Numerical Methods for Partial Differential Equations

Lecture 2: Analytical Solutions

Lecture Outline

- Linear Equations & simple geometries;
- Three kind of equations: Hyperbolic, Parabolic & Elliptic
- Wave equation, diffusion equation, Laplace & Poisson Equations
- Differences between the solutions for the three kinds, in particular Wave equation and diffusion equation
- Methods of solutions: separation of variables, Fourier series, Green functions

Analytical solutions: Hyperbolic PDE

Wave Equation

- Consider the wave PDE $u_{tt} = c^2 u_{xx}$ in $-\infty < x < \infty$
- Split it as $$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$
- Characteristic coordinates: $\xi = x - \mathrm{ct},\ \eta = x + ct,\ u_{\xi\eta} = 0$
- Characteristic curves: $u(x,t) = f(x+ct) + g(x-ct)$

Characteristic curves

- Initial conditions: $u(x,0) = \phi(x),\quad u_t(x,0) = \psi(x)$
- Since we expect a unique solution, we need to specify f and g in accordance with the I.C.

Wave Equation

- Doing the math:

$$\phi(x) = f(x) + g(x)$$

$$\frac{1}{c}\psi(x) = f'(x) - g'(x)$$

$$\phi' = f' + g'$$

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right)$$

$$g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right)$$

- Integrating

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A$$

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B$$

$$A + B = 0$$

(A & B constants)

$$f(s) \rightarrow f(x + ct)$$

$$g(s) \rightarrow g(x - ct)$$

$$u(x,t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

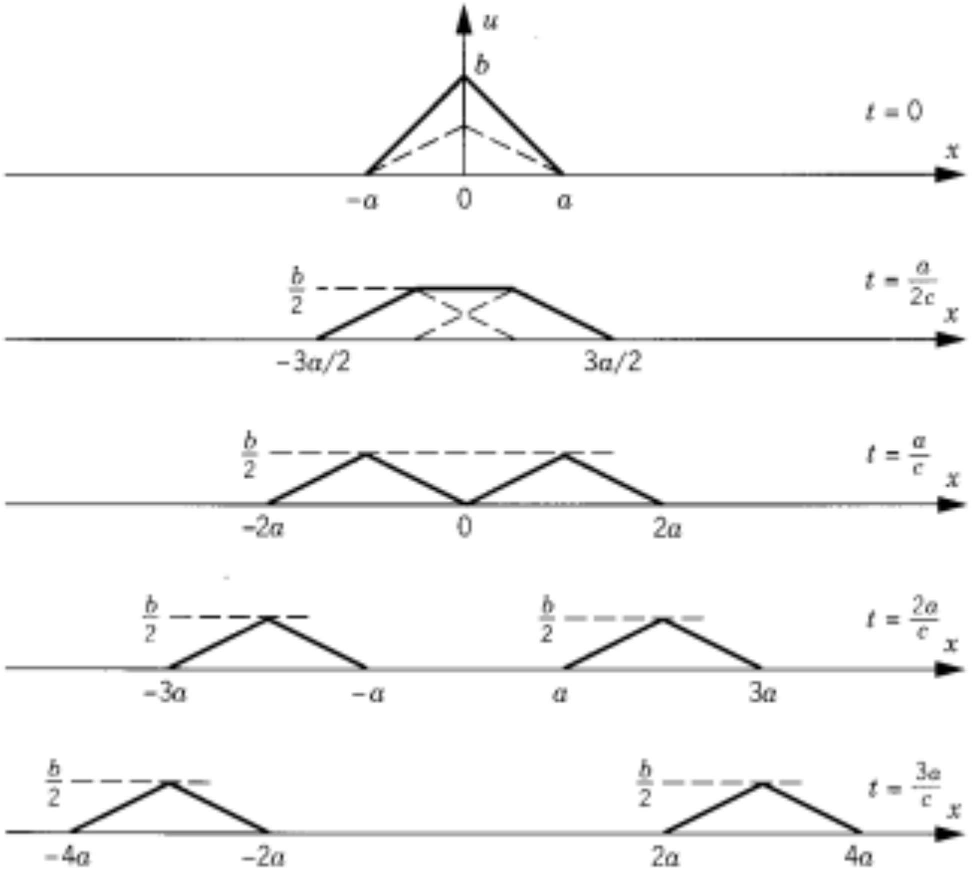
Example #1

- Two triangular functions one moving to the right and the other to the left

$$u(x,0) = \begin{cases} b - \dfrac{b|x|}{a} & for \quad |x| < a \\ 0 & for \quad |x| > a \end{cases}$$

