

SOFT RADIATION BEYOND LEADING POWER

Lorenzo Magnea

University of Torino - INFN Torino

SLAC - 06/27/2015



Outline

- Introduction
- Threshold resummations at leading power
- Gathering evidence beyond leading power
- Next-to-eikonal approximation
- Solving a collinear conundrum
- Outlook

GATEWAY



Logarithms

📌 **Multi-scale** problems in **renormalizable** quantum field theories have perturbative corrections of the form $\alpha_s^n \log^k (Q_i^2/Q_j^2)$, which may **spoil** the reliability of the perturbative expansion. However, they **carry important physical information**.

- **Renormalization** and **factorization** logs: $\alpha_s^n \log^n (Q^2/\mu^2)$
- **High-energy logs**: $\alpha_s^n \log^{n-1} (s/t)$
- **Sudakov** logs: $\alpha_s^n \log^{2n-1} (1-z)$, $1-z = W^2/Q^2, 1-M^2/\hat{s}, Q_\perp^2/Q^2, \dots$

📌 **Logarithms** encode **process-independent** features of perturbation theory. For **Sudakov** logs: the structure of **infrared** and **collinear** divergences.

$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \implies \ln(m^2/Q^2)$$

- For **inclusive** observables: **analytic** resummation to high logarithmic accuracy.
- For **exclusive** final states: **parton shower** event generators, **(N(N))LL** accuracy. 

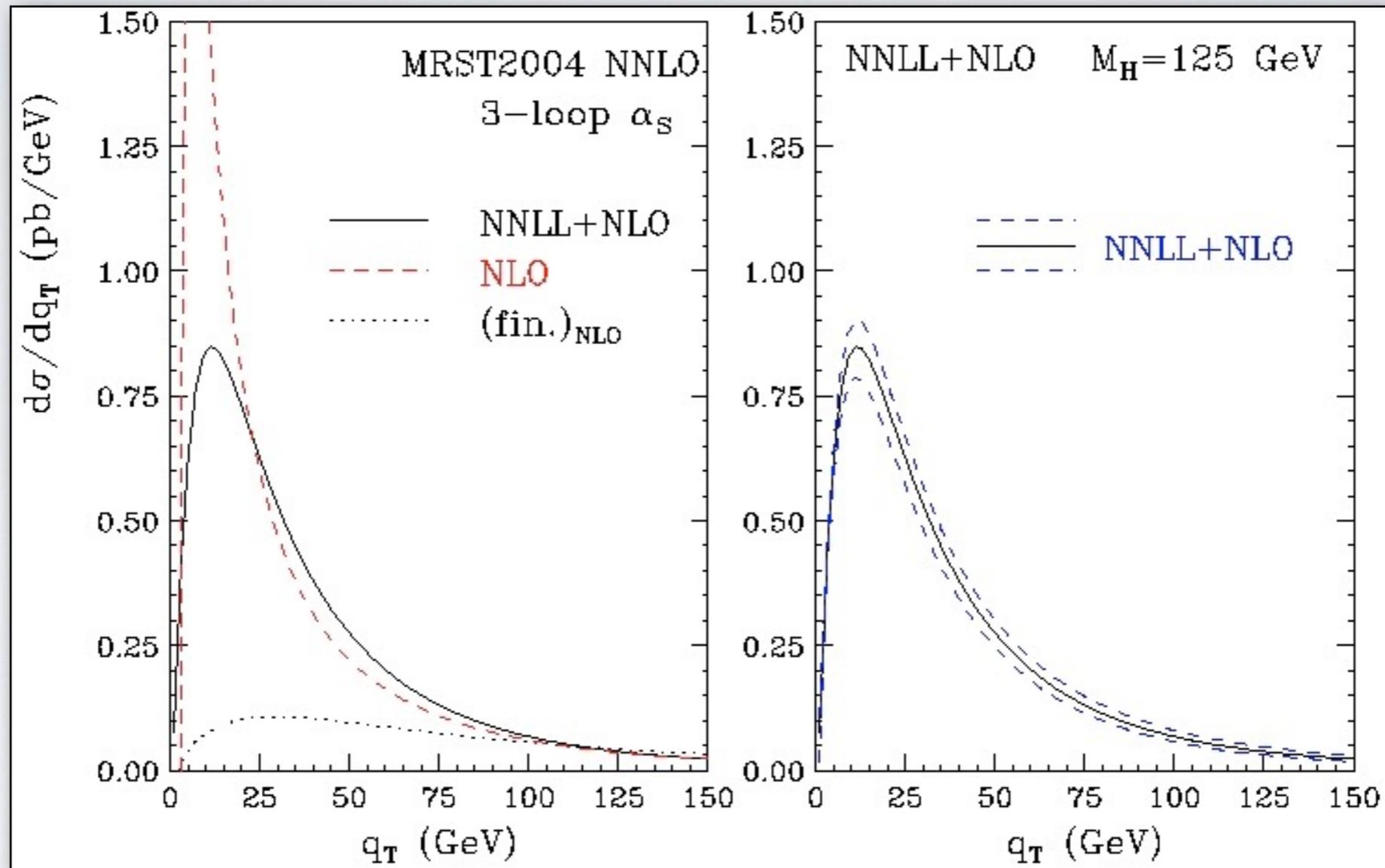
Gradually merging

📌 **Resummation** probes the **all-order structure** of perturbation theory.

- **Non-perturbative** contributions to QCD cross sections can be estimated.
- Links to the **strong coupling** regime can be established for special gauge theories.

Logarithms at work

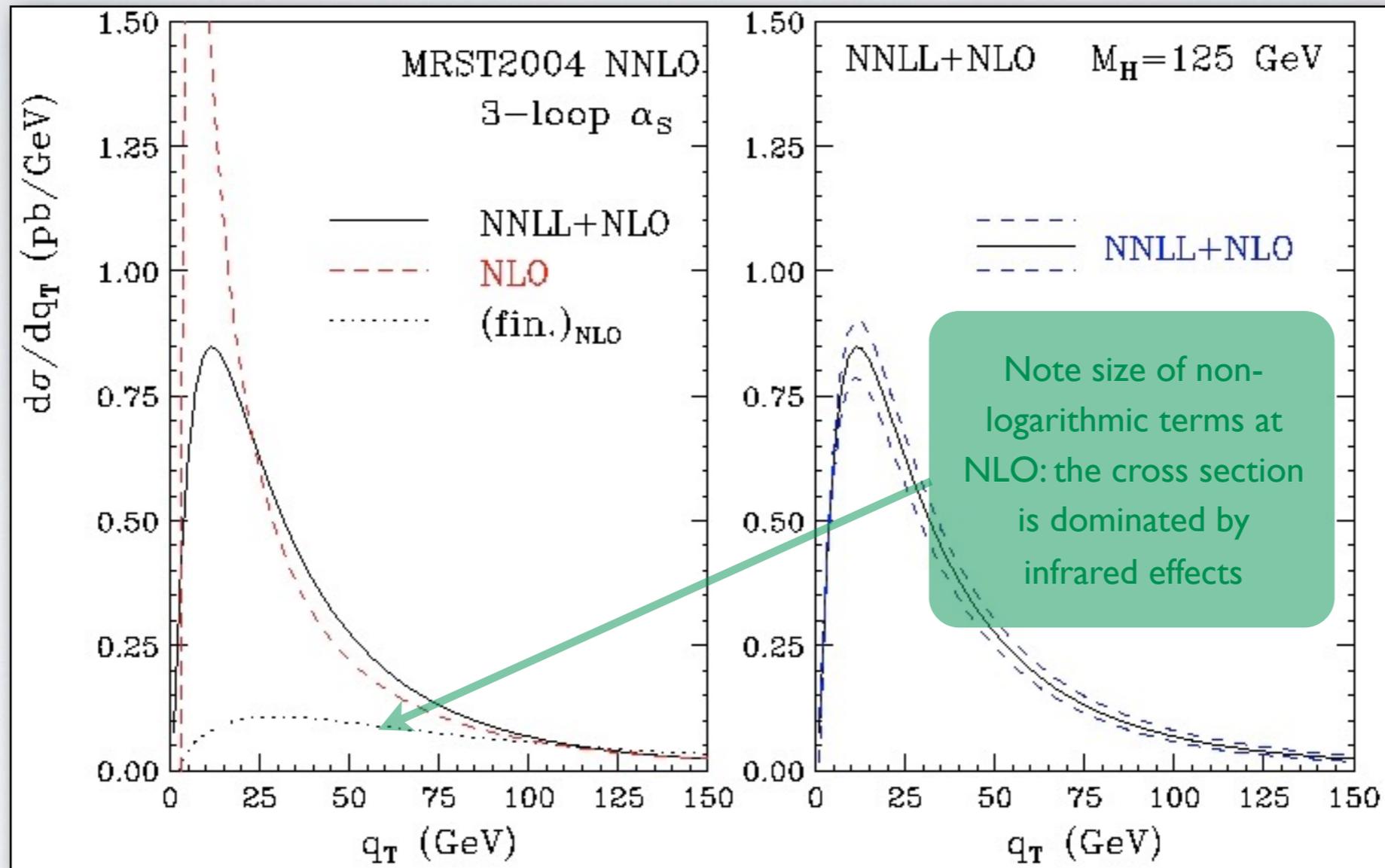
Predictions for the Higgs boson q_T spectrum at LHC (M. Grazzini, 05)



Predictions for the q_T spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.

Logarithms at work

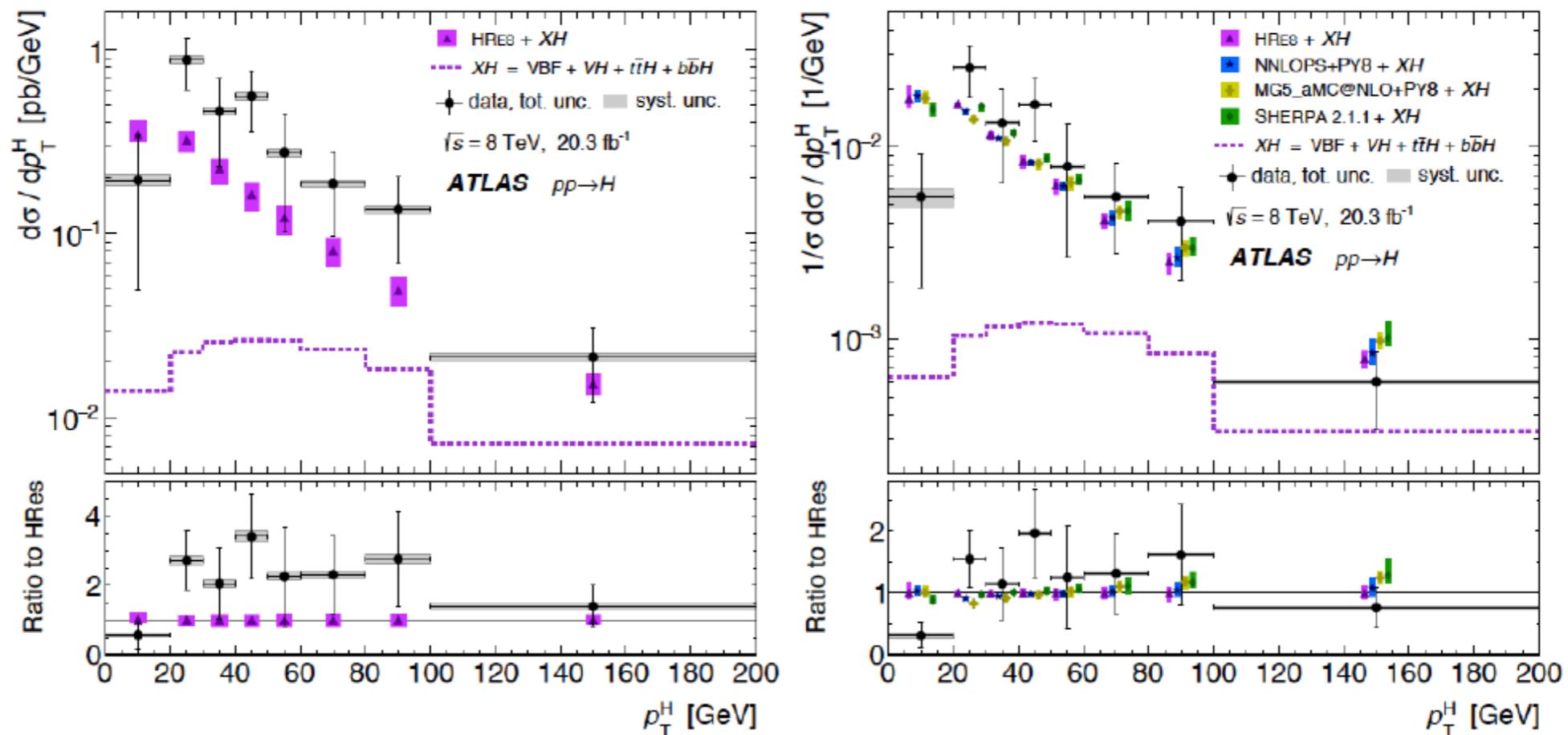
Predictions for the Higgs boson q_T spectrum at LHC (M. Grazzini, 05)



Predictions for the q_T spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.

