = \mathrm{diag}\Big(1, e^{i\lambda_2}, e^{i\lambda_3}\Big)$

- CP is conserved if all the elements of each column of the mixing matrix are either real or purely imaginary:
$$\delta_{13} = 0 \text{ or } \pi \quad \text{and} \quad \lambda_k = 0 \text{ or } \pi/2 \text{ or } \pi \text{ or } 3\pi/2$$

Dirac-Majorana Mass Term

- Dirac Neutrino Masses and Mixing
- Majorana Neutrino Masses and Mixing
- Dirac-Majorana Mass Term
  - One Generation
  - Real Mass Matrix
  - Maximal Mixing
  - Dirac Limit
  - Pseudo-Dirac Neutrinos
  - See-Saw Mechanism
  - Majorana Neutrino Mass?
  - Fundamental Fields in QFT
  - Right-Handed Neutrino Mass Term
  - Singlet Majoron Model
  - Three-Generation Mixing

One Generation

If ν_R exists, the most general mass term is the

Dirac-Majorana Mass Term

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \mathscr{L}_{\text{mass}}^{\text{D}} + \mathscr{L}_{\text{mass}}^{L} + \mathscr{L}_{\text{mass}}^{R}$$

$$\mathscr{L}_{\text{mass}}^{\text{D}} = -m_{\text{D}} \, \overline{\nu_R} \, \nu_L + \text{H.c.}$$ Dirac Mass Term

$$\mathscr{L}_{\text{mass}}^{L} = \frac{1}{2} \, m_L \, \nu_L^T \, \mathcal{C}^\dagger \, \nu_L + \text{H.c.}$$ Majorana Mass Term

$$\mathscr{L}_{\text{mass}}^{R} = \frac{1}{2} \, m_R \, \nu_R^T \, \mathcal{C}^\dagger \, \nu_R + \text{H.c.}$$ New Majorana Mass Term!

- Column matrix of left-handed chiral fields: $N_L = \begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix} = \begin{pmatrix} \nu_L \\ \mathcal{C}\,\overline{\nu_R}^T \end{pmatrix}$

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2}\, N_L^T\, \mathcal{C}^\dagger\, M\, N_L + \text{H.c.} \qquad M = \begin{pmatrix} m_L & m_{\text{D}} \\ m_{\text{D}} & m_R \end{pmatrix}$$

- The Dirac-Majorana Mass Term has the structure of a Majorana Mass Term for two chiral neutrino fields coupled by the Dirac mass

- Diagonalization: $n_L = U^\dagger\, N_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix}$

$$U^T\, M\, U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \text{Real } m_k \geq 0$$

- $\mathscr{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2} \sum_{k=1,2} m_k\, \nu_{kL}^T\, \mathcal{C}^\dagger\, \nu_{kL} + \text{H.c.} = -\frac{1}{2} \sum_{k=1,2} m_k\, \overline{\nu_k}\, \nu_k$

$$\nu_k = \nu_{kL} + \nu_{kL}^C$$

- Massive neutrinos are Majorana! $\qquad \nu_k = \nu_k^C$

Real Mass Matrix

- CP is conserved if the mass matrix is real: $M = M^*$
- $M = \begin{pmatrix} m_L & m_{\text{D}} \\ m_{\text{D}} & m_R \end{pmatrix}$ we consider real and positive m_R and m_{D} and real m_L
- A real symmetric mass matrix can be diagonalized with $U = \mathcal{O}\,\rho$

$$\mathcal{O} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \qquad \rho_k^2 = \pm 1$$

- $\mathcal{O}^T\, M\, \mathcal{O} = \begin{pmatrix} m'_1 & 0 \\ 0 & m'_2 \end{pmatrix} \qquad \tan 2\vartheta = \dfrac{2 m_{\text{D}}}{m_R - m_L}$

$$m'_{2,1} = \frac{1}{2}\left[m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4\, m_{\text{D}}^2} \right]$$