$$u_t(x,0) = 0$$

$$\rightarrow \; u(x,t) = \; \ldots \qquad \text{(Try yourself !)}$$

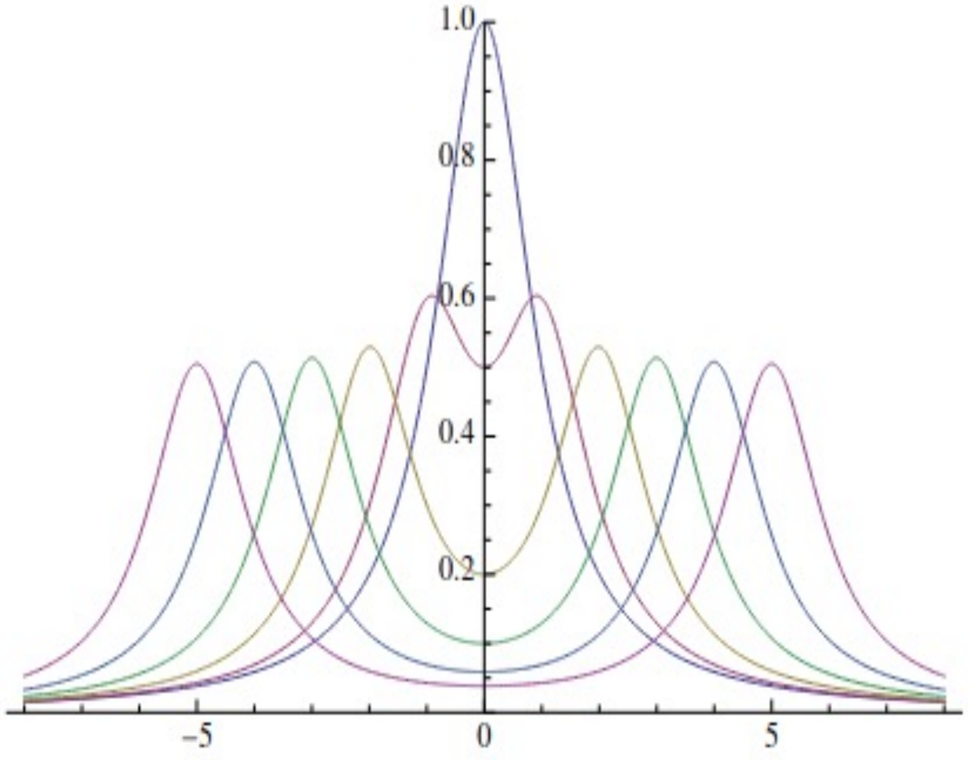


Example #2

- Find the analytical solution of the wave equation, given the initial condition

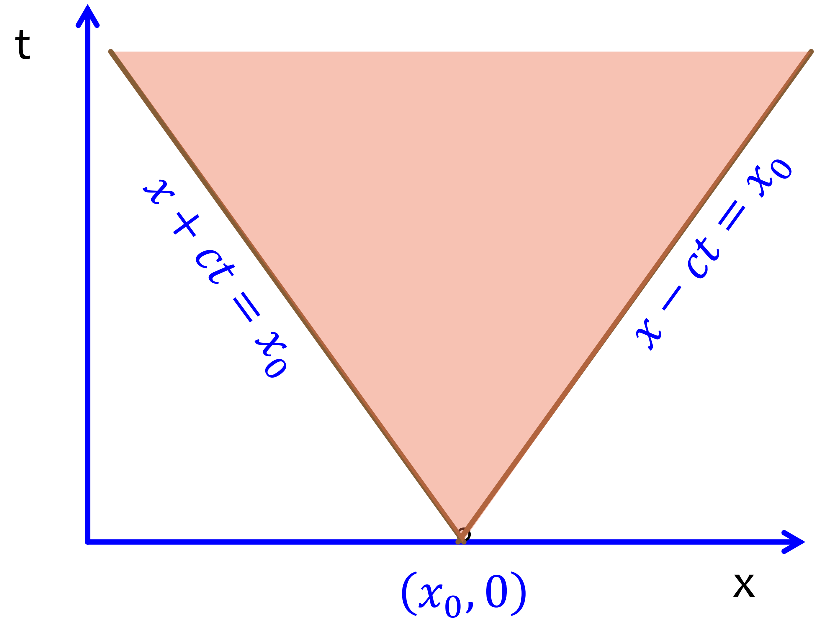
$$u(x,0) = \frac{1}{1+x^2}$$

$$u_t(x,0) = 0$$



Causality

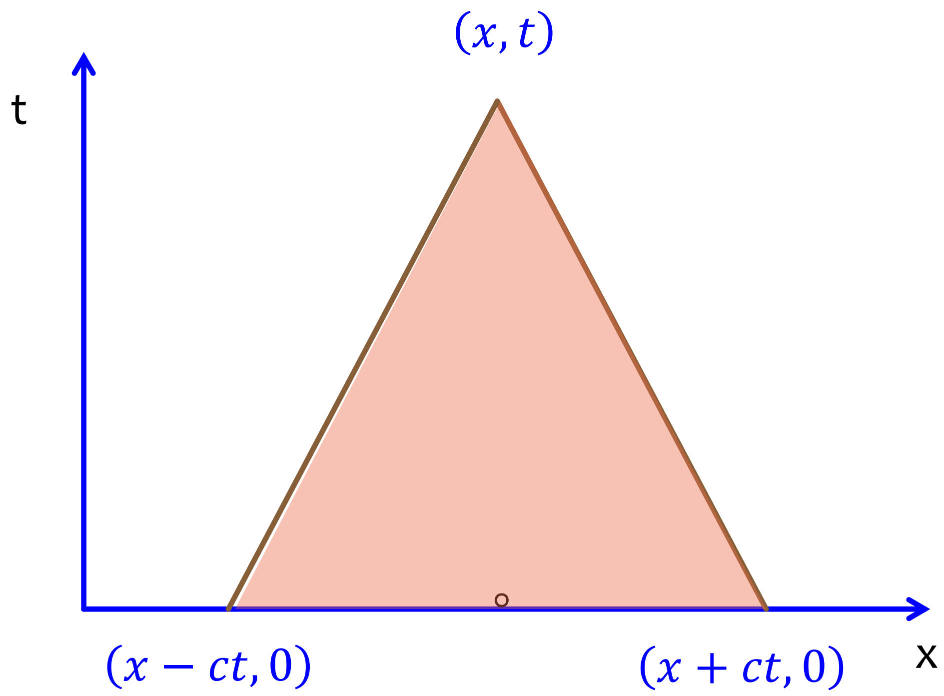
- The effect of an initial pulse is a pair of waves traveling in either direction at speed c and at half the original amplitude.
- The effect of an initial velocity is a wave spreading out at speed $\leq c$.
- Part of the wave may lag behind (if there is an initial velocity) but no part goes faster than c.



An initial condition at the point $(x_0, 0)$ can affect the solution for $t > 0$ only in the shaded sector, which is called the *domain of influence*

Domain of Dependence

- Fix a point (x, t) for $t > 0$.
- The value $u(x, t)$ depends on $u(x - ct, 0)$ and $u(x + c, 0)$ and u_t in interval $[x - ct, x + ct]$ at $t = 0$



$[x - ct, x + ct]$ is the *interval of dependence*

The entire shaded area is called the *domain of dependence* or the *past history* of the point (x, t)

Conservation of Energy

$$KE = \frac{1}{2}\int u_t^{\,2}\,dx \quad\longrightarrow\quad \frac{d\,KE}{dt} = \int u_t u_{tt} dx$$

$$u_{tt} = c^2 u_{xx} \quad\longrightarrow\quad \frac{d\,KE}{dt} = c^2 \int u_t u_{xx} dx$$

integrate by parts

$$\frac{d\,KE}{dt} = c^2 \int u_t u_{xx} dx \quad\longrightarrow\quad \frac{d\,KE}{dt} = c^2 u_t u_x - c^2 \int u_{tx} u_x dx$$