Logarithms at work

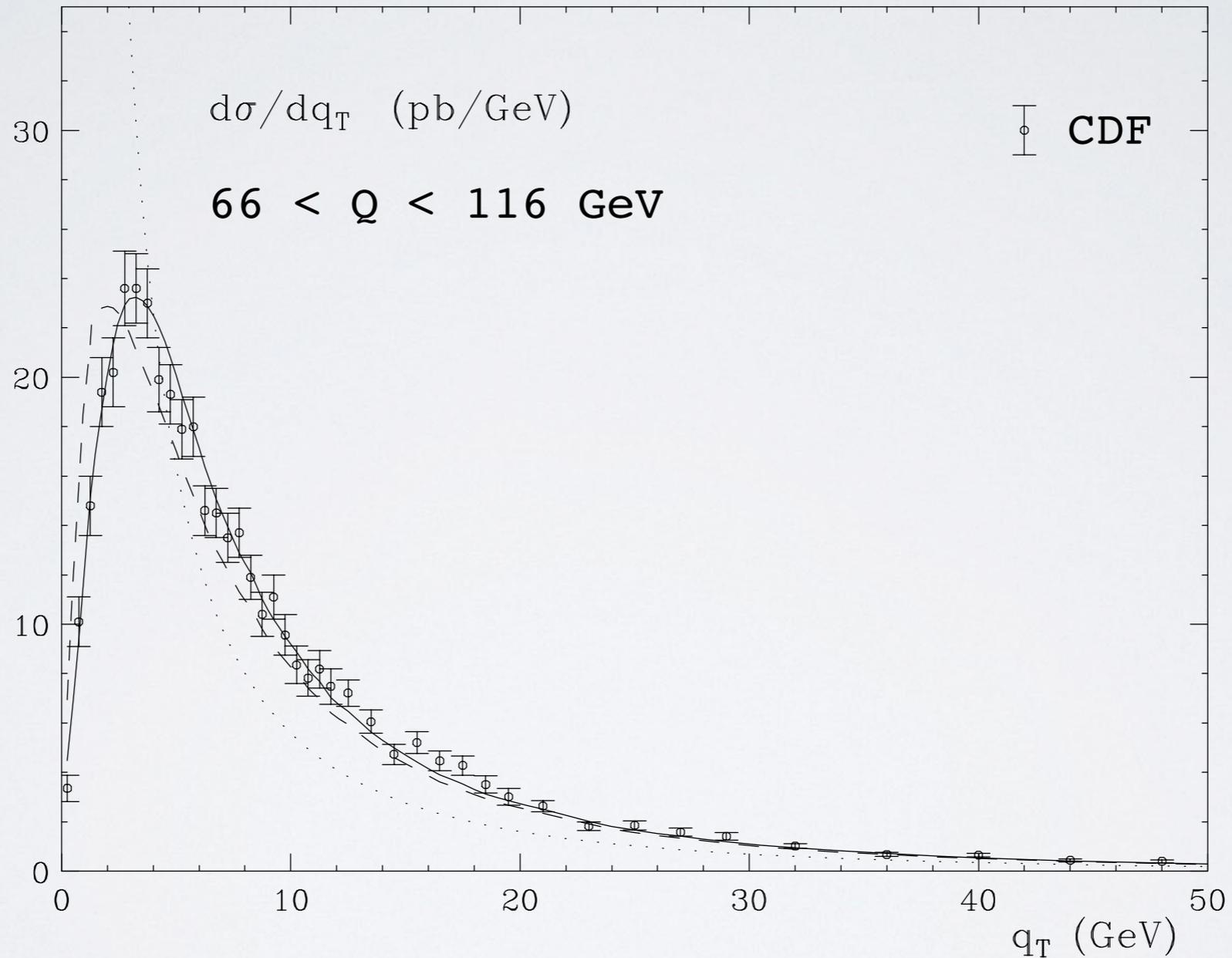
Higgs p_T distribution somewhat shifted to higher transverse momentum but of course statistically limited.



Preliminary ATLAS data on the Higgs p_T distribution from Run I
(shown by Joey Huston at Radcor-Loopfest 2015)

Logarithms at work

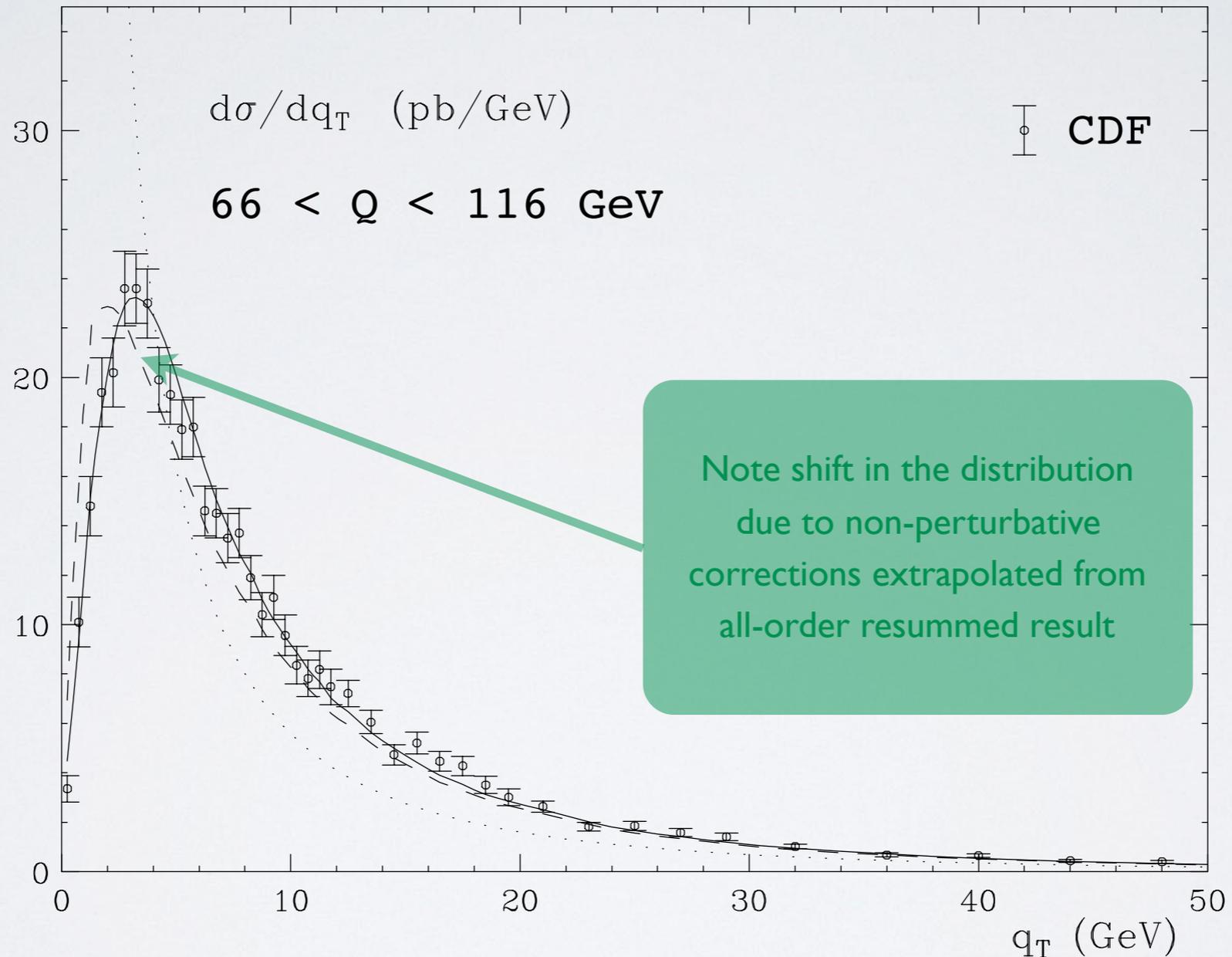
Z-boson q_T spectrum at Tevatron (Kulesza et al. 03)



CDF data on Z production compared with QCD predictions at fixed order (dotted), with joint resummation (dashed), and with the inclusion of power corrections (solid).

Logarithms at work

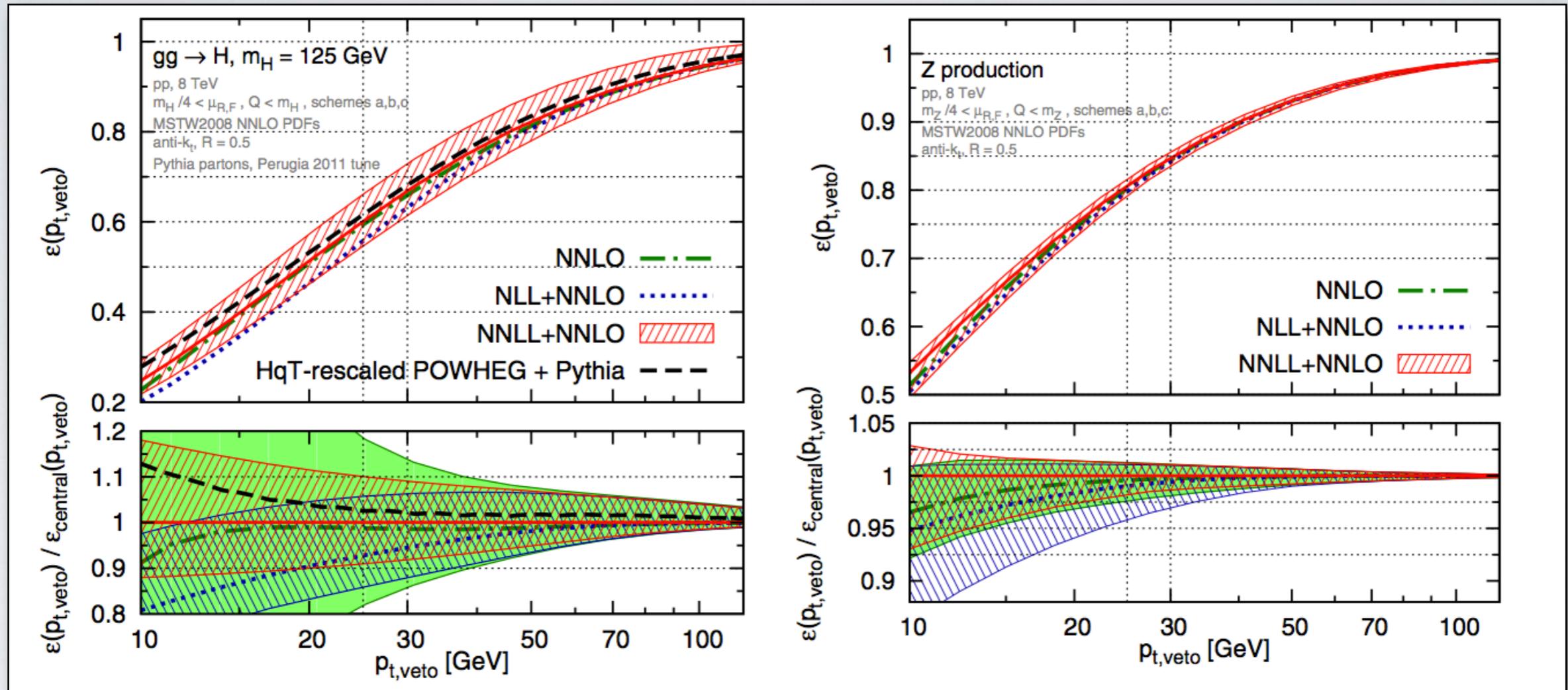
Z-boson q_T spectrum at Tevatron (Kulesza et al. 03)



CDF data on Z production compared with QCD predictions at fixed order (dotted), with (joint) resummation (dashed), and with the inclusion of power corrections (solid).

