The Quark Parton Model

In the proton rest frame (the laboratory frame)

\[ p' = (m, \mathbf{0}) \, , \]
\[ k'' = (E, \mathbf{k}) = (E, 0, 0, k) \, , \]
\[ k'' = (E', \mathbf{k}') = (E', k' \sin \theta, 0, k' \cos \theta) \]
\[ \nu = \frac{p \cdot q}{m} = (E - E') \]
\[ x = \frac{-q^2}{2mv} = \frac{Q^2}{2mv} = \frac{Q^2}{2m(E - E')} \]
\[ y = \frac{q \cdot p}{k \cdot p} = 1 - \frac{E'}{E} \]

Lepton scattering plane is chosen as x-z plane with z-axis along beam and x-axis along the transverse to z-axis component of scattered lepton momentum.
We will systematically ignore the masses of the leptons in the high-energy limit. In this limit we have the familiar result

$$Q^2 = -q^2 = 2EE'(1 - \cos \theta) = 4EE' \sin^2 \frac{\theta}{2}.$$  

The mass excitation (the change from the initial proton mass) is given by

$$W^2 - m^2 = (q + p)^2 - m^2 = 2mv - Q^2$$
$$= 2mv (1 - x) = Q^2 (\omega - 1),$$ \hspace{1cm} \omega = 1/x

The elastic limit corresponds to $x = 1$.

We can write the inclusive cross section for $e + p \rightarrow e' + X$ in the form

$$\frac{d\sigma}{dE'd\Omega'} = \frac{4\alpha^2 E'^2}{Q^4} \left[ 2W(Q^2, \nu) \sin^2 \frac{\theta}{2} + W_2(Q^2, \nu) \cos^2 \frac{\theta}{2} \right],$$
or, terms of invariants,

\[
\frac{d\sigma}{d\nu dQ^2} = \frac{4\pi \alpha^2 E}{Q^2 E} \left[ 2\mathcal{W}_1(Q^2, \nu) \sin^2 \theta + \mathcal{W}_2(Q^2, \nu) \cos^2 \frac{\theta}{2} \right].
\]

We recognize the explicit angular factors as characteristic of spin-flip and non-flip terms. In these expressions the invariants functions have dimensions of $1/E$. It is conventional, and informative, to rewrite this cross section in terms of the dimensionless variables $x$ and $y$ and also dimensionless functions given by

\[
F_1(x, Q^2) = m\mathcal{W}_1(\nu, Q^2),
\]
\[
F_2(x, Q^2) = \nu\mathcal{W}_2(\nu, Q^2).
\]

Note that the invariant functions describing the inelastic scattering of the proton can only depend on $\nu$ and $Q^2$, and implicitly on $m^2$. In a dimensionless expression, this dependence can only be in terms of dimensionless ratios.
Bjorken limit (Bj, bj): \( Q^2, \nu \to \infty \) with fixed x

One expects that if there is no large hadronic scale then

because naively any explicit dependence on \( Q^2 \) would have to be of the form \( m^2/Q^2 \) and therefore vanish. Thus, in the high-energy scaling limit, we expect to describe deeply inelastic scattering in the form

\[
\frac{d\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[ y^2 F_1(x) + \frac{1-y}{x} F_2(x) \right]
= \frac{4\pi\alpha^2}{Q^4} \left[ 1 + (1-y)^2 \right] F_1(x)
+ \left[ \frac{1-y}{x} \right] (F_2(x) - 2xF_1(x)),
\]
Or, for \( m \ll E \)

\[
\frac{Q^4}{s} \frac{d\sigma}{dx dy} = 4\pi\alpha^2 \left[ xy^2 F_1(x) + (1 - y) F_2(x) \right] \\
= 4\pi\alpha^2 \left[ x \left[ 1 + (1 - y)^2 \right] F_1(x) \right. \\
\left. + [1 - y] \left( F_2(x) - 2xF_1(x) \right) \right].
\]

The last form is informative because the function \( F_1 \) describes the absorption of transversely polarized photons while the combination \( F_2 - 2xF_1 \) describes the absorption of longitudinally polarized photons.