Vanishes because $u \to 0$ as $|x| \to \infty$

$$\frac{d\,KE}{dt} = -c^2 \int u_{tx} u_x dx = -c^2 \frac{d}{dt}\int \frac{1}{2} u_x^{\,2} dx$$

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (u_t^{\,2} + c^2 u_x^{\,2}) \quad\longrightarrow\quad \frac{dE}{dt} = 0$$

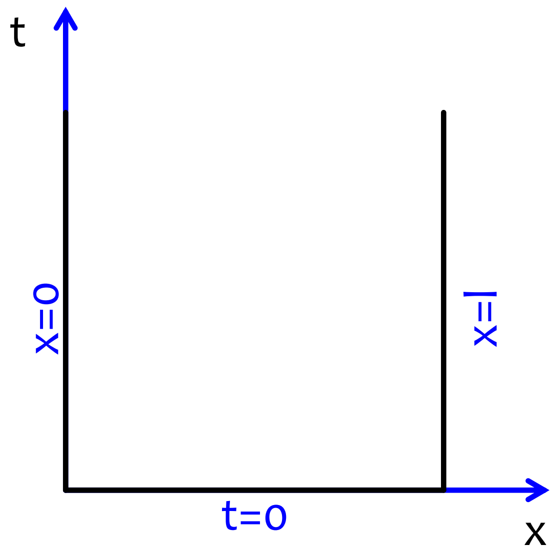
Conservation of energy is one of the most basic facts of the wave equation

Analytical solutions: Parabolic PDE

Diffusion Equation: the Maximum Principle

$$u_t = ku_{xx}$$

- Properties of the solutions very different with respect to the wave equation;
- Maximum principle: if $u(x,t)$ satisfies the diffusion equation in a rectangle $0 \le x \le l,\ \ 0 \le t \le T$ in space-time, then the maximum value of $u(x,t)$ is assumed either initially $(t = 0)$ or on the lateral sides $(x = 0 \quad or \quad x = l)$.
- The minimum value has the same property.
- Rod with non internal heat sources, the hottest spot and the coldest spot can occur only initially or at one of the two ends.



- The differential equation tend to smooth the solution.

Diffusion on the whole line

$$u_t = ku_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

$$u(x,0) = \varphi(x)$$

- Our method is to solve it for a particular $\varphi(x)$ and then build the general solution from this particular one.

- The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the special initial condition

$$Q(x,0) = 1 \qquad for \qquad x > 0$$

$$Q(x,0) = 0 \qquad for \qquad x < 0$$

- The reason for this choice is that this initial condition does not change under dilation.

Solution behavior under dilation

- If $u(x,t)$ is a solution, so is the dilated function $u(\sqrt{a}x, at)$

$$v(x,t) = u(\sqrt{a}x, at) \longrightarrow \begin{array}{l} v_t = au_t \\ v_{xx} = au_{xx} \end{array}$$

- We'll look for $Q(x, t)$ of the special form $Q(x,t) = g(p) \quad with \quad p = \dfrac{x}{\sqrt{4kt}}$

- This combination makes Q invariant under dilation, self-similar solution

- We can then convert our PDE into an ODE:

$$g'' + 2pg' = 0 \qquad Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} \exp(-p^2)\ dp\ +\ c_2$$

Solution behavior under dilation

- The constants c_1 and c_2 are determined by the initial conditions

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} \exp(-p^2)\ dp$$

- This solution is valid only for $t > 0$.
- Define $S = \frac{\partial Q}{\partial x}$. Then S is also a solution, but it does not satisfy our initial condition;
- What is the initial condition for S ? → A δ function
- Given any function φ, we also define

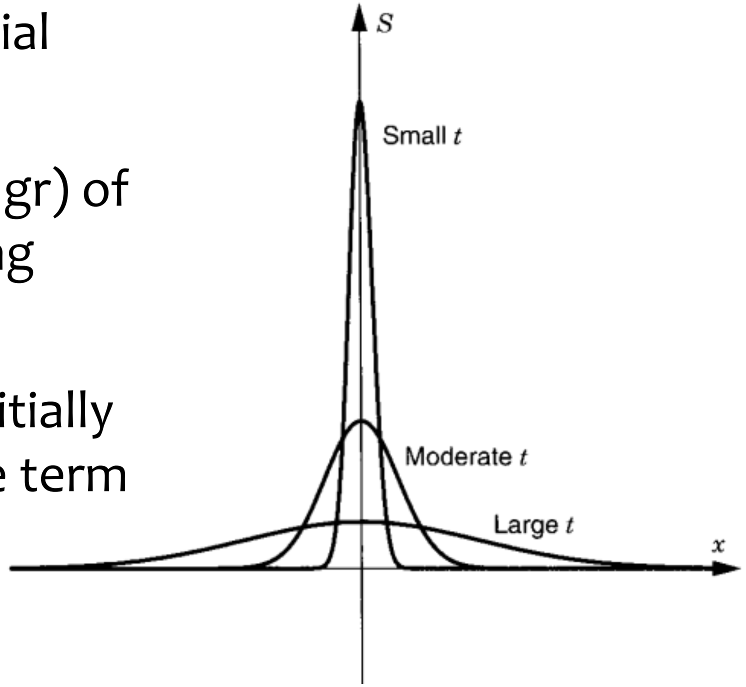
$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\,\varphi(y)dy \qquad for \qquad t > 0$$

- Where *u* is another solution that satisfies the initial condition $u(x,0) = \varphi(x)$