Logarithms working harder

Jet veto efficiency in Higgs and Z production (Banfi et al. 12)

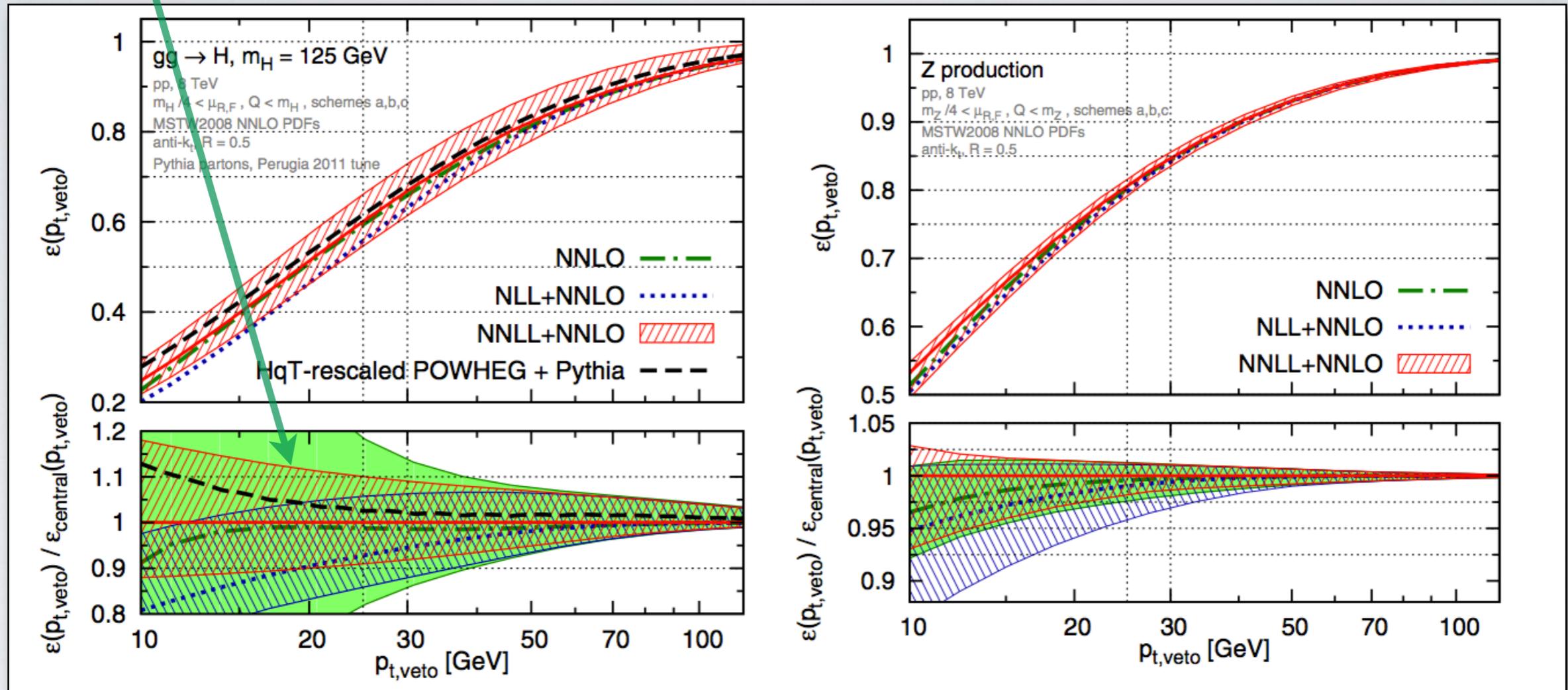


Comparison of NNLO fixed order results and matched resummed NNLL-NNLO results for Higgs production with a jet veto (left) and Z production with a jet veto (right). See also Becher, Neubert, Rothen, 13 for a SCET approach, and further improvements including treatment of quark masses (Banfi et al. 13).

Logarithms working harder

Note the significant reduction of the theoretical uncertainty upon resummation ...

Jet veto efficiency in Higgs and Z production (Banfi et al. 12)



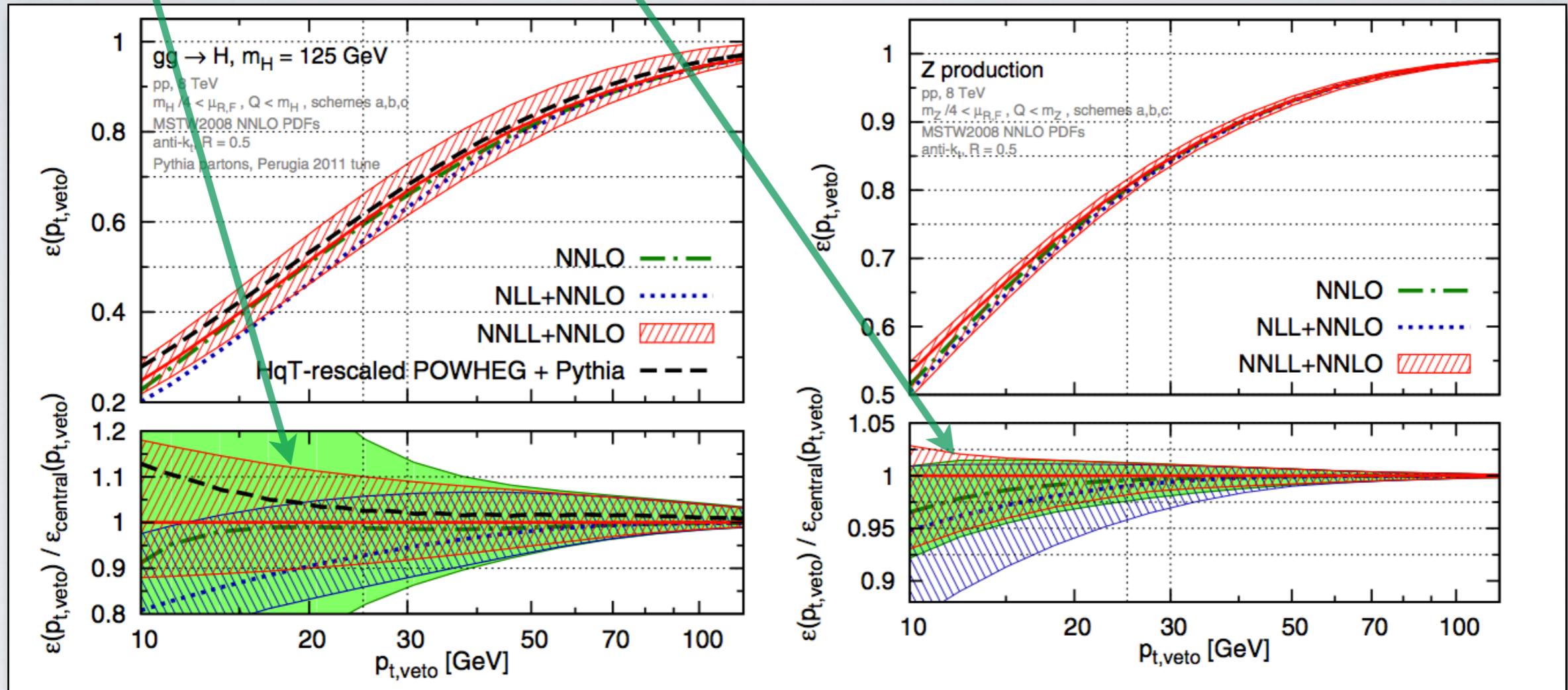
Comparison of NNLO fixed order results and matched resummed NNLL-NNLO results for Higgs production with a jet veto (left) and Z production with a jet veto (right). See also Becher, Neubert, Rothen, 13 for a SCET approach, and further improvements including treatment of quark masses (Banfi et al. 13).

Logarithms working harder

... which does not always take place!

Note the significant reduction of the theoretical uncertainty upon resummation ...

Jet veto efficiency in Higgs and Z production (Banfi et al. 12)



Comparison of NNLO fixed order results and matched resummed NNLL-NNLO results for Higgs production with a jet veto (left) and Z production with a jet veto (right). See also Becher, Neubert, Rothen, 13 for a SCET approach, and further improvements including treatment of quark masses (Banfi et al. 13).

More logarithms

- **Threshold logarithms** are associated with kinematic variables ξ that **vanish** at **Born level** and get **corrections** that are **enhanced** because **phase space** for real radiation is **restricted** near **partonic** threshold: examples are $1 - T$, $1 - M^2/\hat{s}$, $1 - x_{BJ}$.
- At **leading power** in the threshold variable ξ logarithms are **directly related** to **soft** and **collinear divergences**: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+p\epsilon}} = -\frac{1}{p\epsilon} \delta(\xi) + \left(\frac{1}{\xi}\right)_+ - p\epsilon \left(\frac{\log \xi}{\xi}\right)_+ + \dots$$

Cancels virtual IR poles

Leading power threshold logs

- **Beyond** the **leading power**, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_n^{(\delta)} \delta(\xi) + c_{nm}^{(0)} \log^m \xi + \dots \right]$$

Resummed to high accuracy

All-order structure in some cases

NLP threshold logs

- The **structure** of **NLP** threshold logarithms may be understood to **all orders**.

LEADING POWER



Electroweak annihilation

We will **focus** on processes involving **parton annihilation** into **electroweak final states** (**Drell-Yan, Higgs, di-boson** final states): very well **understood** at **LP**, **simpler** at **NLP**.

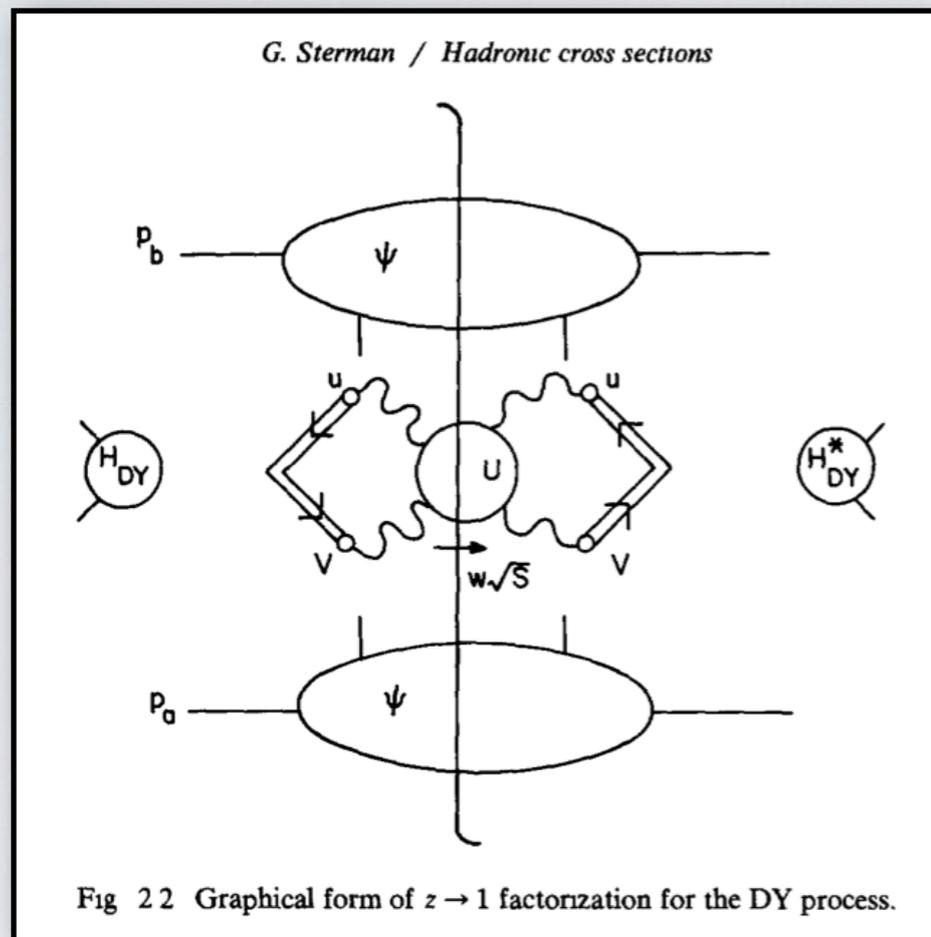
- **LP** threshold **resummation** is based on **factorization**: the **Mellin-space** partonic cross section reads

$$\omega(N, \epsilon) = |H_{\text{DY}}|^2 \psi(N, \epsilon)^2 U(N) + \mathcal{O}\left(\frac{1}{N}\right).$$

- **Collinear** poles can be **subtracted** with suitable parton distributions,

$$\hat{\omega}_{\overline{\text{MS}}}(N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\overline{\text{MS}}}(N, \epsilon)^2}$$

- Each factor in ω obeys **evolution equations** near **threshold**, leading to **exponentiation**.