The fact that this limit yields a non-vanishing experimental result is called scaling and was observed to be a feature of the first data on DIS at SLAC. The measured values of \( F_2 \) for different values of \( Q^2 \) and \( \nu \) lie essentially on top of each other when plotted versus \( x \) as illustrated in the figure.
Given that the proton is not elementary but clearly exhibits internal structure as evidenced by the rapidly falling form factor $\propto 1/Q^4$ we discussed in the case of elastic scattering, we expect on general grounds that there are essentially two possibilities for this scaling behavior. If the electric charge of the proton is distributed uniformly throughout its volume ($\propto fm^3$, as in a naïve “pudding” analogous to early models for the atom), we would expect that the electric charge within a volume of order $1/Q^3$ would vanish in the scaling limit,

$$F_{1,2}^{\text{Uniform Distribution}}(x,Q^2) \xrightarrow{Q^2 \to \infty, x \text{ fixed}} 0.$$  

On the other hand, if the electrically charged constituents of the proton (the charged partons) are elementary (structure less) particles, we anticipate a nontrivial scaling limit, as observed. Such scaling behavior is characteristic of the scattering of elementary point particles because point particles have no relevant internal scale (by definition).
Consider elastic scattering from a single elementary fermion of electric charge $e_f$. To stay on the mass shell (as the particle must do as it has no way to become excited) we must require that 

\[(p+q)^2 = W^2 = p^2 = m^2\]

The delta function arises directly from the mass shell condition 

\[\delta(W^2 - m^2) \Rightarrow \delta(1-x).\]

Thus a point fermion contributes to the invariant functions in the form

\[
\begin{align*}
F_1 &= \frac{e_f^2}{2} \delta(1-x) \\
F_2 &= e_f^2 x \delta(1-x)
\end{align*}
\]

For comparison an elementary scalar particle would contribute

\[
\begin{align*}
F_1 &= 0 \\
F_2 &= e_s^2 \delta(1-x)
\end{align*}
\]
The nontrivial scaling limit for result for $F_1$ and $F_2$ suggested a simple and appealing interpretation – the proton is composed of essentially free, point-like quarks – what could be simpler! This idea was also consistent with the picture of purely hadronic interaction that was being popularized by Feynman at the time (late 1960’s and early 1970’s).

In this Feynman picture we can identify $x$ as the fraction of the proton’s moment carried by the scattered quark. The momentum of the quark is thus

$$p_q^\mu = x_P p^\mu$$

So we imagine that the we can define the state of the proton in terms a probability distribution to find a quark of type $q$ with momentum fraction $x_F$, $F_{qfp}(x_F)$. Since a point quark makes the following scaling contribution

$$\hat{F}_2(x_{qf}) = 2x\hat{F}_1(x_{qf}) = e^2 x_q \delta(x_P - x_{qf}),$$

where $x_{qf} = Q^2/2mv$ and the delta function is again the on-shell constraint for the massless final quark

$$\left(q^\mu + x_P p^\mu\right)^2 = 0 \Rightarrow x_F = -\frac{q^2}{2q \cdot p} = \frac{Q^2}{2mv} = x_{qf}.\]
Thus, in terms of the assumed distributions of quarks, we can write the parton model DIS result for the proton as

\[
F_2(x) = 2x F_1(x) = \sum_{q,\bar{q}} e_q^2 x F_{q/p}(x).
\]

We view the electron and quark as interacting on a short time scale (~1/Q), while the quarks confined inside the proton interact on long time scale (~1/m_p).

It is also often informative to consider the “infinite momentum frame” where the proton momentum \( p \to \infty \). In this frame the proton is “mostly” contracted and the internal interactions are “frozen” (dilated or slowed down).
A common notation is to label the individual quark distribution functions in the proton as

\[ F_{q/p} (x) = q(x). \]

The parton model predicts that the invariant functions describing the proton should obey the relation,

\[ 2xF_1(x) = F_2(x) \]

which arises from the fact that the quarks are fermions and is called the Callan-Gross relation. It is experimentally confirmed and provides evidence that partons are quarks (or at least fermions).

We can more fully develop this model by defining distributions for the valence quarks, which carry the quantum numbers of the hadron, and the flavor neutral sea (or “ocean”) quarks. In conventional notation we have

\[
\begin{align*}
F_{u/p} (x) &= u(x) = u_v(x) + S(x), \\
F_{d/p} (x) &= d(x) = d_v(x) + S(x), \\
S(x) &= \bar{u}(x) = \bar{d}(x) = s(x) = \bar{s}(x).
\end{align*}
\]

Thus we have (ignoring the role of virtual pairs of the even more massive quarks) that

\[
F_2^* (x) = x \left[ \frac{4}{9} (u(x) + \bar{u}(x)) + \frac{1}{9} (d(x) + \bar{d}(x) + s(x) + \bar{s}(x)) \right]
\]
Normalization

\[ \int_0^1 dx \, u_\nu(x) = 2 ; \quad \int_0^1 dx \, d_\nu(x) = 1. \]

The total momentum carried by quarks

\[ \sum \int_0^1 dx \, \left[ q(x) + \bar{q}(x) \right] \approx 0.5 \text{ data}. \]

Only approximately 50% of the momentum of the proton is carried by quarks!