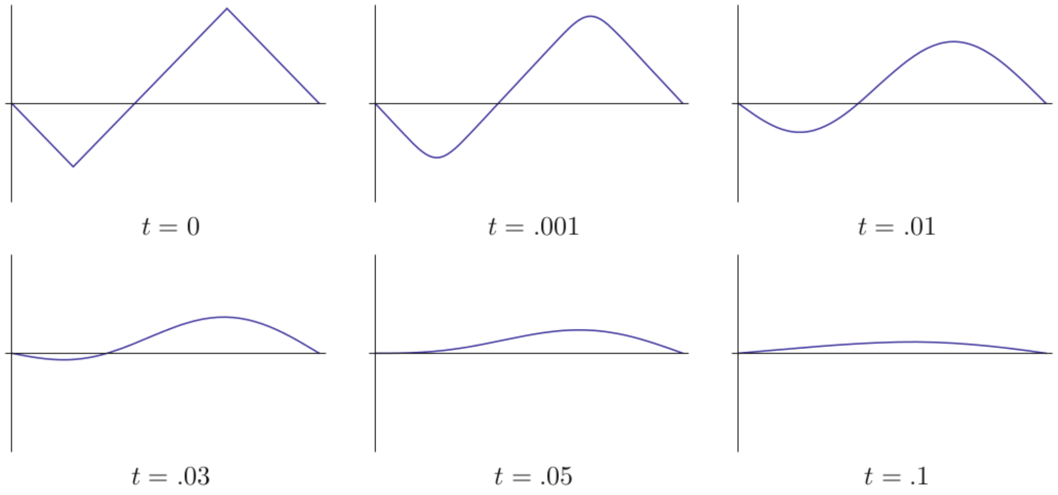
Solution behavior under dilation

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}}\, e^{-x^2/4kt} \quad for \quad t > 0 \qquad\qquad u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi(y)\, dy$$

- S is called Green function or source function
- The area under the curves are constant in time
- S(x – y, t) represents the result of a "hot spot" at y at time t = 0.
- The value of the solution $u(x, t)$ is a kind of weighted average of the initial values around the point x.
- Consider diffusion. S(x – y, t) represents the result of a unit mass (say, 1 gr) of substance located at time zero exactly at the position y which is diffusing (spreading out) as time advances.
- For any initial distribution of concentration, the amount of substance initially in the interval δy spreads out in time and contributes approximately the term S(x – y , t)φ(y) δy .

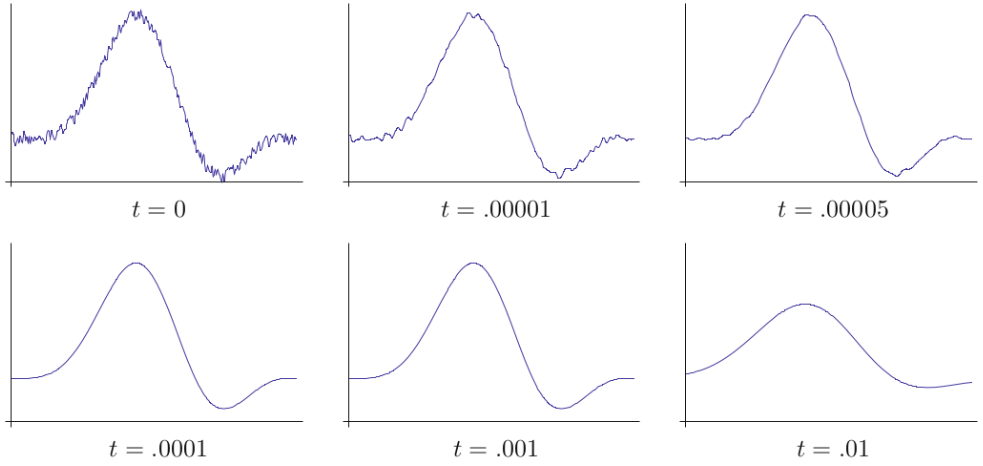


Diffusion Eq: Solutions



Diffusion tends to smooth out singularities

Short wavelengths disappear faster



Comparison Between Wave and Diffusion equations

- The wave equation has no maximum principle (it transfer information along the characteristic curves)
- Singularities:
  - wave equation: transported along characteristics
  - diffusion equation: lost immediately (the solution is differentiable at all orders even if the initial data are not) for $t > 0$ S is a Gaussian.
- The value of u(x, t) depends on the values of the initial datum φ(y) for all y, where −∞ < y < ∞.
- Conversely, the value of φ at a point x_0 has an immediate effect everywhere (for t > 0), even though most of its effect is only for a short time** near x_0.
- The energy decays to zero (if φ integrable)
- → Infinite speed of propagation of information
- Irreversibility, information is lost

Wave Eq: Boundary Problems**

- We now consider solutions for the wave equation on a finite interval $0 < x < l$.
- Homogeneous Dirichlet conditions for the wave equation

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad 0 < x < l$$

Initial conditions: $u(x,0) = \varphi(x)$ $\quad u_t(x,0) = \psi(x)$

Boundary Conditions: $u(0,t) = 0$ $\quad u(l,t) = 0$

- The method we shall use consists of building up the general solution as a linear combination of special ones that are easy to find.
- We look for solutions of the form $u(x,t) = X(x)T(t)$
- Plugging into the wave equation, we get $X(x)T''(t) = c^2 T(t)X''(x)$, dividing by $-c^2XT$ we get:

$$-\frac{T''(t)}{c^2T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

here λ has to be a constant

Wave Eq: Boundary Problems**

- We will see that $\lambda > 0$ then we let $\lambda = \beta^2$. Then the equations above are a pair of separate ordinary differential equations for X(x) and T(t):

eigenvalues

$$X'' + \beta^2 X = 0 \qquad\qquad T'' + c^2\beta^2 T = 0$$

- These ODEs are easy to solve. The solutions have the form

Eigenfunctions, normal modes

$$X(x) = C\cos(\beta x) + D\sin(\beta x) \qquad\qquad \mathrm{T}(x) = A\cos(c\beta t) + B\sin(c\beta t)$$

where A, B, C, and D are constants.

- The second step is to impose the boundary conditions on the separated solution. They simply require that $X(0) = X(l) = 0''$. Thus

$$C = 0 \qquad D\sin(\beta l) = 0 \longrightarrow \beta l = n\pi \qquad \lambda_n = \frac{n\pi}{l}$$

not all wavelengths are allowed

We have an infinite number of solutions $X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$

Wave Eq: Boundary Problems**

- Putting together with the time dependence $u(x,t) = \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}$

- The sum of solutions is again a solution (superposition principle)

$$u(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}$$

- We can now impose the initial conditions

$$\varphi(x) = u(x,0) = \sum_n A_n \sin\frac{n\pi x}{l} \qquad \psi(x) = u_t(x,0) = \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}$$

Fourier series

$\frac{n\pi ct}{l}$ are the frequencies

Wave Eq: Boundary Problems**

- Why are all the eigenvalues of this problem positive?