The original factorization near threshold

- **Real** and **virtual** contributions can be treated **separately**.

$$\psi_R(N, \epsilon) = \exp \left\{ \int_0^1 dz \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \kappa_\psi(\bar{\alpha}((1-y)^2 Q^2), \epsilon) \right\}.$$

Color singlet hard scattering

A well-established formalism exists for **distributions** in processes that are **electroweak at tree level** (Gardi, Grunberg 07). For an observable r **vanishing in the two-jet limit**

$$\frac{d\sigma}{dr} = \delta(r) [1 + \mathcal{O}(\alpha_s)] + C_R \frac{\alpha_s}{\pi} \left\{ \left[-\frac{\log r}{r} + \frac{b_1 - d_1}{r} \right]_+ + \mathcal{O}(r^0) \right\} + \mathcal{O}(\alpha_s^2)$$

The Mellin (Laplace) transform, $\sigma(N) = \int_0^1 dr (1-r)^{N-1} \frac{d\sigma}{dr}$

exhibits **log N** singularities that can be organized in **exponential form**

$$\sigma(\alpha_s, N, Q^2) = H(\alpha_s) \mathcal{S}(\alpha_s, N, Q^2) + \mathcal{O}(1/N)$$

where the exponent of the '**Sudakov factor**' is in turn a Mellin transform

$$\mathcal{S}(\alpha_s, N, Q^2) = \exp \left\{ \int_0^1 \frac{dr}{r} \left[(1-r)^{N-1} - 1 \right] \mathcal{E}(\alpha_s, r, Q^2) \right\}$$

and the general form of the **kernel** is

$$\mathcal{E}(\alpha_s, r, Q^2) = \int_{r^2 Q^2}^{r Q^2} \frac{d\xi^2}{\xi^2} A(\alpha_s(\xi^2)) + B(\alpha_s(r Q^2)) + D(\alpha_s(r^2 Q^2))$$

where **A** is the **cusp** anomalous dimension, and **B** and **D** have **distinct physical characters**.

The perturbative exponent

A classic way to **organize** Sudakov logarithms in terms of the **Mellin (Laplace) transform** of the momentum space cross section (**Catani et al. 93**) is to write

$$\begin{aligned} d\sigma(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=0}^{2n} c_{nk} \log^k N + \mathcal{O}(1/N) \\ &= H(\alpha_s) \exp \left[\log N g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s g_3(\alpha_s \log N) + \dots \right] + \mathcal{O}(1/N) \end{aligned}$$

This displays the main **features of Sudakov resummation**

- Predictive:** a **k**-loop calculation determines **g_k** and thus a whole **tower** of logarithms to all orders in perturbation theory.
- Effective:**
 - the **range of applicability** of perturbation theory is **extended** (finite order: **α_s log²N** small. NLL resummed: **α_s** small);
 - the renormalization **scale dependence** is naturally **reduced**.
- Theoretically interesting:** resummation **ambiguities** related to the **Landau pole** give access to non-perturbative **power-suppressed corrections**.
- Well understood:**
 - NLL** Sudakov resummations **exist** for most **inclusive** observables at hadron colliders, **NNLL** and approximate **N³LL** in simple cases.

Non-logarithms

Delta-function terms arise from **virtual** corrections and **phase space** integration. They yield **constants** in Mellin space (“ π^2 ”) which can be controlled and “**exponentiate**” for simple processes (Parisi 80; Sterman 87; Eynck, Laenen, LM 03; Ahrens, Becher, Neubert, Yang 08),

- For **EW annihilation**, virtual terms reconstruct the **full form factor**.
- In **dimensional regularization**, each term exponentiates with **no prefactor**.

$$\begin{aligned}\omega(N, \epsilon) &= |H_{\text{DY}} \mathcal{R}(\epsilon) \sqrt{U_V(\epsilon)}|^2 \psi_R(N, \epsilon)^2 U_R(N) + \mathcal{O}\left(\frac{1}{N}\right) \\ &= |\Gamma(Q^2, \epsilon)|^2 \psi_R(N, \epsilon)^2 U_R(N, \epsilon) + \mathcal{O}\left(\frac{1}{N}\right),\end{aligned}$$

- **Real** and **virtual** factors are **separately** finite.

$$\hat{\omega}_{\overline{\text{MS}}}(N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\overline{\text{MS}}}(N, \epsilon)^2} = \left(\frac{|\Gamma(Q^2, \epsilon)|^2}{\phi_V(\epsilon)^2}\right) \left(\frac{\psi_R(N, \epsilon)^2 U_R(N, \epsilon)}{\phi_R(N, \epsilon)^2}\right) + \mathcal{O}\left(\frac{1}{N}\right)$$

- An **improved** resummation formula can be written for **DY, DIS** and **Higgs total rates**: **all** constants are **defined** in the exponent.

$$\begin{aligned}\hat{\omega}_{\overline{\text{MS}}}(N) &= \left|\frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)}\right|^2 \left(\frac{\Gamma(-Q^2, \epsilon)}{\phi_V(\epsilon)}\right)^2 \exp\left[F_{\overline{\text{MS}}}(\alpha_s)\right] \times \\ &\times \exp\left[\int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left\{ 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + \right. \right. \\ &\left. \left. + D(\alpha_s((1-z)^2 Q^2)) \right\}\right] + \mathcal{O}\left(\frac{1}{N}\right).\end{aligned}$$

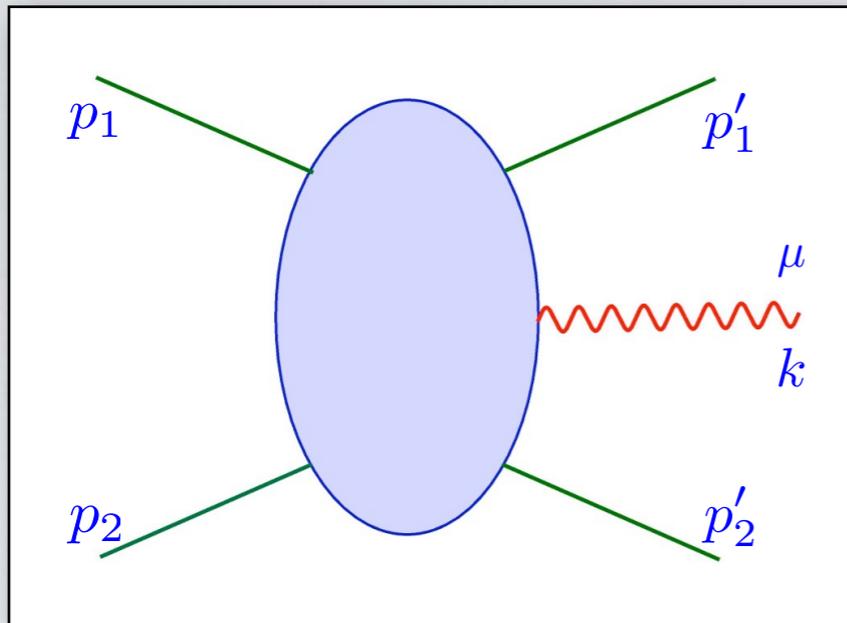
Less predictive than conventional resummation: the exponent receives corrections **order by order**. **Empirically**, exponentiated **lower-order** constants provide **much** of the exact result.

SIFTING FOR EVIDENCE



The LBKD Theorem

The **earliest evidence** that infrared effects can be **controlled** at **NLP** is **Low's theorem** (Low 58)



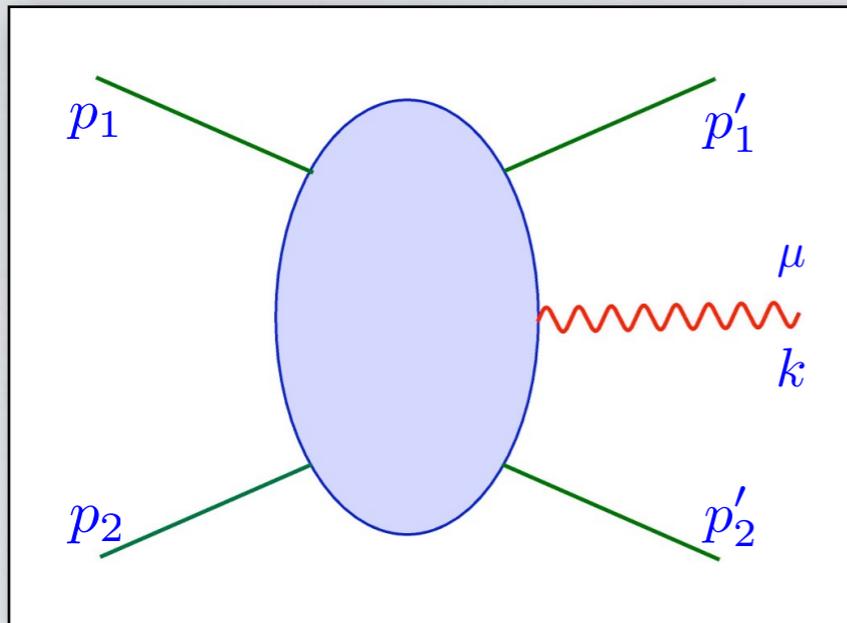
A radiative matrix element

$$M_\mu = e \left(\frac{p_{1\mu}'}{p_1' \cdot k} - \frac{p_{1\mu}}{p_1 \cdot k} \right) T(\nu, \Delta) + e \left(\frac{p_{1\mu}' p_{2\mu}' \cdot k}{p_1' \cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_{2\mu} \cdot k}{p_1 \cdot k} - p_{2\mu} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k),$$

Low's original expression for the radiative matrix element

The LBKD Theorem

The **earliest evidence** that infrared effects can be **controlled** at **NLP** is **Low's theorem** (Low 58)



A radiative matrix element

$$M_\mu = e \left(\frac{p_{1\mu}'}{p_1' \cdot k} - \frac{p_{1\mu}}{p_1 \cdot k} \right) T(\nu, \Delta) + e \left(\frac{p_{1\mu}' p_{2\mu}' \cdot k}{p_1' \cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_{2\mu} \cdot k}{p_1 \cdot k} - p_{2\mu} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k),$$

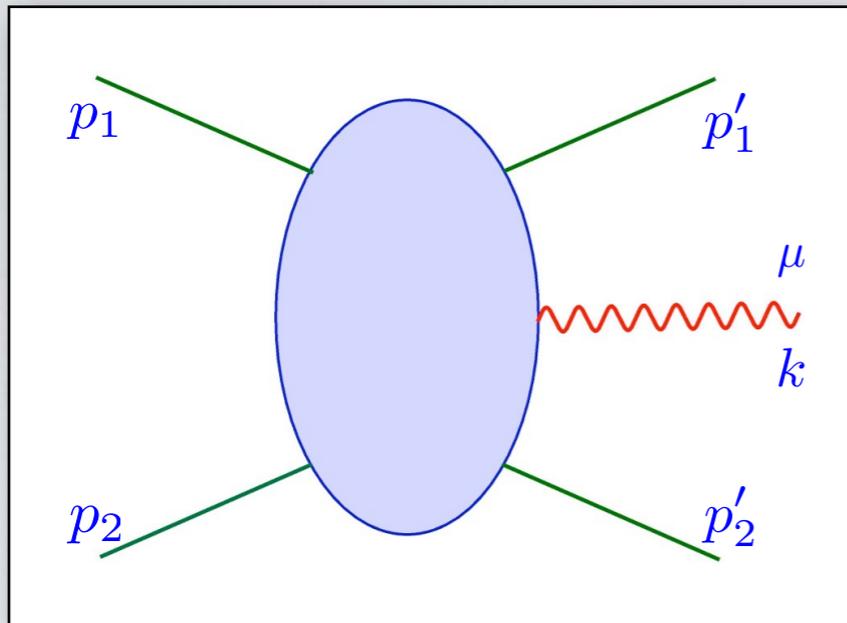
Eikonal approximation

Low's original expression for the radiative matrix element

Next-to-eikonal contribution

The LBKD Theorem

The **earliest evidence** that infrared effects can be **controlled** at **NLP** is **Low's theorem** (Low 58)



A radiative matrix element

$$M_\mu = e \left(\frac{p_{1\mu}'}{p_1' \cdot k} - \frac{p_{1\mu}}{p_1 \cdot k} \right) T(\nu, \Delta) + e \left(\frac{p_{1\mu}' p_{2\mu}' \cdot k}{p_1' \cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_{2\mu} \cdot k}{p_1 \cdot k} - p_{2\mu} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k),$$

Eikonal approximation

Low's original expression for the radiative matrix element

Next-to-eikonal contribution

The **radiative matrix element** for the emission of a (next-to-) soft photon is **determined** by the **Born amplitude T** and its **first derivative** with respect to external momenta.

