Factorization and Universality for massless gauge theory amplitudes

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Outline

Introduction
  History
  Motivation
  Tools

Form factors
  Detailed factorization
  Evolution equations

Results
  Form factors
  Maximal SYM
  Single poles

Perspective
Ancient History

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Note on the Radiation Field of the Electron

F. BLOCH AND A. NORDSIECK*
Stanford University, California
(Received May 14, 1937)

Previous methods of treating radiative corrections in non-
stationary processes such as the scattering of an electron in
an atomic field or the emission of a β-ray, by an expansion
in powers of $e^2/\hbar c$, are defective in that they predict infinite
low frequency corrections to the transition probabilities.
This difficulty can be avoided by a method developed here
which is based on the alternative assumption that $e^2\omega/mc^3$,
$\hbar\omega/mc^2$ and $\hbar\omega/\epsilon\Delta p$ ($\omega =$ angular frequency of radiation,
$\Delta p =$ change in momentum of electron) are small compared
to unity. In contrast to the expansion in powers of $e^2/\hbar c$,
this permits the transition to the classical limit $\hbar = 0$.

External perturbations on the electron are treated in the
Born approximation. It is shown that for frequencies such
that the above three parameters are negligible the quantum
mechanical calculation yields just the directly reinterpreted
results of the classical formulae, namely that the total
probability of a given change in the motion of the electron
is unaffected by the interaction with radiation, and that
the mean number of emitted quanta is infinite in such a way
that the mean radiated energy is equal to the energy
radiated classically in the corresponding trajectory.

A remarkable achievement, before quantum field theory was born.
Modern History

Factorization

\[
\mathcal{M}^{[f]} \left( \beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \sum_{L=1}^{N^{[f]}} \mathcal{M}^{[f]}_L \left( \beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) (c_L)^{\{r_i\}} 
\]

\[
\mathcal{M}^{[f]}_L \left( \beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \prod_{i=1}^{n+2} J^{[i]} \left( \frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) 
\times S^{[f]}_{LI} \left( \beta_j, \frac{Q'^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2), \epsilon \right) H^{[f]}_I \left( \beta_j, \frac{Q^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2) \right) 
\]

Progress

- Exponentiation applies to non-abelian gauge theories.
- Exponentiation extends to collinear divergences.
- Exponentiation is performed at the amplitude level.
- An optimal regularization scheme is used.
Motivation: LHC phenomenology

Higgs boson spectrum at LHC  (M. Grazzini, hep-ph/0512025)

Predictions for the $q_T$ spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation.
Motivation: LHC phenomenology

Z boson spectrum at Tevatron  (A. Kulesza et al., hep-ph/0207148)

CDF data on Z production compared with QCD predictions at fixed order (dotted), with resummation (dashed), and with the inclusion of power corrections (solid).
Motivation: gauge field theories

- **Remarkable progress** has been achieved in *techniques* to compute *finite order* gauge theory amplitudes.

- *Supersymmetric versions* of *Yang-Mills theory* and *QCD* have remarkable properties.
  
  **Example:** $\mathcal{N} = 4$ SYM is *conformal invariant*: $\beta_{\mathcal{N}=4}(\alpha_s) = 0$.
  
  - *Exponentiation* of IR/C poles in QCD amplitudes *simplifies*
    
    **Note:** at most *double* poles in the exponent.
  
  - *AdS/CFT* suggests that $\mathcal{N} = 4$ SYM must ‘be simple’ *at strong coupling*. Can this be seen in *perturbation theory*?
  
  - *Exponentiation* has been observed for *MHV* amplitudes with up to *five legs* (Z. Bern *et al.*).
  