- m'_1 is negative if $m_L m_R < m_{\text{D}}^2$

$$U^T M U = \rho^T \mathcal{O}^T M \mathcal{O} \rho = \begin{pmatrix} \rho_1^2 m'_1 & 0 \\ 0 & \rho_2^2 m'_2 \end{pmatrix} \quad \Longrightarrow \quad \boxed{m_k = \rho_k^2\, m'_k}$$

- m'_2 is always positive:

$$m_2 = m'_2 = \frac{1}{2} \left[m_L + m_R + \sqrt{(m_L - m_R)^2 + 4\, m_D^2} \right]$$

- If $m_L m_R \geq m_D^2$, then $m'_1 \geq 0$ and $\rho_1^2 = 1$

$$m_1 = \frac{1}{2} \left[m_L + m_R - \sqrt{(m_L - m_R)^2 + 4\, m_D^2} \right]$$

$$\rho_1 = 1 \text{ and } \rho_2 = 1 \quad \Longrightarrow \quad U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

- If $m_L m_R < m_D^2$, then $m'_1 < 0$ and $\rho_1^2 = -1$

$$m_1 = \frac{1}{2} \left[\sqrt{(m_L - m_R)^2 + 4\, m_D^2} - (m_L + m_R) \right]$$

$$\rho_1 = i \text{ and } \rho_2 = 1 \quad \Longrightarrow \quad U = \begin{pmatrix} i\cos\vartheta & \sin\vartheta \\ -i\sin\vartheta & \cos\vartheta \end{pmatrix}$$

- If Δm^2 is small, there are oscillations between active ν_a generated by ν_L and sterile ν_s generated by ν_R^C:

$$P_{\nu_a \to \nu_s}(L, E) = \sin^2 2\vartheta \, \sin^2 \left(\frac{\Delta m^2 \, L}{4 \, E} \right)$$

$$\Delta m^2 = m_2^2 - m_1^2 = (m_L + m_R) \sqrt{(m_L - m_R)^2 + 4 \, m_{\mathrm{D}}^2}$$

- It can be shown that the CP parity of ν_k is $\xi_k^{\mathrm{CP}} = i \, \rho_k^2$:

$$\mathrm{U}_{\mathrm{CP}} \nu_k(x) \mathrm{U}_{\mathrm{CP}}^{-1} = i \, \rho_k^2 \, \gamma^0 \, \nu_k(x_{\mathrm{P}})$$

- Special cases:
  - $m_L = m_R \quad \Longrightarrow \quad$ Maximal Mixing
  - $m_L = m_R = 0 \quad \Longrightarrow \quad$ Dirac Limit
  - $|m_L|, m_R \ll m_{\mathrm{D}} \quad \Longrightarrow \quad$ Pseudo-Dirac Neutrinos
  - $m_L = 0 \quad m_{\mathrm{D}} \ll m_R \quad \Longrightarrow \quad$ See-Saw Mechanism

Maximal Mixing

$$\boxed{m_L = m_R}$$

$$\vartheta = \pi/4$$

$$m'_{2,1} = m_L \pm m_{\mathrm{D}}$$

$$\begin{cases} \rho_1^2 = +1\,, \quad m_1 = m_L - m_{\mathrm{D}} \quad \text{if} \quad m_L \geq m_{\mathrm{D}} \\ \rho_1^2 = -1\,, \quad m_1 = m_{\mathrm{D}} - m_L \quad \text{if} \quad m_L < m_{\mathrm{D}} \end{cases}$$

$$m_2 = m_L + m_{\mathrm{D}}$$

$$\underline{m_L < m_{\mathrm{D}}}$$

$$\begin{cases} \nu_{1L} = \dfrac{-i}{\sqrt{2}} \left(\nu_L - \nu_R^C \right) \\ \nu_{2L} = \dfrac{1}{\sqrt{2}} \left(\nu_L + \nu_R^C \right) \end{cases}$$

$$\begin{cases} \nu_1 = \nu_{1L} + \nu_{1L}^C = \dfrac{-i}{\sqrt{2}} \left[(\nu_L + \nu_R) - \left(\nu_L^C + \nu_R^C \right) \right] \\ \nu_2 = \nu_{2L} + \nu_{2L}^C = \dfrac{1}{\sqrt{2}} \left[(\nu_L + \nu_R) + \left(\nu_L^C + \nu_R^C \right) \right] \end{cases}$$