- Low's result established for a **single charged scalar** particle, follows from **gauge invariance**.
- It **generalizes** the well known properties of soft emissions in the **eikonal approximation**.
- The theorem was **extended** by (Burnett, Kroll 68) to particles with **spin**.
- The **LBK** theorem applies to **massive particles** and uses the **mass** as a **collinear cutoff**.
- It was **extended** to **massless** particles by (Del Duca 90), as discussed **below**.

Modified DGLAP

An **important source** of known **NLP** logarithms is the **DGLAP anomalous dimension**. Non-trivial **connections** between **LP** and **NLP** logarithms in **DGLAP** were **uncovered** (**Moch, Vermaseren, Vogt 08**) and made **systematic** (**Dokshitzer, Marchesini, Salam 08**).

Conventional DGLAP for a quark distribution reads

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}, \mu^2\right) P_{qq}(z, \alpha_s(\mu^2)).$$



$$\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2),$$

The **large-N** behavior of the anomalous dimension is **single-logarithmic** in the MS scheme. **NLP** terms **suppressed** by **N** are **related** to **LP**

$$\begin{aligned} \gamma_N(\alpha_s) = & -A(\alpha_s) \ln \bar{N} + B_\delta(\alpha_s) \\ & - C_\gamma(\alpha_s) \frac{\ln \bar{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned}$$

MVV relations

$$C_1 = 0, \quad C_2 = 4C_F A_1, \quad C_3 = 8C_F A_2. \quad (3.12)$$

Especially the relation for C_3 is very suggestive and seems to call for a structural explanation.

These relation **extend** to the function **D**, and recursively **to all orders: modified** splitting functions can be **defined** which **vanish** at **large x beyond one loop**.

Modified DGLAP

DMS, with refinements implied by (Basso, Korchemsky 06) propose to **modify DGLAP** as

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi\left(\frac{x}{z}, z^\sigma \mu^2\right) \mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right).$$

applying to **both PDF's** and **fragmentation**, with $\sigma = \pm 1$ respectively, and the **same kernel** (Gribov-Lipatov reciprocity). The resulting kernel \mathcal{P} and is claimed to **vanish** as $z \rightarrow 1$ **beyond one loop** in the “physical” MC scheme where $\alpha_s = \gamma_{\text{cusp}}$. Therefore

$$\mathcal{P}(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s)\delta(1-z) + \mathcal{O}(1-z).$$

The modified equation **cannot be diagonalized** by Mellin transform: it must be solved **by iteration**, using a formal translation operator

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} e^{-\ln z (\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2})} \times \psi\left(\frac{x}{z}, \mu^2\right) \mathcal{P}(z, \alpha_s(\mu^2)),$$

In practice, this procedure **constructs** a **modified kernel** where high-order terms are **generated by shifts** of lower-orders

An educated guess

Available **NLP information** can be combined in an **ansatz** for generalized threshold resummation applicable to **EW annihilation** processes and **DIS** (Laenen, LM, Stavenga 08).

$$\ln[\hat{\omega}(N)] = \mathcal{F}_{\text{DY}}(\alpha_s(Q^2))$$

$$+ \int_0^1 dz z^{N-1} \left\{ \frac{1}{1-z} D \left[\alpha_s \left(\frac{(1-z)^2 Q^2}{z} \right) \right] \right.$$

$$\left. + 2 \int_{Q^2}^{(1-z)^2 Q^2/z} \frac{dq^2}{q^2} P_s[z, \alpha_s(q^2)] \right\}_+,$$

Exponentiation of constants

Refinement of phase space

DMS kernel

$$\frac{2}{1-z} \longrightarrow \frac{2z}{1-z}$$

This expression, and similar ones for **DIS** and **Higgs** production via gluon fusion, incorporate

- The **exponentiation** of **N-independent** terms.
- A treatment of **phase space** consistent up to $\mathcal{O}(1-z)$, including **running coupling** effects.
- The **DMS modification** of the **DGLAP** kernel, including the **NLP term** in the **LO kernel**.
- **Note:** **DMS** brings **to the exponent** a C_F^2 contribution **crucial** to **fit** two-loop **NLP** logs.

An educated guess: Drell-Yan

Parametrizing DY with

$$\hat{\omega}(N) = \sum_{i=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n \left[\sum_{m=0}^{2n} a_{nm} \ln^m \bar{N} + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m \bar{N}}{N} \right] + \mathcal{O} \left(\frac{\ln^p N}{N^2} \right),$$

Table 1

Comparison of exact and resummed 2-loop coefficients for the Drell-Yan cross section. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

	C_F^2	$C_A C_F$	$n_f C_F$
b_{23}	4	0	0
b_{22}	$\frac{7}{2}$	$\frac{11}{6}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$

Table 2

Comparison of exact and resummed 2-loop coefficients for the DIS structure function. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

	C_F^2	$C_A C_F$	$n_f C_F$
d_{23}	$\frac{1}{4}$	0	0
d_{22}	$\frac{39}{16}$	$\frac{11}{48}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2 - \frac{431}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2 - \frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2 - \frac{953}{1728}$

- Only **one-loop NLP** and **DMS input** has been used in the resummation formula.
- Leading NLP** logarithms are **reproduced exactly** for **all color structures** at two loops.
- NLL** and **NNLL NLP** logarithms are **well approximated** but **not exact**.
- Similar** results hold for **three-loop DIS**, using **two-loop information** in the exponent.

Towards systematics

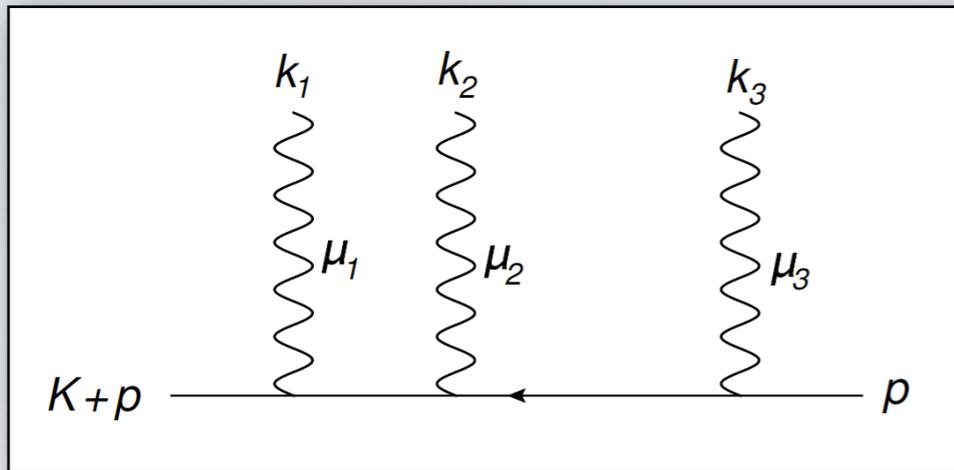
The problem of **NLP** threshold logarithms has been of interest for a **long time**, and several **different approaches** have been proposed. Recent years have seen a **resurgence of interest**, both from a **theoretical** point of view and for **phenomenology**.

- 🎧 **Early attempts** include a study of the impact of **NLP** logs on the **Higgs** cross section by **Kraemer, Laenen, Spira (98)**; work on **F_L** by **Akhoury and Sterman (99)** (logs without plus distributions are however **leading**) and work by **Grunberg et al. (07-09)** on **DIS**.
- 🎧 **Important results** can be obtained by using **physical kernels** (**Vogt et al. 09-14**) which are conjectured to be **single-logarithmic** at large **z** , which poses **constraints** on their **factorized** expression. Note in particular a **recent application** to **Higgs** production by **De Florian, Mazzitelli, Moch, Vogt (14)**.
- 🎧 **Useful approximations** can be obtained by combining **constraints** from **large N** with **high-energy** constraints for **$N \sim 1$** and **analyticity** (**Ball, Bonvini, Forte, Marzani, Ridolfi, 13**), together with **phase space** refinements.
- 🎧 **SCET techniques** can be applied and indeed may be **well-suited** to the problem: a thorough **one-loop** analysis was given in (**Larkoski, Neill, Stewart, 15**).
- 🎧 A lot of recent **formal work** on the behavior of **gauge** and **gravity** scattering **amplitudes beyond the eikonal** limit was triggered by a link to **asymptotic symmetries** of the **S** matrix (**many authors from A(ndy Strominger) to Z(vi Bern), 14-15**).

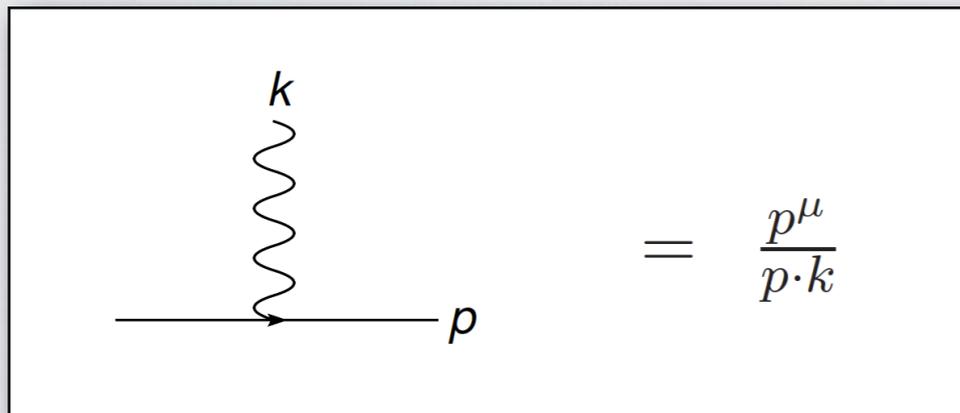
NEXT-TO-SOFT



On the eikonal approximation



A fast particle emitting soft photons



Eikonal Feynman rule

- Taking the **soft approximation** at **leading power** on emissions from an **energetic** (or very **massive**) particle yields a set of **simplified Feynman rules**.
- These rules correspond to emissions from a **Wilson line** oriented **along the trajectory** of the energetic particle, in the **same color** irrep.

$$\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n) \right] .$$

- The results **do not depend on the energy** and **spin** of the emitter, only on its **direction** and **color charge**.
- **Physically**, we are **neglecting** the **recoil** of the emitter: the only effect of interaction with soft radiation is that the **emitter acquires a phase**.
- The **soft** limit of a **multi-particle** amplitude is a **correlator of Wilson lines**

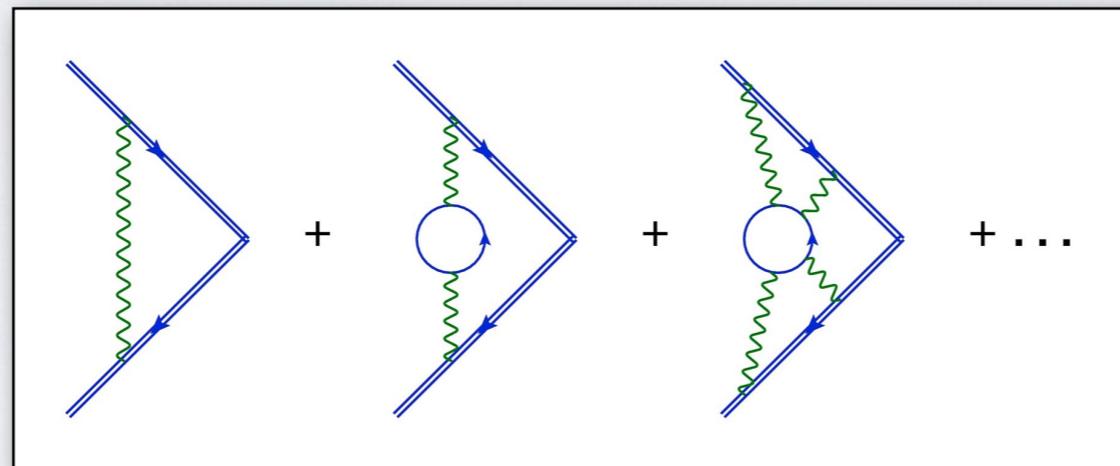
Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in **exponential form**.

$$S_n \equiv \langle 0 | \Phi_1 \otimes \dots \otimes \Phi_n | 0 \rangle = \exp(\omega_n)$$