$\lambda = 0 \longrightarrow X'' = 0 \quad X = C + Dx$ cannot satisfy the BC

$\lambda < 0 \longrightarrow X'' - \gamma^2 X = 0 \quad X = C\cosh\gamma x + D\sinh\gamma x$

cannot again satisfy the BC

Diffusion Eq: Boundary Problems

$$u_t = ku_{xx} \quad \text{for} \quad 0 < x < l$$

- Consider

Initial Condition: $u(x,0) = \varphi(x)$

Boundary Condition: $u(0,t) = 0 \qquad u(l,t) = 0$

- Using variable separation

$$u(x,t) = X(x)T(t) \longrightarrow \begin{matrix} X'' + \lambda X = 0 \\ T' = -\lambda kT \end{matrix}$$

- This is precisely the same problem for X(x) as before and so has the same solutions.
- For T(t) we have instead an exponential solution

$$u(x,t) = \sum_n A_n exp(-(n\pi/l)^2 kt) \sin\frac{n\pi x}{l}$$

wavenumber

Diffusion Eq: Boundary Problems

- For the initial conditions we have

$$\varphi(x) = u(x,0) = \sum_n A_n \sin\frac{n\pi x}{l}$$

- Each normal mode has an exponential decay
- The decay rate depends on the square of the wavenumber.

- For example, consider the diffusion of a substance in a tube of length l. each end of the tube opens up into a very large empty vessel. So the concentration $u(x, t)$ at each end is essentially zero. Given an initial concentration $\varphi(x)$ in the tube, the concentration at all later times is given by formula above.
- Notice that as $t \to \infty$, each term in the sumgoes to zero. Thus the substance gradually empties out into the two vessels and less and less remains in the tube

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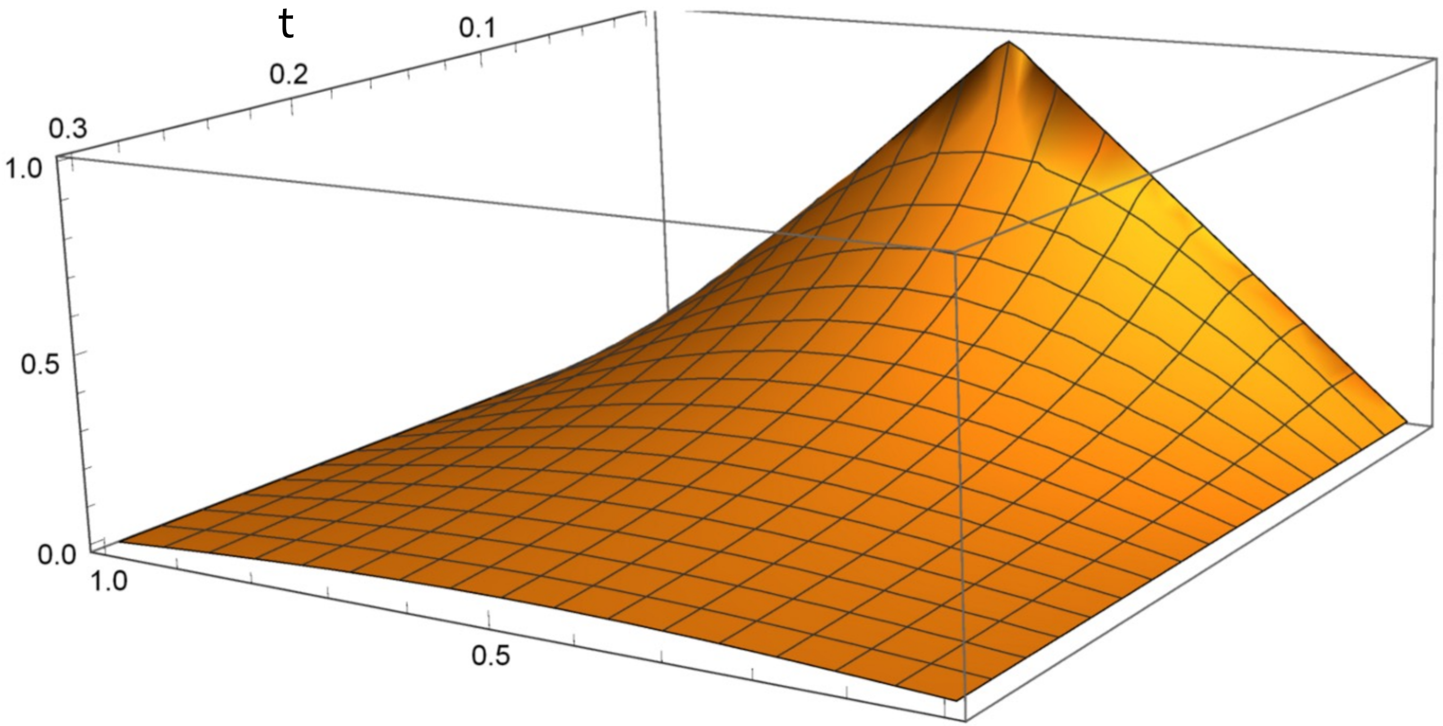


Initial triangular temperature distribution
T fixed at the two boundaries

Neumann b.c.

- Zero derivatives at both boundaries
- Eigenfunctions are cosines, the eigenvalue $\lambda = 0$ is allowed

$$u(x,t) = \frac{1}{2}A_0 + \sum_n A_n exp(-(n\pi/l)^2 kt) \cos\frac{n\pi x}{l}$$

$$\varphi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_n A_n \cos\frac{n\pi x}{l}$$

- Eigenvalue $\lambda = 0$
- What is the behavior of $u(x, t)$ as $t \rightarrow +\infty$? → Since all but the first term contains an exponentially decaying factor, the solution decays quite fast to the first constant term.
- Since these boundary conditions correspond to insulation at both ends, this agrees perfectly with our intuition.