Dirac Limit

$$\boxed{m_L = m_R = 0}$$

- $m'_{2,1} = \pm m_{\mathrm{D}} \quad \Longrightarrow \quad \begin{cases} \rho_1^2 = -1\,, & m_1 = m_{\mathrm{D}} \\ \rho_2^2 = +1\,, & m_2 = m_{\mathrm{D}} \end{cases}$
- The two Majorana fields ν_1 and ν_2 can be combined to give one Dirac field:

$$\nu = \frac{1}{\sqrt{2}} \left(i\nu_1 + \nu_2 \right) = \nu_L + \nu_R$$

- A Dirac field ν can always be split in two Majorana fields:

$$\begin{aligned} \nu &= \frac{1}{2} \left[\left(\nu - \nu^C \right) + \left(\nu + \nu^C \right) \right] \\ &= \frac{i}{\sqrt{2}} \left(-i \, \frac{\nu - \nu^C}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} \left(\frac{\nu + \nu^C}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} \left(i\nu_1 + \nu_2 \right) \end{aligned}$$

- A Dirac field is equivalent to two Majorana fields with the same mass and opposite CP parities

Pseudo-Dirac Neutrinos

$$\boxed{|m_L|\,,\; m_R \ll m_{\mathrm{D}}}$$

- $m'_{2,1} \simeq \dfrac{m_L + m_R}{2} \pm m_{\mathrm{D}}$
- $m'_1 < 0 \quad \Longrightarrow \quad \rho_1^2 = -1 \quad \Longrightarrow \quad m_{2,1} \simeq m_{\mathrm{D}} \pm \dfrac{m_L + m_R}{2}$
- The two massive Majorana neutrinos have opposite CP parities and are almost degenerate in mass
- The best way to reveal pseudo-Dirac neutrinos are active-sterile neutrino oscillations due to the small squared-mass difference

$$\Delta m^2 \simeq m_{\mathrm{D}}\,(m_L + m_R)$$

- The oscillations occur with practically maximal mixing:

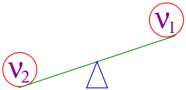
$$\tan 2\vartheta = \frac{2 m_{\mathrm{D}}}{m_R - m_L} \gg 1 \quad \Longrightarrow \quad \vartheta \simeq \pi/4$$

See-Saw Mechanism

[Minkowski, PLB 67 (1977) 42; Yanagida (1979); Gell-Mann, Ramond, Slansky (1979); Mohapatra, Senjanovic, PRL 44 (1980) 912]

$$\boxed{m_L = 0 \qquad m_{\mathrm{D}} \ll m_R}$$

- $\mathscr{L}^L_{\mathrm{mass}}$ is forbidden by SM symmetries $\Longrightarrow m_L = 0$
- $m_{\mathrm{D}} \lesssim v \sim 100\,\mathrm{GeV}$ is generated by SM Higgs Mechanism (protected by SM symmetries)
- m_R is not protected by SM symmetries $\Longrightarrow m_R \sim \mathcal{M}_{\mathrm{GUT}} \gg v$
- $$\left.\begin{array}{l} m'_1 \simeq -\dfrac{m_{\mathrm{D}}^2}{m_R} \\ m'_2 \simeq m_R \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} \rho_1^2 = -1\,, & m_1 \simeq \dfrac{m_{\mathrm{D}}^2}{m_R} \\ \rho_2^2 = +1\,, & m_2 \simeq m_R \end{array}\right.$$



- Natural explanation of smallness of neutrino masses
- Mixing angle is very small: $\tan 2\vartheta = 2\,\dfrac{m_{\mathrm{D}}}{m_R} \ll 1$
- ν_1 is composed mainly of active ν_L: $\nu_{1L} \simeq -i\,\nu_L$
- ν_2 is composed mainly of sterile ν_R: $\nu_{2L} \simeq \nu_R^C$