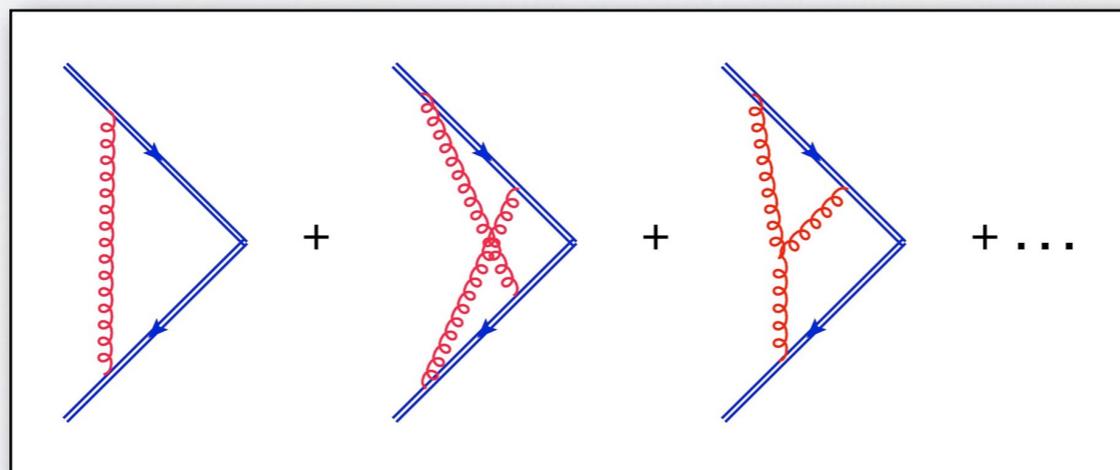
Diagrammatic rules exist to compute **directly the logarithm** of the correlators.

$$\omega_{2,\text{QED}} =$$



Only **connected** photon **subdiagrams** contribute to the logarithm.

$$\omega_{2,\text{QCD}} =$$



Only gluon **subdiagrams** which are **two-eikonal irreducible** contribute to the logarithm. They have **modified color factors**.

For **eikonal form factors**, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).

Multiparticle webs

The concept of **web** generalizes non-trivially to the case of **multiple Wilson lines**.
(Gardi, Smillie, White, et al).

A **web** is a **set of diagrams** which **differ** only by the **order** of the **gluon attachments** on each Wilson line. They are **weighted** by **modified color factors**.

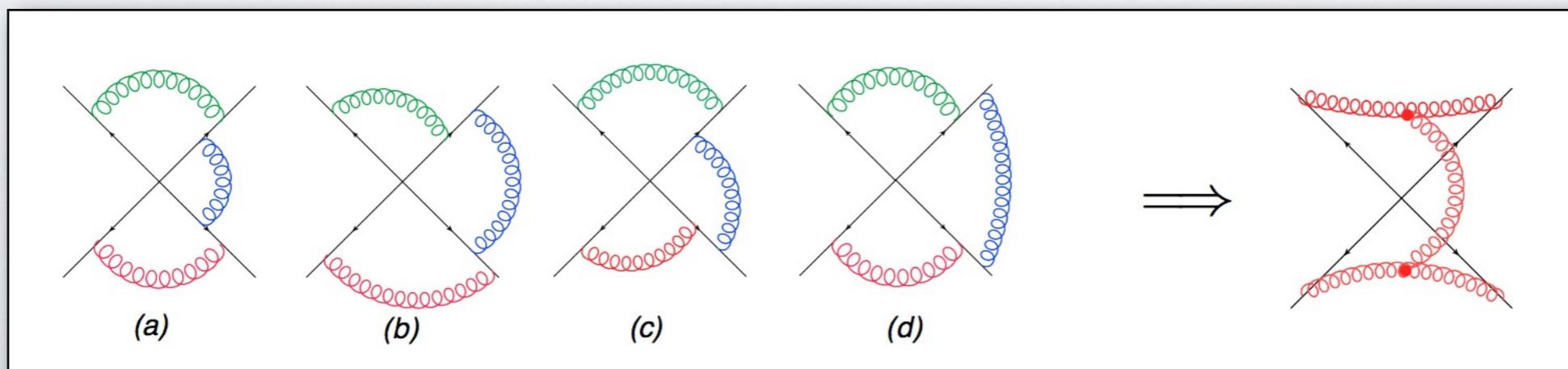
Writing each diagram as the product of its natural **color** factor and a **kinematic** factor

$$D = C(D)\mathcal{F}(D)$$

a **web W** can be expressed as a **sum of diagrams** in terms of a **web mixing matrix R**

$$W = \sum_D \tilde{C}(D)\mathcal{F}(D) = \sum_{D,D'} C(D')R(D',D)\mathcal{F}(D)$$

The **non-abelian exponentiation theorem** holds: each web has the color factor of a **fully connected** gluon subdiagram (Gardi, Smillie, White).



Beyond the eikonal

The **soft expansion** can be **organized beyond leading power** using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic **idea** is **simple**, but the combinatorics **cumbersome**. For **spinors**

$$\frac{\not{p} + \not{k}}{2p \cdot k + k^2} \gamma^\mu u(p) = \left[\frac{p^\mu}{p \cdot k} + \frac{\not{k} \gamma^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} \right] u(p) + \mathcal{O}(k)$$



Eikonal



NE, spin-dependent



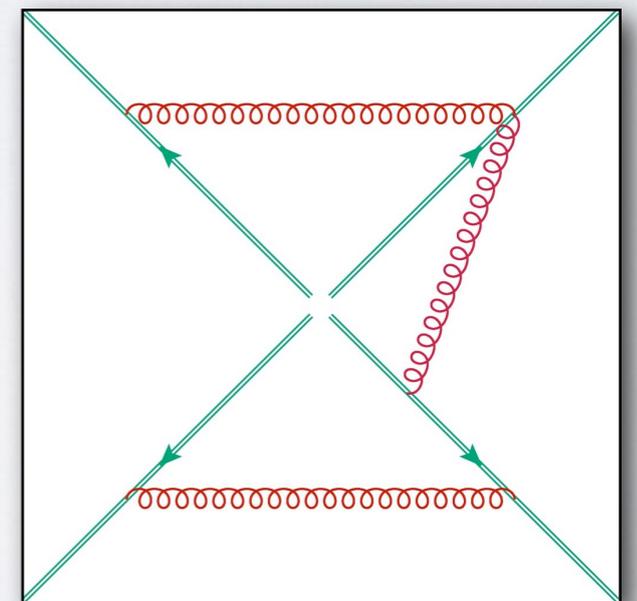
NE, spin-independent

- A class of **factorizable** contributions **exponentiate** via **NE webs**

$$\mathcal{M} = \mathcal{M}_0 \exp \left[\sum_{D_{\text{eik}}} \tilde{C}(D_{\text{eik}}) \mathcal{F}(D_{\text{eik}}) + \sum_{D_{\text{NE}}} \tilde{C}(D_{\text{NE}}) \mathcal{F}(D_{\text{NE}}) \right].$$

- **Feynman rules** exist for the **NE** exponent, including “**seagull**” vertices.

$$\mathcal{M} = \mathcal{M}_0 \exp [\mathcal{M}_{\text{eik}} + \mathcal{M}_{\text{NE}}] (1 + \mathcal{M}_r) + \mathcal{O}(\text{NNE}).$$



A next-to-eikonal web

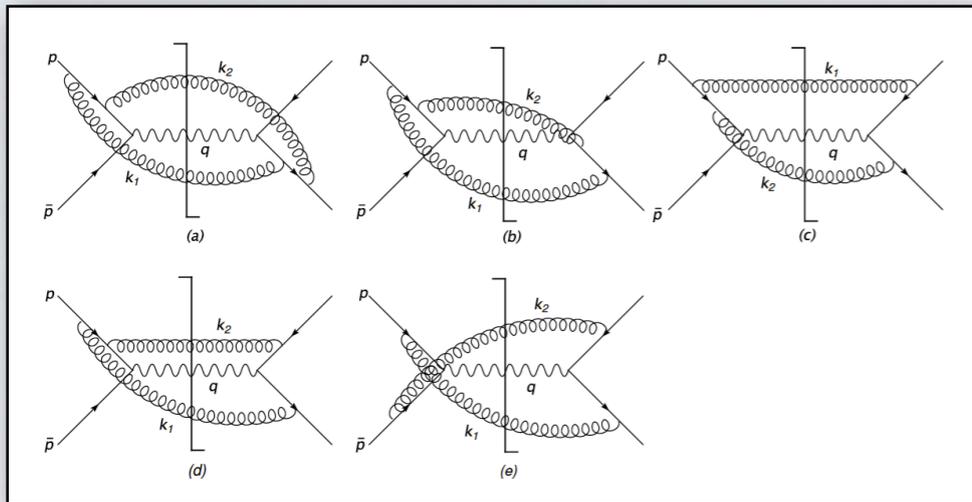
- **Non-factorizable** contributions involve **single gluon emission** from inside the **hard function**, and must be studied using **LBDK's theorem**.

Double real two-loop Drell-Yan

Multiple real emission contributions to EW annihilation processes involve **only factorizable** contributions. **NE Feynman rules** can be **tested** this level.

Defining the **Drell-Yan K-factor** as

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz},$$



Real emission Feynman diagrams for the abelian part of the NNLO K-factor.

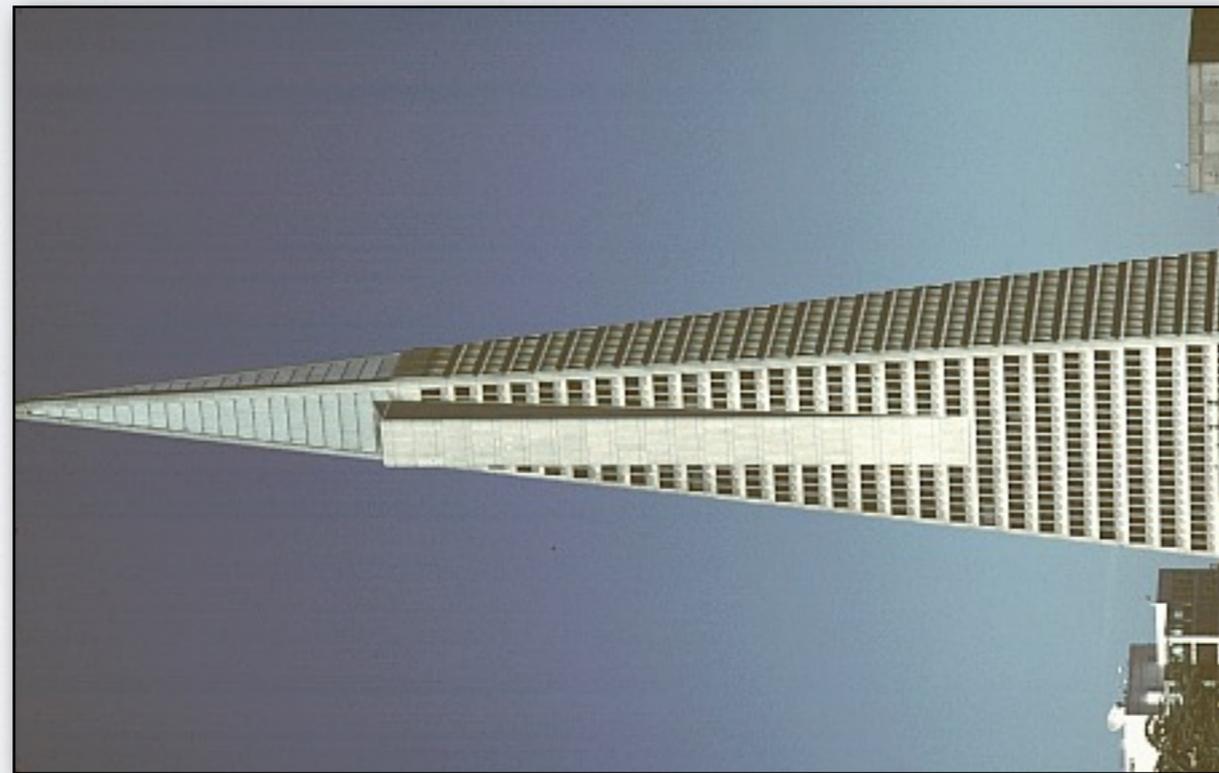
As a **test**, we (re)computed the C_F^2 part of **K** at **NNLO** from **ordinary Feynman diagrams**, and then **using NE Feynman rules**, finding complete **agreement**. As expected, plus distributions arise from the eikonal approximation.