Typical parton distribution functions derived from data are illustrated in the figure.

The sea is \textit{not} really SU(3) or even SU(2) symmetric.
The validity and the power of these parton model results arise from our ability to apply them to other processes. For example, the identification of the electrically charged partons with quarks means that we can immediately relate the electron-proton DIS results with neutrino-proton DIS results. Nearly the only change is the replacement of the electric charge of a given quark flavor with its weak current charge.

Applying the simple parton model ideas to the process \( e^+ e^- \rightarrow \text{hadrons} \) (inclusively), we are led to consider the process: \( e^+ e^- \rightarrow q\bar{q} \)

\[
\sigma(e^+ e^- \rightarrow q\bar{q}) = \sigma(e^+ e^- \rightarrow \mu^+ \mu^-) \cdot e_q^2
\]

\[
R_q = \frac{\sigma(e^+ e^- \rightarrow q\bar{q})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} = e_q^2
\]

At “long distances” the scattered quarks pull more quarks and anti-quarks out of the vacuum that (somehow) reassemble into color singlet hadrons. This picture introduces two new concepts – fragmentation and jets.
Quarks

Bottom \quad \text{Charge} \quad -1/3 \quad \text{Top} \quad \text{Charge} \quad 2/3

Strange \quad \text{Charge} \quad -1/3 \quad \text{Charm} \quad \text{Charge} \quad 2/3

Down \quad \text{Charge} \quad -1/3 \quad \text{Up} \quad \text{Charge} \quad 2/3

\text{R}_{\text{hadrons}} = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N_c \cdot \sum_{q} R_q = 3 \cdot \frac{2}{3}

\text{Fig. 11.3} \quad \text{Ratio} \ R \ \text{of (11.6) as a function of the total} \ c^+c^- \ \text{center-of-mass energy. (The sharp peaks correspond to the production of narrow} \ 1^- \ \text{resonances just below or near the flavor thresholds.)}
Quark jets

Example of the hadron production in $e^+e^-$ annihilation in the JADE detector at the PETRA $e^+e^-$ collider at DESY, Germany.

- cms energy 30 GeV.
- Lines of crosses - reconstructed trajectories in drift chambers (gas ionisation detectors).
- Photons - dotted lines - detected by lead-glass Cerenkov counters.
- Two opposite jets.
In analogy to the parton distribution function introduced above to characterize the probability to find a parton in a hadron, we now define a fragmentation function \( D_{h/q}(z) \) to describe the probability to find a hadron \( h \) in the essentially collinear debris of the fragmenting quark \( q \). Here \( z \) is the fraction of the momentum of the original quark carried by the final hadron.

The word jet is used to label the “spray” of essentially collinear hadrons whose total momentum and even flavor quantum numbers track (but do not precisely equal) those of the fragmenting quark. The angular cross section for electrons to quarks, \( i.e., \) spin \( \frac{1}{2} \) fermions, should describe the angular distribution of the jets (or at least the leading hadrons)

\[
\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2Q^2} (1 + \cos^2\theta) \sum_q e^2_q.
\]

The momentum distribution of the final hadrons along the jet direction can be expressed in terms of the fragmentation functions:

\[
\frac{d\sigma}{ds_h}(e^+e^- \rightarrow hX) = \sum_q \sigma(e^+e^- \rightarrow q\bar{q}) \left[ D_{h/q}(z) + D_{h/q}(z) \right],
\]
The energy/momentum fraction variable can (and is) be expressed in several ways

\[
z = \frac{E_h}{E_q} = \frac{2E_h}{Q} = \frac{2 \hat{p}_h \cdot \hat{j}}{Q} = \frac{E_h + \hat{p}_h \cdot \hat{j}}{E_{\text{jet}} + p_{\text{jet}}}
\]

The momentum conservation suggests the following sum rule

\[
\sum_h \int_0^1 dz z D_{h/q} (z) = 1_z
\]

and crudely we expect

\[
D_{h/q} \sim N_{h/q} \frac{(1-z)^{a_{h/q}}}{z}
\]

The 1 – z factor provides the expected constraint (vanishing) at the edge of phase space, which is expected to depend on the specific quark and hadron and the 1/z factor describes the expected (and observed) wee hadrons.
Parton model in hadron-hadron collisions.