  - A *stringy calculation* at *strong coupling* is consistent with the *perturbative result* (L. Alday and J. Maldacena).
Tools: dimensional regularization

Nonabelian exponentiation of IR poles requires *d-dimensional* evolution equations. The *running coupling* in \( d = 4 - 2\epsilon \) obeys

\[
\mu \frac{\partial \bar{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \bar{\alpha}) = -2\epsilon \bar{\alpha} + \hat{\beta}(\bar{\alpha}) \quad , \quad \hat{\beta}(\bar{\alpha}) = -\frac{\bar{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left( \frac{\bar{\alpha}}{\pi} \right)^n .
\]

The *one-loop* solution is

\[
\bar{\alpha}(\mu^2) = \alpha_s(\mu_0^2) \left[ \left( \frac{\mu^2}{\mu_0^2} \right)^\epsilon - \frac{1}{\epsilon} \left( 1 - \left( \frac{\mu^2}{\mu_0^2} \right)^\epsilon \right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} .
\]

The \( \beta \) function develops an *IR free* fixed point, so that \( \bar{\alpha}(0, \epsilon) = 0 \) for \( \epsilon < 0 \). The Landau pole is at

\[
\mu^2 = \Lambda^2 \equiv Q^2 \left( 1 + \frac{4\pi \epsilon}{b_0 \alpha_s(Q^2)} \right)^{-1/\epsilon} .
\]
Tools: factorization

All factorizations separating dynamics at different energy scales lead to resummation of logarithms of the ratio of scales.

- Renormalization group logarithms.
  Renormalization factorizes cutoff dependence

\[
G_0^{(n)}(p_i, \Lambda, g_0) = \prod_{i=1}^{n} Z_i^{1/2}(\Lambda/\mu, g(\mu)) \cdot G_R^{(n)}(p_i, \mu, g(\mu)),
\]

\[
\frac{dG_0^{(n)}}{d\mu} = 0 \quad \rightarrow \quad \frac{d\log G_R^{(n)}}{d\log \mu} = -\sum_{i=1}^{n} \gamma_i(g(\mu)).
\]

- RG evolution resums \( \alpha_s^n(\mu^2) \log^n \left( Q^2 / \mu^2 \right) \) into \( \alpha_s(Q^2) \).

Note: Factorization is the difficult step . . .
Tools: factorization

- **Collinear factorization** logarithms.

  Mellin moments of partonic DIS structure functions factorize

  \[
  \widetilde{F}_2 \left( N, \frac{Q^2}{m^2}, \alpha_s \right) = \widetilde{C} \left( N, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \widetilde{f} \left( N, \frac{\mu_F^2}{m^2}, \alpha_s \right)
  \]

  \[
  \frac{d\widetilde{F}_2}{d\mu_F} = 0 \quad \rightarrow \quad \frac{d\log \widetilde{f}}{d\log \mu_F} = \gamma_N (\alpha_s) .
  \]

- **Altarelli-Parisi** evolution resums collinear logarithms into evolved parton distributions.

  Note: *Sudakov* logarithms are more difficult. Ordinary renormalization group is not sufficient. Gauge invariance plays a key role. *Or:* use effective field theory (SCET).
Gauge theory form factors

Consider as an example the **quark form factor**

\[
\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_\mu(0) | p_1, p_2 \rangle = \bar{v}(p_2) \gamma_\mu u(p_1) \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right).
\]

- The form factor obeys the **evolution equation**

\[
Q^2 \frac{\partial}{\partial Q^2} \log \left[ \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = \frac{1}{2} \left[ K(\epsilon, \alpha_s(\mu^2)) + G \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right],
\]

- **Renormalization group invariance** requires

\[
\mu \frac{dG}{d\mu} = -\mu \frac{dK}{d\mu} = \gamma_K(\alpha_s(\mu^2)),
\]

**Note:** \( \gamma_K(\alpha_s) \) is the **cusp anomalous dimension**.

- **Dimensional regularization** provides a trivial initial condition for evolution if \( \epsilon < 0 \) (for \( \text{IR} \) regularization).

\[
\bar{\alpha}(\mu^2 = 0, \epsilon < 0) = 0 \quad \rightarrow \quad \Gamma \left( 0, \alpha_s(\mu^2), \epsilon \right) = \Gamma \left( 1, \bar{\alpha}(0, \epsilon), \epsilon \right) = 1.
\]
Detailed factorization