Diffusion with a source

$$u_t - ku_{xx} = f(x,t) \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

$$u(x,0) = \varphi(x)$$

- For instance, if u(x, t) represents the temperature of a rod, then φ(x) is the initial temperature distribution and f (x, t) is a source (or sink) of heat provided to the rod at later times
- Solution to the homogeneous equation

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} S(x-y,t)\, \varphi(y) dy \qquad S(x-y,t) = exp(-(x-y)^2/4kt)$$

- We have to add a term that takes into account the source term

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} S(x-y,t)\, \varphi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s) dy ds$$

Diffusion with a source

- Proof:

$$\frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)dyds = \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s)\, f(y,s)dyds +$$

$$+\lim_{s\to t}\int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)dy = \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)\, f(y,s)dyds +$$

$$+\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} S(x-y,\varepsilon)\ f(y,t)dy = k\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

This identity is exactly the PDE

Wave Equation in 2 Spatial Dimensions

Wave Equation in 2D

- Wave equation in two spatial dimension

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) \qquad R = \{0 < x < a, \quad 0 < y < b\}$$

$$u(t, x = 0, y) = 0 \qquad u(t, x = a, y) = 0$$
$$u(t, x, y = 0) = 0 \qquad u(t, x, y = b) = 0$$

- Using Separation of variables: $u(t, x, y) = X(x)Y(y)T(t)$ $\qquad 0 = -\frac{1}{c^2 T}\frac{d^2T}{dt^2} + \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$

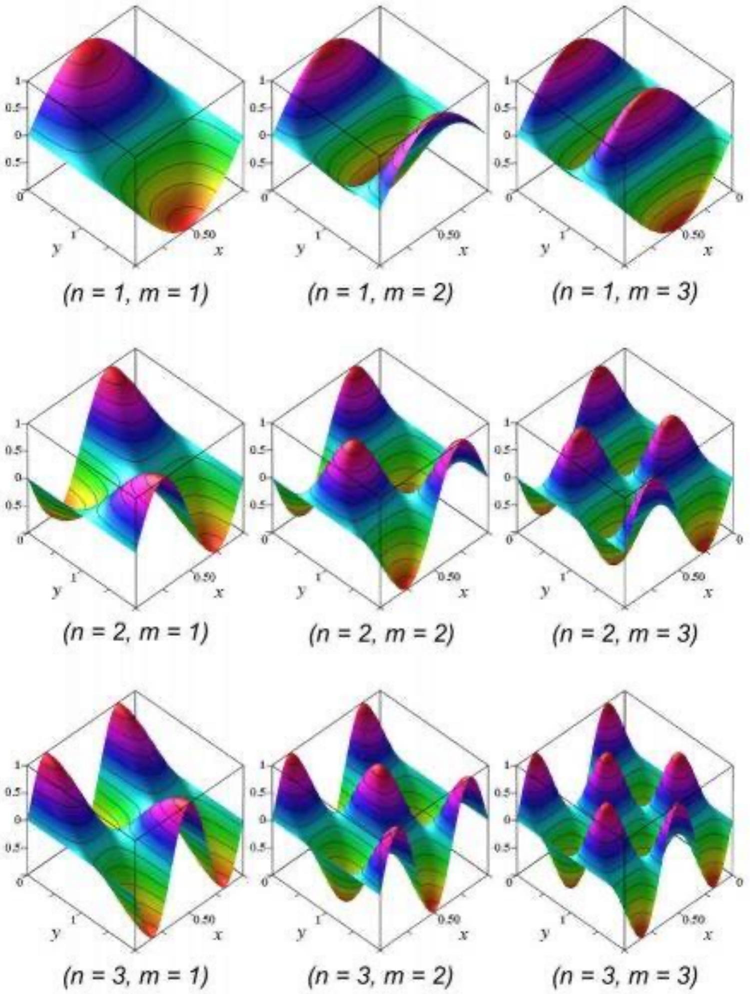
- Each term must be a constant: $\frac{1}{X}\frac{d^2X}{dx^2} = -{k_x}^2 \qquad \frac{1}{Y}\frac{d^2Y}{dx^2} = -{k_y}^2$