Connection with Effective Lagrangian Approach

- Dirac–Majorana neutrino mass term with $m_L = 0$:

$$\mathscr{L}^{\text{D+M}} = -m_{\text{D}} \left(\overline{\nu_R}\, \nu_L + \overline{\nu_L}\, \nu_R \right) + \frac{1}{2}\, m_R \left(\nu_R^T\, \mathcal{C}^\dagger\, \nu_R + \nu_R^\dagger\, \mathcal{C}\, \nu_R^* \right)$$

- Above the electroweak symmetry-breaking scale:

$$\mathscr{L}^{\text{D+M}} = -y^\nu \left(\overline{\nu_R}\, \widetilde{\Phi}^\dagger\, L_L + \overline{L_L}\, \widetilde{\Phi}\, \nu_R \right) + \frac{1}{2}\, m_R \left(\nu_R^T\, \mathcal{C}^\dagger\, \nu_R + \nu_R^\dagger\, \mathcal{C}\, \nu_R^* \right)$$

- If $m_R \gg v \Longrightarrow \nu_R$ is static $\Longrightarrow$ kinetic term in equation of motion can be neglected:

$$0 \simeq \frac{\partial \mathscr{L}^{\text{D+M}}}{\partial \nu_R} = m_R\, \nu_R^T\, \mathcal{C}^\dagger - y^\nu\, \overline{L_L}\, \widetilde{\Phi}$$

$$\nu_R \simeq -\frac{y^\nu}{m_R}\, \widetilde{\Phi}^T\, \mathcal{C}\, \overline{L_L}^T$$

$$\mathscr{L}^{\text{D+M}} \rightarrow \mathscr{L}_5^{\text{D+M}} \simeq -\frac{1}{2}\, \frac{(y^\nu)^2}{m_R} (L_L^T\, \sigma_2\, \Phi)\, \mathcal{C}^\dagger\, (\Phi^T\, \sigma_2\, L_L) + \text{H.c.}$$

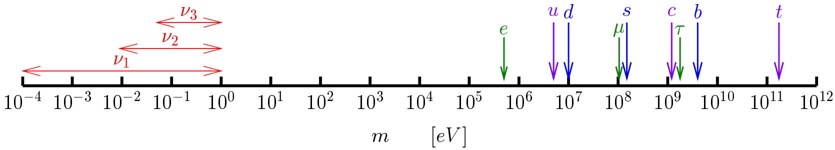
$$\mathscr{L}_5 = \frac{g}{\mathcal{M}} \, (L_L^T \, \sigma_2 \, \Phi) \, \mathcal{C}^\dagger \, (\Phi^T \, \sigma_2 \, L_L) + \text{H.c.}$$

$$\mathscr{L}_5^{\text{D+M}} \simeq -\frac{1}{2} \, \frac{(y^\nu)^2}{m_R} (L_L^T \, \sigma_2 \, \Phi) \, \mathcal{C}^\dagger \, (\Phi^T \, \sigma_2 \, L_L) + \text{H.c.}$$

$$g = -\frac{(y^\nu)^2}{2} \qquad\qquad \mathcal{M} = m_R$$

- See-saw mechanism is a particular case of the effective Lagrangian approach.
- See-saw mechanism is obtained when dimension-five operator is generated only by the presence of ν_R with $m_R \sim \mathcal{M}$.
- In general, other terms can contribute to $\mathscr{L}_5$.

Majorana Neutrino Mass?



known natural explanation of smallness of ν masses

New High Energy Scale $\mathcal{M}$ $\Rightarrow$ $\begin{cases} \text{See-Saw Mechanism (if } \nu_R\text{'s exist)} \\ \text{5-D Non-Renormaliz. Eff. Operator} \end{cases}$

both imply $\begin{cases} \text{Majorana } \nu \text{ masses} \iff |\Delta L| = 2 \iff \beta\beta_{0\nu} \text{ decay} \\ \text{see-saw type relation } m_\nu \sim \dfrac{\mathcal{M}_{\text{EW}}^2}{\mathcal{M}} \end{cases}$

Majorana neutrino masses provide the most accessible
window on New Physics Beyond the Standard Model

Fundamental Fields in QFT

- Each elementary particle is described by a field which is an irreducible representation of the Poincaré group (Lorentz group + space-time translations).
- In this way
  - Under Poincaré transformation an elementary particle remains itself.
  - Lagrangian is constructed with invariant products of elementary fields.
- Spinorial structure of a particle is determined by its representation under the restricted Lorentz group of proper and orthochronous Lorentz transformation (no space or time inversions).