Operator factorization of the Sudakov form factor, with subtractions.
Operator definitions

The functional form of this graphical factorization is

\[
\Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = C \left( \frac{Q^2}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \times S \left( \beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon \right) \times \prod_{i=1}^{2} \left[ J \left( \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) \right].
\]

We introduced factorization vectors \( n_i^\mu \), with \( n_i^2 \neq 0 \), to define the jets,

\[
J \left( \frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle.
\]

where \( \Phi_n \) is the Wilson line operator along the direction \( n_i^\mu \).

\[
\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A(\lambda n) \right],
\]

The jet \( J \) has collinear divergences only along \( p \).


**Operator definitions**

The *soft function* $S$ is the *eikonal limit* of the massless form factor

$$S \left( \beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon \right) = \langle 0 | \Phi_{\beta_2}(\infty, 0) \Phi_{\beta_1}(0, -\infty) | 0 \rangle .$$

*Soft-collinear* regions are *subtracted* dividing by *eikonal jets* $J$.

$$J \left( \left( \frac{\beta_1 \cdot n_1}{n_1^2} \right)^2, \alpha_s(\mu^2), \epsilon \right) = \langle 0 | \Phi_{n_1}(\infty, 0) \Phi_{\beta_1}(0, -\infty) | 0 \rangle ,$$

- $S$ and $J$ are *pure counterterms* in dimensional regularization.
- $S$ *only* depends on kinematics through the *cusp anomaly*.
- A *single pole* function where the cusp anomaly *cancels* is

$$\overline{S} \left( \rho_{12}, \alpha_s(\mu^2), \epsilon \right) \equiv \frac{S \left( \beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon \right)}{\prod_{i=1}^{2} J \left( \left( \frac{\beta_i \cdot n_i}{n_i^2} \right)^2, \alpha_s(\mu^2), \epsilon \right)}$$

It can *only* depend on the *scaling variable*

$$\rho_{12} \equiv \frac{(-\beta_1 \cdot \beta_2)^2 n_1^2 n_2^2}{(-\beta_1 \cdot n_1)^2 (-\beta_2 \cdot n_2)^2} .$$
Jet evolution

The full form factor does not depend on the factorization vectors $n_i^\mu$. Defining $x_i \equiv (-\beta_i \cdot n_i)^2 / n_i^2$,

$$x_i \frac{\partial}{\partial x_i} \log \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = 0.$$ 

This dictates the evolution of the jet $J$

$$x_i \frac{\partial}{\partial x_i} \log J_i = - x_i \frac{\partial}{\partial x_i} \log C + x_i \frac{\partial}{\partial x_i} \log \bar{J}_i$$

$$\equiv \frac{1}{2} \left[ G_i (x_i, \alpha_s(\mu^2), \epsilon) + \mathcal{K}(\alpha_s(\mu^2), \epsilon) \right],$$

Imposing RG invariance of the form factor

$$\gamma_S (\rho_{12}, \alpha_s) + \gamma_C (\rho_{12}, \alpha_s) + 2 \gamma_J (\alpha_s) = 0.$$ 

leads to the final evolution equation

$$Q \frac{\partial}{\partial Q} \log \Gamma = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log C - \gamma_S - 2 \gamma_J + \sum_{i=1}^{2} (G_i + \mathcal{K}).$$
Collinear evolution

It is useful to establish a connection to conventional collinear factorization. Define a parton-in-parton distribution as

\[ \phi_{q/q}(x, \epsilon) = \frac{1}{4N_c} \int \frac{d\lambda}{2\pi} e^{-i\lambda x p \cdot \beta} \langle p| \psi_q(\lambda \beta) \gamma \cdot \beta \Phi_\beta(\lambda, 0) \psi_q(0)|p \rangle , \]

The virtual contribution can be isolated. At the amplitude level it is

\[ \Gamma_{q/q}(\frac{p \cdot \beta}{\mu}, \alpha_s(\mu^2), \epsilon) \equiv \langle 0|\Phi_\beta(\infty, 0) \psi_q(0)|p \rangle , \]