Next-to-eikonal terms arise from **single-gluon** corrections: **seagull-type** contributions **vanish** for the **inclusive** cross section.

$$K_{\text{NE}}^{(2)}(z) = \left(\frac{\alpha_s}{4\pi} C_F\right)^2 \left[-\frac{32}{\epsilon^3} \mathcal{D}_0(z) + \frac{128}{\epsilon^2} \mathcal{D}_1(z) - \frac{128}{\epsilon^2} \log(1-z) \right. \\ \left. - \frac{256}{\epsilon} \mathcal{D}_2(z) + \frac{256}{\epsilon} \log^2(1-z) - \frac{320}{\epsilon} \log(1-z) \right. \\ \left. + \frac{1024}{3} \mathcal{D}_3(z) - \frac{1024}{3} \log^3(1-z) + 640 \log^2(1-z) \right],$$

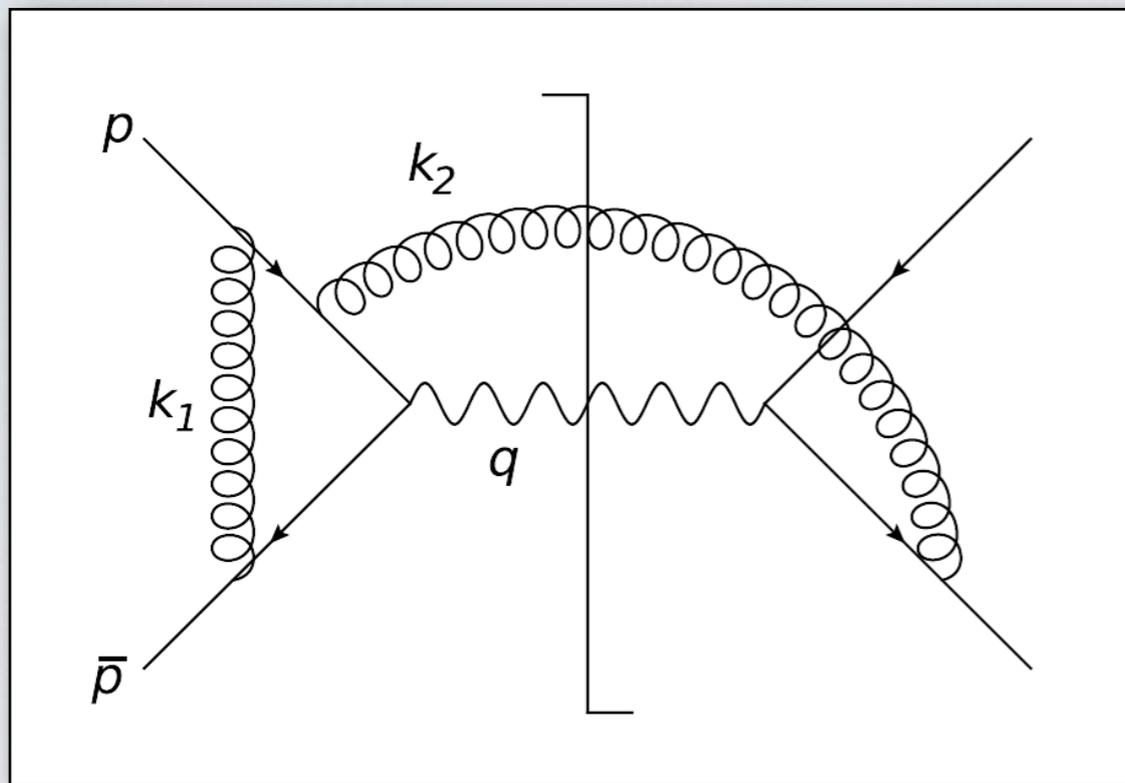
The abelian part of the NNLO K-factor from real emission, omitting constants

COLLINEAR EMISSION



A collinear problem

Non-factorizable contributions start at **NNLO**. For **massive** particles they can be traced to the **original LBK** theorem. For **massless** particles a **new contribution** to **NLP** logs **emerges**.

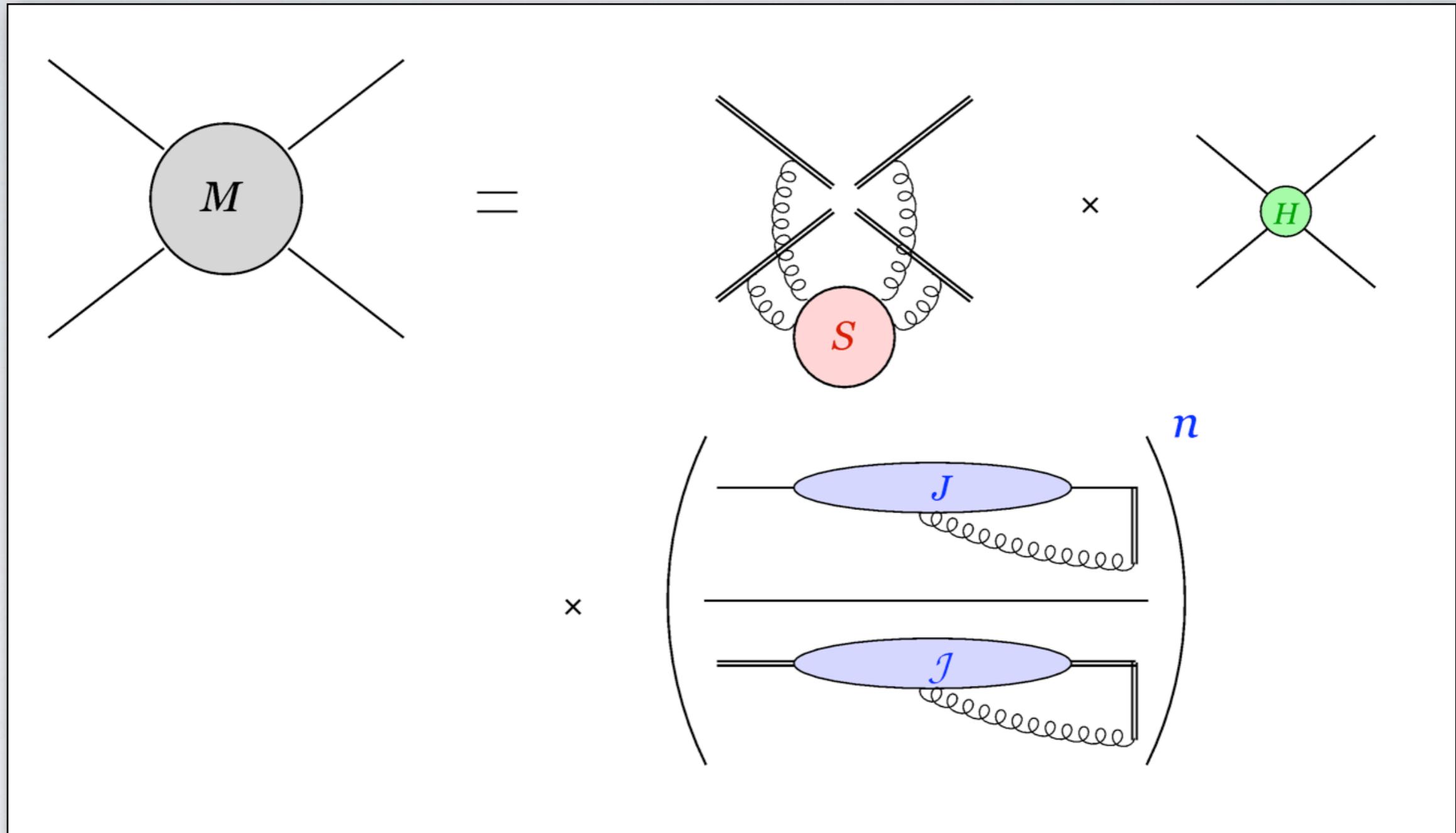


A Feynman diagram containing a collinear enhancement

- Gluon k_2 is **always** (next-to) **soft** for **EW** annihilation **near threshold**.
- When k_1 is (next-to) **soft** all logs are **captured** by **NE** rules.
- Contributions with k_1 **hard** and **collinear** are **missed** by the soft expansion.
- The **collinear pole** interferes with **soft emission** and generates **NLP** logs.
- The problem **first arises** at **NNLO**

- These contributions are **missed** by the **LBK** theorem: it applies to an **expansion** in E_k/m .
- They can be **analyzed** using the **method of regions**: the relevant **factor** is $(p \cdot k_2)^{-\epsilon}/\epsilon$.
- They **cause** the **breakdown** of **next-to-soft theorems** for amplitudes **beyond tree level**.
 ➡ the **soft** expansion and the limit $\epsilon \rightarrow 0$ **do not commute**.
- They **require** an **extension** of **LBK** to $m^2/Q < E_k < m$. It was **provided** by **Del Duca (90)**.

LP factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

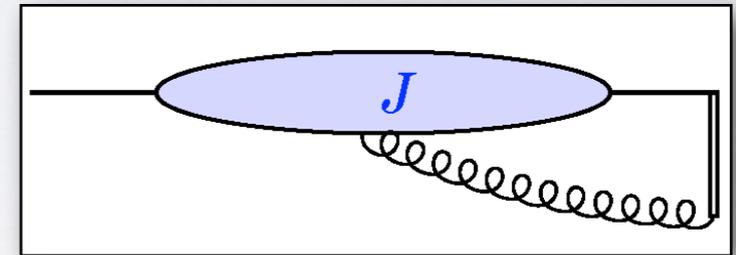
LP factorization: operators

The precise **functional form** of this graphical factorization is

$$\mathcal{M}_L(p_i/\mu, \alpha_s(\mu^2), \epsilon) = \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right) \times \prod_{i=1}^n \left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) / \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right) \right],$$

Here we introduced dimensionless **four-velocities** $\beta_i^\mu = Q p_i^\mu$, $\beta_i^2 = 0$, and **factorization vectors** n_i^μ , $n_i^2 \neq 0$ to define the jets,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle.$$

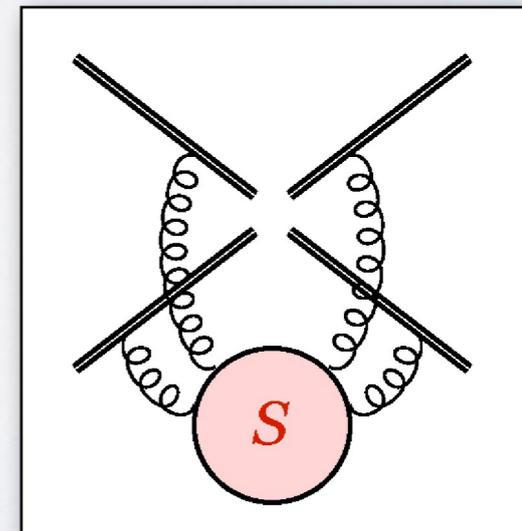


where Φ_n is the **Wilson line** operator along the direction n^μ .

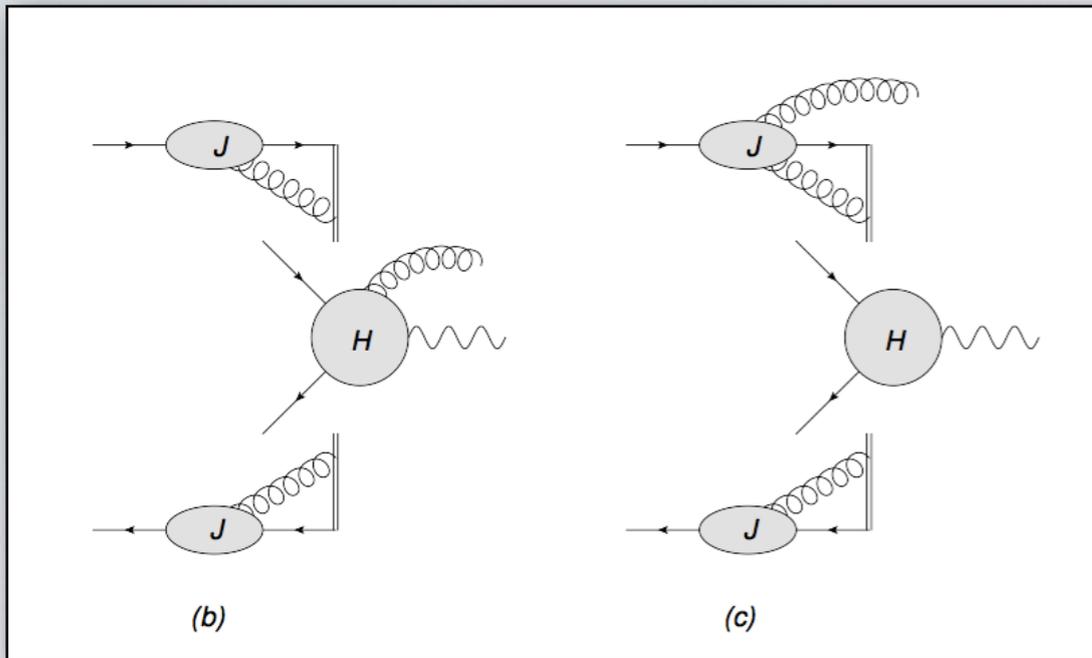
The **soft function S** is a **matrix**, mixing the available color tensors. It is defined by a correlator of **Wilson lines**.

$$(c_L)_{\{a_k\}} \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \epsilon) = \langle 0 | \prod_{k=1}^n [\Phi_{\beta_k}(\infty, 0)]_{a_k}^{b_k} | 0 \rangle (c_K)_{\{b_k\}},$$

where the c_L are a **basis** of **color tensors** for the process at hand.