- Restricted Lorentz group is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.
- Classification of fundamental representations:

$(0,0)$	scalar φ
$(1/2,0)$	left-handed Weyl spinor χ_L (Majorana if massive)
$(0,1/2)$	right-handed Weyl spinor χ_R (Majorana if massive)

- All representations are constructed combining the two fundamental Weyl spinor representations.

$(1/2,1/2)$	four-vector v^μ (irreducible)
$(1/2,0)+(0,1/2)$	four-component Dirac spinor ψ (reducible)

- Two-component Weyl (Majorana if massive) spinor is more fundamental than four-component Dirac spinor.

- Two-component left-handed Weyl (Majorana if massive) spinor:

$$\chi_L = \begin{pmatrix} \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

- Two-component right-handed Weyl (Majorana if massive) spinor:

$$\chi_R = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \end{pmatrix}$$

- Four-component Dirac spinor: $\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{R1} \\ \chi_{R2} \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$

- Lorentz transformation: $v^\mu \to v'^\mu = \Lambda^\mu{}_\nu \, v^\nu$

$$g_{\mu\nu} \, \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma = g_{\rho\sigma} \qquad g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$

- Restricted Lorentz transformation: $\Lambda^\mu{}_\nu = [e^\omega]^\mu{}_\nu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ -v_1 & 0 & \theta_3 & -\theta_2 \\ -v_2 & -\theta_3 & 0 & \theta_1 \\ -v_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

- 6 parameters:
  - 3 for rotations: $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$
  - 3 for boosts: $\vec{v} = (v_1, v_2, v_3)$

$$\begin{aligned} \chi_L \to \chi'_L &= \Lambda_L \chi_L \qquad & \Lambda_L &= e^{i\left(\vec{\theta} - i\vec{v}\right)\cdot\vec{\sigma}/2} \\ \chi_R \to \chi'_R &= \Lambda_R \chi_R \qquad & \Lambda_R &= e^{i\left(\vec{\theta} + i\vec{v}\right)\cdot\vec{\sigma}/2} \end{aligned}$$

- Four-component form of two-component left-handed Weyl (Majorana if massive) spinor:

$$\psi_L = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \chi_{L1} \\ \chi_{L2} \end{pmatrix}$$

- Majorana mass term:

$$\mathscr{L}^L_{\text{mass}} = \underbrace{\frac{1}{2} m_L \psi_L^T \mathcal{C}^\dagger \psi_L}_{\text{four-component form}} + \text{H.c.} = \underbrace{-\frac{1}{2} m_L \chi_L^T i\sigma^2 \chi_L}_{\text{two-component form}} + \text{H.c.}$$

$$(1/2,0) \times (1/2,0) = \underbrace{(1,0)}_{\text{symmetric}} + \underbrace{(0,0)}_{\text{antisymmetric}}$$

σ^2 is antisymmetric!

- Anticommutativity of spinors is necessary, otherwise

$$\chi_L^T i\sigma^2 \chi_L = \left(\chi_L^T i\sigma^2 \chi_L\right)^T = -\chi_L^T i\sigma^2 \chi_L = 0$$

Right-Handed Neutrino Mass Term

Majorana mass term for ν_R respects the $\text{SU}(2)_L \times \text{U}(1)_Y$ Standard Model Symmetry!

$$\mathcal{L}_R^{\text{M}} = -\frac{1}{2}\, m \left(\overline{\nu_R^c}\, \nu_R + \overline{\nu_R}\, \nu_R^c \right)$$

Majorana mass term for ν_R breaks Lepton number conservation!