Comparing factorizations for this amplitude and the jet $J$, and enforcing Altarelli-Parisi evolution one finds

\[ \frac{[J \left( \frac{(\beta p \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon \right)]_{\text{pole}}}{J \left( \frac{(\beta p \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon \right)} = \frac{\Gamma_{q/q}(\beta p \cdot \beta, \alpha_s(\mu^2), \epsilon)}{\Gamma_{q/q}(\beta p \cdot \beta, \alpha_s(\mu^2), \epsilon)} \]

\[ = \exp \left[ \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} B^{[q]}_\delta \left( \overline{\alpha}(\xi^2, \epsilon) \right) \right] . \]
Results for Sudakov form factors

- In dimensional regularization \((\epsilon < 0)\) one has the boundary value \(\Gamma(0, \epsilon) = 1\). Then

\[
\log \left[ \Gamma \left( Q^2, \epsilon \right) \right] = \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[ K(\epsilon) + G(\bar{\alpha}(\xi^2), \epsilon) + \frac{1}{2} \int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K(\bar{\alpha}(\lambda^2)) \right]
\]

- The functions \(K\) and \(\gamma_K\) are not independent

\[
\mu \frac{d}{d\mu} K(\epsilon, \alpha_s) = -\gamma_K(\alpha_s) \implies K(\epsilon, \alpha_s(\mu^2)) = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K(\bar{\alpha}(\lambda^2, \epsilon)).
\]

- The form factor can be written in terms of just \(G\) and \(\gamma_K\),

\[
\Gamma \left( Q^2, \epsilon \right) = \exp \left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[ G(-1, \bar{\alpha}(\xi^2, \epsilon), \epsilon) \right. \\
- \left. \frac{1}{2} \gamma_K(\bar{\alpha}(\xi^2, \epsilon)) \log \left( \frac{-Q^2}{\xi^2} \right) \right] \right\}.
\]
**Implications**

- The exponent is *not affected* by the *Landau pole* for $\epsilon < 0$. $\Gamma$ is an *analytic function* of the coupling and $\epsilon$. At *one loop* in QCD

$$\log \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \log \Gamma \left( 1, \alpha_s(Q^2), \epsilon \right)$$

$$= -\frac{1}{b_0} \left\{ \frac{\gamma_{K}^{(1)}}{\epsilon} \right. \text{Li}_2 \left( a(Q^2) + \frac{a(Q^2)}{a(Q^2) + \epsilon} \right) + 2 G^{(1)}(\epsilon) \log \left[ 1 + \frac{a(Q^2)}{\epsilon} \right] \left\} \right.$$  

- The *ratio* of the *timelike* to the *spacelike* form factor admits a simple representation

$$\log \left[ \frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)} \right] = i \frac{\pi}{2} K(\epsilon) + \frac{i}{2} \int_0^\pi \left[ G \left( \alpha \left( e^{i\theta} Q^2 \right), \epsilon \right) - \frac{i}{2} \int_0^\theta d\phi \gamma_K \left( \alpha \left( e^{i\phi} Q^2 \right) \right) \right]$$

which is *physically relevant* for resummed EW annihilation processes.
Form factors in $\mathcal{N} = 4$ SYM

- In $d = 4 - 2\epsilon$ conformal invariance is \textit{broken} and $\beta(\alpha_s) = -2\epsilon\alpha_s$.

- All integrations are trivial. The exponent has \textit{only double} and \textit{single} poles to \textit{all orders}.

\[
\log \left[ \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n \left( \frac{\mu^2}{-Q^2} \right)^{n\epsilon} \left[ \frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right] \\
= -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\alpha_s(Q^2)}{\pi} \right)^n e^{-i\pi n\epsilon} \left[ \frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right],
\]

- In the \textit{planar limit} this captures \textit{all singularities} of fixed-angle amplitudes in $\mathcal{N} = 4$ SYM.