NLP factorization: a new jet



Soft radiation can arise either from the jets or from the hard function

$$\mathcal{A}_\mu \epsilon^\mu(k) = \mathcal{A}_\mu^J \epsilon^\mu(k) + \mathcal{A}_\mu^H \epsilon^\mu(k),$$

The amplitude for emission from the jets can be precisely defined in terms of a new jet function

$$\mathcal{A}_\mu^J = \sum_{i=1}^2 H(p_i - k; p_j, n_j) J_\mu(p_i, k, n_i) \prod_{j \neq i} J(p_j, n_j) \equiv \sum_{i=1}^2 \mathcal{A}_\mu^{J_i}.$$

Factorized contributions to the radiative amplitude

$$J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) u(p) = \int d^d y e^{-i(p-k) \cdot y} \langle 0 | \Phi_n(y, \infty) \psi(y) j_\mu(0) | p \rangle,$$

defines the radiative jet.

- At tree level the radiative jet displays the expected dependence on spin.
- Dependence on the gauge vector n^μ starts at loop level: simplifications arise for $n^2 = 0$.

$$\begin{aligned} J^{\nu(0)}(p, n, k) &= \frac{\not{k} \gamma^\nu}{2p \cdot k} - \frac{p^\nu}{p \cdot k} \\ &= -\frac{p^\nu}{p \cdot k} + \frac{k^\nu}{2p \cdot k} - \frac{i k_\alpha \Sigma^{\alpha\mu}}{2p \cdot k}. \end{aligned}$$

Beyond Low's theorem

A **slightly modified** version of **Del Duca's** result gives the **radiative amplitude** in terms of the **non-radiative** one, its **derivatives**, and the **two "jet"** functions.

$$\mathcal{A}^\mu(p_j, k) = \sum_{i=1}^2 \left\{ q_i \left(\frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \right) + G_i^{\nu\mu} \left[\frac{J_\nu(p_i, k, n_i)}{J(p_i, n_i)} - q_i \frac{\partial}{\partial p_i^\nu} \left(\ln J(p_i, n_i) \right) \right] \right\} \mathcal{A}(p_i; p_j).$$

The tensors $G^{\mu\nu}$ **project out the eikonal** contribution present in the first term.

$$\eta^{\mu\nu} = G^{\mu\nu} + K^{\mu\nu}, \quad K^{\mu\nu}(p; k) = \frac{(2p - k)^\nu}{2p \cdot k - k^2} k^\mu,$$

The **factorized** expression for the **radiative** amplitude can be **simplified**.

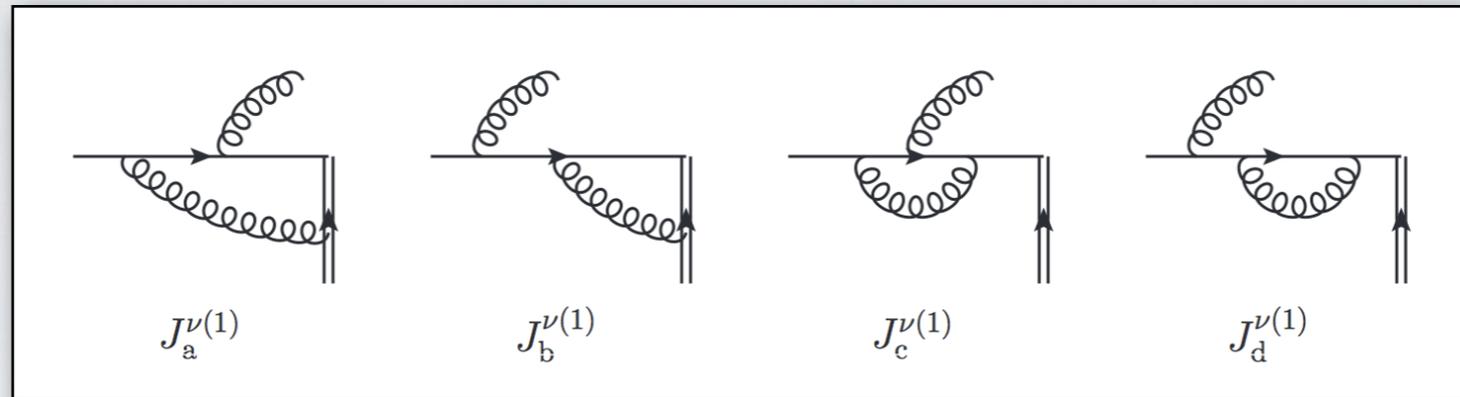
- The **jet factor** is **RG invariant**: it can be computed in **bare** perturbation theory.
- With this choice one can use that $J(p, n) = 1$ for $n^2 = 0$: it is a **pure counterterm**.
- The **choice of reference vectors** is then **physically motivated** (and **confirmed** by a complete analysis using the **method of regions**): we take $n_1 = p_2$ and $n_2 = p_1$.

$$\mathcal{A}^\mu(p_j, k) = \sum_{i=1}^2 \left(q_i \frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + q_i G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} + G_i^{\nu\mu} J_\nu(p_i, k) \right) \mathcal{A}(p_i; p_j).$$

For **general amplitudes**, a **full subtraction** of the residual **n** dependence should be **aimed at**.

The one-loop radiative jet

To achieve **NNLO accuracy** at **NLP**, we need the **radiative jet** function at **one loop**.
As a **test**, we compute the C_F^2 contributions. They **simplify** considerably for $n^2 = 0$.



Abelian-like Feynman diagrams for the bare one-loop radiative jet

The result has a **simple structure**, with characteristic **scale** and **spin** dependence

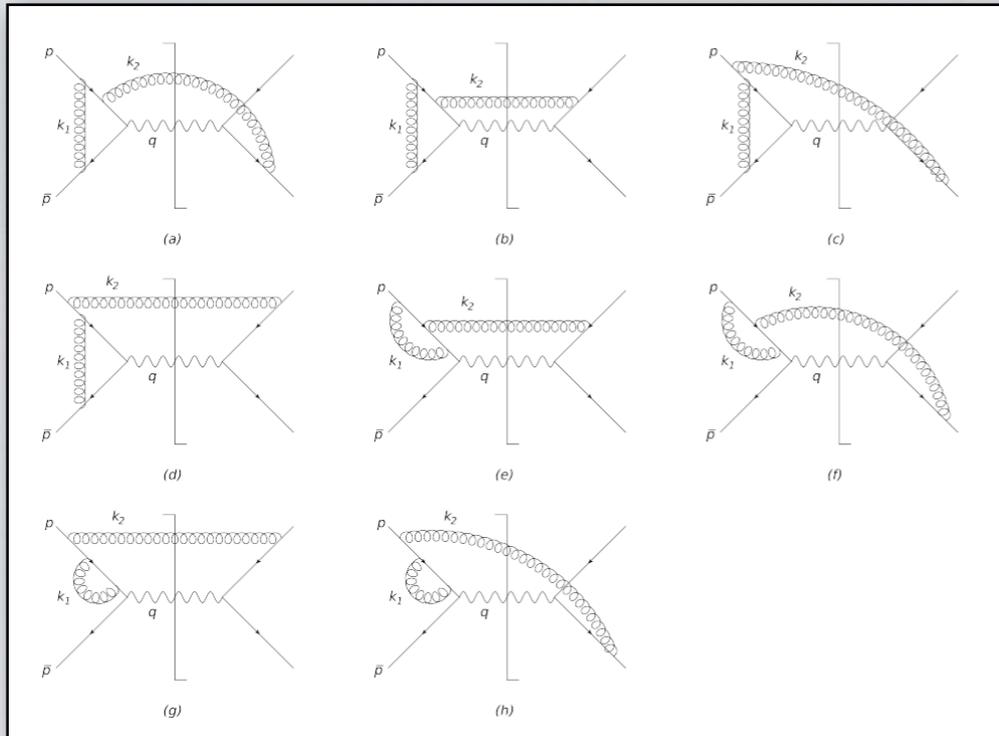
$$J^{\nu(1)}(p, n, k; \epsilon) = (2p \cdot k)^{-\epsilon} \left[\left(\frac{2}{\epsilon} + 4 + 8\epsilon \right) \left(\frac{n \cdot k}{p \cdot k} \frac{p^\nu}{p \cdot n} - \frac{n^\nu}{p \cdot n} \right) - (1 + 2\epsilon) \frac{i k_\alpha \Sigma^{\alpha\nu}}{p \cdot k} \right. \\ \left. + \left(\frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon \right) \frac{k^\nu}{p \cdot k} + (1 + 3\epsilon) \left(\frac{\gamma^\nu \not{n}}{p \cdot n} - \frac{p^\nu \not{k} \not{n}}{p \cdot k p \cdot n} \right) \right] + \mathcal{O}(\epsilon^2, k),$$

As a test, it **obeys** the **Ward identity**

$$k^\mu J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) = q J(p, n, \alpha_s(\mu^2), \epsilon),$$

Real-virtual two-loop Drell-Yan

Real-virtual corrections to EW annihilation processes involve non-factorizable contributions. NE rules cannot reproduce the perturbative result at NLP, due to collinear interference.



Real-virtual Feynman diagrams for the abelian part of the NNLO K-factor.

As a test of the LBDK factorization, we computed the C_F^2 part of the real-virtual K-factor at NNLO from ordinary Feynman diagrams, and then using the radiative amplitude integrated over phase space. As expected, plus distributions arise from the eikonal approximation, fully determined by the dressed non-radiative amplitude. Derivative terms and the projected radiative jet contribute at NLP.

- All NLP terms are correctly reproduced, including those with no logarithms.
- The radiative jet reproduces exactly the NLP collinear contribution derived by the method of regions.

$$K_{\text{rv}}^{(2)}(z) = \left(\frac{\alpha_s}{4\pi} C_F\right)^2 \left\{ \frac{32}{\epsilon^3} [\mathcal{D}_0(z) - 1] + \frac{16}{\epsilon^2} [-4\mathcal{D}_1(z) + 3\mathcal{D}_0(z) + 4L(z) - 6] \right. \\ \left. + \frac{4}{\epsilon} [16\mathcal{D}_2(z) - 24\mathcal{D}_1(z) + 32\mathcal{D}_0(z) - 16L^2(z) + 52L(z) - 49] \right. \\ \left. - \frac{128}{3}\mathcal{D}_3(z) + 96\mathcal{D}_2(z) - 256\mathcal{D}_1(z) + 256\mathcal{D}_0(z) \right. \\ \left. + \frac{128}{3}L^3(z) - 232L^2(z) + 412L(z) - 408 \right\},$$

The abelian part of the NNLO K-factor from real-virtual diagrams, omitting constants

OUTLOOK



A Perspective

- Perturbation theory continues to display **new** and **unexplored** structures.
- Leading power threshold resummation is **highly developed** and provides some of the **most precise** predictions in perturbative **QCD**.
- Mellin-space **constants** naturally **reside in the exponent** for simple processes.
- Low's theorem is the first of **many hints** that **NLP logs** can be understood and **organized**.
- Different approaches catch a number of **towers** of **NLP logs** in simple processes.
- The **next-to-soft** approximation is **well understood**, using both **diagrammatic** and **path integral** approaches, even for **multi-parton** processes.
- **Hard collinear** emissions **spoil** Low's theorem: a **new** radiative **jet function** emerges.
- A **complete treatment** of **NLP threshold logs** is at hand.
- **Much work to do** to organize a true **resummation** formula, even for **EW** annihilation: we have a more **intricate** “**factorization**”, we must make sure to control **double countings**.
- In order to achieve **complete generality**, we will need to include **final state jets**.

THANK YOU!