Three possibilities:

- Lepton number can be explicitly broken
- Lepton number is spontaneously broken locally, with a massive vector boson coupled to the lepton number current
- Lepton number is spontaneously broken globally and a massless Goldstone boson appears in the theory (Majoron)

Singlet Majoron Model

[Chikashige, Mohapatra, Peccei, Phys. Lett. B98 (1981) 265, Phys. Rev. Lett. 45 (1980) 1926]

$$\mathcal{L}_\Phi = -y_d \left(\overline{L_L}\, \Phi\, \nu_R + \overline{\nu_R}\, \Phi^\dagger\, L_L \right) \xrightarrow[\langle\Phi\rangle\neq 0]{} -m_\mathrm{D} \left(\overline{\nu_L}\, \nu_R + \overline{\nu_R}\, \nu_L \right)$$

$$\mathcal{L}_\eta = -y_s \left(\eta\, \overline{\nu_R^c}\, \nu_R + \eta^\dagger\, \overline{\nu_R}\, \nu_R^c \right) \xrightarrow[\langle\eta\rangle\neq 0]{} -\tfrac{1}{2}\, m_R \left(\overline{\nu_R^c}\, \nu_R + \overline{\nu_R}\, \nu_R^c \right)$$

$$\eta = 2^{-1/2} \left(\langle\eta\rangle + \rho + i\,\chi \right) \qquad \mathcal{L}_\mathrm{mass} = -\frac{1}{2} \left(\overline{\nu_L^c} \;\; \overline{\nu_R} \right) \begin{pmatrix} 0 & m_\mathrm{D} \\ m_\mathrm{D} & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \mathrm{H.c.}$$

$$\underset{\text{scale of } L \text{ violation}}{m_R} \gg \underset{\text{EW scale}}{m_\mathrm{D}} \implies \text{See-Saw:} \quad \boxed{m_1 \simeq \frac{m_\mathrm{D}^2}{m_R}}$$

ρ = massive scalar, χ = Majoron (massless pseudoscalar Goldstone boson)

The Majoron is weakly coupled to the light neutrino

$$\mathcal{L}_{\chi-\nu} = \frac{i y_s}{\sqrt{2}}\, \chi \left[\overline{\nu_2}\gamma^5\nu_2 - \frac{m_\mathrm{D}}{m_R} \left[\overline{\nu_2}\gamma^5\nu_1 + \overline{\nu_1}\gamma^5\nu_2 \right) + \left(\frac{m_\mathrm{D}}{m_R} \right)^2 \overline{\nu_1}\gamma^5\nu_1 \right]$$

Three-Generation Mixing

$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \mathscr{L}_{\text{mass}}^{\text{D}} + \mathscr{L}_{\text{mass}}^{L} + \mathscr{L}_{\text{mass}}^{R}$$

$$\mathscr{L}_{\text{mass}}^{\text{D}} = -\sum_{s=1}^{N_S} \sum_{\alpha=e,\mu,\tau} \overline{\nu'_{sR}}\, M_{s\alpha}^{\text{D}}\, \nu'_{\alpha L} + \text{H.c.}$$

$$\mathscr{L}_{\text{mass}}^{L} = \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} \nu'^{T}_{\alpha L}\, \mathcal{C}^{\dagger}\, M_{\alpha\beta}^{L}\, \nu'_{\beta L} + \text{H.c.}$$

$$\mathscr{L}_{\text{mass}}^{R} = \frac{1}{2} \sum_{s,s'=1}^{N_S} \nu'^{T}_{sR}\, \mathcal{C}^{\dagger}\, M_{ss'}^{R}\, \nu'_{s'R} + \text{H.c.}$$

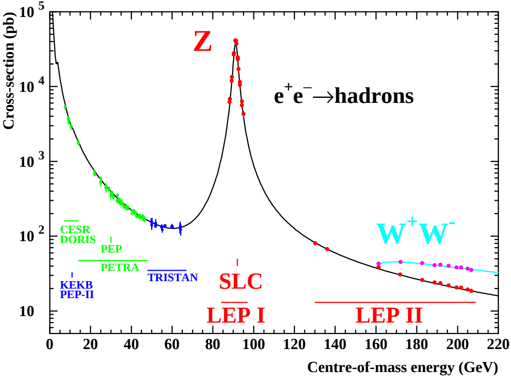
$$\mathbf{N}'_L \equiv \begin{pmatrix} \boldsymbol{\nu}'_L \\ \boldsymbol{\nu}'^{C}_R \end{pmatrix} \qquad \boldsymbol{\nu}'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix} \qquad \boldsymbol{\nu}'^{C}_R \equiv \begin{pmatrix} \nu'^{C}_{1R} \\ \vdots \\ \nu'^{C}_{N_S R} \end{pmatrix}$$