$$\frac{1}{T}\frac{d^2T}{dt^2} = -c^2({k_x}^2 + {k_y}^2)$$

- Solutions are sines and cosines

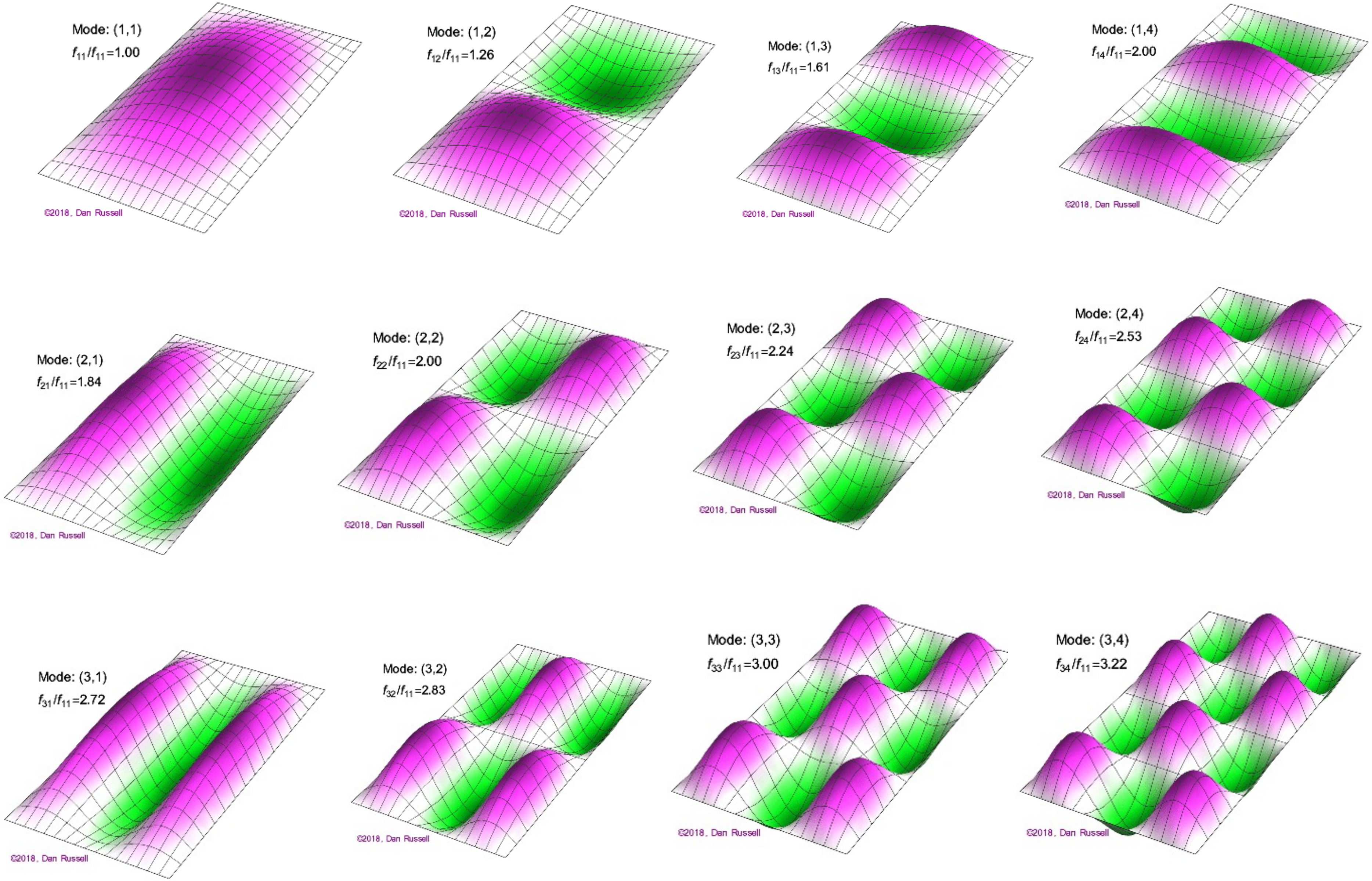
Wave Equation in 2D

- Imposing the BC: $$u_{n,m}(t,x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\begin{Bmatrix}\sin\omega_{nm}t\\ \cos\omega_{nm}t\end{Bmatrix}$$ $$\omega_{nm} = \pi c\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$$



- The functions $u_{nm}(t,x,y)$ are the normal modes of vibration of the rectangular membrane, and ω_{nm} are the membrane's natural frequencies .

- Modes of vibration of a rectangular membrane with a=1 and b=2 , plotted as functions of x and y.
- The modes are labelled with two positive Integers, n and m and they are evaluated at t=0.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f_{11}/f_{11}=1.00
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Mode: (1,2)
f_{12}/f_{11}=1.26
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Mode: (1,3)
f_{13}/f_{11}=1.61
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Mode: (1,4)
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General Solution

- General solution: $$u(t,x,y) = \sum_n \sum_m \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}(a_n \sin\omega_{nm}t + b_n \cos\omega_{nm}t)$$

The coefficients a_{nm} and b_{nm} are determined by the initial conditions.

$$u(0,x,y) = f(x,y) = \sum_n \sum_m b_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$u_t(0,x,y) = g(x,y) = \sum_n \sum_m a_{nm}\omega_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$b_{nm} = \frac{4}{ab}\int_0^a \int_0^b f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$$

$$a_{nm} = \frac{4}{ab\omega_{nm}}\int_0^a \int_0^b g(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$$

Elliptic Equations in 2 Spatial Dimensions

Laplace & Poisson Equation in 2D

$$u_{xx} + u_{yy} = 0 \qquad \text{or} \qquad u_{xx} + u_{yy} = f(x, y)$$

- A real-valued solution *u(x, y)* to the Laplace equation is known as a *harmonic function*.
- Besides their theoretical importance, the Laplace and Poisson equations arise as the basic equilibrium equations in a remarkable variety of physical systems.
- For example, we may interpret u(x, y) as the displacement of a membrane; the inhomogeneity f (x, y) in the Poisson equation represents an external forcing over the surface of the membrane.
- Another example is in the thermal equilibrium of flat plates; here u(x, y) represents the temperature and f (x, y) an external heat source.
- In fluid mechanics, u(x, y) represents the potential function whose gradient $\boldsymbol{v} = \nabla u$ is the velocity vector field of a steady planar fluid flow.
- Similar considerations apply to two-dimensional electrostatic and gravitational potentials.
- Since both the Laplace and Poisson equations describe equilibrium configurations, theyalmost always appear the context of boundary value problems.

Laplace & Poisson Equation in 2D

- We seek a solution $u(x, y)$ to the partial differential equation defined at points (x, y) belonging to a bounded, open domain . The solution is required to satisfy suitable conditions on the boundary of the domain, denoted by $\partial\Omega$.
- Our approach will be based on the method of separation of variables $u(x, t) = v(x)w(y)$

$$v''(x) = +\lambda v(x),$$

$$w''(y) = -\lambda w(y)$$

λ	$v(x)$	$w(y)$	$u(x, y) = v(x)\, w(y)$
$\lambda = -\omega^2 < 0$	$\cos\omega x,\ \sin\omega x$	$e^{-\omega y},\ e^{\omega y},$	$e^{\omega y}\cos\omega x,\quad e^{\omega y}\sin\omega x,$ $e^{-\omega y}\cos\omega x,\quad e^{-\omega y}\sin\omega x$
$\lambda = 0$	$1,\ x$	$1,\ y$	$1,\ x,\ y,\ xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x},\ e^{\omega x}$	$\cos\omega y,\ \sin\omega y$	$e^{\omega x}\cos\omega y,\quad e^{\omega x}\sin\omega y,$ $e^{-\omega x}\cos\omega y,\quad e^{-\omega x}\sin\omega y$

- Linear combinations of solutions are still solutions ← Superposition principle.
- We use this property to satisfy boundary conditions.