To ensure interactions at short distances and the validity of our factorized picture, we focus on interactions that produce hadrons with large transverse momentum.

\[ \sigma \sim \int F(x_1) F(x_2) \hat{\sigma} D(z_1) D(z_2). \]

\( \hat{\sigma} \) -- hard parton-parton scattering cross-section
Some inequalities

Consider the structure function $F_2$ defined in terms of the individual quark distributions in the proton

$$\frac{F_{2}^{ep}}{x} = \frac{4}{9} \left( u(x) + \bar{u}(x) \right) + \frac{1}{9} \left( d(x) + \bar{d}(x) \right) + \frac{1}{9} \left( s(x) + \bar{s}(x) \right) + \ldots$$

We can use (strong) isospin invariance to relate the parton distributions in the proton to those in the neutron, $u_p = d_n$, $d_p = u_n$, $s_p = s_n$ and similarly for the antiquarks. Thus for the neutron

$$\frac{F_{2}^{en}}{x} = \frac{4}{9} \left( d(x) + \bar{d}(x) \right) + \frac{1}{9} \left( u(x) + \bar{u}(x) \right) + \frac{1}{9} \left( s(x) + \bar{s}(x) \right) + \ldots$$

proton. A little thought about the coefficients (the charges squared) leads to the conclusion that the extremes of the ratio of neutron to proton will arise when either the $u$ or $d$ quarks dominates. These two limits yield
\[
\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow{d \gg u, s, \ldots} \frac{4/9}{1/9} = 4,
\]
\[
\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow{u \gg d, s, \ldots} \frac{1/9}{4/9} = \frac{1}{4}.
\]

The general situation must lie somewhere in between and we have

\[
\frac{1}{4} \leq \frac{F_2^{en}}{F_2^{ep}} \leq 4
\]
Assuming an SU(2) (but not SU(3)) symmetric and $C$ invariant sea, we expect

$$x \to 0, u(x) \approx \bar{u}(x) \approx d(x) \approx \bar{d}(x) \gg s(x) = \bar{s}(x)$$

Thus the neutron to proton ratio is expected to have the limit

$$\frac{F_{2\,en}^e(x)}{F_{2\,ep}^e(x)} \xrightarrow{x \to 0} \frac{\left(\frac{2}{9} + \frac{1}{9}\right)}{\left(\frac{2}{9} + \frac{1}{9}\right)} = 1$$

In the limit of large $x$, we would expect the valence quarks to be dominant. We know that there are twice as many $u$ as $d$ valence quarks in the proton, and vice versa in the neutron. If this ratio is approximately true at all (large) $x$, we might expect
On the other hand, it might be that the u-quark distribution is much larger than the d-quark distribution in the $x \to 1$ limit. In this case we have

$$\frac{d(x)}{u(x)} \xrightarrow{x \to 1} \frac{1}{2},$$

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \xrightarrow{x \to 1} \frac{4d+u}{4u+d} \approx \frac{4+2}{8+1} = \frac{2}{3} = \frac{2}{3} \iff \frac{d(x)}{u(x)} \xrightarrow{x \to 1} \frac{1}{2}.$$
Going the other direction we can write

\[
\left. \frac{d}{u} \right|_{x \to 1} = 4 \left( \frac{F_2^{en}}{F_2^{ep}} \right)^{-1} - 1
\]

The data suggest that \( F_2^{en} / F_2^{ep} \xrightarrow{x \to 1} 1/4 \) and thus that \( d/u \xrightarrow{x \to 1} 0 \).

Finally consider the so-called Gottfried Sum Rule

\[
I_{\text{Gottfried}}(x) = \int_x^1 \frac{dx'}{x'} \left[ F_2^{ep}(x') - F_2^{en}(x') \right]
\]
Assuming SU(2) symmetry in the sea means that the sea contributions in the integrand will cancel between the proton and neutron and that only the valence quarks will contribute. We define the valence distribution as, for example, \( u_V(x) = u(x) - \bar{u}(x) \)

\[
I_{\text{Gottfried}}(0) = \int_0^1 dx' \left[ \frac{4}{9} u_v(x') + \frac{1}{9} d_v(x') - \frac{4}{9} d_v(x') - \frac{1}{9} u_v(x') \right]
\]

\[
= \left( \frac{4}{9} - \frac{1}{9} \right) \int_0^1 dx' \left\{ u_v(x') - d_v(x') \right\}
\]

\[
= \frac{1}{3} \{ 2 - 1 \} = \frac{1}{3}.
\]

Recent experimental results, e.g., those from the EMC and NMC collaborations, are systematically smaller than this expectation (~0.234) suggesting a variety of possible interpretations. In particular, it seems that the sea is not SU(2) invariant as already noted. Since the valence distributions are not SU(2) invariant, the exclusion principle could work to suppress the anti-\( u \) distribution in the sea, \( \bar{u} < \bar{d} \), which could explain the Gottfried Sum Rule result.