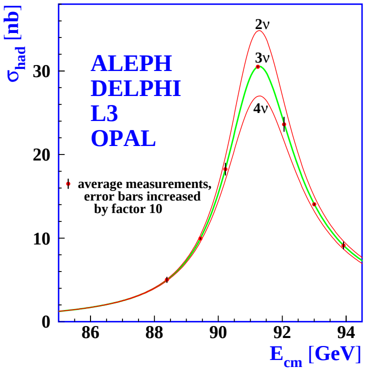
$$\mathscr{L}_{\text{mass}}^{\text{D+M}} = \frac{1}{2}\, \mathbf{N}'^{T}_L\, \mathcal{C}^{\dagger}\, M^{\text{D+M}}\, \mathbf{N}'_L + \text{H.c.} \qquad M^{\text{D+M}} = \begin{pmatrix} M^L & M^{\text{D}\,T} \\ M^{\text{D}} & M^R \end{pmatrix}$$

- Diagonalization of the Dirac-Majorana Mass Term $\Longrightarrow$ massive Majorana neutrinos
- See-Saw Mechanism $\Longrightarrow$ right-handed neutrinos have large Majorana masses and are decoupled from the low-energy phenomenology.
- If all right-handed neutrinos have large Majorana masses, at low energy we have an effective mixing of three Majorana neutrinos.
- It is possible that not all right-handed neutrinos have large Majorana masses: some right-handed neutrinos may correspond to low-energy Majorana particles which belong to new physics beyond the Standard Model (as a light neutralino in supersymmetric models).
- Light anti-ν_R are called sterile neutrinos

$$\nu_R^c \to \nu_s \qquad \text{(left-handed)}$$

Number of Flavor and Massive Neutrinos?





[LEP, Phys. Rept. 427 (2006) 257, arXiv:hep-ex/0509008]

$$\Gamma_Z = \sum_{\ell=e,\mu,\tau} \Gamma_{Z\to\ell\bar{\ell}} + \sum_{q\neq t} \Gamma_{Z\to q\bar{q}} + \Gamma_{\text{inv}} \qquad \Gamma_{\text{inv}} = N_\nu \, \Gamma_{Z\to\nu\bar{\nu}}$$

$$\boxed{N_\nu = 2.9840 \pm 0.0082}$$

$$e^+ e^- \to Z \xrightarrow{\text{invisible}} \sum_{a=\text{active}} \nu_a \bar{\nu}_a \quad \Longrightarrow \quad \nu_e \; \nu_\mu \; \nu_\tau$$

3 light active flavor neutrinos

mixing $\Rightarrow$ $$\nu_{\alpha L} = \sum_{k=1}^{N} U_{\alpha k} \nu_{kL} \qquad \alpha = e, \mu, \tau$$

$N \geq 3$
no upper limit!

Mass Basis:	ν_1	ν_2	ν_3	ν_4	ν_5	$\cdots$
Flavor Basis:	ν_e	ν_μ	ν_τ	ν_{s_1}	ν_{s_2}	$\cdots$
	ACTIVE			STERILE		

$$\boxed{\nu_{\alpha L} = \sum_{k=1}^{N} U_{\alpha k} \nu_{kL} \qquad \alpha = e, \mu, \tau, s_1, s_2, \ldots}$$

Sterile Neutrinos

- Sterile means no standard model interactions
- Obviously no electromagnetic interactions as normal active neutrinos
- Thus sterile means no standard weak interactions
- But sterile neutrinos are not absolutely sterile:
  - Gravitational Interactions
  - New non-standard interactions of the physics beyond the Standard Model which generates the masses of sterile neutrinos
- Active neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ can oscillate into sterile neutrinos (ν_s)
- Observables:
  - Disappearance of active neutrinos
  - Indirect evidence through combined fit of data
- Powerful window on new physics beyond the Standard Model