- The \textit{analytic continuation} yields a \textit{finite} result in \textit{four dimensions}, arguably \textit{exact}.

\[
\left| \frac{\Gamma(Q^2)}{\Gamma(-Q^2)} \right|^2 = \exp \left[ \frac{\pi^2}{4} \gamma_K \left( \alpha_s(Q^2) \right) \right].
\]
Characterizing $G(\alpha_s, \epsilon)$

The *single pole* function $G(\alpha_s, \epsilon)$ is a sum of *anomalous dimensions*

$$G(\alpha_s) = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log C - \gamma \bar{S} - 2 \gamma J + \sum_{i=1}^{2} G_i ,$$

In $d = 4 - 2\epsilon$ *finite remainders* can be *neatly exponentiated*

$$C(\alpha_s(Q^2), \epsilon) = \exp \left[ \int_0^{Q^2} \frac{d \xi^2}{\xi^2} \left\{ \frac{d \log C(\bar{\alpha}(\xi^2, \epsilon), \epsilon)}{d \ln \xi^2} \right\} \right] \equiv \exp \left[ \frac{1}{2} \int_0^{Q^2} \frac{d \xi^2}{\xi^2} G_C(\bar{\alpha}(\xi^2, \epsilon), \epsilon) \right]$$

The *soft function* exponentiates *like* the full form factor

$$S(\alpha_s(\mu^2), \epsilon) = \exp \left\{ \frac{1}{2} \int_0^{\mu^2} \frac{d \xi^2}{\xi^2} \left[ G_{eik}(\bar{\alpha}(\xi^2, \epsilon)) - \frac{1}{2} \gamma_K(\bar{\alpha}(\xi^2, \epsilon)) \log \left( \frac{\mu^2}{\xi^2} \right) \right] \right\} .$$

$G(\alpha_s, \epsilon)$ is then *simply related* to *collinear splitting functions* and to the *eikonal approximation*

$$G(\alpha_s, \epsilon) = 2 B_\delta(\alpha_s) + G_{eik}(\alpha_s) + G_C(\alpha_s, \epsilon) ,$$
Single logarithms in resummation

A deeper characterization of infrared and collinear single poles in amplitudes should be reflected in single logarithms in cross sections.

For a resummation as in the Drell-Yan cross section

\[
\tilde{\omega}_{\overline{\text{MS}}} (N) = \left| \frac{\Gamma(Q^2, \epsilon)}{\phi_V(Q^2, \epsilon)} \right|^2 \exp \left[ F_{\text{DY}}(\alpha_s) + \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \right. \\
\left. \left\{ 2 \int_Q^{(1-z)Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + D(\alpha_s((1-z)^2Q^2)) \right\} \right],
\]

the function \( D(\alpha_s) \) should be related to \( B_\delta(\alpha_s) \) and to \( G(\alpha_s) \). Indeed,

\[
A(\alpha_s) = \gamma K(\alpha_s)/2, \\
D(\alpha_s) = 4 B_\delta(\alpha_s) - 2 \tilde{G}(\alpha_s) + \hat{\beta}(\alpha_s) \frac{d}{d\alpha_s} F_{\text{DY}}(\alpha_s), \\
B(\alpha_s) = B_\delta(\alpha_s) - \tilde{G}(\alpha_s) + \hat{\beta}(\alpha_s) \frac{d}{d\alpha_s} F_{\text{DIS}}(\alpha_s),
\]

where \( B(\alpha_s) \) is the single log function for DIS resummation.
Perspective

• The all-order analysis of infrared and collinear divergences in gauge theories has a long history.

• This history is entering a new phase
  
  • New motivations from LHC phenomenology
  • New (non)-perturbative input from $\mathcal{N} = 4$ SYM.

• We are beginning to unravel all-order structure in the resummed exponent.

• Exact results are in sight in $\mathcal{N} = 4$ SYM.

• For infrared and collinear divergences in fixed-angle massless gauge theory amplitudes only three functions play a role: $\gamma_K(\alpha_s)$, $G_{eik}(\alpha_s)$ and $B_5(\alpha_s)$, possibly even beyond the planar limit.