Laplace & Poisson Equation in 2D

- Not so easy. The only bounded domains on which we can explicitly solve boundary value problems using the preceding separable solutions are rectangles

$$\Delta u = 0 \quad \text{on a rectangle} \quad R = \{0 < x < a, \quad 0 < y < b\}$$

- We use Dirichlet boundary conditions:

$$\longrightarrow \qquad u(x,0) = f(x) \qquad u(x,b) = 0 \qquad u(0,y) = 0 \qquad u(a,y) = 0$$

$$v(0) = 0 \qquad v(a) = 0 \qquad w(b) = 0$$

So that:

$$v(x) = \begin{cases} \sin \omega x, & \lambda = -\omega^2 < 0, \\ x, & \lambda = 0, \\ \sinh \omega x, & \lambda = \omega^2 > 0, \end{cases}$$

Laplace & Poisson Equation in 2D

- The 2nd and 3rd cases cannot satisfy the second boundary condition
- 1st case: $v(a) = \sin \omega a = 0$ $\qquad \omega a = n\pi$

$$w(y) = c_1 exp(\omega y) + c_2 exp(-\omega y) \qquad \lambda_n = -\omega^2 = -\frac{n^2\pi^2}{a^2}$$

- 3rd boundary condition: $w(b) = 0 \rightarrow w_n = \sinh \frac{n\pi(b-y)}{a}$
- The complete solution is: $u_n(x, y) = \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$
- In order to satisfy the inhomogeneous BC we consider the infinite series

$$u(x, y) = \sum_n c_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$$

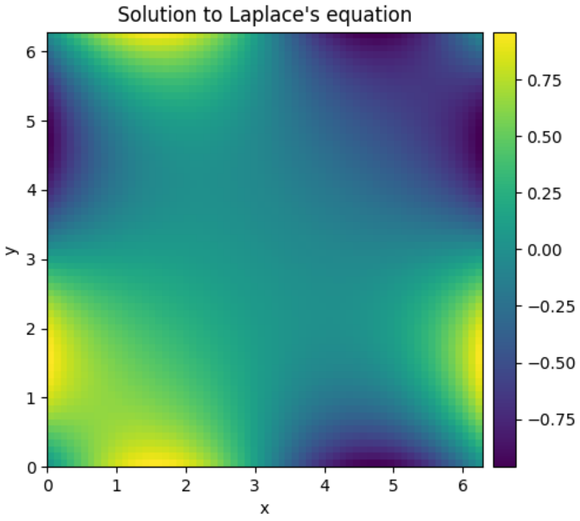
- At the bottom edge

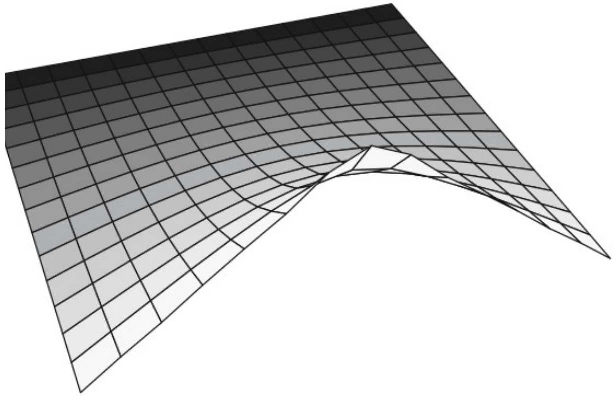
$$u(x, 0) = f(x) = \sum_n c_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

Fourier sine series

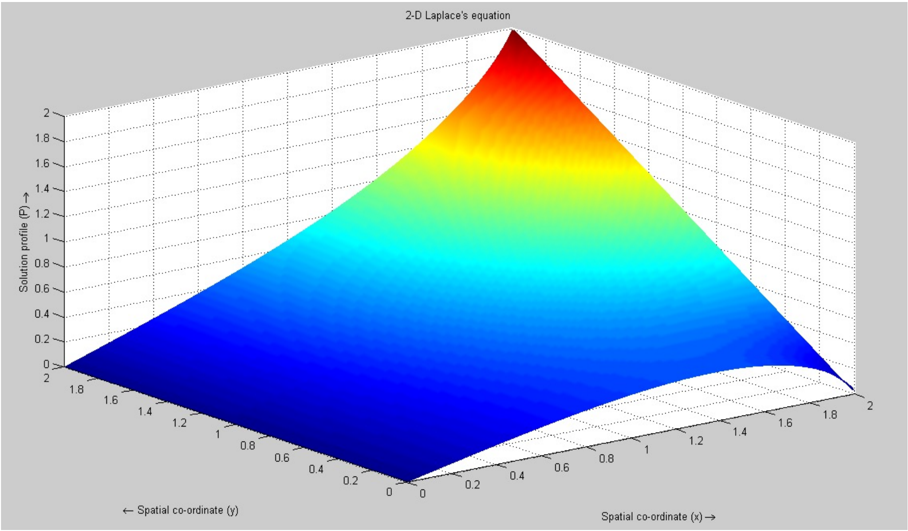
Example of Solutions

- Maximum principle: the highest and lowest points are necessarily on the boundary of the domain





- This reconfirms our physical intuition: if we think of an elastic membrane, the restoring force exerted by the stretched membrane will serve to flatten any bump, and hence a membrane with a local maximum or minimum cannot be in equilibrium.



Properties of Harmonic Functions

- Mean value theorem: the value of a harmonic function at a point is equal to its average value over circles or spheres centered at that point.

- Minimization of energy: An harmonic function minimizes the quantity $E = \frac{1}{2}\int_{\Omega} |\nabla u|^2 dV$ between all function in Ω that satisfies the same BC on $\partial\Omega$.

- Laplace equation typically describes equilibrium problems, at equilibrium energy is minimized.

Green function for the 3D Poisson Equation

- Consider $\nabla^2 V = -f$
- Green function satisfies $\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}')$
- Formal solution: $V(\boldsymbol{r}) = \int G(\boldsymbol{r}, \boldsymbol{r}') f(\boldsymbol{r}') dV'$
- Take the vector field $v = -\nabla r^{-1} = \frac{\hat{\boldsymbol{r}}}{r^2}$
- This has zero divergence everywhere except at r=0. Therefore $\Delta \frac{1}{r} = 0$ everywhere except r = 0

Green function for the 3D Poisson Equation

$$\nabla^2 \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}'), \qquad G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi}\int \frac{f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, dV',$$

- These results take a familiar turn, and they can be given a compelling physical interpretation
- Green's equation is identical to Poisson's equation for a point charge of unit strength
- To obtain the potential *V* of a distribution of charge density *f*, we invoke the superposition principle